Optimal Monetary Policy with Heterogeneous Firms*

Beatriz González, Galo Nuño, Dominik Thaler, Silvia Albrizio†
Banco de España

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Abstract

We analyze optimal monetary policy in a New Keynesian model with heterogeneous firms. Firms differ in their productivity and net worth and face collateral constraints that cause capital misallocation. TFP depends on the time-varying distribution of firms. We introduce a new algorithm to compute optimal policies in continuous-time heterogeneous-agent models. Our results show that a central bank without pre-commitments engineers an unexpected monetary expansion to increase the profits of high-productivity firms, allowing them to relax their financial constraints. This reduces capital misallocation and increases TFP. Contrary to the case with complete markets, in the event of a cost-push shock, the central bank *leans with the wind* to increase demand and reduce misallocation. We provide empirical evidence based on Spanish granular data supporting the main mechanism at play, that is, that high-productivity firms increase their investment relatively more following an expansionary monetary policy shock.

Keywords: Monetary policy, firm heterogeneity, financial frictions, misallocation.
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†Corresponding e-mails: beatrizgonzalez@bde.es (Beatriz González), galo.nuno@bde.es (Galo Nuño), dominik.thaler@bde.es (Dominik Thaler) and silvia.albrizio@bde.es (Silvia Albrizio).

1 Introduction

Monetary policy affects capital allocation. If real interest rates decrease, low productivity projects can turn into profitable firms due to their lower costs of capital. In an economy with financial frictions and sticky prices, this mechanism may lead to scarce capital being redirected towards less productive firms after a monetary expansion, thus reducing average productivity. If this is the case, the central bank faces a non-trivial trade-off between short-term inflation/output stabilization and aggregate productivity. What does this trade-off imply for the optimal design of monetary policy? Should the central bank renounce to stabilize the economy after a negative shock on the grounds that it would decrease allocative efficiency? Do concerns about fueling misallocation justify a more hawkish monetary policy stance? These are the important questions that we try to answer in this paper.

To this end, we propose a continuous-time New Keynesian model with heterogeneous firms. The model features a continuum of firms operated by entrepreneurs with idiosyncratic time-varying productivity levels and subject to borrowing constraints. Each entrepreneur decides whether to operate a firm or not: if her idiosyncratic productivity is above a certain threshold, the entrepreneur hires workers and rents capital in order to produce, otherwise she does not operate. Due to financial frictions, a continuum of firms at the top of the productivity distribution will operate, introducing dispersion in the marginal product of capital (MPK), i.e. capital misallocation.

Firm heterogeneity and incomplete markets change the behavior of the economy relative to its complete-market representative-agent New-Keynesian (RANK) counterpart in a meaningful way.[1] The borrowing constraint implies that only self-financing can undo financial frictions, such that the distribution of net worth across firms matters for allocative efficiency. While in the RANK economy aggregate TFP is exogenous, in our economy TFP evolves endogenously as a result of the heterogeneous investment decisions of firms. Due to nominal rigidities, by changing nominal rates the central bank can influence the cut-off level of productivity above which operating a firm is profitable. We call this mechanism the *productivity threshold channel* of monetary policy. The central bank also affects firms' profits and net worth, thus relaxing or tightening the borrowing constraint and changing the dynamics of the

[1] Kaplan et al. (2018) refer to HANK and RANK to distinguish between heterogeneous and representative household models. Here we use the same names to differentiate between heterogeneous and representative firms.

firm distribution, in what we call the *net-worth distribution channel* of monetary policy. The net impact of these two channels constitutes the *'misallocation channel'* of monetary policy.

Finding the optimal monetary policy in models with heterogeneous firms is a challenge, as the productivity-capital distribution is an infinite-dimensional object. To overcome this problem, we propose a novel methodology to compute optimal policies in models featuring non-trivial heterogeneity. We approximate the original continuous-time, continuous-space problem by a discrete-time, discrete-space problem using a finite difference method similar to the one introduced in Achdou et al. (2017). Then we compute the first-order conditions of the Ramsey planner on the modified, finite-dimensional, problem using standard software packages for symbolic differentiation. Finally, we solve the resulting system of nonlinear equilibrium conditions in the sequence space using a Newton solver. We provide a proposition that shows how this methodology can be applied to a general class of Ramsey problems in heterogeneous-agent models. Our algorithm is easy to code using Dynare, for instance.

We first analyze time-0 Ramsey optimal policy and uncover a new source of time inconsistency. Though zero inflation is optimal in the long run, the central bank engineers a temporary monetary expansion in the short run.[2] This policy surprise increases the profits of high-productivity firms, allowing them to accumulate more capital and thus to grow. This, in turn, reduces capital misallocation and increases TFP in the medium term. Firm heterogeneity thus represents a new source of time inconsistency that is absent in the complete-market representative-firm New Keynesian model. In order to understand this result, we decompose the effects of monetary policy on misallocation into *direct effects*, i.e., those operating through interest rates, and *indirect or general equilibrium effects*, i.e. those operating through product prices and wages. We show how, ceteris paribus, a reduction in real rates increases misallocation by reducing the cut-off, thus allowing low-productivity firms to start operating (the productivity threshold channel). However, when all prices are allowed to change, the general equilibrium effects increase the cut-off and shift the firm distribution towards high-productivity firms, thus decreasing misallocation. We decompose the relative impact of the two channels on aggregate TFP and find that the *net-worth distribu-*

[2]We introduce taxes and subsidies such that optimal policy in the RANK model is time consistent.

tion channel accounts for the bulk of TFP dynamics. The threshold channel is thus negligible in general equilibrium.

We turn next to optimal monetary policy from a 'timeless perspective' (Woodford, 2003), in which the central bank has to honor its pre-commitments when the economy is hit by a shock. We consider a cost-push shock. The prescription in the RANK model is that the central bank should lean *against* the wind (Gali, 2008) – by tightening the monetary policy stance but tolerating some inflation to minimize the reduction in output. In the case of firm heterogeneity, the central bank should instead lean *with* the wind. It loosens monetary policy despite the rise in inflation, as the increase in demand boosts firms' profits and increases TFP, amplifying the expansionary demand effect on output. The misallocation channel of monetary policy thus makes optimal policy more dovish.

Finally, we present empirical evidence supporting the main mechanism through which optimal monetary policy operates in our model: we show how high-productivity firms invest more relative to low-productivity ones in response to an expansionary monetary policy shock. We use micro panel data for the quasi-universe of Spanish firms during the period 2000-2016, and construct the monetary policy shocks using the high-frequency event-study approach of Jarociński and Karadi (2020). Our empirical estimation follows closely that of Ottonello and Winberry (2020), with the difference that we focus on the heterogeneous impact of monetary policy depending on firms' productivity, proxied by the marginal revenue product of capital (MRPK). We find that having one standard deviation higher MRPK implies a further 20 pp increase in capital in response to a 100 bp cut in interest rates.This confirms the main mechanism of the model, and reinforces the messages that the optimal policy exercises deliver.

Related literature. This paper contributes to several strands of the literature. First, three recent papers have focused on the role of financial frictions and firm heterogeneity in monetary policy transmission. Ottonello and Winberry (2020) analyze the effect of monetary policy on firm investment in a model with endogenous default. They find that expansionary monetary policy causes an increase in investment both because it affects the cost of capital and because it relaxes the borrowing constraint for riskier firms. Jeenas (2020) analyzes the role of firms' balance sheet liquidity in the transmission of monetary policy to investment. Koby and Wolf (2020) study the conditions under which the lumpiness of firm-level investment matters for aggregate investment dynamics and, as an application, analyze monetary policy transmission with heterogeneous firms. We contribute to this nascent literature on two fronts. First, we focus on capital misallocation. Second, and more importantly, our paper is normative, not positive. We analyze optimal monetary policy in a model with non-trivial firm heterogeneity.[3]

Second, we add to the literature analyzing optimal monetary policies in models with heterogeneous agents. First, Nuño and Thomas (2016) employ calculus of variations to analyze optimal monetary policy in a model with heterogeneous agents, incomplete markets and Fisherian redistribution through long-term nominal debt. Bhandari et al. (2021) analyze optimal monetary and fiscal policies in a HANK model using perturbation techniques. Bilbiie and Ragot (2020), Acharya et al. (2019), and Le Grand et al. (2020) analyze optimal monetary policy in HANK models in which the wealth distribution is finite-dimensional, and hence tractable using standard techniques. Beyond the fact that we focus on heterogeneous firms and not households, the key difference with these papers is that we introduce a new method that is both easy to code and can deal with relatively complex models with heterogeneous agents, including exogenous borrowing limits or other nonlinear features that cannot be tackled with perturbation techniques.

Finally, our model is related to the extensive literature on capital misallocation, and the different channels that may affect it, such as Hsieh and Klenow (2009) or Midrigan and Xu (2014) – see Restuccia and Rogerson (2017) for a review on this literature. Our paper builds on Moll (2014), who introduces a heterogeneous-firm model to study how the nature of the idiosyncratic shocks impacts the speed of transitions. We enrich his model by introducing a New Keynesian monetary block, since our focus is to understand how monetary policy affects aggregates through its impact on heterogeneous firms.[4] Focusing on the impact of lower interest rates in a small open economy, Reis (2013), Gopinath et al. (2017) and Asriyan et al. (2021) analyze how an exogenous increase in the availability of cheap foreign funds or an exogenous decrease in real interest rates may increase capital misallocation among firms facing financial frictions. Here, instead, we focus on a closed-economy general

[3]Other strands of the literature have analyzed the links between monetary policy and firm heterogeneity through heterogeneity in markups and entry-exit (e.g. Meier et al., 2020, Bilbiie et al., 2014, Zanetti and Hamano (2020), Andrés et al., 2021, Nakov and Webber, 2021 or Baqaee et al., 2021), in cyclicality (David and Zeke, 2021) or in firm-level productivity trends (e.g, Adam and Weber, 2019).

[4]Buera and Nicolini (2020) employ a discrete-time version of Moll (2014) with cash-in-advance constraints to analyze the impact of different monetary and fiscal policies after a credit crunch.

equilibrium setting where real rates depend on the endogenous reaction of the central bank.

2 Model

We propose a New Keynesian closed economy model with heterogeneous firms based on Moll (2014). Time is continuous and there is no aggregate uncertainty. The economy is populated by five types of agents: households, the central bank, input good firms, retail, and final goods producers. The representative household is composed by two type of members: workers and entrepreneurs. Workers rent their labor whereas entrepreneurs operate the input good firms, which combine capital and labor to produce the input good. Entrepreneurs are heterogeneous in their net worth and productivity. The input good is differentiated by imperfectly competitive retail goods producers facing sticky prices, whose output is aggregated by the final goods producer. The latter two firms are standard in New Keynesian models.

2.1 Heterogeneous input good firms

There is a continuum of entrepreneurs. Each entrepreneur owns some net worth, which they hold in units of capital. They can use this capital for production in their own input-good producing firm – firm for short – or rent it out to other firms. Similar to Gertler and Karadi (2011), we assume that entrepreneurs are members of the representative household, to whom they may transfer dividends.[5]

Entrepreneurs are heterogeneous in two dimensions: their net worth a_t and in their idiosyncratic productivity z_t.[6] Each entrepreneur owns a technology which uses capital k_t and labor l_t to produce input good y_t:

$$y_t = f_t(z_t, k_t, l_t) = (\Gamma z_t k_t)^{\alpha} (l_t)^{1-\alpha}. \tag{1}$$

[5] This assumption is the only relevant difference between the real side of our model and the model of Moll (2014). We consider it to avoid having to deal with redistributive issues between households and entrepreneurs when analyzing optimal monetary policy. It can be shown that the general equilibrium outcomes of these two models are equivalent when the discount factor of the entrepreneurs in Moll (2014), ρ^{ent}, equals $(1-\psi)\eta$ in our model.

[6] For notational simplicity, we use x_t instead of $x(t)$ for the variables depending on time. Furthermore, we suppress the input goods firm's index.

The labor share $\alpha \in (0,1)$ and the aggregate productivity level Γ are the same across entrepreneurs. Idiosyncratic productivity z_t follows a diffusion process,

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t, \tag{2}$$

where $\mu(z)$ is the drift and $\sigma(z)$ the diffusion of the process.

Entrepreneurs can use their technology to produce or not. If they do, we say they run a firm and call them active. If they do not, they lend their net worth to firms owned by other entrepreneurs. Firms hire workers at the real wage w_t, and rent capital at the real rental rate of capital R_t. Capital is rented from the agents which save, i.e. both households and inactive entrepreneurs. Firms sell the input good at the real price $m_t = p_t^y/P_t$, which is the inverse of the gross markup associated to retail products over input goods, being p_t^y the nominal price of the input good and P_t the price of the final good, i.e. the numeraire. Entrepreneurs uses the return on their activities to distribute (non-negative) dividends d_t to the household and to invest in additional capital at the real price q_t. Capital depreciates at rate δ. An entrepreneur's flow budget constraint can be expressed as follows

$$\dot{a}_t = \frac{1}{q_t}\left[\underbrace{m_t f_t(z_t, k_t, l_t) - w_t l_t - R_t k_t}_{\text{Firm's profits}} + \underbrace{(R_t/q_t - \delta)}_{\text{Return on net worth}} q_t a_t - \underbrace{d_t}_{\text{Dividends}}\right]. \tag{3}$$

Note that we have rearranged the budget constraint to yield the law of motion of net worth *in units of capital*.

Entrepreneurs can borrow additional capital $b_t = k_t - a_t$ for use in production. However, they face a collateral constraint, such that the value of capital used in production cannot exceed $\gamma > 1$ of their net worth,

$$q_t k_t \leq \gamma q_t a_t. \tag{4}$$

Entrepreneurs retire and return to the household according to an exogenous Poisson process with arrival rate η. Upon retirement they pay all their assets, valued $q_t a_t$, to the household, and they are replaced by a new entrepreneur. Entrepreneurs maximize the discounted flow of dividends, which is given by

$$V_0(z,a) = \max_{k_t,l_t,d_t} \mathbb{E}_0 \int_0^\infty e^{-\eta t}\Lambda_{0,t}\left(\underbrace{d_t}_{\text{Dividends}} + \underbrace{\eta\, q_t a_t}_{\text{Terminal value}}\right) dt, \tag{5}$$

subject to the budget constraint (3), the collateral constraint (4), and the process followed by productivity (2). Future profits are discounted by the household's stochastic discount factor $\Lambda_{0,t}$. Below we show that $\Lambda_{0,t} = e^{-\int_0^t r_s ds}$, where r_t is the real interest rate.

We can split the entrepreneurs' problem into two parts: a static profit maximization problem and a dynamic dividend-distribution problem. First, entrepreneurs maximize firm profits given their productivity and net worth,

$$\max_{k_t,l_t} \left\{m_t f_t(z_t,k_t,l_t) - w_t l_t - R_t k_t\right\}, \tag{6}$$

subject to the collateral constraint (4). Since the production function has constant returns to scale, entrepreneurs find it optimal to operate a firm at the maximum scale defined by the borrowing constraint whenever their idiosyncratic productivity is high enough. Else they remain inactive, because they cannot run a profitable firm given their low productivity. Factor demands and profits of operating firms are thus linear in net worth, and there exists a productivity cut-off z_t^* below which entrepreneurs remain inactive. Firm's demand for capital and labor is :

$$k_t(z_t,a_t) = \begin{cases} \gamma a_t, & \text{if } z_t \geq z_t^*, \\ 0, & \text{if } z_t < z_t^*, \end{cases} \tag{7}$$

$$l_t(z_t,a_t) = \left(\frac{(1-\alpha)m_t}{w_t}\right)^{1/\alpha} \Gamma z_t k_t(z_t,a_t). \tag{8}$$

Firm's profits are then given by

$$\Phi_t(z_t,a_t) = \max\left\{\Gamma z_t \varphi_t - R_t, 0\right\}\gamma a_t, \quad \text{where} \quad \varphi_t = \alpha\left(\frac{(1-\alpha)}{w_t}\right)^{(1-\alpha)/\alpha} m_t^{\frac{1}{\alpha}}, \tag{9}$$

and the productivity cut-off, above which firms are profitable, is given by

$$\Gamma z_t^* \varphi_t = R_t. \tag{10}$$

Second, entrepreneurs decide the dividends d_t that they pay to households. The law of motion of an entrepreneur's net worth (in units of capital) (3) can be rewritten as

$$\begin{aligned} \dot{a}_t &= \frac{1}{q_t}\left[\Phi_t(z_t, a_t) + (R_t - \delta q_t)a_t - d_t\right] \\ &= \frac{1}{q_t}\left[(\gamma \max\left\{\Gamma z_t \varphi_t - R_t, 0\right\} + R_t - \delta q_t)a_t - d_t\right]. \end{aligned} \tag{11}$$

The solution to this problem is shown in Appendix A.1. There we show how entrepreneurs never distribute dividends until retirement, when they bring all their capital home to the household. The intuition is simple: one unit of capital in the hands of the entrepreneur receives *at least* a return of $(R_t - \delta q_t)$, while for the household the return of this unit of capital is *exactly* $(R_t - \delta q_t)$. Since the terminal value is all their net worth, $q_t a_t$, it is always worthwhile for entrepreneurs to keep their funds. The household collects all these funds as dividends once the entrepreneur retires. To keep things simple, we assume the representative household uses a fraction ψ of these dividends to finance the net worth of the new entrepreneurs, so net dividends are (1-ψ) of the net worth of retiring entrepreneurs.

2.2 Final good producers

As usual in new Keynesian models, a competitive representative final goods producer aggregates a continuum of output produced by retailer $j \in [0, 1]$,

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{12}$$

where $\varepsilon > 0$ is the elasticity of substitution across goods. Cost minimization implies

$$y_{j,t}\left(p_{j,t}\right) = \left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t, \text{ where } P_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}.$$

2.3 Retailers

We assume that monopolistic competition occurs at the retail level. Retailers purchase input goods from the input good firms, differentiate them and sell them to final good producers. Each retailer j chooses the sales price $p_{j,t}$ to maximize profits subject to price adjustment costs as in Rotemberg (1982), taking as given the demand curve $y_{j,t}(p_{j,t})$ and the price of input goods, p_t^y. We assume the government pays a proportional constant subsidy τ on input good, so that the net real price for the retailer is $\tilde{m}_t = m_t(1-\tau)$. This subsidy is financed by a lump-sum tax on the retailer Ψ_t.[7] The adjustment costs are quadratic in the rate of price change $(\dot{p}_{j,t}/p_{j,t})$ and expressed as a fraction of output (Y_t),

$$\Theta_t\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right) = \frac{\theta}{2}\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right)^2 Y_t,$$

where $\theta > 0$. Suppressing notational dependence on j, each retailers chooses $\{p_t\}_{t\geq 0}$ to maximize the expected profit stream, discounted at the stochastic discount factor of the household,

$$\int_0^\infty \Lambda_{0,t}\left[\Pi_t(p_t) - \Theta_t\left(\frac{\dot{p}_t}{p_t}\right)\right] dt, \tag{13}$$

where

$$\Pi_t(p_t) = \left(\frac{p_t}{P_t} - \tilde{m}_t\right)\left(\frac{p_t}{P_t}\right)^{-\varepsilon} Y_t - \Psi_t$$

are per-period profits gross of price adjustment costs.

The symmetric solution to the pricing problem yields the New Keynesian Phillips curve (see Appendix A.2), which is given by

$$\left(r_t - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\varepsilon}{\theta}\left(\tilde{m}_t - m^*\right) + \dot{\pi}_t, \qquad m^* = \frac{\varepsilon - 1}{\varepsilon}. \tag{14}$$

where π_t denotes the inflation rate $\pi_t = \dot{P}_t/P_t$. Here again we exploit the fact that, given the lack of aggregate risk, the household's stochastic discount factor can be expressed as $\Lambda_{0,t} = e^{-\int_0^t r_s ds}$. The total profit of retailers, net of the lump-sum tax,

[7] This fiscal scheme is introduced to eliminate the distortions caused by imperfect competition in steady state, as common in the optimal policy literature.

which is transferred to the households lump sum, is

$$\Pi_t = (1 - m_t) Y_t - \frac{\theta}{2}\pi_t^2 Y_t. \tag{15}$$

2.4 Capital producers

A representative capital producer owned by the representative household produces capital and sells it to the household and the firms at price q_t, which he takes as given. His cost function is given by $(\iota_t + \Phi(\iota_t)) K_t$ where ι_t is the investment rate and $\Phi(\iota_t)$ is a capital adjustment cost function. He maximizes the expected profit stream, discounted at the stochastic discount factor of the household. Profits are paid to the household.

$$W_t = \max_{\iota_t, K_t} \mathbb{E}_0 \int_0^\infty \Lambda_{0,t} \left(q_t \iota_t - \iota_t - \Phi(\iota_t)\right) K_t dt. \tag{16}$$

$$\text{s.t.} \quad \dot{K}_t \;=\; (\iota_t - \delta) K_t. \tag{17}$$

The optimality conditions imply (see Appendix A.3)

$$r_t = (\iota_t - \delta) + \frac{\dot{q}_t - \Phi''(\iota_t)\, \dot{\iota}_t}{q_t - 1 - \Phi'(\iota_t)} - \frac{q_t \iota_t - \iota_t - \Phi(\iota_t)}{q_t - 1 - \Phi'(\iota_t)}.$$

We assume adjustment costs are quadratic, i.e.,

$$\Phi(\iota_t) \;=\; \frac{\phi^k}{2} (\iota_t - \delta)^2. \tag{18}$$

2.5 Households

There is a representative household, composed of workers and entrepreneurs, that saves in capital D_t or in nominal instantaneous bonds whose real value is denoted by B_t^N.Nominal bonds B_t^N are in zero net supply. Workers supply labor L_t. The household maximizes

$$W_t = \max_{C_t, L_t, B_t^N, D_t} \mathbb{E}_0 \int_0^\infty e^{-\rho_t^h t} u(C_t, L_t) dt. \tag{19}$$

$$s.t. \quad \dot{D}_t q_t \quad + \dot{B}_t^N + C_t = \quad \left(R_t - \delta q_t\right) D_t + \left(i_t - \pi_t\right) B_t^N + w_t L_t + T_t, \tag{20}$$

and T_t are the profits received by the household, which is the sum of the profits of the capital producer ($\left[\iota_t q_t - \iota_t - \frac{\phi^k}{2}(\iota_t - \delta)^2\right] K_t$), the profits from retail goods producers (Π_t from equation 15) and net dividends received from entrepreneurs ($(1-\psi)\eta A_t$).

We assume separable utility of CRRA form, i.e., $u(C_t, L_t) = \frac{C_t^{1-\zeta}}{1-\zeta} - \Upsilon \frac{L_t^{1+\vartheta}}{1+\vartheta}$. Solving this problem (see Appendix (A.4) for details), we obtain the Euler equation, the labor supply condition and the Fisher equation, respectively:

$$\frac{\dot{C}_t}{C_t} = \frac{r_t - \rho_t^h}{\zeta}, \tag{21}$$

$$w_t = \frac{\Upsilon L_t^{\vartheta}}{C_t^{-\zeta}}, \tag{22}$$

$$r_t = i_t - \pi_t, \tag{23}$$

where, for convenience, we have made use of the following definition of the real rate

$$r_t \equiv \frac{R_t - \delta q_t + \dot{q}_t}{q_t}. \tag{24}$$

Integrating the Euler equation (21), we can verify that the stochastic discount factor results in

$$\Lambda_{0,t} \equiv e^{-\int_0^t \rho_t^h ds} \frac{u_c'\left(C_t\right)}{u_c'\left(C_0\right)} = e^{-\int_0^t r_s ds}.$$

2.6 Distribution

We assume that for each entrepreneur returning to the household, a new entrepreneur arrives operating the same technology, that is, with the same productivity level. This new entrepreneur receives a startup capital stock from the household in a lump-sum fashion. As previously explained, we assume that the initial net worth of each new entrepreneur is equal to a fraction $\psi < 1$ of the net worth of the entrepreneur she replaces. The evolution of the joint distribution of net worth and productivity $g_t(z, a)$ is then given by the Kolmogorov Forward equation

$$\frac{\partial g_t(z,a)}{\partial t} = \underbrace{-\frac{\partial}{\partial a}[g_t(z,a)s_t(z)a]}_{\text{Retained earnings}} \underbrace{-\frac{\partial}{\partial z}[g_t(z,a)\mu(z)] + \frac{1}{2}\frac{\partial^2}{\partial z^2}[g_t(z,a)\sigma^2(z)]}_{\text{Productivity changing randomly}}$$

$$\underbrace{-\eta g_t(z,a)}_{\text{Entrepreneurs retiring}} \quad \underbrace{+\frac{\eta}{\psi} g_t(z,\frac{a}{\psi}))}_{\text{Entrepreneurs entering}} \quad , \tag{25}$$

where$s_t(z)$ is the entrepreneurs' investment rate (11)

$$s_t(z) \equiv \frac{1}{q_t}(\gamma \max \{\Gamma z_t \varphi_t - R_t, 0\} + R_t - \delta q_t), \tag{26}$$

and $1/\psi g_t(z, a/\psi)$ is the distribution of new entrepreneurs entering.

Note that we can express the distribution also in terms of *net worth shares* defined as $\omega_t(z) \equiv \frac{1}{A_t} \int_0^\infty a g_t(z,a) da$. Given this definition and the structure of the problem, wealth shares are non-negative, continuous, once differentiable everywhere and they integrate up to 1. The law of motion of net worth shares is given by (see in Appendix A.5)

$$\frac{\partial \omega_t(z)}{\partial t} = \left[s_t(z) - \frac{\dot{A}_t}{A_t} - (1-\psi)\eta \right] \omega_t(z) - \frac{\partial}{\partial z} \mu(z)\omega_t(z) + \frac{1}{2} \frac{\partial^2}{\partial z^2} \sigma^2(z)\omega_t(z). \tag{27}$$

2.7 Market Clearing and Aggregation

Borrowing of an input good firm is the extra capital used for production in excess of their net worth, $b_t = k_t - a_t$, where $b_t > 0$ if the firm is borrowing and $b_t < 0$ if it is saving. Adding up, we get

$$\underbrace{\int k_t(z,a) dG_t(z,a)}_{\text{Agg. capital } K_t} = \underbrace{\int b_t(z,a) dG_t(z,a)}_{\text{Firms' net debt } B_t} + \underbrace{\int a_t dG_t(z,a)}_{\text{Firms' net worth } A_t}, \tag{28}$$

Asset market clearing requires that net borrowing of entrepreneurs, B_t, equals net savings of the household, D_t,

$$B_t = D_t. \tag{29}$$

Let $\Omega(z)$ be the cumulative distribution of net-worth shares, i.e. $\Omega_t(z) = \int_0^z \omega_t(x)\, dx$. By combining equations (28), (29), aggregating capital used by firms (7), and solving for A_t, we obtain

$$A_t = \frac{D_t}{\gamma(1-\Omega_t(z_t^*)) - 1}. \tag{30}$$

Labor market clearing implies

$$L_t = \int_0^\infty l_t(z,a)dG_t(z,a). \tag{31}$$

Aggregating up firms, one can express output as a function of aggregate factors and aggregate TFP

$$Y_t = \tilde{Z}_t K_t^\alpha {L_t}^{1-\alpha}, \tag{32}$$

where aggregate TFP $\tilde{Z}_t$ is an endogenous variable given by

$$\tilde{Z}_t = \left(\Gamma \mathbb{E}\left[z \mid z > z_t^*\right]\right)^\alpha = \left(\Gamma \frac{\int_{z_t^*}^\infty \omega_t\left(x\right)xdx}{1-\Omega_t(z_t^*)}\right)^\alpha \tag{33}$$

This highlights that in terms of output the model is isomorphic to a standard representative agent model with TFP $\tilde{Z}_t$. The financial frictions faced by entrepreneurs imply that capital is not optimally allocated. The entrepreneur operating the most productive firm does not have enough net worth to operate the whole capital stock, hence less productive firms operate as well. The degree to which capital is misallocated is endogenous and implies that aggregate TFP, $\tilde{Z}_t$, fluctuates over time and, importantly, depends on monetary policy.[8]

Factor prices are

$$w_t = (1-\alpha)m_t \tilde{Z}_t K_t^\alpha {L_t}^{-\alpha}, \tag{34}$$

$$R_t = \alpha m_t \tilde{Z}_t K_t^{\alpha-1} {L_t}^{1-\alpha} \frac{z_t^*}{\mathbb{E}\left[z \mid z > z_t^*\right]}. \tag{35}$$

Finally, the law of motion of aggregate wealth of entrepreneurs is given by

$$\frac{\dot{A}_t}{A_t} = \frac{1}{q_t}\left[\gamma(1-\Omega_t(z_t^*))\left(\alpha m_t \tilde{Z}_t K_t^{\alpha-1}{L_t}^{1-\alpha} - R_t\right) + R_t - \delta q_t - q_t(1-\psi)\eta)\right]. \tag{36}$$

Appendix A.6 derives step by step these aggregate formulas.

[8]This mechanism linking aggregate TFP and monetary policy differs from the one in Benigno and Fornaro (2018) or Moran and Queralto (2018), who focus instead on endogenous R&D.

2.8 Central Bank

The central bank controls nominal interest rates i_t on nominal bonds held by households. The central bank solves the following Ramsey problem

$$\max_{\left\{\omega_t(z),s_t(z),w_t,r_t,q_t,\varphi_t,K_t,A_t,L_t,C_t,D_t,\tilde{Z}_t,\mathbb{E}_t[z|z>z_t^*],\Omega_t,z_t^*,\iota_t,\pi_t,m_t,\tilde{m}_t,i_t,Y_t,T_t\right\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho^h t} u(C_t, L_t) dt$$

subject to the private equilibrium conditions derived above and listed in Appendix A.7 and initial conditions. The private equilibrium conditions include the law of motion of the productivity-net worth distribution (equation 27). Notice that $\omega_t(z)$ and $s_t(z)$ not only depend on time, but also on the idiosyncratic state variables. This poses some difficulties when solving optimal monetary policy. In the next section we deal with them in a general environment.

3 Computing optimal policies in heterogeneous-agent models

Solving for the optimal policy in models with heterogeneous agents poses a certain challenge since the state in such a model contains a distribution, which is an infinite-dimensional object. In this section, we explain how such models can be solved in a relatively straightforward manner. Our approach relies on three main conceptual ingredients: (i) finite difference approximation of continuous time and continuous idiosyncratic states, (ii) symbolic derivation of the planner's first-order conditions, and (iii) use of a Newton algorithm to solve the optimal policy problem non-linearly in the sequence space.

We start by reviewing the three existing approaches to analyze optimal monetary policy in models with non-trivial heterogeneity. Le Grand et al. (2020) employ the finite-memory algorithm proposed by Ragot (2019). It requires changing the original problem such that, after K periods, the state of each agent is reset. This way the cross-sectional distribution becomes finite-dimensional. Bhandari et al. (2021), instead, make the continuous cross-sectional distribution finite-dimensional by assuming that there are N agents instead of a continuum. They then derive standard first-order conditions (FOCs) for the planner. In order to cope with the large dimensionality of their problem, they employ a perturbation technique. This precludes the use of their algorithm to problems with kinks, such as the one presented here, or with exogenous borrowing limits, as in the standard Aiyagari-Bewley-Hugget framework. Nuño and Thomas (2016) deal with the full infinite-dimensional problem in continuous time. This implies that the continuous Kolmogorov forward (KF) and the Hamilton-Jacobi-Bellman (HJB) equations form part of the constraints faced by the central bank. They derive the planner's FOCs using calculus of variations, thus expanding the original problem to also include the Lagrange multipliers, which in this case may take the form of distribution and (social) value functions. They then solve the problem using the upwind finite-difference method of Achdou et al. (2017). The problem with this approach is that it requires solving by hand the first-order conditions, which can be demanding in medium-scale models such as the one presented in this paper.

The algorithm proposed here is distinct from the previous ones. If any, it can be seen as the mirror image of Nuño and Thomas (2016). Instead of first computing by hand the planner's FOCs and then discretizing them using finite differences, we propose to first discretize the private equilibrium conditions using finite differences, and then to find the planner's FOCs by symbolic differentiation. This avoids the cumbersome mathematical derivations and allows us to solve the dynamic problem *nonlinearly* in a few seconds using Dynare.

(i) Finite difference approximation A continuous-time, continuous-space heterogeneous-agent model discretized using an upwind finite-difference method becomes a discrete-time, discrete-space model. In this discretized model the dynamics of the (now finite-dimensional) distribution $\boldsymbol{\mu}_t$ at period t are given by

$$\boldsymbol{\mu}_t = \left(\mathbf{I} - \Delta t \boldsymbol{A}_t^T\right)^{-1} \boldsymbol{\mu}_{t-1}, \tag{37}$$

where Δt is the time step between periods and $\boldsymbol{A}_{\mathbf{t}}$ is a matrix whose entries depend *nonlinearly* and in *closed form* on the idiosyncratic and aggregate variables in period t.[9] Similarly, the HJB equation is approximated as

$$\rho \mathbf{v}_{t+1} = \boldsymbol{u}_{t+1} + \mathbf{A}_{t+1}\mathbf{v}_{t+1,} - \left(\mathbf{v}_{t+1} - \mathbf{v}_t\right)/\Delta t. \tag{38}$$

[9]Technically, this matrix results from the discretization of the *infinitesimal generator* of the idiosyncratic states. In the example of Section 2, $\boldsymbol{\mu}_t = \boldsymbol{\omega}_{\mathbf{t}}$ and $\boldsymbol{A}_t = \mathbf{B}_{\mathbf{t}}$.

Together with additional static equations, such as market clearing conditions or budget constraints, and aggregate dynamic equations, including the Euler equations of representative agents (if any) and the dynamics of aggregate states, they define the discretized model.

Though we have ended up with a discrete-time approximation, casting the original model in continuous time is central to our method. The discretized dynamics of the distribution (37) and Bellman equation (38) present two advantages compared to their counterparts in the discrete-time continuous-state formulation typically employed in the literature. First, the analytical tractability of the original continuous-time model implies that the agents' optimal choices in the discretized version are always "on the grid", avoiding the need for interpolation, and are "one step at a time" making the matrix Π_t sparse.[10] Second, the private agent's FOCs hold with equality even at the exogenous boundaries (see Achdou et al. (2017) for a detailed discussion of these advantages).

(ii) Symbolic derivation of planner's FOCs Once we have a finite-dimensional discrete-time discrete-space model, we can derive the planner's FOCs by *symbolic differentiation* using standard software packages. For convenience, we rely on Dynare's toolbox for Ramsey optimal policy to do this task for us. To this end, we simply provide the discretized version of our model's private equilibrium conditions to Dynare (the discretized counterpart to the equations in Appendix A.7), making use of loops for the heterogeneous-agent block, as in Winberry (2018). We furthermore provide the discretized objective function, and Dynare then takes symbolic derivatives to construct the set of optimality conditions of the planner for us.

A natural question at this stage is under which conditions the optimal policies of the discrete-time, discrete-space problem coincide with those of the original problem. The following proposition shows that, if the time interval is small enough (the standard condition when approximating continuous-time models), then the two solutions coincide.

Proposition 1 : *Provided that the Lagrange multipliers are continuous, the solution of the "discretize-optimize" and the "optimize-discretize" algorithms converge to each other as the time step Δt goes towards 0.*

Proof: See Appendix D.

[10] The introduction of Poisson shocks would not change the sparsity of matrix Π_t.

The proposition guarantees that both strategies coincide when Δt goes towards zero and provides an error bound that depends on the value of the maximum change in the Lagrange multipliers. This proposition is quite general, as most continuous-time, perfect-foresight, general equilibrium models do not feature discontinuities for $t > 0$.

The model presented in Section 2 is arguably simpler than the general heterogenous-agent model covered by Proposition 1, as it features an analytic solution for the HJB equation. To get an idea of the performance of our method in a case in which the HJB is also a constraint in the planner's problem, as well as to showcase its generality in dealing with different problems, we compute the optimal monetary policy in the HANK model of Nuño and Thomas (2016) using our method in Dynare (see Appendix D). We compare our results with those using their "optimize-discretize" algorithm at monthly frequency $\triangle t = 1/12$. We conclude that both approaches essentially coincide.

(iii) Newton algorithm to solve the optimal policy problem non-linearly in the sequence space Finally, we use the discretized optimality conditions of the planner to compute non-linearly the *optimal responses* a temporary change in parameters (an "MIT shock") using a Newton algorithm. Instead of time iterations over guesses for aggregate sequences, as is common in the literature, we use a global relaxation algorithm. This approach has been made popular in discrete-time models by Juillard et al. (1998) thanks to Dynare, but it is somewhat less common in continuous-time models (e.g. Trimborn et al., 2008). This approach helps to overcome the curse of dimensionality since in the sequence space the complexity of the problem grows only linearly in the number of aggregate variables, whereas the complexity of the state-space solution grows exponentially in the number of state variables. Recently Auclert et al. (2019) have exploited a particularly efficient variant of this approach in the context of heterogeneous-agent models.[11] We build on these contributions when we compute the optimal transition path. Again we make use of Dynare. We use its nonlinear Newton solver to compute both the steady state of the Ramsey problem and the optimal transition path under perfect foresight.[12] Our hope is that the con-

[11] Compared to Auclert et al. (2020), who break the solution procedure into two steps, first solving for the idiosyncratic variables given the aggregate variables, we solve for the path of all aggregate and idiosyncratic variables at once. Note that, besides the nonlinear perfect foresight method we refer to here (see their Section 6), they also propose a linear method.

[12] To find the steady state, we provide Dynare with the steady state of the private equilibrium conditions as a function of the policy instrument.

venience of using Dynare will make optimal policy problems in heterogeneous-agent models easily accessible to a large audience of researchers.

The solution to the perfect foresight problem can be easily adapted to the case with aggregate shocks. As Boppart et al. (2018) show, the perfect-foresight transitional dynamics to an "MIT shock" coincides with the solution of the model with aggregate uncertainty using a first-order perturbation approach. We follow this approach to analyze the optimal response to a cost-push shock below.

Finally, it is important to highlight that our solution approach is different from the one in Winberry (2018) or Ahn et al. (2018). These papers expand the seminal contribution by Reiter (2009), based on a two-stage algorithm that (i) first finds the nonlinear solution of the steady state of the model and (ii) then applies perturbation techniques to produce a linear system of equations describing the dynamics around the steady state. Winberry (2018) illustrates how this can be also implemented using Dynare and Ahn et al. (2018) extend the methodology to continuous-time problems. However, these methods were not created to deal with the problem of finding the optimal policies, the focus of our algorithm, as the first stage requires the computation of the steady state, which in our case is the steady state of the problem under optimal policies. Our algorithm finds the steady state of the planner's problem, including the Lagrange multipliers. Naturally, this steady does not need to coincide with the steady state that can be found by looking for the value of the planner's policy that maximizes steady-state welfare.

4 Calibration

We solve the model using the method described above. Table 1 summarizes our calibration. We work at quarterly frequency (time period $\Delta t = 1/4$). The rate of time preference of the household ρ^h is 0.025, which targets an average real rate of return of 2.5 percent. The capital depreciation rate δ is set at 0.065, equal to the aggregate depreciation rate in NIPA. The fraction of assets of exiting entrepreneurs reinvested (ψ) is 0.1, so that the average size of entrants is 10 percent of that of incumbents, in line with US data (OECD, 2001). Entrepreneurs' exit rate (η) is 0.12 which, together with ψ, implies an average real return on equity of 11 percent, the return of the S&P500 from 2009 to 2019. The borrowing constraint parameter γ is 1.43, implying that entrepreneurs can borrow up to 43% of their net worth, which

Table 1: **Calibration**

	Parameter	**Value**	**Source/target**
ρ^{hh}	Rate of time preference of HH	0.025	Av. 10Y bond return of 2.5% (FRED)
δ	Capital depreciation rate	0.065	Aggregate depreciation rate (NIPA)
ψ	Fraction firms' assets at entry	0.1	Av. size at entry 10% (OECD, 2001)
η	Firms' death rate	0.12	Av. real return on equity 11% (S&P500)
γ	Borrowing constraint parameter	1.43	Corporate debt to net worth of 43% (FRED)
α	Capital share in production function	0.3	Standard
ζ	Relative risk aversion parameter HH	1	Log utility in consumption
ϑ	Inverse Frisch Elasticity	1	Kaplan et al. (2018)
Υ	Constant in disutility of labor	0.71	Normalization $L = 1$
ϕ^k	Capital adjustment costs	10	VAR evidence
ϵ	Elasticity of substitution retail goods	10	Mark-up of 11%
θ	Price adjustment costs	100	Slope of PC of 0.1
Γ	SS Aggregate Productivity	1	Normalization
ς_z	Mean reverting parameter	0.8	Persistence Gilchrist et al. (2014)
σ_z	Volatility of the shock	0.30	Volatility Gilchrist et al. (2014)

targets the level of aggregate US corporate debt as a percentage of net worth from 2009 to 2019. The capital share α is set at a standard value of 0.3. We assume log-utility in consumption ($\zeta = 1$) and the inverse Frisch elasticity ϑ is also set to 1, standard values in the literature. We set the constant multiplying the disutility of labor Υ such that aggregate labor supply in steady state is equal to one. Capital adjustment costs, ϕ^k, are set to 10, such that the peak response of investment to output after a monetary policy shock is around 2, in line with the VAR evidence of Christiano et al. (2016).

Regarding the New Keynesian block, the elasticity of substitution of retailer goods ϵ is set to 10, so that the steady state mark-up is $1/(1-\epsilon) = 0.11$. The Rotemberg cost parameter θ is set to 100, so that the slope of the Phillips curve is $\epsilon/\theta = 0.1$ as in Kaplan et al. (2018)

The aggregate productivity term Γ is normalized to 1 in SS. We assume that individual productivity z follows an Ornstein-Uhlenbeck process in logs

$$d\log(z) = -\varsigma_z \log(z)dt + \sigma_z dW_t. \tag{39}$$

By Ito's lemma, this implies that z in levels follows the diffusion process

$$dz = \mu(z)dt + \sigma(z)dW_t, \tag{40}$$

where $\mu(z) = z\left(-\varsigma_z \log z + \frac{\sigma^2}{2}\right)$ and $\sigma(z) = \sigma_z z$. We calibrate the productivity process using the estimates from Gilchrist et al. (2014), who find quarterly persistence of 0.8 and volatility of 0.15 (0.3 annualized).

5 Optimal monetary policy

We now analyze optimal policy under commitment, i.e. we solve the central bank's Ramsey problem. We compare the optimal responses in our heterogeneous-firm (HANK) problem to the responses in the representative-agent version (RANK). The RANK economy is the standard New Keynesian model with capital. It is a special case of the HANK economy where the borrowing constraint is set to infinite, so that the productivity-net worth distribution becomes irrelevant and only the most productive firm operates. In this case, capital allocation is efficient (no misallocation) and TFP is exogenous. This contrasts with the HANK economy, in which the distribution across firms matters due to financial frictions and determines the endogenous component of TFP (see Appendix A.8 for more details regarding the RANK versus HANK model). We stress the fact that the central bank's only instrument is the nominal interest rate. The way monetary policy affects real allocations is through its impact on prices in the New Keynesian Phillips curve. For simplicity, we calibrate the tax/subsidy τ such that it undoes the New Keynesian mark-up distortion in the steady state of both economies.

We start by analyzing how the central bank would behave if it is allowed to re-optimize without pre-commitments, starting at the steady state of the Ramsey solution. This is the "time-0 optimal policy" (Woodford, 2003). Next, we analyze the optimal policy response when an unexpected (MIT) mark-up shock hits the economy that was previously in its steady state. In this case, we adopt a "timeless perspective". The timeless perspective means that the central bank cannot exploit the initial state of the economy, but rather stick to its pre-commitments, implementing the policy that it would have chosen to implement if it had been optimizing from a time period far in the past. This is a concept that only makes sense in the presence of aggregate

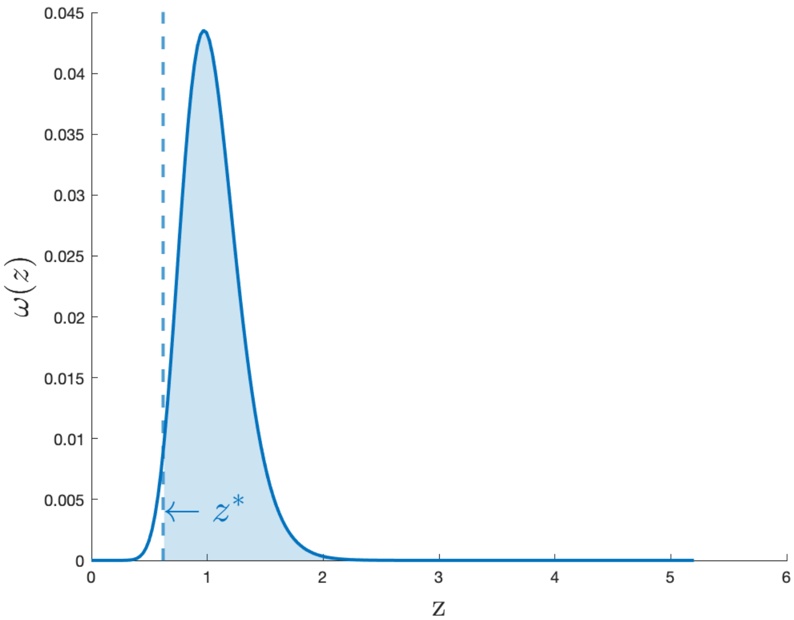

Figure 1: Net worth shares in steady state.

Notes: The figure shows the net-worth distribution $\omega(z)$ in steady state.

risk. As discussed above, building on the argument by Boppart et al. (2018) one can reinterpret the timeless response to MIT shocks as a first order approximation to the response in a model with aggregate uncertainty under the ex-ante optimal time-invariant state-contingent policy rule.

Before we get to the dynamics, we analyze the steady state of the Ramsey problem. It is well known that the RANK economy features zero inflation in steady state under the optimal policy, since the zero inflation steady state is first best. We find numerically that, for our calibration and several robustness checks, the HANK economy also features zero inflation in the steady state of the Ramsey problem. This result mirrors a similar result from the textbook New Keynesian model with a distorted steady state (Woodford, 2003, Gali, 2008). Though the long-run Phillips curve allows monetary policy to affect misallocation in the long run through positive inflation, the benefits of this policy are compensated for by the cost of the anticipation of this policy. The net worth share distribution in steady state is shown in Figure 1, with a dashed vertical line showing the productivity cut-off z^*. Entrepreneurs at the left of z^* remain inactive and rent out their net worth to active entrepreneurs at the right of the cut-off (those in the shaded area).

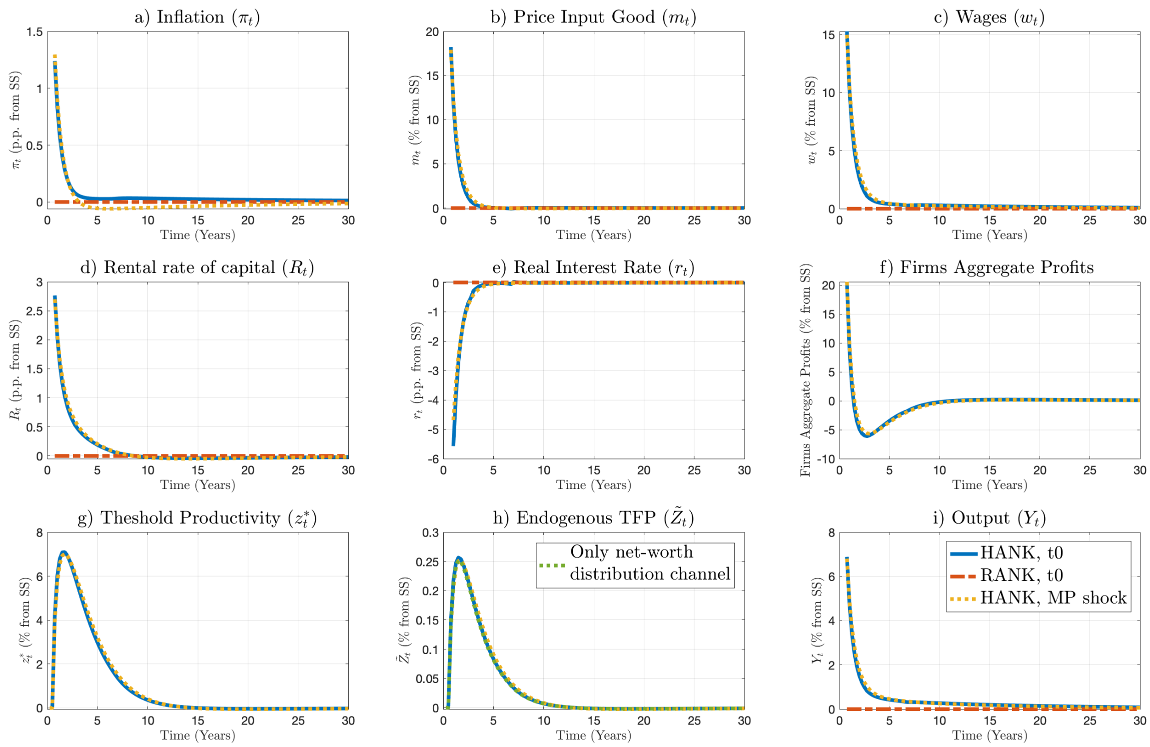

Figure 2: Time 0 optimal monetary policy.

Notes: The figure shows the deviations from steady state of the economy when the planner is allowed to re-optimize with no pre-commitments in response to no shock. RANK is red dashed line, HANK is the solid blue line. The dotted yellow line is the response of the HANK model in general equilibrium to a monetary policy shock of 500 basis points, where the central bank follows the Taylor rule $di = -\upsilon\left(i_t - \left(\rho_t^h + \phi\left(\pi_t - \bar{\pi}\right) + \bar{\pi}\right)\right)dt$, with $\upsilon = 0.8$, which implies a quarterly persistence of 0.8, and $\phi = 1.25$. In panel h, the green dotted line shows the change in endogenous TFP due to the net-worth distribution channel.

5.1 Time-0 optimal policy

Aggregate dynamics. Firm heterogeneity causes a time inconsistency problem that does not exist in the standard RANK. Figure 2 shows this time-inconsistency problem: starting at the steady state of the Ramsey problem, if the central bank is allowed to re-optimize, it takes advantage of the lack of pre-commitments and engineers a monetary expansion. It does so by reducing the nominal rate (not shown) which leads to a reduction in the real rate (blue solid line, panel e) and an increase in inflation and output (panels a and i) through the standard New Keynesian channels. In the RANK, absent capital misallocation, the steady state is first best, so the central bank does nothing. But why exactly is it optimal to engineer an expansion in HANK?

The surprise expansion is socially optimal because it shifts factor prices in such a way that the allocation of resources improves, which leads to a temporary increase in aggregate TFP (panel g). This improvement is the result of two channels. First, the threshold z^* moves up (panel g), making the least productive entrepreneurs abstain from producing. This is what we call the *productivity threshold channel*. Second, firms' profits increase (panel f), such that firms can accumulate more net worth, partially undoing financial frictions. This is the *net-worth distribution channel*. The net effect of these two channels, which we call the *misallocation channel* of monetary policy, leads to an increase in endogenous TFP (panel h), which amplifies the boom generated by the initial monetary policy expansion.

Quantitatively, the net-worth distribution channel explains almost all of the movement in TFP. We isolate the contribution of this channel as follows. We plug the simulated path of z^* and $\omega(z)$ into the derivative $\frac{d\tilde{Z}}{dz^*}$ (see equation 41) to compute the contribution of changes in the threshold to the dynamics of endogenous TFP. We subtract this contribution from the simulated path for $\tilde{Z}_t$ to obtain a counterfactual path for TFP that is purely driven by the net-worth distribution channel. The green dotted line in panel h of Figure 2 plots this counterfactual path. It is evident that the effect of the threshold channel is negligible.

Notice how the optimal monetary policy can be described as an expansionary monetary policy shock. The yellow dotted line in Figure 2 displays the dynamics after an expansionary monetary policy shock in the case of a central bank following a Taylor rule. These results almost coincide with those under the Ramsey policy.

Heterogeneity in the response. Next, we dig into the heterogeneity of the responses that drive the aggregate responses just explained. We define the firm's excess return on capital $\tilde{\Phi}_t(z)$ as

$$\tilde{\Phi}_t(z) \equiv \frac{\Phi_t}{k_t} = \max\left\{\Gamma z_t \alpha \left(\frac{(1-\alpha)}{w_t}\right)^{(1-\alpha)/\alpha} m_t^{\frac{1}{\alpha}} - \underbrace{\left(q_t\left(r_t+\delta\right)-\dot{q}_t\right)}_{R_t}, 0\right\},$$

where the latter equality comes from equation (9). We speak of the *excess* return here since it is the return that a firm makes net of the cost of capital R_t. Since entrepreneurs do not distribute dividends until they retire, these returns are reinvested. Hence we can understand this excess return as the investment rate of firm with productivity z in excess of the investment rate of the marginal firm with productivity z^*.[13] Because of this, we refer to $\tilde{\Phi}_t(z)$ as the *excess investment rate* from now on.

[13]Note that the investment rate of the marginal firm with productivity z^* is equal to the risk free

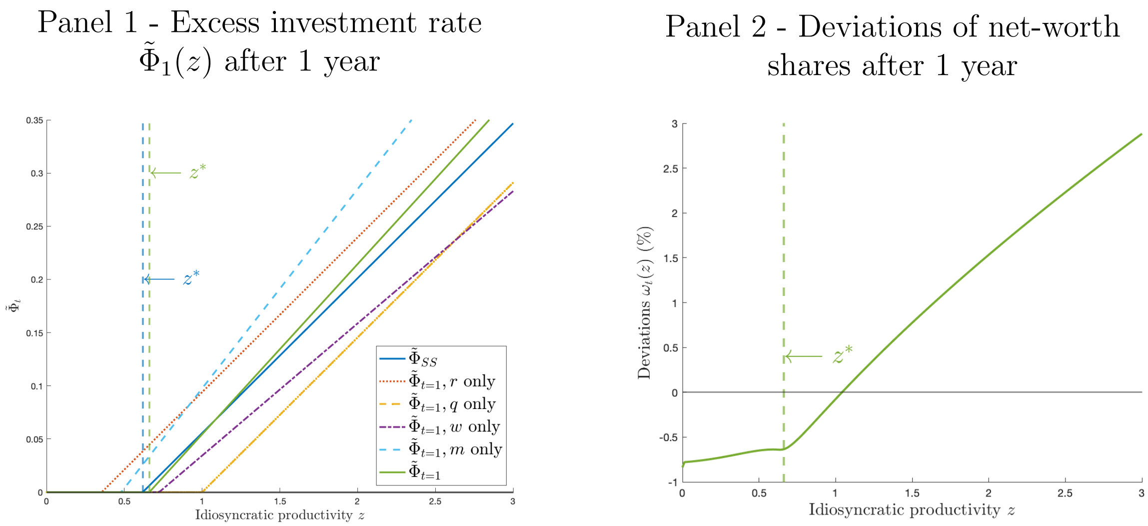

Figure 3: Heterogeneity one year after the shock hits.

Notes: Panel a): Idiosyncratic productivity is shown on the X-axis, the excess investment rate $\tilde{\Phi}(z)$ on the Y-axis. The solid blue line is the excess investment rate function in the SS, and the solid green line is the same function in year 1. The rest of the lines show the excess investment rate function when only one price is changed at a time to its year 1 value, keeping the rest of the prices constant to the SS value. Panel b): deviations from steady state of the net-worth shares for each idiosyncratic productivity level z 1 year after the shock, i.e. $\frac{\omega_{t=1}(z)-\omega_{ss}(z)}{\omega_{ss}(z)}$.

The blue solid line of panel a) in Figure 3 shows the excess investment rate $\tilde{\Phi}_{SS}(z)$ in steady state. For low values of productivity, this value is 0, since entrepreneurs with such low productivity prefer to remain inactive. From z^* onwards, entrepreneurs operate firms, and their profits increase linearly in productivity. The green solid line shows the excess investment rate one year after the implementation of the time-0 optimal policy. The cut-off z_1^* moves to the right. However, as already discussed, the impact of this threshold channel on TFP is quantitatively negligible. More importantly, the slope of the excess investment rate $\tilde{\Phi}_1(z)$ increases with the monetary policy shock. This implies that investment increases *relatively more* the more productive an entrepreneur is. That is, as high-productivity firms accumulate more profits they can undo financial frictions faster and operate at a larger scale, which improves the allocation of resources through the net-worth distribution channel. Panel 2 of Figure 3 displays the percent deviations of net-worth shares, $[\omega_1(z) - \omega_{ss}(z)] / \omega_{ss}(z)$, after 1 year. Firms with productivities slightly above one see their shares increase, whereas those below that threshold experience a decline. As a result, production now concentrates more on high-productivity firms.

Direct versus indirect effects. We decompose the impact of monetary policy

return on the net worth of the firm.

into direct and indirect effects, following Kaplan et al. (2018). *Direct effects* are those directly operating through the real interest rate. We discuss them first.

Holding everything else constant, a decrease in real interest rates increases misallocation on impact: a lower cost of capital makes production cheaper, but since the most productive entrepreneurs are constrained by the borrowing constraint, this reduction can only stimulate investment by entrepreneurs that would otherwise find it unprofitable to operate. This is the direct effect of monetary policy through the productivity threshold channel. A similar result was first illustrated numerically by Gopinath et al. (2017). In our simpler framework, we can prove it analytically: a fall in real interest rates decreases the productivity cut-off z^*, which, in turn, induces a decline in aggregate TFP. To see this, we plug the definition of z^* (equation 10) into the definition of TFP (equation 33), and take the partial derivative of TFP with respect to r_t, holding the other prices constant ($\varphi_t = \varphi$, $q_t = q$, $\dot{q} = 0$):

$$\frac{\partial \tilde{Z}_t}{\partial r_t} = \frac{\partial \tilde{Z}_t}{\partial z_t^*}\frac{\partial z_t^*}{\partial r_t} = \overbrace{\alpha\Gamma\left(\Gamma\mathbb{E}\left[z \mid z > z_t^*\right]\right)^{\alpha-1}\frac{\omega\left(z_t^*\right)}{\left(1-\Omega_t\left(z_t^*\right)\right)}\left(\mathbb{E}\left[z \mid z > z_t^*\right] - z_t^*\right)}^{\frac{\partial \tilde{Z}_t}{\partial z_t^*}}\overbrace{\frac{q}{\varphi\Gamma}}^{\frac{\partial z_t^*}{\partial r_t}} \geq 0. \tag{41}$$

The derivative of TFP with respect to the interest rate is always non-negative, and it is strictly positive as long as the distribution $\omega(z)$ is non-zero for $z > z_t^*$. This means that, *ceteris paribus*, if interest rates decrease so does TFP. Note that the term $\left(\mathbb{E}\left[z \mid z > z_t^*\right] - z_t^*\right)$ is a measure of the dispersion of productivity of active firms: the larger the difference between the average productivity of active firms and the cut-off productivity, the larger the impact of a change in interest rates is.

The previous result concerns the direct effect of monetary policy on TFP through the threshold channel, which operates exclusively through changes in the cut-off z_t^*. At impact, this is the only direct effect, since the net-worth distribution cannot change at impact. However, over time monetary policy may also have a second direct effect through the net-worth distribution channel, if it changes the net-worth distribution $\omega_t(z)$. It is easily verified that a real rate reduction increases the excess profit rate and hence the excess investment rate. However it does so by the same amount for all firms. Hence the real rate does not change the shape of the distribution of active firms. The direct effect of monetary policy on TFP though the net-worth distribution channel thus turns out to be 0.

Panel 1 of Figure 3 shows a decomposition of the partial-equilibrium impact of each of the prices on the excess investment rate $\tilde{\Phi}(z)$ one year after the shock. The red dotted line illustrates the two direct effects of monetary policy just discussed: The decrease in the real rate r_t shifts the excess investment rate function parallel to the left. The reduction in the real rate *ceteris paribus* makes capital cheaper and stimulates investment across all firms. This leads to no change in the average productivity among the firms above the steady state productivity threshold indicated by the vertical blue dashed (net-worth distribution effect) but crowds in less productive firms (threshold effect). The direct effect monetary policy is thus an increase in misallocation or, equivalently, a reduction in TFP.

In general equilibrium, the response of TFP depends not only on the direct effects of monetary policy through the real rate r_t explained above, but also on the *indirect effects* coming from changes in the other factor prices, namely the wage w_t, the price of capital q_t, and that of the input good m_t. Direct and indirect effects work both through the threshold channel and the net-worth distribution channel. Under our calibration, the change in the price of capital q_t (yellow dashed line in 3) has the exact opposite partial equilibrium effect to that of real rates. The increase in wages w_t (purple dashed-dotted line) both shifts the kink of the excess investment rate function to the right and decreases its slope, $\frac{\partial \tilde{\Phi}}{\partial w} < 0$. This reflects the reduction in returns as wages increase. However, the increase in the price of the input good m_t shifts z^* to the left and increases the slope of the return function significantly, $\frac{\partial \tilde{\Phi}}{\partial m} > 0$ (light blue dashed line). As the input-good price increases, firms' returns and investment go up, especially those of high-productivity firms. Which of these channels prevails is a quantitative question. For our particular calibration, the result (green solid line) is a tilt to the right, implying a rise in the threshold z_t^* and an increase in the slope of the profit function. Expansionary monetary policy thus reduces misallocation.[14] This quantitative result is robust to alternative realistic calibrations of the model. The bottom line is that, by expanding demand through a more accommodative monetary policy stance, the central bank increases the share of production carried out by high-productivity firms, reducing misallocation and increasing TFP. We test this model prediction below.

[14] It does so through both channels the threshold and the net-worth distribution channel, though the latter dominates as we showed before.

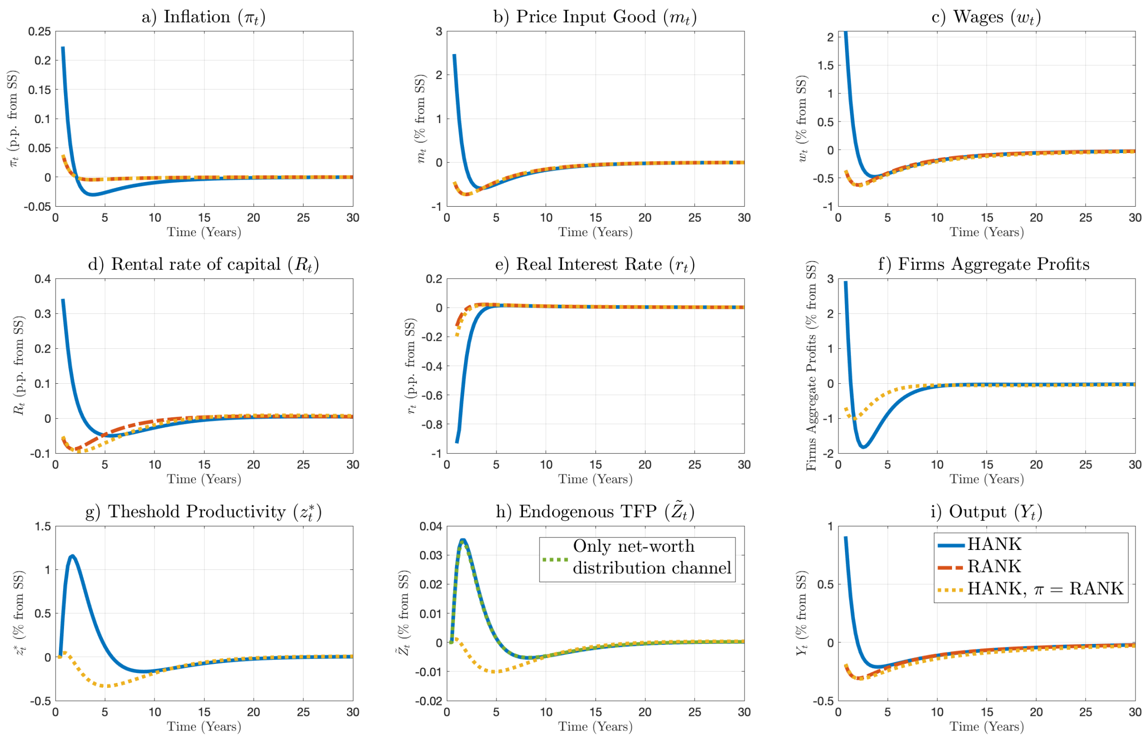

Figure 4: Optimal monetary policy response to a cost-push shock.

Notes: The figure shows the optimal response from a timeless perspective (in deviations from steady state) to a 10% decrease in the elasticity of substitution ϵ that is mean reverting with a yearly persistence of 0.8. RANK is the dashed red line, HANK is the blue solid line, the response of the HANK model when fed exogenously the path of Π obtained in the optimal policy of RANK is the yellow dotted line. In panel h, the green dotted line shows the change in endogenous TFP due to the net-worth distribution channel.

5.2 Timeless optimal policy response to a cost-push shock

We turn next to the timeless optimal response to shocks. Figure 4 shows the optimal timeless response of the central bank to a cost-push shock caused by a sudden unexpected temporary decrease in the elasticity of substitution (ϵ) of 10% that is mean-reverting with yearly persistence of 0.8. This shock increases retailers' markup, reducing the price of the goods sold by heterogeneous firms. Each panel shows the response of different equilibrium variables. The dashed red line in Figure 4 shows the optimal response in the RANK economy to this cost-push shock. This shock induces inflationary pressures due to the increase in markups (panel a). This induces a trade off between inflation and output gap stabilization. The central bank reacts optimally by driving output below its efficient level to dampen inflation (panels a and i). This is the well-known policy of *leaning against the wind* (Gali, 2008).

The optimal response of the monetary authority is, however, very different in the

HANK economy. In addition to the short-run trade-off between inflation and output gap, the central bank also influences misallocation and TFP in the medium run. This motivates the central bank to adopt a *leaning with the wind* policy. Instead of containing inflation at the cost of a fall in output, the central bank allows inflation to rise well above the optimal level in the RANK (solid blue line panel a). Thus, output increases (panel i). By increasing inflation, the central bank generates a demand expansion, increasing input good prices, wages, and real rental rates (panels b, c and d). This, in turn, increases the excess returns to capital (panel f), particularly for the most productive firms, allowing them to undo financial frictions faster, which increases endogenous TFP (panel h), aggregate capital and aggregate output.

To isolate the difference driven by the difference in policy from that driven by the different model structures of RANK and HANK, we also consider a scenario where the central bank in the HANK economy targets the suboptimal path for inflation of the RANK (dashed yellow dotted line). The behaviour of the economy in this case differs significantly from that under the optimal policy and is very similar to the RANK economy. Instead of an expansion, aggregate output falls below its steady state value, and so do profits, capital, and endogenous TFP. The policy conclusion is that the misallocation channel of monetary policy calls for a more dovish policy stance in the presence of cost push shocks.

6 Empirical evidence on the main mechanism

As outlined in Section 5.1, the difference in the optimal conduct of monetary policy relative to the RANK model is driven by the effect of monetary policy on misallocation, and thus on endogenous TFP. According to our model, a monetary expansion increases the profits of high-productivity firms relatively more than the profits of low-productivity ones. This net-worth distribution channel allows the most productive firms to increase their investment relatively more, which reduces misallocation and increases TFP. This result is, however, of quantitative nature: the overall effect of a monetary expansion on firm profitability and investment depends on the responses of different equilibrium prices, whose individual effects can be positive or negative. For the calibration considered here, the net effect is positive. Since this is the mechanism that drives our policy prescriptions, we now test empirically whether, in response to a monetary expansion, high-productivity firms indeed increase investment relatively more compared to low-productivity ones.

To address this question we consider an empirical application to the case of Spain, combining firm-level panel data with a time series measure of exogenous monetary policy shocks. We use yearly balance-sheet and cash-flow data of Spanish firms from 2000 to 2016 from the *Central de Balances Integrada* (see Appendix B.1 for further details on the data). This dataset covers the quasi-universe of Spanish firms, including large firms with access to stock and bond markets, but especially medium and small firms more reliant on bank credit and internal financing.

We use the marginal revenue product of capital ($MRPK_{j,t-1}$) as a proxy for firm level productivity. Note that, in our model,

$$MRPK_t = \frac{\partial m_t f_t(z,k,l^*)}{\partial k} = \left[\Gamma \left(\frac{1-\alpha}{w_t} \right)^{\frac{1-\alpha}{\alpha}} m_t^{\frac{1}{\alpha}} \right] z \propto z.$$

Since m and w are the same for all firms in our model, ranking firms according to MRPK is equivalent to ranking firms according to raw productivity z. We use MRPK for two reasons. First, it is a measure directly linked to capital productivity, and hence to investment in capital. Second, its computation from the data is straightforward and it does not rely on estimation.

The monetary policy shock ε_t^{MP} is taken from Jarociński and Karadi (2020). They use high-frequency data and sign restrictions in a SVAR to identify monetary policy shocks in the Euro area at the monthly level. The key idea behind their identification strategy is that movements of interest rates and stock markets within a narrow window around monetary policy announcements can help disentangle monetary policy shocks from information surprises. While an unexpected policy tightening raises interest rates and reduces stock prices, a positive central bank information shock (i.e. unexpected positive assessment of the economic outlook) raises both. We need to aggregate their shocks to yearly frequency, as in our data. We follow the methodology employed by Ottonello and Winberry (2020) to aggregate at a quarterly frequency. Appendix B.2 provides more details on the construction of the monetary policy shock.

In order to test whether productive firms' investment is more responsive, we follow the same specification as Ottonello and Winberry (2020), but focusing on heterogeneity in productivity instead of leverage,

$$\Delta log\ k_{j,t} = \alpha_j + \alpha_{s,t} + \beta \left(MRPK_{j,t-1} - \mathbb{E}_j \left[MRPK_j \right] \right) \varepsilon_t^{MP} + \Lambda' Z_{j,t-1} + u_{j,t}. \tag{42}$$

The dependent variable $\Delta log\ k_{j,t}$ is the log increase in the capital stock of firm j from $t-1$ to t. The key parameter of interest in equation (42) is the coefficient β multiplying the interaction term between productivity and the monetary policy shock . We demean $MRPK_{j,t-1}$ by the firm average across time $\mathbb{E}_j[MRPK_j]$ to ensure that the results are not driven by permanent heterogeneity in responsiveness across firms, as suggested in Ottonello and Winberry (2020). We lag $MRPK_{jt-1}$ to address reverse causality concerns. A positive value of β indicates that investment responds more to a monetary expansion in the case of high-productivity firms. We also include firm fixed effects (α_j) to capture permanent differences in investment patterns, sector-year fixed effects $(\alpha_{s,t})$ to control for aggregate shocks at the sector level, and a vector of controls $Z_{j,t-1}$ that includes the demeaned MRPK measure, total assets, sales growth, net financial assets as a share of total assets, and the interaction of demeaned MRPK with lagged GDP growth. The specification and the cleaning of the data is done following very closely Ottonello and Winberry (2020), see Appendix B.1 for further details.

Table 2: **Heterogeneous responses of investment to monetary policy in MRPK**

	(1)	(2)
ε_t^{MP} x MRPK_{t-1}	0.141**	0.201**
	(0.06)	(0.08)
Observations	5567706	3532022
R^2	0.267	0.304
MRPK control	YES	YES
Controls	NO	YES
Time-sector FE	YES	YES
Time-sector clustering	YES	YES

Notes: The table shows the coefficient β that results of estimating equation (42). Column (1) only includes the standardized demeaned MRPK as control, while column (2) introduces the all the controls $Z_{j,t-1}$ (standardized demeaned MRPK, total assets, sales growth, and net financial assets as a share of total assets; and the interaction of demeaned MRPK with lagged GDP growth). Standard errors are clustered at the sector-year level. We have normalized the sign of the monetary shock ε_t^{MP} so that a positive shock corresponds to a decrease in interest rates. We have standardize $(MRPK_{jt-1} - \mathbb{E}_j[MRPK])$ over the entire sample.

Table 2 shows the main results of the estimation. We perform the same normalization as in Ottonello and Winberry (2020), so that the coefficient of interest, β, is easily interpretable. First, we standardize $(MRPK_{jt-1} - \mathbb{E}_j[MRPK])$ over the entire sample, which implies that the units are standard deviations in our sample.

Second, we normalize the shock, so that the interpretation of β can be read as the response to an expansionary monetary policy shock of 100bps (or in other words, a decrease of 100bps in the EONIA rate). Results show that firms with high productivity, proxied by high MRPK, respond more to expansionary monetary policy shocks. Our baseline specification, column (2), shows that an expansionary monetary policy shock implies a 0.2 higher semi-elasticity of investment when it affects a firm one standard deviation more productive than the average in our sample (in terms of MRPK). When we do not include firm controls (column 1), this effect is still positive and significant, although of lower magnitude. Appendix B.3 shows that this result is robust to several alternative specifications. It is worth noticing that this heterogeneous response is not driven by changes in the composition of firms in the data, since keeping a balanced sample of firms, we finding even larger results (see Appendix B.3). This points at the heterogeneous changes in investment of incumbent firms being the main driver of the results.

Summing up, the empirical evidence supports the prediction that the impact of monetary policy on investment is increasing in the productivity of the firms, which is the key mechanism behind our optimal policy results.

7 Conclusions

This paper analyzes optimal monetary policy in a model with heterogeneous firms, financial frictions, and nominal rigidities. The model features a link between monetary policy and capital misallocation.

We identify a new source of time-inconsistency in monetary policy: though zero inflation is optimal in the long run, a benevolent central bank without pre-commitments engineers a temporary surprise expansion. It does so because the surprise expansion modifies equilibrium prices in such a way that capital misallocation is reduced. This is even though a drop in the real rate has an unambiguously negative *direct effect* on capital misallocation by crowding in low-productivity firms on impact (threshold channel). However, the associated general equilibrium changes in other factor prices favor high-productivity firms, allowing them to increase investment and grow faster (net-worth distribution channel). Overall the latter indirect effects dominate, and the capital allocation becomes more efficient. Using granular information about Spanish firms, we provide empirical evidence that this mechanism is also present in the data:

high-productivity firms are more responsive to monetary policy shocks. We illustrate how, when faced with a cost-push shock, the optimal prescription from a timeless perspective is to lean with the wind, tolerating more inflation in exchange for a boom in demand that will raise TFP further down the road.

The model presented in this paper abstracts from several relevant mechanisms driving firm dynamics, such as endogenous default, size-varying capital constraints, or decreasing returns to scale, among many others. This helps us to provide a clear understanding of the different forces shaping optimal monetary policy, as well as highlighting the similarities and differences with the standard representative agent New Keynesian model. A natural extension would be to add more of these features to study their impact on the optimal conduct of monetary policy.

The paper also makes what we deem as an important methodological contribution. It introduces a new algorithm to compute optimal policies in heterogeneous-agent models. The algorithm leverages on the numerical advantages of continuous time and will allow researchers to solve optimal policy in heterogeneous-agent models with or without aggregate shocks in an efficient and simple way using Dynare. It is our hope that this will spur a new wave of research on the normative implications of heterogeneous-agent models in the years to come.

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Online appendix

A Further details on the model

A.1 Entrepreneur's intertemporal problem

The Hamilton-Jacobi-Bellman (HJB) equation of the entrepreneur is given by

$$r_t V_t(z,a) = \max_{d_t \geq 0} d_t + s_t^a(z,a,d)\frac{\partial V}{\partial a} + \mu(z)\frac{\partial V}{\partial z} + \frac{\sigma^2(z)}{2}\frac{\partial^2 V}{\partial z^2} + \eta\left(q_t a_t - V_t(z,a)\right) + \frac{\partial V}{\partial t}.$$

We guess and verify a value function of the form $V_t(z,a) = \kappa_t(z)\, q_t a$. The first order condition is

$$\kappa_t(z) - 1 = \lambda_d \text{ and } \min\{\lambda_d, d_t\} = 0,$$

where $\lambda_d = 0$ if $\kappa_t(z) = 1$. If $\kappa_t(z) > 1\ \forall z, t$, then $d_t = 0$ and the firm does not pay dividends until it closes down. If this is the case, then the value of $\kappa_t(z)$ can be obtained from

$$\begin{aligned}&(r_t+\eta)\,\kappa_t(z)\, q_t = \\ &\eta q_t + (\gamma \max\{\Gamma z_t \varphi_t - R_t, 0\} + R_t - \delta q_t)\kappa_t(z) + \mu(z) q_t \frac{\partial \kappa_t}{\partial z} + \frac{\sigma^2(z)}{2} q_t \frac{\partial^2 \kappa_t}{\partial z^2} + \frac{\partial (q_t \kappa_t)}{\partial t}.\end{aligned} \tag{43}$$

Lemma. $\kappa_t(z) > 1\ \forall z, t$

Proof. The drift of the entrepreneur's capital holdings is

$$s_t^a = \frac{1}{q_t}\left[(\gamma \max\{\Gamma z_t \varphi_t - R_t, 0\} + R_t - \delta q_t\right] \geq \frac{R_t - \delta q_t}{q_t}$$

which is expected to hold with strict inequality eventually if $\exists\, \mathbb{P}(z_t \geq z_t^*) > 0$ (which is satisfied in equilibrium since z is unbounded), and hence

$$\mathbb{E}_0 a_t = \mathbb{E}_0 a_0 e^{\int_0^t s_u^a du} > a_0 e^{\int_0^t \frac{R_s - \delta q_s}{q_s} ds}. \tag{44}$$

The value function is then

$$\begin{aligned}\kappa_{t_0}(z)\, q_{t_0} a_{t_0} &= V_{t_0}(z, a_{t_0}) \\ &= \mathbb{E}_{t_0}\int_0^\infty e^{-\int_{t_0}^t (r_s+\eta) ds}\left(d_t + \eta q_t a_t\right) dt\end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{E}_{t_0}\int_0^\infty e^{-\int_{t_0}^t (r_s+\eta)ds}\eta q_t a_t dt = \mathbb{E}_{t_0}\int_0^\infty e^{-\int_{t_0}^t\left(\overbrace{\frac{R_s-\delta q_s+\dot{q}_s}{q_s}}^{r_s}+\eta\right)ds}\eta q_t a_t dt \\
&= \mathbb{E}_{t_0}\int_0^\infty e^{-\int_{t_0}^t\left(\frac{R_s-\delta q_s}{q_s}+\eta\right)ds-\log\frac{q_t}{q_{t_0}}}\eta q_t a_t dt = \mathbb{E}_{t_0}\int_0^\infty e^{-\int_{t_0}^t\left(\frac{R_s-\delta q_s}{q_s}+\eta\right)ds}\eta q_{t_0} a_t dt \\
&> \mathbb{E}_{t_0}\int_0^\infty e^{-\int_{t_0}^t\left(\frac{R_s-\delta q_s}{q_s}+\eta\right)ds}\eta q_{t_0} a_{t_0} e^{\int_0^t \frac{R_s-\delta q_s}{q_s}ds} dt = \int_0^\infty e^{-\eta t}\eta q_{t_0} a_{t_0} dt = q_{t_0} a_{t_0},
\end{aligned}
$$

where in the first equality we have employed the linear expression of the value function, in the second equation (5), in the third the fact that dividends are non-negative, in the fourth the definition of the real rate 24 and in the last line the inequality (44). Hence $\kappa_{t_0}(z) > 1$ for any t_0.

A.2 New Keynesian Philips curve

The proof is similar to that of Lemma 1 in Kaplan et al. (2018). The Hamilton-Jacobi-Bellman (HJB) equation of the retailer's problem is

$$
r_t V_t^r(p) = \max_\pi \left(\frac{p-P_t^y(1-\tau)}{P_t}\right)\left(\frac{p}{P_t}\right)^{-\varepsilon} Y_t - \frac{\theta}{2}\pi^2 Y_t + \pi p\frac{\partial V^r}{\partial p} + \frac{\partial V^r}{\partial t},
$$

where where $V_t^r(p)$ is the real value of a retailer with price p. The first order and envelope conditions for the retailer are

$$
\begin{aligned}
\theta\pi Y_t &= p\frac{\partial V^r}{\partial p}, \\
(r-\pi)\frac{\partial V^r}{\partial p} &= \left(\frac{p}{P_t}\right)^{-\varepsilon}\frac{Y_t}{P_t} - \varepsilon\left(\frac{p-P_t^y(1-\tau)}{P_t}\right)\left(\frac{p}{P_t}\right)^{-\varepsilon-1}\frac{Y_t}{P_t} + \pi p\frac{\partial^2 V^r}{\partial p^2} + \frac{\partial^2 V^r}{\partial t\partial p}.
\end{aligned}
$$

In a symmetric equilibrium we will have $p = P$, and hence

$$
\begin{aligned}
\frac{\partial V^r}{\partial p} &= \frac{\theta\pi Y_t}{p}, \qquad (45) \\
(r-\pi)\frac{\partial V^r}{\partial p} &= \frac{Y_t}{p} - \varepsilon\left(\frac{p-P_t^y(1-\tau)}{p}\right)\frac{Y_t}{p} + \pi p\frac{\partial^2 V^r}{\partial p^2} + \frac{\partial^2 V^r}{\partial t\partial p}.
\end{aligned}
$$

Deriving (45) with respect to time gives

$$\pi p \frac{\partial^2 V^r}{\partial p^2} + \frac{\partial^2 V^r}{\partial t \partial p} = \frac{\theta \pi \dot{Y}}{p} + \frac{\theta \dot{\pi} Y}{p} - \frac{\theta \pi^2 Y}{p},$$

and substituting into the envelope condition and dividing by $\frac{\theta Y}{p}$ we obtain

$$\left(r - \frac{\dot{Y}}{Y}\right)\pi = \frac{1}{\theta}\left(1 - \varepsilon\left(1 - \frac{P_t^y(1-\tau)}{p}\right)\right) + \dot{\pi}.$$

Finally, rearranging we obtain the New Keynesian Phillips curve

$$\left(r - \frac{\dot{Y}}{Y}\right)\pi = \frac{\varepsilon}{\theta}\left(\frac{1-\varepsilon}{\varepsilon} + \tilde{m}\right) + \dot{\pi}.$$

A.3 Capital producers

The problem of the capital producer is

$$W_t = \max_{\iota_t, K_t} \mathbb{E}_0 \int_0^\infty e^{-\int_0^t r_s ds} \left(q_t \iota_t - \iota_t - \Phi\left(\iota_t\right)\right) K_t dt. \tag{46}$$

$$\dot{K}_t = \left(\iota_t - \delta\right) K_t, \tag{47}$$

We construct the Hamiltonian

$$H = \left(q_t \iota_t - \iota_t - \Phi\left(\iota_t\right)\right) K_t + \lambda_t \left(\iota_t - \delta\right) K_t$$

with first-order conditions

$$\left(q_t - 1 - \Phi'\left(\iota_t\right)\right) + \lambda_t = 0 \tag{48}$$

$$\left(q_t \iota_t - \iota_t - \Phi\left(\iota_t\right)\right) + \lambda_t \left(\iota_t - \delta\right) = r_t \lambda_t - \dot{\lambda}_t \tag{49}$$

Taking the time derivative of equation (48)

$$\dot{\lambda}_t = -\left(\dot{q}_t - \Phi''\left(\iota_t\right) i_t\right)$$

which, combined with (49), yields

$$\left(q_t\iota_t-\iota_t-\Phi\left(\iota_t\right)\right)-\left(q_t-1-\Phi'\left(\iota_t\right)\right)\left(\iota_t-\delta-r_t\right)=\left(\dot{q}_t-\Phi''\left(\iota_t\right)i_t\right)$$

Rearranging we get

$$r_t=\left(\iota_t-\delta\right)+\frac{\dot{q}_t-\Phi''\left(\iota_t\right)i_t}{q_t-1-\Phi'\left(\iota_t\right)}-\frac{q_t\iota_t-\iota_t-\Phi\left(\iota_t\right)}{q_t-1-\Phi'\left(\iota_t\right)}.$$

A.4 Household's problem

We can rewrite the household's problem as

$$W_t=\max_{C_t,L_t,D_t,B_t^N,S_t^N}\mathbb{E}_0\int_0^{\infty}e^{-\rho_t^h t}\left(\frac{C_t^{1-\zeta}}{1-\zeta}-\Upsilon\frac{L_t^{1+\vartheta}}{1+\vartheta}\right)dt. \tag{50}$$

$$\begin{aligned} s.t.\quad \dot{D}_t &= \left[\left(R_t-\delta q_t\right)D_t+w_tL_t-C_t-S_t^N+\Pi_t\right]/q_t, &(51)\\ \dot{B_t^N} &= S_t^N+\left(i_t-\pi_t\right)B_t^N, &(52)\end{aligned}$$

The Hamiltonian is

$$\begin{aligned}H&=\left(\frac{C_t^{1-\zeta}}{1-\zeta}-\Upsilon\frac{L_t^{1+\vartheta}}{1+\vartheta}\right)\\ &+\varrho_t\left[\left(\left(R_t-\delta q_t\right)D_t+w_tL_t-C_t-S_t^N+\left(q_t\iota_t-\iota_t-\Phi\left(\iota_t\right)\right)K_t+\Pi_t\right)/q_t\right]+\eta_t\left[S_t^N+\left(i_t-\pi_t\right)B_t^N\right]\end{aligned}$$

The first order conditions are

$$C_t^{-\zeta}-\varrho_t/q_t=0 \tag{53}$$

$$-\Upsilon L_t^{\vartheta}+\varrho_t w_t/q_t=0 \tag{54}$$

$$-\varrho_t/q_t+\eta_t=0 \tag{55}$$

$$\dot{\varrho}_t=\rho_t^h\varrho_t-\varrho_t\left(R_t-\delta q_t\right)/q_t \tag{56}$$

$$\dot{\eta}_t = \rho_t^h \eta_t - \eta_t \left[(i_t - \pi_t) \right] \tag{57}$$

(53) and (54) combine to the optimality condition for labor

$$w_t = \frac{L_t^{\vartheta}}{C_t^{-\eta}},$$

(53) can be rewritten as

$$\varrho_t = C_t^{-\eta} q_t$$

Now take derivative with respect to time

$$\dot{\varrho}_t = -\eta C_t^{-\eta-1} \dot{C}_t q_t + C_t^{-\eta} \dot{q}_t$$

and plug this into (56) and rearrange to get the first Euler equation

$$\frac{\dot{C}_t}{C_t} = \frac{\frac{R_t - \delta q_t + \dot{q}_t}{q_t} - \rho_t^h}{\eta}$$

(55) can be rewritten as

$$\eta_t = \varrho_t / q_t$$

Now take derivative with respect to time

$$\dot{\eta}_t = \frac{\dot{\varrho}_t q_t - \varrho_t \dot{q}}{q_t^2}$$

Use these two expressions and the definition of $\dot{\varrho}_t$ in (57) to get the second Euler equation

$$\frac{\dot{C}_t}{C_t} = \frac{(i_t - \pi_t) - \rho_t^h}{\eta}$$

Combining the two Euler equations, we get the Fisher equation

$$\frac{R_t - \delta q_t + \dot{q}_t}{q_t} = (i_t - \pi_t)$$

Finally using the definition of $r_t \equiv \frac{R_t - \delta q_t + \dot{q}_t}{q_t}$ we can rewrite the first Euler equation and the Fisher equation as in the main text.

A.5 Distribution

The joint distribution of net worth and productivity is given by the Kolmogorov Forward equation

$$\frac{\partial g_t(z,a)}{\partial t} = -\frac{\partial}{\partial a}[g_t(z,a)s_t(z)a] - \frac{\partial}{\partial z}[g_t(z,a)\mu(z)] + \frac{1}{2}\frac{\partial^2}{\partial z^2}[g_t(z,a)\sigma^2(z)] - \eta g_t(z,a) + \eta/\psi g_t(z,a/\psi), \tag{58}$$

where $1/\psi g_t(z,a/\psi)$ is the distribution of entry firms.

To characterize the law of motion of net-worth shares, defined as $\omega_t(z) = \frac{1}{A_t}\int_0^\infty a g_t(z,a)da$, first we take the derivative of $\omega_t(z)$ wrt time

$$\frac{\partial \omega_t(z)}{\partial t} = -\frac{\dot{A}_t}{A_t^2}\int_0^\infty a g_t(z,a)da + \frac{1}{A_t}\int_0^\infty a\frac{\partial g_t(z,a)}{\partial t}da. \tag{59}$$

Next, we plug in the derivative of $g_t(z,a)$ wrt time from equation(58) into equation (59),

$$\begin{aligned}\frac{\partial \omega_t(z)}{\partial t} =& -\frac{\dot{A}_t}{A_t^2}\int_0^\infty a g_t(z,a)da + \frac{1}{A_t}\int_0^\infty a\left(-\frac{\partial}{\partial a}[g_t(z,a)s_t(z)a]\right)da \\ &- \frac{\partial}{\partial z}\mu(z)\frac{1}{A_t}\int_0^\infty a g_t(z,a)da + \frac{1}{2}\frac{\partial^2}{\partial z^2}\sigma^2(z)\frac{1}{A_t}\int_0^\infty a g_t(z,a)da \\ &- \frac{1}{A_t}\int_0^\infty \eta a g_t(z,a)da + \frac{1}{A_t}\int_0^\infty \eta a/\psi g_t(z,a/\psi)da.\end{aligned}$$

Using integration by parts and the definition of net worth shares, we obtain the second order partial differential equation that characterizes the law of motion of net-worth shares,

$$\frac{\partial \omega_t(z)}{\partial t} = \left[s_t(z) - \frac{\dot{A}_t}{A_t} - (1-\psi)\eta\right]\omega_t(z) - \frac{\partial}{\partial z}\mu(z)\omega_t(z) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\sigma^2(z)\omega_t(z). \tag{60}$$

The stationary distribution is therefore given by the following second order partial differential equation,

$$0 = (s(z) - (1-\psi)\eta)\,\omega(z) - \frac{\partial}{\partial z}\mu(z)\omega(z) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\sigma^2(z)\omega(z). \tag{61}$$

Remember that $s_t^a(z_t, a_t, c_t) = \frac{1}{q_t}\left[\Phi_t(z_t, a_t) + (R_t - \delta q_t)a_t\right]$, since entrepreneurs distribute zero dividends while active.

A.6 Market clearing and aggregation

Define the cumulative function of net-worth shares as

$$\Omega_t(z) = \int_0^z \omega_t(z)dz. \tag{62}$$

Using the optimal choice for k_t from equation (7), we obtain

$$K_t = \int k_t(z,a)dG_t(z,a) = \int_{z_t^*}^{\infty}\int \gamma a \frac{1}{A_t} g_t(z,a)dadzA_t = \gamma(1-\Omega(z_t^*))A_t. \tag{63}$$

By combining equations (28), (29) and (63), and solving for A_t,we obtain

$$A_t = \frac{D_t}{(1-\Omega(z_t^*))-1}, \tag{64}$$

Labor market clearing implies

$$L_t = \int_0^{\infty} l_t(z,a)dG_t(z,a). \tag{65}$$

Define the following auxiliary variable,

$$X_t \equiv \int_{z_t^*}^{\infty} z\omega_t(z)dz = \mathbb{E}\left[z \mid z > z_t^*\right](1-\Omega(z_t^*)). \tag{66}$$

Using labor demand from (8) , X_t and using the definition of φ_t, we obtain

$$L_t = \int_0^{\infty}\left(\frac{\varphi_t}{\alpha m_t}\right)^{\frac{1}{1-\alpha}}\Gamma z_t \gamma a_t dG_t(z,a) = \left(\frac{\varphi_t}{\alpha m_t}\right)^{\frac{1}{1-\alpha}}\gamma\Gamma A_t X_t. \tag{67}$$

Plugging in (8) into production function (1), and using again the definition of shares, we obtain

$$Y_t = \int \underbrace{\frac{\Gamma z_t \varphi_t}{\alpha m_t}\gamma a}_{y_t(z,a)} dG_t(z,a) = \Gamma\frac{\varphi_t}{\alpha m_t}X_t\gamma A_t = Z_t A_t^{\alpha} L_t{}^{1-\alpha}, \tag{68}$$

where in the last equality we have used equation (67), and we have defined

$$Z_t = (\Gamma \gamma X_t)^{\alpha}. \tag{69}$$

Aggregate profits of retailers are given by

$$\Phi_t^{Agg} = \int \gamma \max\{\Gamma z_t \varphi_t - R_t, 0\} a_t dG_t(z,a) = [\Gamma \varphi_t X_t - R_t (1-\Omega(z^*))] \gamma A_t. \tag{70}$$

We can also write the aggregate production in terms of physical capital,

$$Y_t = \tilde{Z}_t K_t^{\alpha} L_t^{1-\alpha}, \tag{71}$$

where the TFP term $\tilde{Z}_t$ si defined as

$$\tilde{Z}_t = \left(\frac{\Gamma X_t}{(1-\Omega(z_t^*))}\right)^{\alpha} = (\Gamma \mathbb{E}[z \mid z > z_t^*])^{\alpha}. \tag{72}$$

Aggregating the budget constraint of all input good firms, using the linearity of savings policy (11) and using (64), we obtain

$$\begin{aligned}\dot{A}_t &= \int \dot{a} dG(z,a,t) - \eta \int (1-\psi) a_t dG(z,a,t) \; = \\ &= \int_0^{\infty} \frac{1}{q_t}(\gamma \max\{\Gamma z_t \varphi_t - R_t, 0\} + R_t - \delta q_t - q_t(1-\psi)\eta) a_t dG(z,a),\end{aligned}$$

Dividing by A_t both sides of this equation, using the definition of net worth shares and the fact that these integrate up to one, we obtain

$$\frac{\dot{A}_t}{A_t} = \frac{1}{q_t}(\gamma \varphi_t \Gamma X_t - R_t \gamma (1-\Omega(z_t^*)) + R_t - \delta q_t - q_t(1-\psi)\eta). \tag{73}$$

Using the definition of X_t, and substituting φ_t using equation (67), we can simplify equation (73) as

$$\frac{\dot{A}_t}{A_t} = \frac{1}{q_t}(\alpha m_t Z_t A_t{}^{\alpha-1} L_t{}^{1-\alpha} - R_t \gamma (1-\Omega(z_t^*)) + R_t - \delta q_t - q_t(1-\psi)\eta). \tag{74}$$

Finally, we can obtain factor prices

$$
\begin{aligned}
w_t =&(1-\alpha) m_t Z_t {A_t}^{\alpha} {L_t}^{-\alpha} && (75)\\
R_t =&\alpha m_t Z_t A_t^{\alpha-1} {L_t}^{1-\alpha} \frac{z_t^*}{\gamma X_t} && (76)
\end{aligned}
$$

where wages come from substituting the definition of φ_t into equation (67); and interest rates come from plugging in the wage expression (75) into the cut-off rule (10) and using equation (64). We could equivalently write equation (76) in terms of real rate of return r_t :

$$
r_t = \frac{1}{q_t}\left(\alpha m_t Z_t A_t^{\alpha-1} {L_t}^{1-\alpha} \frac{z_t^*}{\gamma X_t}\right) - \delta + \frac{\dot{q}}{q_t} \tag{77}
$$

We can easily get these equations in terms of capital instead of net worth by simply using equation (63), i.e. $A_t = \frac{K_t}{\gamma(1-\Omega(z_t^*))}$, and using that $\mathbb{E}\left[z \mid z > z_t^*\right] = \frac{X_t}{(1-\Omega(z_t^*))} = \frac{\int_{z_t^*}^{\infty} z\omega_t(z)dz}{(1-\Omega(z_t^*))}$ (see equation (69) and (72)).

A.7 Full set of equations

The competitive equilibrium economy is described by the following 22 equations, for the 22 variables $\left\{\omega(z), s(z), w, r, q, \varphi, K, A, L, C, D, \tilde{Z}, \mathbb{E}\left[z \mid z > z_t^*\right], \Omega, z^*, \iota, \pi, m, \tilde{m}, i, Y, T\right\}$. Remember that $\mu(z) = z\left(-\varsigma_z \log z + \frac{\sigma^2}{2}\right)$ and $\sigma(z) = \sigma_z z$, and that government bonds are in zero net supply ($B_t^N = 0$, hence $X_t = 0$). Except from the last equation (Taylor rule), the other 21 equations are the constraints of the Ramsey problem described in Section 2.8.

$$
\begin{aligned}
\frac{\partial \omega_t\left(z\right)}{\partial t} &= \left(s_t(z) - (1-\psi)\eta - \frac{\dot{A}_t}{A_t}\right)\omega_t\left(z\right) - \frac{\partial}{\partial z}\left[\mu(z)\omega_t\left(z\right)\right] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[\sigma^2(z)\omega_t\left(z\right)\right] \\
s_t(z) &= \frac{1}{q_t}(\gamma \max\left\{\Gamma z_t \varphi_t - R_t, 0\right\} + R_t - \delta q_t) \\
\Omega_t(z^*) &= \int_0^{z^*} \omega_t\left(z\right) dz \\
\varphi_t &= \alpha\left(\frac{(1-\alpha)}{w_t}\right)^{(1-\alpha)/\alpha} m_t^{\frac{1}{\alpha}} \\
\tilde{m}_t &= m_t(1-\tau) \\
w_t &= (1-\alpha)m_t\tilde{Z}_t {K_t}^{\alpha} {L_t}^{-\alpha} \\
R_t &= \alpha m_t \tilde{Z}_t K_t^{\alpha-1} {L_t}^{1-\alpha}\frac{z_t^*}{\mathbb{E}\left[z \mid z > z_t^*\right]} \\
\frac{\dot{A}_t}{A_t} &= \frac{1}{q_t}\left[\gamma(1-\Omega(z_t^*))\left(\alpha m_t \tilde{Z}_t K_t^{\alpha-1}{L_t}^{1-\alpha} - R_t\right) + R_t - \delta q_t - q_t(1-\psi)\eta)\right] \\
K_t &= A_t + D_t \\
\dot{K}_t &= (\iota_t - \delta)K_t \\
A_t &= \frac{D_t}{\gamma(1-\Omega(z_t^*)) - 1} \\
\tilde{Z}_t &= \left(\Gamma\mathbb{E}\left[z \mid z > z_t^*\right]\right)^{\alpha} \\
\mathbb{E}\left[z \mid z > z_t^*\right] &= \frac{\int_{z_t^*}^{\infty} z\omega_t(z)dz}{(1-\Omega(z_t^*))} \\
\frac{\dot{C}_t}{C_t} &= \frac{r_t - \rho_t^h}{\eta} \\
w_t &= \frac{\Upsilon L_t^{\vartheta}}{C_t^{-\eta}} \\
\dot{D}_t &= \left[\left(R_t - \delta q_t\right)D_t + w_t L_t - C_t + T_t\right]/q_t \\
r_t &= i_t - \pi_t \\
r_t &= \frac{R_t - \delta q_t + \dot{q}_t}{q_t} \\
\left(q_t - 1 - \Phi'\left(\iota_t\right)\right)\left(r_t - \left(\iota_t - \delta\right)\right) &= \dot{q}_t - \Phi''\left(\iota_t\right)\dot{\iota}_t - \left(q_t\iota_t - \iota_t - \Phi\left(\iota_t\right)\right) \\
\left(r_t - \frac{\dot{Y}_t}{Y_t}\right)\pi_t &= \frac{\varepsilon}{\theta}\left(\tilde{m}_t - m^*\right) + \dot{\pi}_t, \qquad m^* = \frac{\varepsilon - 1}{\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
Y_t &= \tilde{Z}_t K_t^{\alpha} {L_t}^{1-\alpha} \\
T_t &= (1-m_t)\, Y_t - \frac{\theta}{2}\pi_t^2 Y_t + (1-\psi)\eta A_t + \left[\iota_t q_t - \iota_t - \frac{\phi^k}{2}(\iota_t - \delta)^2\right] K_t \\
di &= -\upsilon \left(i_t - \left(\rho_t^h + \phi\left(\pi_t - \bar{\pi}\right) + \bar{\pi}\right)\right) dt.
\end{aligned}
$$

A.8 RANK vs HANK

In this appendix we want to highlight the differences between the heterogeneous-agent New Keynesian model (HANK) presented in this paper and the standard representative agent New Keynesian model with capital (RANK). Note first that the HANK economy collapses to the standard RANK economy if the borrowing constraint is made infinitely slack (assuming that the support of entrepreneurs productivity distribution is bounded above). In that case entrepreneurial net worth becomes irrelevant and only the entrepreneur with the highest level of productivity z_t produces, since she can frictionlessly rent all the capital in the economy. Her productivity determines aggregate productivity $\tilde{Z}_t = (z_t^{max}\Gamma)^{\alpha}$.In contrast, in the HANK model with incomplete markets, entrepreneurs' firms can only use capital up to a multiple γ of their net worth , i.e. $\gamma a_t \leq k_t$. Thus entrepreneurs need to accumulate net worth (in units of capital) to alleviate these financial frictions. Hence, in the HANK model, the distribution of aggregate capital across entrepreneurs and the representative household matters and aggregate productivity depends on the expected productivity of active firms, $\tilde{Z} = (\Gamma\mathbb{E}\left[z \mid z > z_t^*\right])^{\alpha}$. The rest of the agents (retailers, final good producers, capital producers) are identical in both economies.

Below we report the equilibrium conditions in the RANK economy. Comparing them with those of the HANK economy reveals that they are identical up to the fact that in HANK $\tilde{Z}_t$ is endogenous (and determined by a bunch of extra equations) and up to a term in the condition equating the rental rate of capital R_t with the marginal return on capital.

The competitive equilibrium of the RANK model with capital consists of the following equations 16 equations, for the 16 variables $\left\{w, r, q, \varphi, K, L, C, D, \tilde{Z}, \iota, \pi, m, \tilde{m}, i, Y, T\right\}$:

$$
\begin{aligned}
\varphi_t &= \alpha \left(\frac{(1-\alpha)}{w_t} \right)^{(1-\alpha)/\alpha} m_t^{\frac{1}{\alpha}} \\
\tilde{m}_t &= m_t(1-\tau) \\
w_t &= (1-\alpha) m_t \tilde{Z}_t {K_t}^{\alpha} {L_t}^{-\alpha} \\
R_t &= \alpha m_t \tilde{Z}_t K_t^{\alpha-1} {L_t}^{1-\alpha} \\
K_t &= D_t \\
\dot{K}_t &= (\iota_t - \delta) K_t \\
\tilde{Z}_t &= (\Gamma_t)^{\alpha} \\
\frac{\dot{C}_t}{C_t} &= \frac{r_t - \rho_t^h}{\eta} \\
w_t &= \frac{\Upsilon L_t^{\vartheta}}{C_t^{-\eta}} \\
\dot{D}_t &= \left[\left(R_t - \delta q_t \right) D_t + w_t L_t - C_t + T_t \right] / q_t \\
r_t &= i_t - \pi_t \\
r_t &= \frac{R_t - \delta q_t + \dot{q}_t}{q_t} \\
\left(q_t - 1 - \Phi'\left(\iota_t\right) \right) \left(r_t - \left(\iota_t - \delta \right) \right) &= \dot{q}_t - \Phi''\left(\iota_t\right) i_t - \left(q_t \iota_t - \iota_t - \Phi\left(\iota_t\right) \right) \\
\left(r_t - \frac{\dot{Y}_t}{Y_t} \right) \pi_t &= \frac{\varepsilon}{\theta} \left(\tilde{m}_t - m^* \right) + \dot{\pi}_t, \qquad m^* = \frac{\varepsilon - 1}{\varepsilon} \\
Y_t &= \tilde{Z}_t K_t^{\alpha} {L_t}^{1-\alpha} \\
T_t &= \left(1 - m_t \right) Y_t - \frac{\theta}{2} \pi_t^2 Y_t + \left[\iota_t q_t - \iota_t - \frac{\phi^k}{2} (\iota_t - \delta)^2 \right] K_t \\
di &= -\upsilon \left(i_t - \left(\rho_t^h + \phi \left(\pi_t - \bar{\pi} \right) + \bar{\pi} \right) \right) dt.
\end{aligned}
$$

B Empirical Appendix

B.1 Firm level data

The empirical exercise relies on annual firm balance-sheet data from the *Central de Balances Integrada* database (Integrated Central BalanceSheet Data Office Survey). Being a detailed administrative dataset, the main advantage is that it covers the quasi-universe of Spanish firms (see Almunia et al., 2018 for further details on the representativeness of this dataset). Our dependent variable, the investment rate, is defined as the log difference of firm's tangible capital between periods t and $t-1$. Firm's marginal revenue product of capital (MRPK) is the log of the ratio of value added over tangible capital. Leverage is computed as total debt (short-term plus long-term debt) divided by total assets. Net financial assets are constructed as the log difference between financial assets and financial liabilities, where financial assets include short-term financial investment, trade receivables, inventories and cash holdings; and financial liabilities include short-term debt, trade payables and long-term debt. We proxy for size using log total assets. Real revenue growth is defined as the log difference of revenue in two consecutive years. Variables are deflated using industry price level to preserve the firm's level price changes and consider a revenue-based measure of MRPK (Foster et al., 2008). We use the value-added price deflator for value added and revenues, and the investment price deflator for capital and total assets. Descriptive statistics are reported in Table 3.

Data is cleaned following closely Ottonello and Winberry (2020). In particular, (i) observations with negative capital or value added are dropped; (ii) the investment rate and MRPK are winsorized at 0.5%; (ii) we use net financial assets over as a share of total assets to control for firms' savings, following Armenter and Hnatkovska, 2017, instead of net current assets (as Ottonello and Winberry (2020) do), and we drop values in absolute terms greater than 10; and (iii) negative values of leverage are dropped, as well as values higher than 10. While Ottonello and Winberry (2020) drop firms for which the time spell is shorter than 10 years, we prefer to consider the full sample of firms without imposing an arbitrary threshold, and we show that our results are robust considering a balanced sample where we keep only firms that are present in our dataset for the whole time period considered.

Table 3: **Descriptive statistics**

	mean	sd	min	max
ε_t^{MP}	-3.06	7.40	-17.99	7.94
ε_t^{MP} x $MRPK_{t-1}$	-0.00	0.08	-1.60	1.82
$MRPK_t$	-0.00	1.00	-10.09	10.25
g^{tGDP} x $MRPK_{t-1}$	0.22	3.07	-40.36	46.81
$MRPK_t$ (lev)	0.51	2.09	-5.47	6.22
Sales growth	0.00	1.00	-17.84	13.56
Total assets	0.00	1.00	-5.57	7.07
Leverage	-0.00	1.00	-0.57	25.95
Observations	9485676			

Notes: The table shows the mean (column 1), standard deviation (column 2), minimum and maximum value (column 3 and 4 respectively) of the main variables used in the analysis. ε_t^{MP} is the annualized monetary policy shock, renormalized so that a positive value is an expansionary shock.MRPK stands for the demeaned measure of MRPK explained in Section 6. MRPK, sales growth, total assets and leverage are standardized, as in Ottonello and Winberry (2020). MRPK (lev) is the raw variable of MRPK. g_t^{GDP} stands for GDP growth.

B.2 Monetary policy shocks

We construct our yearly monetary policy shocks aggregating the monthly monetary policy shocks of Jarociński and Karadi (2020). Since firms have less time to react to shocks happening at the end of the year, ignoring this issue would lead to biased estimates. Therefore, similar to Ottonello and Winberry (2020), but on a month-year level instead of month-quarter, we apply a weighting scheme that aggregates the shocks happening in the fourth quarter of the previous year with increasing linear weight, and uses linear and decreasing weights in the current year. Namely, we add them using decreasing weights within the year $\omega_a(m)$, and increasing weights in the last quarter of the previous year $\omega_b(m)$, i.e.

$$\varepsilon_t^{MP} = \sum_{m \in t} \omega_a(m) \varepsilon_m^{MP} + \sum_{m \in q4_{t-1}} \omega_b(m) \varepsilon_m^{MP}.$$

This is equivalent to say that a shock in January of period t has more weight than a shock in December of the same year, exactly because firms take time to adjust their investment plans. Panel 1 of Figure 5 shows the time series of the shock built in this way.

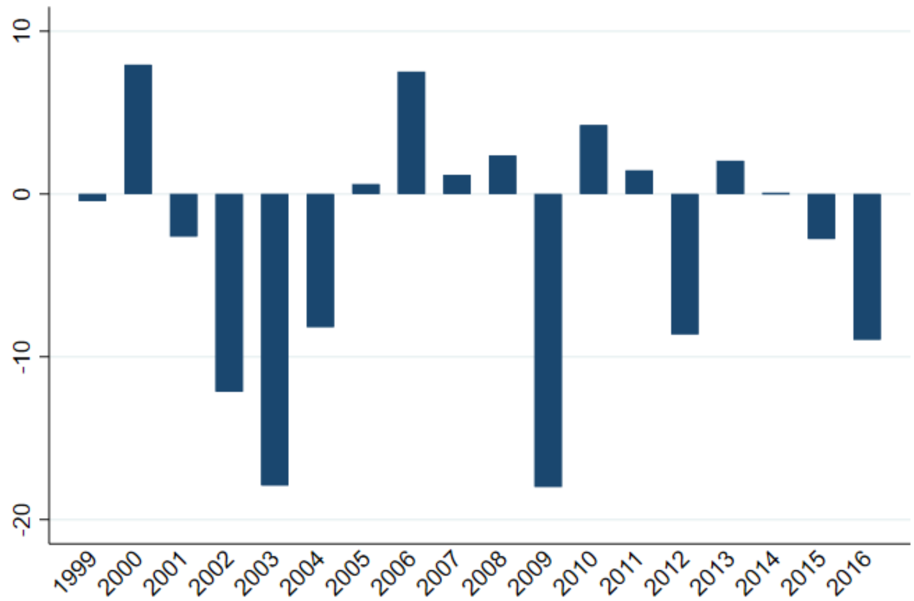

Figure 5: Monetary policy shocks at annual frequency.

Notes: The figure shows the monetary policy shocks at an annual frequency, applying a weighting scheme at aggregation that includes the shock in the fourth quarter of the previous period with an increasing linear weight and uses linear and decreasing weights in the current year.

B.3 Robustness

In this section we check the robustness of our empirical results. We perform variations of the main empirical specification explained in the main text, equation (42), which we repeat here for the sake of completeness.

$$\Delta log\ k_{j,t} = \alpha_j + \alpha_{s,t} + \beta \left(MRPK_{j,t-1} - \mathbb{E}_j \left[MRPK_j \right] \right) \varepsilon_t^{MP} + \Gamma' Z_{j,t-1} + u_{j,t}.$$

Following Ottonello and Winberry (2020) and Eberly et al. (2012), we control for the lagged of the dependent variable, i.e. firms' lagged investment rate, since it has been shown that it is a good predictor of a firm's current investment. Columns (1) and (2) in Table 4 show that results are robust to adding this variable, even stronger in magnitude, and R^2 does not change significantly. Columns (3) and (4) in Table 4 show the results using the baseline monetary policy shock ε^{MP}, but interacting this shock with the lagged MRPK in levels, instead of the demeaned standardized measure. We see that the coefficients are still positive and significant. Finally, we estimate the baseline equation (42) considering the balanced panel, i.e. keeping only firms that we observe during the entire time sample period, in order to focus on pure incumbents. Columns (5) and (6) in Table 4 not only confirm the baseline results, but show that the effect can be even larger for incumbent firms. All these robustness exercises point at a higher heterogeneous response of investment for high MRPK firms.

Table 4: **Robustness**

	(1)	(2)	(3)	(4)	(5)	(6)
ε_t^{MP} x $MRPK_{t-1}$	0.238***	0.332***			0.177**	0.250***
	(0.06)	(0.10)			(0.07)	(0.09)
inv_{t-1}	-0.0310***	-0.0280***				
	(0.00)	(0.00)				
ε_t^{MP} x $MRPK_{t-1}$ (lev)			0.419***	0.458***		
			(0.05)	(0.06)		
Observations	4162114	3023427	5551870	3527360	283835	225976
R^2	0.279	0.314	0.275	0.313	0.153	0.181
MRPK control	YES	YES	YES	YES	YES	YES
Controls	NO	YES	NO	YES	NO	YES
Time-sector FE	YES	YES	YES	YES	YES	YES
Time-sector clustering	YES	YES	YES	YES	YES	YES
Panel	FULL	FULL	FULL	FULL	BALANCED	BALANCED

Notes: Results of estimating equation (42), departing from some of the specifications of the estimation in the main text (Section 6). Columns (1) and (2) include as control the lag of the investment rate ($log(k_{t-1}) - log(k_{t-2})$). Columns (3) and (4) use MRPK in levels, $MRPK$ (lev), instead of the demeaned standardized value. Columns (1) , (3) and (5) use only MRPK as controls, while columns (2), (4) and (6) include all the controls: MRPK, total assets, sales growth, and net financial assets as a share of total assets; and the interaction of MRPK with lagged GDP growth. Columns (1),(2), (5) and (6) use the demeaned standardized measure of MRPK explained in the main text, while columns (3)-(4) use MRPK in levels.

C Numerical Appendix

C.1 Finite difference approximation of the Kolmogorov Forward equation

The KF equation is solved by a finite difference scheme following Achdou et al. (2017). It approximates the density $\omega_t(z)$ on a finite grid $z \in \{z_1, ..., z_J\}$, $t \in \{t_1, ..., t_N\}$ with steps Δz and time steps Δt. We use the notation $\omega_j^n := \omega_{n\Delta t}(z_j)$, $j = 1, ..., J$, $n = 0, .., N$. The KF equation is then approximated as

$$\frac{\omega_j^n - \omega_j^{n-1}}{\Delta t} = \left(s_n(z_j) - \frac{\dot{A}_n}{A_n} - (1-\psi)\eta \right) \omega_n(z_j) \\ - \frac{\omega_j^n \mu(z_j) - \omega_{j-1}^n \mu(z_{j-1})}{\Delta z} + \frac{\omega_{j+1}^n \widetilde{\sigma}^2(z_{j+1}) + \omega_{j-1}^n \widetilde{\sigma}^2(z_{j-1}) - 2\omega_j^n \widetilde{\sigma}^2(z_j)}{2\left(\Delta z\right)^2},$$

which, grouping, results in

$$\frac{\omega_j^n - \omega_j^{n-1}}{\Delta t} = \underbrace{\left[\left(s_n(z_j) - \frac{\dot{A}_n}{A_n} - (1-\psi)\eta \right) - \frac{\mu(z_j)}{\Delta z} - \frac{\widetilde{\sigma}^2(z_j)}{\left(\Delta z\right)^2} \right]}_{\beta_j^n} \omega_n(z_j) \\ + \underbrace{\left[\frac{\mu(z_{j-1})}{\Delta z} + \frac{\widetilde{\sigma}^2(z_{j-1})}{2\left(\Delta z\right)^2} \right]}_{\varrho_{j-1}^n} \omega_{j-1}^n + \underbrace{\left[\frac{\widetilde{\sigma}^2(z_{j+1})}{2\left(\Delta z\right)^2} \right]}_{\chi_{j+1}^n} \omega_{j+1}^n.$$

The boundary conditions are the ones associated with a reflected process z at the boundaries:[15]

$$\frac{\omega_1^n - \omega_1^{n-1}}{\Delta t} = \left(\beta_1^n + \chi_1^n\right) \omega_n(z_1) + \chi_2^n \omega_{j+1}^n,$$
$$\frac{\omega_J^n - \omega_J^{n-1}}{\Delta t} = \left(\beta_J^n + \varrho_J^n\right) \omega_n(z_J) + \varrho_{J-1}^n \omega_{j-1}^n.$$

[15] It is easy to check that this formulation preserves the fact that matrix $\mathbf{B}^n$ below is the transpose of the matrix associated with the infinitesimal generator of the process.

If we define matrix

$$\mathbf{B}^n = \begin{bmatrix} \beta_1^n + \chi_1^n & \chi_2^n & 0 & 0 & \cdots & 0 & 0 & 0 \\ \varrho_1^n & \beta_2^n & \chi_3^n & 0 & \cdots & 0 & 0 & 0 \\ 0 & \varrho_2^n & \beta_3^n & \chi_4^n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \varrho_{J-2}^n & \beta_{J-1}^n & \chi_J^n \\ 0 & 0 & 0 & 0 & \cdots & 0 & \varrho_{J-1}^n & \beta_J^n + \varrho_J^n \end{bmatrix},$$

then we can express the KF equation as

$$\frac{\boldsymbol{\omega}^n - \boldsymbol{\omega}^{n-1}}{\Delta t} = \mathbf{B}^{n-1}\boldsymbol{\omega}^n,$$

or

$$\boldsymbol{\omega}^n = \left(\mathbf{I} - \Delta t\mathbf{B}^{n-1}\right)^{-1}\boldsymbol{\omega}^{n-1}, \tag{78}$$

where $\boldsymbol{\omega}^n = \begin{bmatrix} \omega_1^n & \omega_2^n & \ldots & \omega_{J-1}^n & \omega_J^n \end{bmatrix}^T$, and $\mathbf{I}$ is the identity matrix of dimension J.

Extension to non-homogeneous grids Our model has been solved using a homogeneous grid and all the results presented in the paper have been computed using homogeneous grids. However, in some robustness tests that we have performed to assess the accuracy of the method, we have used non-homogeneous grid for the state $z$to economize on grid points. . We could not find a universally applicable way to implement non-homogeneous grids in the economics literature, so we propose the following discretization scheme.[16] We have used this scheme to verify that our numerical results are accurate in the sense that they do not change if we add additional grid points to the ω grid – no matter whether we add them where most of the mass of $\omega(z)$ is located or in the range in which z_t^* moves.

Be $z = \begin{bmatrix} z_1, & z_2, & \ldots & z_{J-1} & z_J \end{bmatrix}$ the grid. Define $\Delta z_{a,b} = z_b - z_a$ and let $\Delta z =$

[16]Our approach builds on the one in the appendix to Achdou et al., 2017. It differs from theirs in two ways. First, it can be derived as a finite difference scheme to the KFE. Their approach delivers a finite difference approximation for the HJB, but not for the KFE, and hence it requires the grid to be constructed such that the step size to both sides of any grid point converge to one another. Furthermore, our approach is not an upwind scheme and has only been tested in the current model, which features no endogenous drift.

$\frac{1}{2}\begin{bmatrix} \Delta z_{1,2}, & \Delta z_{1,3}, & \Delta z_{2,4}, & ..., & \Delta z_{J-2,J} & \Delta z_{J-1,J} \end{bmatrix}$. We approximate the KFE (27) using central difference for both the first derivative and the second derivative.

$$\begin{aligned}
\frac{\omega_j^n - \omega_j^{n-1}}{\Delta t} \quad &= \quad \left(s_n(z) - (1-\psi)\eta - \frac{\dot{A}_n}{A_n}\right)\omega_t\left(z_j\right) - \left[\frac{\mu(z_{j+1})\omega_t\left(z_{j+1}\right) - \mu(z_{j-1})\omega_t\left(z_{j-1}\right)}{\Delta z_{j-1,j+1}}\right] \\
&+ \frac{1}{2}\frac{\Delta z_{j-1,j}\sigma^2(z_{j+1})\omega_t\left(z_{j+1}\right) + \Delta z_{j,j+1}\sigma^2(z_{j-1})\omega_t\left(z_{j-1}\right) - \Delta z_{j-1,j+1}\sigma^2(z_j)\omega_t\left(z_j\right)}{\frac{1}{2}\left(\Delta z_{j-1,j+1}\right)\Delta z_{j,j+1}\Delta z_{j-1,j}}
\end{aligned}$$

which, grouping, results in

$$\begin{aligned}
\frac{\omega_j^n - \omega_j^{n-1}}{\Delta t} \quad &= \quad \underbrace{\left[\left(s_n(z) - (1-\psi)\eta - \frac{\dot{A}_n}{A_n}\right)\omega_t\left(z\right) + \frac{\sigma^2(z_j)\omega_t\left(z_j\right)}{\Delta z_{j,j+1}\Delta z_{j-1,j}}\right]}_{\beta_j^n}\omega_n(z_j) \\
&+ \underbrace{\left[\frac{\mu(z_{j-1})\omega_t\left(z_{j-1}\right)}{\Delta z_{j-1,j+1}} + \frac{\sigma^2(z_{j+1})\omega_t\left(z_{j+1}\right)}{\left(\Delta z_{j-1,j+1}\right)\Delta z_{j,j+1}}\right]}_{\varrho_{j-1}^n}\omega_{j-1}^n \\
&+ \underbrace{\left[-\frac{\mu(z_{j+1})\omega_t\left(z_{j+1}\right)}{\Delta z_{j-1,j+1}} + \frac{\sigma^2(z_{j+1})\omega_t\left(z_{j+1}\right)}{\left(\Delta z_{j-1,j+1}\right)\Delta z_{j,j+1}}\right]}_{\chi_{j+1}^n}\omega_{j+1}^n.
\end{aligned}$$

The law of motion of ω can equivalently be written in matrix form

$$\frac{\boldsymbol{\omega}^n - \boldsymbol{\omega}^{n-1}}{\Delta t} = \mathbf{B}^{n-1}\boldsymbol{\omega}^n$$

where

$$\mathbf{B}^n = \begin{bmatrix}
\beta_1^n + \chi_1^n & \chi_2^n & 0 & 0 & \cdots & 0 & 0 & 0 \\
\varrho_1^n & \beta_2^n & \chi_3^n & 0 & \cdots & 0 & 0 & 0 \\
0 & \varrho_2^n & \beta_3^n & \chi_4^n & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \varrho_{J-2}^n & \beta_{J-1}^n & \chi_J^n \\
0 & 0 & 0 & 0 & \cdots & 0 & \varrho_{J-1}^n & \beta_J^n + \varrho_J^n
\end{bmatrix},$$

Abstracting for brevity from the term $\left(s_n(z) - (1-\psi)\eta - \frac{\dot{A}_n}{A_n}\right)$, which is independent of the grid, and spelling out $\mathbf{B}^n$ we have

$$
\frac{\boldsymbol{\omega}^n - \boldsymbol{\omega}^{n-1}}{\Delta t} = \begin{bmatrix} -\frac{\mu(z_1)}{\Delta z_{1,2}} - \frac{\sigma(z_1)}{\Delta z_{1,2}\Delta z_{1,2}/2} + \frac{\sigma(z_1)}{\Delta z_{1,2}\Delta z_{1,2}} & -\frac{\mu(z_2)}{\Delta z_{1,2}} + \frac{\sigma(z_2)}{\Delta z_{1,2}\Delta z_{1,2}} & 0 & \cdots \\ \frac{\mu(z_1)}{\Delta z_{1,3}} + \frac{\sigma(z_1)}{\Delta z_{1,3}\Delta z_{1,2}} & -\frac{\sigma(z_2)}{\Delta z_{1,2}\Delta z_{2,3}} & -\frac{\mu(z_3)}{\Delta z_{1,3}} + \frac{\sigma(z_3)}{\Delta z_{1,3}\Delta z_{2,3}} & \cdots \\ 0 & \frac{\mu(z_2)}{\Delta z_{2,4}} + \frac{\sigma(z_2)}{\Delta z_{2,4}\Delta z_{2,3}} & -\frac{\sigma(z_3)}{\Delta z_{2,3}\Delta z_{3,4}} & \cdots \\ 0 & 0 & \frac{\mu(z_3)}{\Delta z_{3,5}} + \frac{\sigma(z_3)}{\Delta z_{3,4}\Delta z_{3,5}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \boldsymbol{\omega}^n.
$$

We can rewrite this as follows

$$
\frac{\boldsymbol{\omega}^n - \boldsymbol{\omega}^{n-1}}{\Delta t} = \begin{bmatrix} -\frac{\mu(z_1)}{\Delta z_{1,2}} - \frac{\sigma(z_2)}{\boldsymbol{\Delta z_{1,2}}\Delta z_{1,2}} & -\frac{\mu(z_2)}{\Delta z_{1,2}} + \frac{\Delta z_{2,3}\sigma(z_2)}{\Delta z_{2,3}(\boldsymbol{\Delta z_{1,2}}\Delta z_{1,2})} & 0 & \cdots \\ \frac{\mu(z_1)}{\Delta z_{1,3}} + \frac{\sigma(z_1)}{\boldsymbol{\Delta z_{1,3}}\Delta z_{1,2}} & -\frac{(\Delta z_{1,2}+\Delta z_{2,3})\sigma(z_2)}{\boldsymbol{\Delta z_{1,3}}(\Delta z_{1,2}\Delta z_{2,3})} & -\frac{\mu(z_3)}{\Delta z_{1,3}} + \frac{\Delta z_{3,4}\sigma(z_3)}{\Delta z_{3,4}(\boldsymbol{\Delta z_{1,3}}\Delta z_{2,3})} & \cdots \\ 0 & \frac{\mu(z_2)}{\Delta z_{2,4}} + \frac{\Delta z_{1,2}\sigma(z_2)}{\Delta z_{1,2}(\boldsymbol{\Delta z_{2,4}}\Delta z_{2,3})} & -\frac{(\Delta z_{2,3}+\Delta z_{3,4})\sigma(z_3)}{\boldsymbol{\Delta z_{2,4}}(\Delta z_{2,3}\Delta z_{3,4})} & \cdots \\ 0 & 0 & \frac{\mu(z_3)}{\Delta z_{3,5}} + \frac{\Delta z_{2,3}\sigma(z_3)}{\Delta z_{2,3}(\Delta z_{3,4}\boldsymbol{\Delta z_{3,5}})} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \boldsymbol{\omega}^n.
$$

Note that the bold terms in line i are equal to $1/\Delta z_i$. Thus the columns of $\mathbf{B}^n \Delta z$ sum up to 1 and the operation is mass preserving, in the sense that the above relationship guarantees that

$$
\sum \omega_j^n \Delta z_j = \sum \omega_j^{n-1} \Delta z_j
$$

where $\sum \omega_j^n \Delta z_j$ is a trapezoid approximation of the integral $\int \omega^n(z) dz$.

C.2 Finite difference approximation of the Integrals

To approximate the integrals in $\int_0^z \omega_t(z)\, dz$ and $\int_{z_t^*}^{\infty} z\omega_t(z)dz$ we use the trapezoid rule. I.e. if $f(z)$ is either $\omega_t(z)$ or $z\omega_t(z)$ and $z_j \le \bar{z} \le z_{j+1}$ then the integral from the closest lower gridpoint is given by

$$
\int_{z_j}^{\bar{z}} f(z)\, dz = \left[f(z_j) + \frac{1}{2} \left[f(z_{j+1}) - f(z_j) \right] \frac{(\bar{z} - z_j)}{\Delta z} \right] (\bar{z} - z_j)
$$

We use this formula to construct the integrals over a larger range piecewise. For example:

$$\int_{z_1}^{z_N} f(z)\,dz = \begin{bmatrix} \frac{1}{2} & 1 & 1 & \cdots & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} f(z_1) \\ f(z_2) \\ \vdots \\ f(z_N) \end{bmatrix}$$

and

$$\begin{aligned} \int_{z_1}^{z^*} f(z)\,dz &= \begin{bmatrix} \frac{1}{2} & 1 & 1 & \cdots & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} f(z_1) \\ f(z_2) \\ \vdots \\ f(z_{j^*}) \end{bmatrix} \\ &+ \left[f(z_{j^*-1}) + \frac{1}{2}\left[f(z_{j^*}) - f(z_{j^*-1})\right] \frac{(z^* - z_{j^*-1})}{\Delta z} \right] (z^* - z_{j^*-1}) \\ &\quad \text{where } j^* = \arg\min_j \{ j \le J | z_{j^*} > z^* \} \end{aligned}$$

C.3 Algorithm to solve for the SS

Here we present how to solve for the SS of the private equilibrium, that is for the SS when the central bank sets a certain level of the nominal interest rate in SS i^{ss}.

We know that in SS consumption does not grow, hence from (21)

$$r^{ss} = \rho^h. \tag{79}$$

We also know that in SS, the investment rate is equal to the depreciation,

$$\iota^{ss} = \delta. \tag{80}$$

This means that, from equation (**??**) and the functional form we assumed for the capital adjustment costs (18),

$$(q_t - 1 - \Phi'(\iota_t))(r_t - (\iota_t - \delta)) = \dot{q}_t - \Phi''(\iota_t)\,i_t - (q_t \iota_t - \iota_t - \Phi(\iota_t)) \tag{81}$$

$$\left(q^{ss} - 1 - \phi^k(\iota^{ss} - \delta)\right)\left(\rho^{hh} - (\iota^{ss} - \delta)\right) = 0 - \phi^k * 0 - \left(q^{ss}\iota^{ss} - \iota^{ss} - \phi^k(\iota^{ss} - \delta)\right)$$

$$\rho^{hh}(q^{ss}-1)=\delta(1-q^{ss})$$

.From here we can solve for the steady state value of q^{ss}, which is given by

$$q^{ss}=1. \tag{82}$$

Furthermore, combining (79) with the fisher equation and the fact that the planner sets a certain nominal rate i^{ss} we get that

$$\pi^{ss}=i^{ss}-\rho^h. \tag{83}$$

In SS, $\dot{\pi}_t=0$ and $\dot{Y}_t=0$. Hence, from equation (14) we obtain

$$m^{ss}=\left(m^*+\rho^h\pi^{ss}\frac{\theta}{\varepsilon}\right). \tag{84}$$

Using equation (35) and (79),

$$\rho^h=\frac{1}{q^{ss}}\left(\alpha m_t Z_t A_t^{\alpha-1}L^{1-\alpha}\frac{z_t^*}{\gamma X_t}\right)-\delta \tag{85}$$

From equation (36) and (79),

$$\frac{\dot{A}_t}{A_t}=0=\frac{1}{q_t}(\alpha m_t Z_t {A_t}^{\alpha-1}{L_t}^{1-\alpha}-R_t\Gamma(1-\Omega(z_t^*))+R_t-\delta q_t-q_t(1-\psi)\eta). \tag{86}$$

Plugging the latter equation into the former, using $q^{SS}=1$ and using the definition of r_t we obtain:

$$\rho^h+\delta=\left[(\rho^h+\delta)\left(\gamma(1-\Omega(z_t^*))-1\right)+(1-\psi)\eta+\delta\right]\frac{z^*}{\gamma X^*}. \tag{87}$$

In the algorithm, we use a non-linear equation solver to obtain z^* from this equation.

The Algorithm.

- Get $r^{ss}=\rho^h$, $\pi^{ss}=\bar{\pi}$ and $i^{ss}=\rho^h+\pi^{ss}$ and $R^{ss}=q^{ss}(\rho^h+\delta)$ and $m^{ss}=m^*+\rho^h\pi^{ss}\frac{\theta}{\epsilon}$.

- Given that our calibration target for $L^{ss}=1$, we "guess" $L^{ss}=1$

- Let n now denote the iteration counter. Make an initial guess for the net worth distribution $\boldsymbol{\omega}^0$

  1. Use a non-linear equation solver on equation (87) to obtain z^* from equation (87).
  2. Obtain $Z_n = (\gamma \Gamma_n X_n^*)^\alpha$.
  3. Find A from equation (34),
  $$A^n = \left[\frac{q^{ss}\rho^h + \delta q^{ss}}{\alpha m_n Z_n {L_m}^{1-\alpha} \frac{z_t^*}{\gamma X_t}} \right]^{\frac{1}{\alpha-1}}.$$
  4. Find the stocks $K_n = \gamma(1 - \Omega^n(z^*))A^n$, $D_n = K_n - A_n$.
  5. Compute $w_n = (1-\alpha)m^{ss} Z_n {A_n}^\alpha {L_n}^{-\alpha}$,$\varphi_n = \alpha \left(\frac{(1-\alpha)}{w_n}\right)^{(1-\alpha)/\alpha} {m^{ss}}^{\frac{1}{\alpha}}$.
  6. Get aggregate output $Y = Z_n A_n^\alpha {L_n}^{1-\alpha}$, transfers $T_n = (1 - m^{ss}) Y_n - \frac{\theta}{2} (\pi^{ss})^2 Y_n + (1-\psi)\eta A_t$, and consumption $C_n = w_n L_m + r^{ss} D_n + T_n$.
  7. Update $\hat{s}_j^n = \frac{1}{q^{ss}} (\gamma \max \{\Gamma z \varphi_n - R_n, 0\} + R_n - \delta q^{ss})$ and employ it to construct matrix $\mathbf{B}^{n-1}$.
  8. Update $\boldsymbol{\omega}^{n+1}$ using equation $\frac{\boldsymbol{\omega}^{n+1} - \boldsymbol{\omega}^n}{\Delta t} = \mathbf{B}^n \boldsymbol{\omega}^{n+1}$.
  9. If the net worth distribution do not coincide with the guess, set $n = n+1$ and return to point 1

- Set $\Upsilon = \left(w_{L=1} C_{L=1}^{-\eta}\right)$ to ensure our "guess" for L^{ss} is correct.

D Proof of proposition 3

Proof: The proof has the following structure. First, we set up a generic planner's problem in a continuous-time heterogeneous-agent economy without aggregate uncertainty. Second, we derive the continuous time optimality conditions of the planner's problem and discretize them. Third, we discretize the planners problem and the derive the optimality conditions. Fourth, we compare the two sets of discretized optimality conditions.

1. The generic problem The planner's problem in an economy with heterogeneity among one agent type (e.g. households or firms) can be written as

$$\max_{Z_t,u_t(x),\mu_t(x),v_t(x)} \int_0^\infty \exp(-\varrho t) f_0(Z_t) dt \tag{88}$$

$$s.t.\ \forall t$$

$$\dot{X}_t = f_1(Z_t) \tag{89}$$

$$\dot{U}_t = f_2(Z_t) \tag{90}$$

$$0 = f_3(Z_t) \tag{91}$$

$$\tilde{U}_t = \int f_4\left(x, u_t(x), Z_t\right) \mu_t\left(x\right)\ dx \tag{92}$$

$$\begin{aligned} \rho v_t(x) = {} & \dot{v}_t(x) + f_5(x, u_t(x), Z_t) \\ & + \sum_{i=1}^{I} b_i\left(x, u_t(x), Z_t\right) \frac{\partial v_t(x)}{\partial x_i} + \sum_{i=1}^{I}\sum_{k=1}^{I} \frac{\left(\sigma(x)\sigma(x)^\top\right)_{i,k}}{2} \frac{\partial^2 v_t(x)}{\partial x_i \partial x_k},\ \forall x \end{aligned} \tag{93}$$

$$0 = \frac{\partial f_5}{\partial u_{j,t}} + \sum_{i=1}^{I} \frac{\partial b_i}{\partial u_{j,t}} \frac{\partial v_t(x)}{\partial x_i}, \quad j = 1, ..., J, \forall x. \tag{94}$$

$$\begin{aligned} \dot{\mu}_t\left(x\right) = {} & -\sum_{i=1}^{I} \frac{\partial}{\partial x_i}\left[b_i\left(x, u_t(x), Z_t\right) \mu_t\left(x\right)\right] \\ & + \frac{1}{2}\sum_{i=1}^{I}\sum_{k=1}^{I} \frac{\partial^2}{\partial x_i \partial x_k}\left[\left(\sigma(x)\sigma(x)^\top\right)_{i,k} \mu_t\left(x\right)\right],\ \forall x \end{aligned} \tag{95}$$

$$X_0 = \bar{X}_0 \tag{96}$$

$$\mu_0\left(x\right) = \bar{\mu}_0\left(x\right) \tag{97}$$

$$lim_{t\to\infty} U = \bar{U}_\infty \tag{98}$$

$$lim_{t\to\infty} v(x) = \bar{v}(x)_\infty \tag{99}$$

where we have adopted the following notation:

- Variables (capitals are reserved for aggregate variables):
  - x individual state vector with I elements
  - u individual control vector with J elements
  - v individual value function vector with 1 element
  - $u(x)$ control vector as function of individual state

- $\mu(x)$ distribution of agents across states
- $v(x)$ value function as function of individual state
- X aggregate state vector (other than μ)
- $\hat{U}$ aggregate control vector of purely contemporaneous variables
- U aggregate control vector of intertemporal variables
- $\tilde{U}$ control vector of aggregator variables
- $Z_t = \left\{\tilde{U}_t, U_t, \bar{U}_t, X_t\right\}$ vector of all aggregate variables

- Functions
  - b function that determines the drift of x
  - f_0 welfare function
  - f_1, f_2, f_3 aggregate equilibrium conditions
  - f_4 aggregator function
  - f_5 individual utility function

Line (88) is the planner's objective function.[17] Equations (89)-(91) are the aggregate equilibrium conditions for aggregate states, jump variables and contemporaneous variables. In our model, examples for each of these three types of equations are the law of motion of aggregate capital, the household's Euler equation and the household's labor supply condition, respectively. Equation (92) links aggregate and individual variables, such as the definition of aggregate TFP in our model. Equations (93) and (94) are the individual agent's value function and first order conditions, which must hold across the whole individual state vector x. In our model we do not have these two types of equations since we can analytically solve the individual optimal choice. The Kolmogorov Forward equation (25) determines the evolution of the distribution of agents. Finally (96)-(99) are the initial and terminal conditions for the aggregate and individual state and dynamic control variables. In our model these are the initial capital stock and firm distribution and the terminal conditions for variables such as consumption.

[17] Notice that the planner's discount factor, ϱ, can be different to that of individual agents, ρ.

2. Optimize, then discretize First we consider the approach introduced in Nuño and Thomas (2016), namely to compute the first order conditions using calculus of variations and then to discretize the problem using an upwind finite difference scheme.

2.a The Lagrangian The Lagrangian for this problem is given by:[18]

$$
\begin{aligned}
\mathcal{L} &= \int_0^\infty \left\{ e^{-\varrho t} \left[\!\left[f_0(Z_t) \right.\right.\right. \\
&+ \lambda_{1,t}\left(\dot{X}_t - f_1(Z_t)\right) \\
&+ \lambda_{2,t}\left(\dot{U}_t - f_2(Z_t)\right) \\
&+ \lambda_{3,t}\left(f_3(Z_t)\right) \\
&+ \lambda_{4,t}\left(\tilde{U}_t - \int f_4\left(x, u_t(x), Z_t\right)\mu_t\left(x\right)\ dx\right) \\
&+ \int \left[\lambda_{5,t}(x)\left(-\rho v_t(x) + \dot{v}_t(x) + f_5(x, u_t(x), Z_t) + \sum_{i=1}^{I} b_i\left(x, u_t(x), Z_t\right)\frac{\partial v_t(x)}{\partial x_i} + \sum_{i=1}^{I}\frac{\sigma_i^2(x)}{2}\frac{\partial^2 v_t(x)}{\partial^2 x_i}\right)\right] dx \\
&+ \sum_{j=1}^{J}\int\left[\lambda_{6,j,t}(x)\left(\frac{\partial f_5}{\partial u_{j,t}} + \sum_{i=1}^{I}\frac{\partial b_i}{\partial u_{j,t}}\frac{\partial v_t(x)}{\partial x_i}\right)\right] dx \\
&+ \left.\left.\left. \int\left[\lambda_{7,t}(x)\left(-\dot{\mu}_t\left(x\right) + \left(-\sum_{i=1}^{I}\frac{\partial}{\partial x_i}\left[b_i\left(x, u_t(x), Z_t\right)\mu_t\left(x\right)\right] + \frac{1}{2}\sum_{i=1}^{I}\frac{\partial^2}{\partial^2 x_i}\left[\sigma_i^2(x)\mu_t\left(x\right)\right]\right)\right)\right] dx \right]\!\right] \right\} dt
\end{aligned}
$$

where λ_1 to λ_7 denote the multipliers on the respective constraints. For convenience, we write the time derivatives in a separate line at the end. The Lagrangian becomes:

$$
\begin{aligned}
\mathcal{L} &= \int_0^\infty \left\{ e^{-\varrho t} \left[\!\left[f_0(Z_t) \right.\right.\right. \\
&+ \lambda_{1,t}\left(-f_1(Z_t)\right) \\
&+ \lambda_{2,t}\left(-f_2(Z_t)\right) \\
&+ \lambda_{3,t}\left(-f_3(Z_t)\right) \\
&+ \lambda_{4,t}\left(\tilde{U}_t - \int f_4\left(x, u_t(x), Z_t\right)\mu_t\left(x\right)\ dx\right) \\
&+ \int \left[\lambda_{5,t}(x)\left(-\rho v_t(x) + f_5(x, u_t(x), Z_t) + \sum_{i=1}^{I} b_i\left(x, u_t(x), Z_t\right)\frac{\partial v_t(x)}{\partial x_i} + \sum_{i=1}^{I}\frac{\sigma_i^2(x)}{2}\frac{\partial^2 v_t(x)}{\partial^2 x_i}\right)\right] dx
\end{aligned}
$$

[18] For simplicity, we assume that the Wiener processes driving the dynamics of the state x are independent, though the proof can be trivially extended to that case, at the cost of a more cumbersome notation.

$$
\begin{aligned}
&+ \quad \sum_{j=1}^{J} \int \left[\lambda_{6,j,t}(x) \left(\frac{\partial f_{5,t}}{\partial u_{j,t}} + \sum_{i=1}^{I} \frac{\partial b_i}{\partial u_{j,t}} \frac{\partial v_t(x)}{\partial x_i} \right) \right] dx \\
&+ \quad \int \left[\lambda_{7,t}(x) \left(-\sum_{i=1}^{I} \frac{\partial}{\partial x_i} \left[b_i \left(x, u_t(x), Z_t\right) \mu_t\left(x\right) \right] + \frac{1}{2} \sum_{i=1}^{I} \frac{\partial^2}{\partial^2 x_i} \left[\sigma_i^2(x) \mu_t\left(x\right) \right] \right) \right] dx \Big]\!\Big] \Bigg\} dt \\
&+ \quad \int_0^\infty \left\{ e^{-\varrho t} \left[\!\left[\lambda_{1,t} \dot{X}_t + \lambda_{2,t} \dot{U}_t + \int \left[\lambda_{5,t} \dot{v}_t(x) \right] dx - \int \left[\lambda_{7,t} \dot{\mu}_t\left(x\right) \right] dx \right]\!\right] \right\} dt.
\end{aligned}
$$

We have ignored the terminal and initial conditions but we will account for them later on. Now we manipulate the Lagrangian using integration by parts in order to bring it into a more convenient form. We start with the last line. Switching the order of integration, the last line becomes

$$
\begin{aligned}
\int_0^\infty e^{-\varrho t} \left[\!\left[\lambda_{1,t} \dot{X}_t \right]\!\right] dt + \int_0^\infty e^{-\varrho t} \left[\!\left[\lambda_{2,t} \dot{U}_t \right]\!\right] dt + \int \left[\!\left[\int_0^\infty \left[e^{-\varrho t} \lambda_{5,t}(x) \dot{v}_t(x) \right] dt \right]\!\right] dx \\
- \int \left[\!\left[\int_0^\infty \left[e^{-\varrho t} \lambda_{7,t}(x) \dot{\mu}_t\left(x\right) \right] dt \right]\!\right] dx
\end{aligned}
$$

Now we integrate this expression by parts with respect to time t, using

$$
\begin{aligned}
\int_0^\infty e^{-\varrho t} \left[\!\left[a_t \dot{b}_t \right]\!\right] dt &= \left[e^{-\varrho t} a_t b_t \right]_0^\infty - \int_0^\infty \left[\!\left[e^{-\varrho t} (\dot{a}_{1,t} - \varrho a_{1,t}) b_t \right]\!\right] dt \\
&= \lim_{t \to \infty} e^{-\varrho t} a_t b_t - a_0 b_0 - \int_0^\infty \left[\!\left[e^{-\varrho t} (\dot{a}_t - \varrho a_t) b_t \right]\!\right] dt
\end{aligned}
$$

to get

$$
\begin{aligned}
& \lim_{t\to\infty} e^{-\varrho t} \lambda_{1,t} X_t - \lambda_{1,0} X_0 - \int_0^\infty e^{-\varrho t} (\dot{\lambda}_{1,t} - \varrho \lambda_{1,t}) X_t dt + \lim_{t\to\infty} e^{-\varrho t} \lambda_{2,t} U_t - \lambda_{2,0} U_0 \\
& - \int_0^\infty e^{-\varrho t} (\dot{\lambda}_{2,t} - \varrho \lambda_{2,t}) U_t dt x \\
+ & \int \left(\lim_{t\to\infty} e^{-\varrho t} \lambda_{5,t}(x) v_t(x) - \lambda_{5,0}(x) v_0(x) \right) dx - \int \int_0^\infty e^{-\varrho t} (\dot{\lambda}_{5,t}(x) - \varrho \lambda_{5,t}(x)) v_t(x) dt dx \\
- & \int \lim_{t\to\infty} e^{-\varrho t} \lambda_{7,t}(x) \mu_t(x) - \lambda_{7,0}(x) \mu_0(x) dx + \int \int_0^\infty e^{-\varrho t} (\dot{\lambda}_{7,t}(x) - \varrho \lambda_{7,t}(x)) \mu_t\left(x\right) dt dx
\end{aligned}
$$

Now we use the initial and terminal conditions to drop some $\lim_{t\to\infty}$ and $t = 0$ terms,

$$
+ \quad \lim_{t\to\infty} e^{-\varrho t} \lambda_{1,t} X_t - \lambda_{2,0} U_0 - \int_0^\infty e^{-\varrho t} (\dot{\lambda}_{1,t} - \varrho \lambda_{1,t}) X_t dt - \int_0^\infty e^{-\varrho t} (\dot{\lambda}_{2,t} - \varrho \lambda_{2,t}) U_t dt
$$

$$
\begin{aligned}
- & \int \lambda_{5,0}(x)v_0(x)dx + \int\int_0^\infty e^{-\varrho t}(\dot{\lambda}_{5,t}(x) - \varrho\lambda_{5,t}(x))v_t(x)dtdx \\
- & \int \lim_{t\to\infty} e^{-\varrho t}\lambda_{7,t}(x)\mu_t(x)dx + \int\int_0^\infty e^{-\varrho t}(\dot{\lambda}_{7,t}(x) - \varrho\lambda_{7,t}(x))\mu_t\,(x)\,dtdx
\end{aligned}
$$

Next we integrate lines 6 to 8 by parts with respect to x. This yields:

$$
\begin{aligned}
+ & \int \Bigg\{ \left[\left(-\rho\lambda_{5,t}(x)v_t(x) + f_5(x,u_t(x),Z_t) - \sum_{i=1}^{I} \frac{\partial b_i\,(x,u_t(x),Z_t)\,\lambda_{5,t}(x)}{\partial x_i} v_t(x)\right)\right] dx \\
& + \int \left[\left(+\frac{1}{2}\sum_{i=1}^{I}\frac{\partial^2}{\partial^2 x_i}\left[\sigma_i^2(x)\lambda_{5,t}(x)\right] v_t(x)\right)\right] dx \\
+ & \sum_{j=1}^{J}\int \left[\lambda_{6,j,t}(x)\frac{\partial f_{5,t}}{\partial u_{j,t}} - \sum_{i=1}^{I}\frac{\partial\left[\lambda_{6,j,t}(x)\frac{\partial b_i}{\partial u_{j,t}}\right]}{\partial x_i} v_t(x)\right] dx \\
+ & \int \left[\left(\sum_{i=1}^{I}\frac{\partial\lambda_{7,t}(x)}{\partial x_i}\left[b_i\,(x,u_t(x),Z_t)\,\mu_t\,(x)\right] + \sum_{i=1}^{I}\frac{\partial^2\lambda_{7,t}(x)}{\partial^2 x_i}\frac{\sigma_i^2(x)}{2}\mu_t\,(x)\right)\right] dx \Bigg\}\, dt
\end{aligned}
$$

Putting this all together the Lagrangian has become:

$$
\begin{aligned}
\mathcal{L} \;=\; & \int_0^\infty \Big\{ e^{-\varrho t} \llbracket f_0(Z_t) \\
+ \; & \lambda_{1,t}\left(-f_1(Z_t)\right) \\
+ \; & \lambda_{2,t}\left(-f_2(Z_t)\right) \\
+ \; & \lambda_{3,t}\left(-f_3(Z_t)\right) \\
+ \; & \lambda_{4,t}\left(\tilde{U}_t - \int f_4\,(x,u_t(x),Z_t)\,\mu_t\,(x)\;\;dx\right) \\
+ \; & \int \left(-\rho\lambda_{5,t}(x)v_t(x) + \lambda_{5,t}(x)f_5(x,u_t(x),Z_t) - \sum_{i=1}^{I}\frac{\partial\left[b_i\,(x,u_t(x),Z_t)\,\lambda_{5,t}(x)\right]}{\partial x_i} v_t(x)\right) dx \\
+ \; & \int \left(\frac{1}{2}\sum_{i=1}^{I}\frac{\partial^2}{\partial^2 x_i}\left[\sigma_i^2(x)\lambda_{5,t}(x)\right] v_t(x)\right) dx \\
+ \; & \sum_{j=1}^{J}\int \left[\lambda_{6,j,t}(x)\frac{\partial f_{5,t}}{\partial u_{j,t}} - \sum_{i=1}^{I}\frac{\partial\left[\lambda_{6,j,t}(x)\frac{\partial b_i}{\partial u_{j,t}}\right]}{\partial x_i} v_t(x)\right] dx \\
+ \; & \int_0^\infty \left[\left(\sum_{i=1}^{I}\frac{\partial\lambda_{7,t}(x)}{\partial x_i}\left[b_i\,(x,u_t(x),Z_t)\,\mu_t\,(x)\right] + \sum_{i=1}^{I}\frac{\partial^2\lambda_{7,t}(x)}{\partial^2 x_i}\frac{\sigma_i^2(x)}{2}\mu_t\,(x)\right)\right] dx \rrbracket \Big\}\, dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \lim_{t\to\infty} e^{-\varrho t}\lambda_{1,t}X_t - \lambda_{2,0}U_0 - \int_0^\infty e^{-\varrho t}(\dot{\lambda}_{1,t} - \varrho\lambda_{1,t})X_t dt - \int_0^\infty e^{-\varrho t}(\dot{\lambda}_{2,t} - \varrho\lambda_{2,t})U_t dt \\
&+ \int -\lambda_{5,0}(x)v_0(x)dx + \int\int_0^\infty e^{-\varrho t}(\dot{\lambda}_{5,t}(x) - \varrho\lambda_{5,t}(x))v_t(x)dtdx \\
&- \int \lim_{t\to\infty} e^{-\varrho t}\lambda_{7,t}(x)\mu_t(x)dx + \int\int_0^\infty e^{-\varrho t}(\dot{\lambda}_{7,t}(x) - \varrho\lambda_{7,t}(x))\mu_t(x)\, dtdx.
\end{aligned}
$$

2.b Optimality conditions in the continuous state space We take the Gateaux derivatives in direction $h_t(x)$ for each endogenous variable x. These derivatives have to be equal to zero for any $h_t(x)$ in the optimum. This implies the following optimality conditions:

Aggregate variables:

$$
\begin{aligned}
U_t:\quad 0 = \; & -(\dot{\lambda}_{2,t} - \varrho\lambda_{2,t}) && (100)\\
& + \frac{\partial f_{0,t}}{\partial U_t} - \lambda_{1,t}\frac{\partial f_{1,t}}{\partial U_t} - \lambda_{2,t}\frac{\partial f_{2,t}}{\partial U_t} - \lambda_{3,t}\frac{\partial f_{3,t}}{\partial U_t} - \lambda_{4,t}\int \frac{\partial f_{4,t}}{\partial U_t}\mu_t(x)\;\; dx && (101)\\
& + \int \left[\lambda_{5,t}(x)\left(\frac{\partial f_{5,t}}{\partial U_t} + \sum_{i=1}^{I}\frac{\partial b_{i,t}}{\partial U_t}\frac{\partial v_t(x)}{\partial x_i}\right)\right] dx && (102)\\
& + \sum_{j=1}^{J}\int \left[\lambda_{6,j,t}(x)\left(\frac{\partial^2 f_{5,t}}{\partial u_{j,t}\partial U_t} + \sum_{i=1}^{I}\frac{\partial b_{i,t}}{\partial u_{j,t}\partial U_t}\frac{\partial v_t(x)}{\partial x_i}\right)\right] dx && (103)\\
& + \int \left[\lambda_{7,t}(x)\left(-\sum_{i=1}^{I}\frac{\partial}{\partial x_i}\left[\frac{\partial b_{i,t}}{\partial U_t}\mu_t(x)\right]\right)\right] dx, && (104)\\
& \forall t > 0, && (105)
\end{aligned}
$$

$$
0 = \lambda_{2,0}. \qquad (106)
$$

$$
\begin{aligned}
X_t:\quad 0 = \; & -(\dot{\lambda}_{1,t} - \varrho\lambda_{1,t}) \\
& + \frac{\partial f_{0,t}}{\partial X_t} - \lambda_{1,t}\frac{\partial f_{1,t}}{\partial X_t} - \lambda_{2,t}\frac{\partial f_{2,t}}{\partial X_t} - \lambda_{3,t}\frac{\partial f_{3,t}}{\partial X_t} - \lambda_{4,t}\int \frac{\partial f_{4,t}}{\partial X_t}\mu_t(x)\;\; dx \\
& + \int \left[\lambda_{5,t}(x)\left(\frac{\partial f_{5,t}}{\partial X_t} + \sum_{i=1}^{I}\frac{\partial b_{i,t}}{\partial X_t}\frac{\partial v_t(x)}{\partial x_i}\right)\right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{J}\int\left[\lambda_{6,j,t}(x)\left(\frac{\partial^2 f_{5,t}}{\partial u_{j,t}\partial X_t}+\sum_{i=1}^{I}\frac{\partial b_{i,t}}{\partial u_{j,t}\partial X_t}\frac{\partial v_t(x)}{\partial x_i}\right)\right]dx\\
&+\int\left[\lambda_{7,t}(x)\left(-\sum_{i=1}^{I}\frac{\partial}{\partial x_i}\left[\frac{\partial b_{i,t}}{\partial X_t}\mu_t\left(x\right)\right]\right)\right]dx,\\
&\forall t\geq 0,
\end{aligned}
$$

$$
0=\lim_{t\to\infty}e^{-\varrho t}\lambda_{1,t}(x).
$$

$$
\begin{aligned}
\hat{U}_t:\quad 0=\;&0\\
&+\frac{\partial f_{0,t}}{\partial \hat{U}_t}-\lambda_{1,t}\frac{\partial f_{1,t}}{\partial \hat{U}_t}-\lambda_{2,t}\frac{\partial f_{2,t}}{\partial \hat{U}_t}-\lambda_{3,t}\frac{\partial f_{3,t}}{\partial \hat{U}_t}-\lambda_{4,t}\int\frac{\partial f_{4,t}}{\partial \hat{U}_t}\mu_t\left(x\right)\;dx\\
&+\int\left[\lambda_{5,t}(x)\left(\frac{\partial f_{5,t}}{\partial \hat{U}_t}+\sum_{i=1}^{I}\frac{\partial b_{i,t}}{\partial \hat{U}_t}\frac{\partial v_t(x)}{\partial x_i}\right)\right]dx\\
&+\sum_{j=1}^{J}\int\left[\lambda_{6,j,t}(x)\left(\frac{\partial^2 f_{5,t}}{\partial u_{j,t}\partial \hat{U}_t}+\sum_{i=1}^{I}\frac{\partial b_{i,t}}{\partial u_{j,t}\partial \hat{U}_t}\frac{\partial v_t(x)}{\partial x_i}\right)\right]dx\\
&+\int\left[\lambda_{7,t}(x)\left(-\sum_{i=1}^{I}\frac{\partial}{\partial x_i}\left[\frac{\partial b_{i,t}}{\partial \hat{U}_t}\mu_t\left(x\right)\right]\right)\right]dx,\\
&\forall t\geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{U}_t:\quad 0=\;&\lambda_{4,t}\\
&+\frac{\partial f_{0,t}}{\partial \tilde{U}_t}-\lambda_{1,t}\frac{\partial f_{1,t}}{\partial \tilde{U}_t}-\lambda_{2,t}\frac{\partial f_{2,t}}{\partial \tilde{U}_t}-\lambda_{3,t}\frac{\partial f_{3,t}}{\partial \tilde{U}_t}-\lambda_{4,t}\int\frac{\partial f_{4,t}}{\partial \tilde{U}_t}\mu_t\left(x\right)\;dx\\
&+\int\left[\lambda_{5,t}(x)\left(\frac{\partial f_{5,t}}{\partial \tilde{U}_t}+\sum_{i=1}^{I}\frac{\partial b_{i,t}}{\partial \tilde{U}_t}\frac{\partial v_t(x)}{\partial x_i}\right)\right]dx\\
&+\sum_{j=1}^{J}\int\left[\lambda_{6,j,t}(x)\left(\frac{\partial^2 f_{5,t}}{\partial u_{j,t}\partial \tilde{U}_t}+\sum_{i=1}^{I}\frac{\partial b_{i,t}}{\partial u_{j,t}\partial \tilde{U}_t}\frac{\partial v_t(x)}{\partial x_i}\right)\right]dx\\
&+\int\left[\lambda_{7,t}(x)\left(-\sum_{i=1}^{I}\frac{\partial}{\partial x_i}\left[\frac{\partial b_{i,t}}{\partial \tilde{U}_t}\mu_t\left(x\right)\right]\right)\right]dx,\\
&\forall t\geq 0.
\end{aligned}
$$

Value function, distribution and policy functions

$$
\begin{aligned}
v_t(x): \quad 0 = \quad & \left(-\lambda_{5,t}(x)\rho - \sum_{i=1}^{I}\frac{\partial\left[\lambda_{5,t}(x)b_i\left(x,u_t(x),Z_t\right)\right]}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{I}\frac{\partial^2}{\partial^2 x_i}\left[\sigma_i^2(x)\lambda_{5,t}(x)\right]\right) \\
& -\sum_{j=1}^{J}\sum_{i=1}^{I}\frac{\partial}{\partial x_i}\left(\lambda_{6,j,t}(x)\frac{\partial b_i\left(x,u_t(x),Z_t\right)}{\partial u_{j,t}}\right) \\
& -(\dot{\lambda}_{5,t}(x) - \varrho\lambda_{5,t}(x)), \\
& \forall t > 0,
\end{aligned}
$$

$$
0 = \quad \lambda_{5,0}(x).
$$

$$
\begin{aligned}
\mu_t(x): \quad 0 = \quad & -\lambda_{4,t}f_4\left(x,u_t(x),Z_t\right) \\
& +\lambda_{7,t}(x)\left(\sum_{i=1}^{I}\frac{\partial\lambda_{7,t}(x)}{\partial x_i}b_i\left(x,u_t(x),Z_t\right) + \sum_{i=1}^{I}\frac{\partial^2\lambda_{7,t}(x)}{\partial^2 x_i}\frac{\sigma_i^2(x)}{2}\right) \\
& +(\dot{\lambda}_{7t}(x) - \varrho\lambda_{7,t}(x)), \\
& \forall t \geq 0,
\end{aligned}
$$

$$
0 = \quad \lim_{t\to\infty} e^{-\varrho t}\lambda_{7,t}(x).
$$

$$
\begin{aligned}
u_{l,t}(x): \quad 0 = \quad & -\lambda_{4,t}\frac{\partial f_4}{\partial u_{l,t}}\mu_t(x) \\
+ \quad & \overbrace{\left[\lambda_{5,t}(x)\left(\frac{\partial f_5}{\partial u_{l,t}} + \sum_{i=1}^{I}\frac{\partial b_i}{\partial u_{l,t}}\frac{\partial v_t(x)}{\partial x_i}\right)\right]}^{=0} \\
+ \quad & \sum_{j=1}^{J}\lambda_{6,k,t}(x)\left(\frac{\partial^2 f_5}{\partial u_{l,t}\partial u_{j,t}} + \sum_{i=1}^{I}\frac{\partial^2 b_i}{\partial u_{l,t}\partial u_{j,t}}\frac{\partial v_t(x)}{\partial x_i}\right) \\
- \quad & \left(\sum_{i=1}^{I}\frac{\partial\lambda_{7,t}(x)}{\partial x_i}\frac{\partial b_{i,t}}{\partial u_{l,t}}\mu_t(x)\right).
\end{aligned}
$$

2.c Discretized optimality conditions Now we discretize these conditions with respect to time and idiosyncratic states.

The idiosyncratic state is discretized by a evenly-spaced grid of size $[N_1, ..., N_I]$ where $1, .., I$ are the dimensions of the state x. We assume that in each dimension there is no mass of agents outside the compact domain $[x_{i,1}, x_{i,N_i}]$. The state step size is Δx_i.We define $x^n \equiv (x_{1,n_1}, ..., x_{i,n_i}, ..., x_{I,n_I})$, where $n_1 \in \{1, N_1\}, ..., n_I \in \{1, N_I\}$. We are assuming that, due to state constraints and/or reflecting boundaries, the dynamics of idiosyncratic states are constrained to the compact set $[x_{1,1}, x_{1,N_1}] \times [x_{2,1}, x_{2,N_2}] \times .... \times [x_{I,1}, x_{I,N_I}]$. We also define $x^{n_i+1} \equiv (x_{1,n_1}, ..., x_{i,n_i+1}, ..., x_{I,n_I})$, $x^{n_i-1} \equiv (x_{1,n_1}, ..., x_{i,n_i-1}, ..., x_{I,n_I})$ $f_t^n \equiv f(x^n, u_t^n, Z_t)$, $f_t^{n_i-1} \equiv f(x^{n_i-1}, u_t^n, Z_t)$ and $f_t^{n_i+1} \equiv f(x^{n_i+1}, u_t^n, Z_t)$. I.e. the superscript n indicates a particular grid point and the superscript n_i+1 and n_i-1 indicate neighboring grid points along dimension i.

To discretize the problem we now replace (i) time derivatives of multipliers by backward derivatives, (ii) integrals by sums (iii) derivatives with respect to x by the upwind derivatives ∇ or $\hat{\nabla}$:

$$
\begin{aligned}
\nabla_i [v_t^n] &\equiv \left[\mathbb{I}_{b_{i,t}^n>0}\frac{v_t^{n_i+1}-v_t^n}{\Delta x_i}+\mathbb{I}_{b_{i,t}^n<0}\frac{v_t^n-v_t^{n_i-1}}{\Delta x_i}\right], \\
\hat{\nabla}_i [\mu_t^n] &\equiv \left[\frac{\mathbb{I}_{b_{i,t}^{n_i+1}<0}\mu_t^{n_i+1}-\mathbb{I}_{b_{i,t}^n<0}\mu_t^n}{\Delta x_i}+\frac{\mathbb{I}_{b_{i,t}^n>0}\mu_t^n-\mathbb{I}_{b_{i,t}^{n_i-1}>0}\mu_t^{n_i-1}}{\Delta x_i}\right],
\end{aligned}
$$

for any discretized functions v_t^n, μ_t^n. We simplify the notation for sums $\sum_n \equiv \sum_{n_1 \in \{1,..,N_1\},..,n_I \in \{1,..,N_I\}}$ .We maintain the subscript t even if it refers now to discrete time with a step Δt, that is, $X_{t+1}$is the shortcut for $X_{t+\triangle t}$. The second-order derivative is approximated as

$$
\triangle_i [v_t^n] \equiv \left[\frac{\left(v_t^{n_i+1}\right)+\left(v_t^{n_i-1}\right)-2\left(v_t^n\right)}{\left(\Delta x_i\right)^2}\right].
$$

We start with the optimality condition for U_t

$$
\begin{aligned}
U_t: \quad 0 = \quad & -\left(\frac{\lambda_{2,t}-\lambda_{2,t-1}}{\Delta t}-\varrho\lambda_{2,t}\right) && (107) \\
& +\frac{\partial f_0}{\partial U_t}-\lambda_{1,t}\frac{\partial f_1}{\partial U_t}-\lambda_{2,t}\frac{\partial f_2}{\partial U_t}-\lambda_{3,t}\frac{\partial f_3}{\partial U_t}-\lambda_{4,t}\sum_{n=1}^{N}\frac{\partial f_4^n}{\partial U_t}\mu_t^n && (108)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_n \left[\lambda_{5,t}^n \left(\frac{\partial f_5^n}{\partial U_t} + \sum_{i=1}^I \frac{\partial b_i^n}{\partial U_t} \nabla_i \left[v_t^n\right]\right)\right] \\
&+\sum_{j=1}^J \sum_n \left[\lambda_{6,j,t}^n \left(\frac{\partial^2 f_5^n}{\partial u_j \partial U_t} + \sum_{i=1}^I \frac{\partial b_i^n}{\partial u_j \partial U_t} \nabla_i \left[v_t^n\right]\right)\right] \\
&+\sum_n \left[-\lambda_{7,t}^n \sum_{i=1}^I \hat{\nabla}_i \left[\frac{\partial b_{i,t}^n}{\partial U_t} \mu_t^n\right]\right] \\
&\forall t \geq 0.
\end{aligned}
\tag{109}
$$

The optimality conditions for the other aggregate variables look very much alike:

$$
\begin{aligned}
X_t: \quad 0 = \quad & -\left(\frac{\lambda_{1,t} - \lambda_{1,t-1}}{\Delta} - \varrho \lambda_{1,t}\right) \\
&+\frac{\partial f_0}{\partial X_t} - \lambda_{1,t} \frac{\partial f_1}{\partial X_t} - \lambda_{2,t} \frac{\partial f_2}{\partial X_t} - \lambda_{3,t} \frac{\partial f_3}{\partial X_t} - \lambda_{4,t} \sum_n \frac{\partial f_4^n}{\partial X_t} \mu_t^n \\
&+\sum_n \left[\lambda_{5,t}^n \left(\frac{\partial f_5^n}{\partial X_t} + \sum_{i=1}^I \frac{\partial b_i^n}{\partial X_t} \nabla_i \left[v_t^n\right]\right)\right] \\
&+\sum_{j=1}^J \sum_n \left[\lambda_{6,j,t}^n \left(\frac{\partial^2 f_5^n}{\partial u_j \partial X_t} + \sum_{i=1}^I \frac{\partial b_i^n}{\partial u_j \partial X_t} \nabla_i \left[v_t^n\right]\right)\right] \\
&+\sum_n \left[-\lambda_{7,t}^n \sum_{i=1}^I \hat{\nabla}_i \left[\frac{\partial b_{i,t}^n}{\partial X_t} \mu_t^n\right]\right] \\
&\forall t > 0.
\end{aligned}
$$

$$
\begin{aligned}
\hat{U}_t: \quad 0 = \quad & 0 \\
&+\frac{\partial f_0}{\partial \hat{U}_t} - \lambda_{1,t} \frac{\partial f_1}{\partial \hat{U}_t} - \lambda_{2,t} \frac{\partial f_2}{\partial \hat{U}_t} - \lambda_{3,t} \frac{\partial f_3}{\partial \hat{U}_t} - \lambda_{4,t} \sum_n \frac{\partial f_4^n}{\partial \hat{U}_t} \mu_t^n \\
&+\sum_n \left[\lambda_{5,t}^n \left(\frac{\partial f_5^n}{\partial \hat{U}_t} + \sum_{i=1}^I \frac{\partial b_i^n}{\partial \hat{U}_t} \nabla_i \left[v_t^n\right]\right)\right] \\
&+\sum_{j=1}^J \sum_n \left[\lambda_{6,j,t}^n \left(\frac{\partial^2 f_5^n}{\partial u_j \partial \hat{U}_t} + \sum_{i=1}^I \frac{\partial b_i^n}{\partial u_j \partial \hat{U}_t} \nabla_i \left[v_t^n\right]\right)\right] \\
&+\sum_n \left[-\lambda_{7,t}^n \sum_{i=1}^I \hat{\nabla}_i \left[\frac{\partial b_{i,t}^n}{\partial \hat{U}_t} \mu_t^n\right]\right]
\end{aligned}
$$

$\forall t \geq 0.$

$$
\begin{aligned}
\tilde{U}_t: \quad 0 = \quad & \lambda_{4,t} \\
& + \frac{\partial f_0}{\partial \tilde{U}_t} - \lambda_{1,t} \frac{\partial f_1}{\partial \tilde{U}_t} - \lambda_{2,t} \frac{\partial f_2}{\partial \tilde{U}_t} - \lambda_{3,t} \frac{\partial f_3}{\partial \tilde{U}_t} - \lambda_{4,t} \sum_{n=1}^{N} \frac{\partial f_4^n}{\partial \tilde{U}_t} \mu_t^n \\
& + \sum_n \left[\lambda_{5,t}^n \left(\frac{\partial f_5^n}{\partial \tilde{U}_t} + \sum_{i=1}^{I} \frac{\partial b_i^n}{\partial \tilde{U}_t} \nabla_i \left[v_t^n \right] \right) \right] \\
& + \sum_{j=1}^{J} \sum_n \left[\lambda_{6,j,t}^n \left(\frac{\partial^2 f_5^n}{\partial u_j \partial \tilde{U}_t} + \sum_{i=1}^{I} \frac{\partial b_i^n}{\partial u_j \partial \tilde{U}_t} \nabla_i \left[v_t^n \right] \right) \right] \\
& + \sum_n \left[-\lambda_{7,t}^n \sum_{i=1}^{I} \hat{\nabla}_i \left[\frac{\partial b_{i,t}^n}{\partial \tilde{U}_t} \mu_t^n \right] \right] \\
& \forall t \geq 0.
\end{aligned}
$$

The discretized optimality condition with respect to the value function $v_t(x)$, the distribution $\mu_t(x)$ and the individual jump variable $u_{j,t}(x)$ are.

$$
\begin{aligned}
v_t(x): \quad 0 = \quad & -\lambda_{5,t}^n \rho - \sum_{i=1}^{I} \hat{\nabla}_i \left[\lambda_{5,t}^n b_{i,t}^n \right] \\
& + \frac{1}{2} \sum_{i=1}^{I} \sum_{k=1}^{I} \nabla_i \left[\sigma_{i,k}^n \lambda_{5,t}^n \right] \\
& - \sum_{j=1}^{J} \sum_{i=1}^{I} \left(\hat{\nabla}_i \left[\lambda_{6,j,t}^n \frac{\partial b_{i,t}^n}{\partial u_{j,t}^n} \right] \right) \\
& - \left(\frac{\lambda_{5,t}^n - \lambda_{5,t-1}^n}{\Delta t} - \varrho \lambda_{5,t}^n \right).
\end{aligned} \tag{110}
$$

$$
\begin{aligned}
\mu_t(x): \quad 0 = \quad & -\lambda_{4,t} f_{4,t}^n \\
& + \lambda_{7,t}(x) \left(\sum_{i=1}^{I} b_i\left(x, u_t(x), Z_t\right) \nabla_i \left[\lambda_{7,t}^n \right] + \frac{1}{2} \sum_{i=1}^{I} \left(\sigma_i^2 \right)^n \triangle_i^2 \left[\lambda_{7,t}^n \right] \right) \\
& + \frac{\lambda_{7,t}^n - \lambda_{7,t-1}^n}{\Delta t} - \varrho \lambda_{7,t}^n
\end{aligned} \tag{111}
$$

$$
\begin{aligned}
u_{l,t}(x): \quad 0 = \quad & -\lambda_{4,t}\frac{\partial f_4}{\partial u_{l,t}}\mu_t^n \\
& + \sum_{j=1}^{J} \lambda_{6,k,t}^n \left(\frac{\partial^2 f_{5,t}^n}{\partial u_{l,t}^n \partial u_{j,t}^n} + \sum_{i=1}^{I} \frac{\partial^2 b_{i,t}^n}{\partial u_{l,t}^n \partial u_{j,t}^n} \nabla_i \left[v_t^n \right] \right) \\
& - \sum_{i=1}^{I} \nabla_i \left[\lambda_{7,t}^n \right] \frac{\partial b_{i,t}^n}{\partial u_{l,t}} \mu_t^n
\end{aligned} \tag{112}
$$

3. Discretize, then optimize We follow here the reverse approach, discretizing first and optimizing next.3.a The discretized planner's problem

Now first discretize the optimization problem with respect to time (time step Δt) and the idiosyncratic state (N grid points, grid step Δx_i). We define the discount factor $\beta \equiv (1 + \varrho \Delta t)^{-1}$.

$$
\begin{aligned}
\max_{Z_t, u_t^n, \mu_t^n, v_t^n} \quad & \quad \sum_t \beta^t f_0(Z_t) \\
& s.t. \ \forall t \\
\frac{X_{t+1} - X_t}{\Delta t} &= f_1(Z_t) && (113) \\
\frac{U_{t+1} - U_t}{\Delta t} &= f_2(Z_t) && (114) \\
0 &= f_3(Z_t) && (115) \\
\tilde{U}_t &= \sum_{n=1}^{N} f_4\left(x^n, u_t^n, Z_t\right) \mu_t^n && (116) \\
\rho v_t^n &= \frac{v_{t+1}^n - v_t^n}{\Delta t} + f_5(x^n, u_t^n, Z_t) + \sum_{i=1}^{I} b_i\left(x^n, u_t^n, Z_t\right) \nabla_i \left[v_t^n\right] && (117) \\
& + \frac{1}{2} \sum_{i=1}^{I} \left(\sigma_i^2\right)^n \triangle_i^2 \left[v_t^n\right], \ \forall n \\
0 &= \frac{\partial f_{5,t}^n}{\partial u_{j,t}^n} + \sum_{i=1}^{I} \frac{\partial b_{i,t}^n}{\partial u_{j,t}^n} \nabla_i \left[v_t^n\right], \quad \forall j, n. && (118) \\
\frac{\mu_{t+1}^n - \mu_t^n}{\Delta t} &= -\sum_{i=1}^{I} \hat{\nabla}_i \left[b_{i,t}^n \mu_t^n\right] && (119) \\
& + \frac{1}{2} \sum_{i=1}^{I} \triangle_i \left[\sigma_i^2 \mu_t^n\right] && (120) \\
X_0 &= \bar{X}_0 && (121) \\
\mu_0^n &= \bar{\mu}_0^n && (122)
\end{aligned}
$$

3.b The Lagrangian The Lagrangian is

$$
\begin{aligned}
L \;=\; & \sum_t \beta^t f_0(Z_t) \\
& + \sum_t \beta^t \lambda_{1,t} \left\{ \frac{X_{t+1} - X_t}{\Delta t} - f_1(Z_t) \right\} \\
& + \sum_t \beta^t \lambda_{2,t} \left\{ \frac{U_{t+1} - U_t}{\Delta t} - f_2(Z_t) \right\} \\
& + \sum_t \beta^t \lambda_{3,t} \left\{ - f_3(Z_t) \right\} \\
& + \sum_t \beta^t \lambda_{4,t} \left\{ \tilde{U}_t - \sum_n f_4\left(x^n, u_t^n, Z_t\right) \mu_t^n \right\} \\
& + \sum_t \sum_n \beta^t \lambda_{5,t}^n \left\{ \begin{array}{c} -\rho v_t^n + \frac{v_{t+1}^n - v_t^n}{\Delta t} + f_5(x^n, u_t^n, Z_t) + \sum_{i=1}^I b_i\left(x^n, u_t^n, Z_t\right) \nabla_i \left[v_t^n\right] \\ + \sum_{i=1}^I \triangle_i^2 \left[v_t^n\right] \end{array} \right\} \\
& + \sum_t \sum_n \sum_{j=1}^J \beta^t \lambda_{6,j,t}^n \left\{ \frac{\partial f_{5,t}^n}{\partial u_{j,t}^n} + \sum_{i=1}^I \frac{\partial b_{i,t}^n}{\partial u_{j,t}^n} \nabla_i \left[v_t^n\right] \right\} \\
& + \sum_t \sum_n \beta^t \lambda_{7,t}^n \left\{ \begin{array}{c} -\frac{\mu_{t+1}^n - \mu_t^n}{\Delta t} - \sum_{i=1}^I \hat{\nabla}_i \left[b_{i,t}^n \mu_t^n\right] \\ + \frac{1}{2} \sum_{i=1}^I \triangle_i \left[\sigma_i^2 \mu_t^n\right] \end{array} \right\}
\end{aligned}
$$

3.c The optimality conditions The FOCs are

$$
\begin{aligned}
\frac{\partial L}{\partial U_t} : 0 \;=\; & \frac{\partial f_{0,t}}{\partial U_t} - \lambda_{1,t} \frac{\partial f_{1,t}}{\partial U_t} + \lambda_{2,t} \left\{ -\frac{1}{\Delta t} - \frac{\partial f_{2,t}}{\partial U_t} \right\} + \beta^{-1} \lambda_{2,t-1} \frac{1}{\Delta t} - \lambda_{3,t} \frac{\partial f_{3,t}}{\partial U_t} - \lambda_{4,t} \sum_n \frac{\partial f_{4,t}^n}{\partial U_t} \mu_t^n \\
& + \sum_n \lambda_{5,t}^n \left\{ + \frac{\partial f_{5,t}^n}{\partial U_t} + \sum_{i=1}^I \frac{\partial b_{i,t}^n}{\partial U_t} \nabla_i \left[v_t^n\right] \right\} \\
& + \sum_n \sum_{j=1}^J \lambda_{6,j,t}^n \left\{ \frac{\partial^2 f_{5,t}^n}{\partial u_{j,t}^n \partial U_t} + \sum_{i=1}^I \frac{\partial^2 b_{i,t}^n}{\partial u_{j,t}^n \partial U_t} \nabla_i \left[v_t^n\right] \right\} \\
& + \sum_n \left\{ \sum_{i=1}^I \left(\lambda_{7,t}^n - \lambda_{7,t}^{n_i - 1}\right) \left[\mathbb{I}_{b_{i,t}^n < 0} \frac{\partial b_{i,t}^n}{\partial U_t} \frac{\mu_t^n}{\Delta x_i} \right] + \sum_{i=1}^I \left(\lambda_{7,t}^{n_i + 1} - \lambda_{7,t}^n\right) \left[\mathbb{I}_{b_{i,t}^n > 0} \frac{\partial b_{i,t}^n}{\partial U_t} \frac{\mu_t^n}{\Delta x_i} \right] \right\} \\
& \forall t \geq 0
\end{aligned} \tag{123}
$$

$$\begin{aligned}
\frac{\partial L}{\partial X_t}:0 \quad = \quad & \frac{\partial f_{0,t}}{\partial X_t} - \lambda_{1,t}\left\{\frac{1}{\Delta t} + \frac{\partial f_{1,t}}{\partial X_t}\right\} + \beta^{-1}\lambda_{1,t-1}\frac{1}{\Delta t} - \lambda_{2,t}\frac{\partial f_{2,t}}{\partial X_t} - \lambda_{3,t}\frac{\partial f_{3,t}}{\partial X_t} - \lambda_{4,t}\sum_n \frac{\partial f^n_{4,t}}{\partial X_t}\mu^n_t \\
& + \sum_n \lambda^n_{5,t}\left\{\frac{\partial f^n_{5,t}}{\partial X_t} + \sum_{i=1}^{I}\frac{\partial b^n_{i,t}}{\partial X_t}\nabla_i\left[v^n_t\right]\right\} \\
& + \sum_n\sum_j \lambda^n_{6,j,t}\left\{\frac{\partial^2 f^n_{5,t}}{\partial u^n_{j,t}\partial X_t} + \sum_{i=1}^{I}\frac{\partial^2 b^n_{i,t}}{\partial u^n_{j,t}\partial X_t}\nabla_i\left[v^n_t\right]\right\} \\
& + \sum_n\left\{\sum_{i=1}^{I}\left(\lambda^n_{7,t} - \lambda^{n_i-1}_{7,t}\right)\left[\mathbb{I}_{b^n_{i,t}<0}\frac{\partial b^n_{i,t}}{\partial X_t}\frac{\mu^n_t}{\Delta x_i}\right] + \sum_{i=1}^{I}\left(\lambda^{n_i+1}_{7,t} - \lambda^n_{7,t}\right)\left[\mathbb{I}_{b^n_{i,t}>0}\frac{\partial b^n_{i,t}}{\partial X_t}\frac{\mu^n_t}{\Delta x_i}\right]\right\} \\
& \forall t > 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \tilde{U}_t}:0 \quad = \quad & \frac{\partial f_{0,t}}{\partial \tilde{U}_t} - \lambda_{1,t}\frac{\partial f_{1,t}}{\partial \tilde{U}_t} - \lambda_{2,t}\frac{\partial f_{2,t}}{\partial \tilde{U}_t} - \lambda_{3,t}\frac{\partial f_{3,t}}{\partial \tilde{U}_t} - \lambda_{4,t}\sum_n \frac{\partial f^n_{4,t}}{\partial \tilde{U}_t}\mu^n_t \\
& + \sum_n \lambda^n_{5,t}\left\{+\frac{\partial f^n_{5,t}}{\partial \tilde{U}_t} + \sum_{i=1}^{I}\frac{\partial b^n_{i,t}}{\partial \tilde{U}_t}\nabla_i\left[v^n_t\right]\right\} \\
& + \sum_n\sum_j \lambda^n_{6,j,t}\left\{\frac{\partial^2 f^n_{5,t}}{\partial u^n_{j,t}\partial \tilde{U}_t} + \sum_{i=1}^{I}\frac{\partial^2 b^n_{i,t}}{\partial u^n_{j,t}\partial \tilde{U}_t}\nabla_i\left[v^n_t\right]\right\} \\
& + \sum_n\left\{\sum_{i=1}^{I}\left(\lambda^n_{7,t} - \lambda^{n_i-1}_{7,t}\right)\left[\mathbb{I}_{b^n_{i,t}<0}\frac{\partial b^n_{i,t}}{\partial \tilde{U}_t}\frac{\mu^n_t}{\Delta x_i}\right] + \sum_{i=1}^{I}\left(\lambda^{n_i+1}_{7,t} - \lambda^n_{7,t}\right)\left[\mathbb{I}_{b^n_{i,t}>0}\frac{\partial b^n_{i,t}}{\partial \tilde{U}_t}\frac{\mu^n_t}{\Delta x_i}\right]\right\} \\
& \forall t \geq 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \hat{U}_t}:0 \quad = \quad & \frac{\partial f_{0,t}}{\partial \hat{U}_t} - \lambda_{1,t}\frac{\partial f_{1,t}}{\partial \hat{U}_t} - \lambda_{2,t}\frac{\partial f_{2,t}}{\partial \hat{U}_t} - \lambda_{3,t}\frac{\partial f_{3,t}}{\partial \hat{U}_t} - \lambda_{4,t}\sum_n \frac{\partial f^n_{4,t}}{\partial \hat{U}_t}\mu^n_t \\
& + \sum_n \lambda^n_{5,t}\left\{+\frac{\partial f^n_{5,t}}{\partial \hat{U}_t} + \sum_{i=1}^{I}\frac{\partial b^n_{i,t}}{\partial \hat{U}_t}\nabla_i\left[v^n_t\right]\right\} \\
& + \sum_n\sum_j \lambda^n_{6,j,t}\left\{\frac{\partial^2 f^n_{5,t}}{\partial u^n_{j,t}\partial \hat{U}_t} + \sum_{i=1}^{I}\frac{\partial^2 b^n_{i,t}}{\partial u^n_{j,t}\partial \hat{U}_t}\nabla_i\left[v^n_t\right]\right\} \\
& + \sum_n\left\{\sum_{i=1}^{I}\left(\lambda^n_{7,t} - \lambda^{n_i-1}_{7,t}\right)\left[\mathbb{I}_{b^n_{i,t}<0}\frac{\partial b^n_{i,t}}{\partial \hat{U}_t}\frac{\mu^n_t}{\Delta x_i}\right] + \sum_{i=1}^{I}\left(\lambda^{n_i+1}_{7,t} - \lambda^n_{7,t}\right)\left[\mathbb{I}_{b^n_{i,t}>0}\frac{\partial b^n_{i,t}}{\partial \hat{U}_t}\frac{\mu^n_t}{\Delta x_i}\right]\right\} \\
& \forall t \geq 0
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial L}{\partial v_t^n} : 0 \;\; = \;\; & \lambda_{5,t}^n \left\{ -\rho - \frac{1}{\Delta t} + \sum_{i=1}^{I} b_{i,t}^n \frac{\mathbb{I}_{b_t^n<0} - \mathbb{I}_{b_t^n>0}}{\Delta x_i} - \sum_{i=1}^{I} \frac{2\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2} \right\} \\
& + \lambda_{5,t-1}^n \beta^{-1} \frac{1}{\Delta t} \\
& + \sum_{i=1}^{I} \lambda_{5,t}^{n_i-1} b_{i,t}^{n_i-1} \frac{\mathbb{I}_{b_{i,t}^{n_i-1}>0}}{\Delta x_i} + \sum_{i=1}^{I} \lambda_{5,t}^{n_i-1} \frac{\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2} \\
& - \sum_{i=1}^{I} \lambda_{5,t}^{n_i+1} b_{i,t}^{n_i+1} \frac{\mathbb{I}_{b_{i,t}^{n_i+1}<0}}{\Delta x_i} + \sum_{i=1}^{I} \lambda_{5,t}^{n_i+1} \frac{\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2} \\
& + \sum_{j=1}^{J} \sum_{i=1}^{I} \left\{ \lambda_{6,j,t}^n \left\{ \frac{\partial b_{i,t}^n}{\partial u_{j,t}^n} \frac{\mathbb{I}_{b_{i,t}^n<0} - \mathbb{I}_{b_{i,t}^n>0}}{\Delta x_i} \right\} + \lambda_{6,j,t}^{n_i-1} \left\{ \frac{\partial b_{i.t}^{n_i-1}}{\partial u_{j,t}^{n_i-1}} \frac{\mathbb{I}_{b_{i,t}^{n_i-1}>0}}{\Delta x_i} \right\} - \lambda_{6,j,t}^{n_i+1} \left\{ \frac{\partial b_{i,t}^{n_i+1}}{\partial u_{j,t}^{n_i+1}} \frac{\mathbb{I}_{b_{i,t}^{n_i+1}<}}{\Delta x_i} \right. \right. \\
& \forall t \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \mu_t^n} : 0 \;\; = \;\; & -\lambda_{4,t} f_{4,t}^n \\
& + \lambda_{7,t}^n \left\{ \frac{1}{\Delta t} - \sum_{i=1}^{I} \left[\left(\mathbb{I}_{b_{i,t}^n>0} - \mathbb{I}_{b_{i,t}^n<0} \right) \frac{b_{i,t}^n}{\Delta x_i} \right] - \sum_{i=1}^{I} \frac{-2\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2} \right\} \\
& + \left\{ -\sum_{i=1}^{I} \lambda_{7,t}^{n_i-1} \left[\frac{\mathbb{I}_{b_{i,t}^n<0} b_{i,t}^n}{\Delta x_i} \right] + \sum_{i=1}^{I} \frac{\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2} \right\} \\
& + \left\{ -\sum_{i=1}^{I} \lambda_{7,t}^{n_i+1} \left[\frac{-\mathbb{I}_{b_{i,t}^n>0} b_{i,t}^n}{\Delta x_i} \right] + \sum_{i=1}^{I} \frac{\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2} \right\} \\
& + \beta^{-1} \lambda_{7,t-1}^n \left\{ -\frac{1}{\Delta t} \right\} \\
& \forall t > 0
\end{aligned}
\tag{125}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial u_{l,t}^n} : 0 \;\; = \;\; & -\lambda_{4,t} \frac{\partial f_{4,t}^n}{\partial u_{l,t}^n} \mu_t^n \\
& + \beta^t \lambda_{5,t}^n \left\{ \frac{\partial f_{5,t}^n}{\partial u_{l,t}^n} + \sum_{i=1}^{I} \frac{\partial b_{i,t}^n}{\partial u_{l,t}^n} \nabla_i \left[v_t^n \right] \right\}
\end{aligned}
\tag{126}
$$

$$
\begin{aligned}
&+\sum_j \lambda_{6,t}^n \left\{ \frac{\partial^2 f_{5,t}^n}{\partial u_{j,t}^n \partial u_{l,t}^n} + \sum_{i=1}^{I} \frac{\partial^2 b_{i,t}^n}{\partial u_{j,t}^n \partial u_{l,t}^n} \nabla_i \left[v_t^n\right] \right\} \\
&+\sum_{i=1}^{I} \left(\lambda_{7,t}^n - \lambda_{7,t}^{n_i-1}\right) \left[\mathbb{I}_{b_{i,t}^n<0} \frac{\partial b_{i,t}^n}{\partial u_{l,t}^n} \frac{\mu_t^n}{\Delta x_i} \right] + \sum_{i=1}^{I} \left(\lambda_{7,t}^{n_i+1} - \lambda_{7,t}^n\right) \left[\mathbb{I}_{b_{i,t}^n>0} \frac{\partial b_{i,t}^n}{\partial u_{l,t}^n} \frac{\mu_t^n}{\Delta x_i} \right] \\
&\forall t \geq 0
\end{aligned}
$$

By the individual agents' optimality condition, line 2 of this expression is equal to 0.

4. Compare Finally, by comparing the respective discretized optimality conditions, we show that the two procedures yield the same equilibrium conditions in the limit. Consider first the condition for U_t. The optimize-discretize condition is given by (107), which we reproduce here

$$
\begin{aligned}
U_t: \quad 0 = \quad & -\left(\frac{\lambda_{2,t} - \lambda_{2,t-1}}{\Delta} - \varrho \lambda_{2,t} \right) \\
& + \frac{\partial f_0}{\partial U_t} - \lambda_{1,t} \frac{\partial f_1}{\partial U_t} - \lambda_{2,t} \frac{\partial f_2}{\partial U_t} - \lambda_{3,t} \frac{\partial f_3}{\partial U_t} - \lambda_{4,t} \sum_{n=1}^{N} \frac{\partial f_4^n}{\partial U_t} \mu_t^n \\
& + \sum_n \lambda_{5,t}^n \left\{ \frac{\partial f_5^n}{\partial U_t} + \sum_{i=1}^{I} \frac{\partial b_i^n}{\partial U_t} \nabla_i \left[v_t^n\right] \right\} \\
& + \sum_n \sum_{j=1}^{J} \lambda_{6,j,t}^n \left\{ \frac{\partial^2 f_{5,t}^n}{\partial u_{j,t}^n \partial U_t} + \sum_{i=1}^{I} \frac{\partial^2 b_t^n}{\partial u_{j,t}^n \partial U_t} \nabla_i \left[v_t^n\right] \right\} \\
& + \sum_n \left[-\lambda_{7,t}^n \sum_{i=1}^{I} \hat{\nabla}_i \left[\frac{\partial b_{i,t}^n}{\partial U_t} \mu_t^n \right] \right] \\
& \forall t \geq 0
\end{aligned}
$$

The discretize-optimize condition (123), rearranges to

$$
\begin{aligned}
\frac{\partial L}{\partial U_t}: 0 \quad = \quad & -\left(\frac{\lambda_{2,t} - \lambda_{2,t-1}}{\Delta t} - \frac{\beta^{-1} - 1}{\Delta t} \lambda_{2,t-1} \right) \\
& + \frac{\partial f_{0,t}}{\partial U_t} - \lambda_{1,t} \frac{\partial f_{1,t}}{\partial U_t} - \lambda_{2,t} \frac{\partial f_{2,t}}{\partial U_t} - \lambda_{3,t} \frac{\partial f_{3,t}}{\partial U_t} - \lambda_{4,t} \sum_{n=1}^{N} \frac{\partial f_{4,t}^n}{\partial U_t} \mu_t^n \\
& + \sum_{n=1}^{N} \lambda_{5,t}^n \left\{ \frac{\partial f_{5,t}^n}{\partial U_t} + \sum_{i=1}^{I} \frac{\partial b_i^n}{\partial U_t} \nabla_i \left[v_t^n\right] \right\}
\end{aligned}
$$

$$+\sum_{n=1}^{N}\sum_{j=1}^{J}\lambda_{6,j,t}^{n}\left\{\frac{\partial^{2}f_{5,t}^{n}}{\partial u_{j,t}^{n}\partial U_{t}}+\frac{\partial^{2}b_{t}^{n}}{\partial u_{j,t}^{n}\partial U_{t}}\nabla_{i}\left[v_{t}^{n}\right]\right\}$$

$$+\sum_{n}\left\{\sum_{i=1}^{I}\left(\lambda_{7,t}^{n}-\lambda_{7,t}^{n_i-1}\right)\left[\mathbb{I}_{b_{i,t}^{n}<0}\frac{\partial b_{i,t}^{n}}{\partial U_{t}}\frac{\mu_{t}^{n}}{\Delta x_{i}}\right]+\sum_{i=1}^{I}\left(\lambda_{7,t}^{n_i+1}-\lambda_{7,t}^{n}\right)\left[\mathbb{I}_{b_{i,t}^{n}>0}\frac{\partial b_{i,t}^{n}}{\partial U_{t}}\frac{\mu_{t}^{n}}{\Delta x_{i}}\right]\right\}$$

$$\forall t\geq 0$$

The second to fourth lines are evidently identical. The last lines also coincide once we take into account the definition of $\hat{\nabla}_i\left[\frac{\partial b_{i,t}^{n}}{\partial U_t}\mu_t^n\right]=\frac{\mathbb{I}_{b_{i,t}^{n_i+1}<0}\frac{\partial b_{i,t}^{n_i+1}}{\partial U_t}\mu_t^{n_i+1}-\mathbb{I}_{b_{i,t}^{n}<0}\frac{\partial b_{i,t}^{n}}{\partial U_t}\mu_t^{n}}{\Delta x_i}+\frac{\mathbb{I}_{b_{i,t}^{n}>0}\frac{\partial b_{i,t}^{n}}{\partial U_t}\mu_t^{n}-\mathbb{I}_{b_{i,t}^{n_i-1}>0}\frac{\partial b_{i,t}^{n_i-1}}{\partial U_t}\mu_t^{n_i-1}}{\Delta x_i}$.

Finally compare the first lines. Since $\beta\equiv(1+\varrho\Delta t)^{-1}$ we have that $\frac{\beta^{-1}-1}{\Delta t}=\varrho$. The difference between these two equations hence is $\|\varrho\left(\lambda_{2,t}-\lambda_{2,t-1}\right)\|$. In the limit as $\Delta t\to 0$, and provided that $\lambda_{2,t}$ features no jumps for $t>0$,this difference converges to zero.The same argument applies to the optimality conditions with respect to X_t with the difference now proportional to $\|\varrho\left(\lambda_{1,t}-\lambda_{1,t-1}\right)\|$. The optimality conditions with respect to $\hat{U}_t$ and $\tilde{U}_t$ are identical, that is, there is no difference.

Next consider the two discretized optimality conditions with respect to v_t^n (110) and (124). After some rearranging they are given by

$$v_t(x):\quad 0=\;-\sum_{i=1}^{I}\left(\frac{\mathbb{I}_{b_{i,t}^{n}>0}\lambda_{5,j,t}^{n}b_{i,t}^{n}-\mathbb{I}_{b_{i,t}^{n_i-1}>0}\lambda_{5,j,t}^{n_i-1}b_{i,t}^{n_i-1}}{\Delta x_i}+\frac{\mathbb{I}_{b_{i,t}^{n_i+1}<0}\lambda_{5,j,t}^{n_i+1}b_{i,t}^{n_i+1}-\mathbb{I}_{b_{i,t}^{n}<0}\lambda_{5,j,t}^{n}b_{i,t}^{n}}{\Delta x_i}\right)$$

$$+\frac{1}{2}\sum_{i=1}^{I}\frac{\left(\sigma_i^2\right)^{n_i+1}\lambda_{5,t}^{n_i+1}+\left(\sigma_i^2\right)^{n_i-1}\lambda_{5,t}^{n_i-1}-2\left(\sigma_i^2\right)^{n}\lambda_{5,t}^{n}}{\left(\Delta x_i\right)^2}$$

$$-\sum_{j=1}^{J}\sum_{i=1}^{I}\left(\frac{\mathbb{I}_{b_{i,t}^{n}>0}\lambda_{6,j,t}^{n}\frac{\partial b_{i,t}^{n}}{\partial u_{j,t}^{n}}-\mathbb{I}_{b_{i,t}^{n_i-1}>0}\lambda_{6,j,t}^{n_i-1}\frac{\partial b_{i,t}^{n_i-1}}{\partial u_{j,t}^{n_i-1}}}{\Delta x_i}+\frac{\mathbb{I}_{b_{i,t}^{n_i+1}<0}\lambda_{6,j,t}^{n_i+1}\frac{\partial b_{i,t}^{n_i+1}}{\partial u_{j,t}^{n_i+1}}-\mathbb{I}_{b_{i,t}^{n}<0}\lambda_{6,j,t}^{n}\frac{\partial b_{i,t}^{n}}{\partial u_{j,t}^{n}}}{\Delta x_i}\right)$$

$$-\lambda_{5,t}^{n}\rho-\left(\frac{\lambda_{5,t}^{n}-\lambda_{5,t-1}^{n}}{\Delta t}-\varrho\lambda_{5,t}^{n}\right)$$

and

$$
\begin{aligned}
\frac{\partial L}{\partial v_t^n}:0 \quad = \quad & \lambda_{5,t}^n\left\{\sum_{i=1}^I b_{i,t}^n\frac{\mathbb{I}_{b_t^n<0}-\mathbb{I}_{b_t^n>0}}{\Delta x_i}-\sum_{i=1}^I\frac{\left(\sigma_i^2\right)^n}{\left(\Delta x_i\right)^2}\right\} \\
& +\left\{\sum_{i=1}^I\lambda_{5,t}^{n_i-1}b_{i,t}^{n_i-1}\frac{\mathbb{I}_{b_{i,t}^{n_i-1}>0}}{\Delta x_i}+\sum_{i=1}^I\lambda_{5,t}^{n_i-1}\frac{\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2}\right\} \\
& +\left\{-\sum_{i=1}^I\lambda_{5,t}^{n_i+1}b_{i,t}^{n_i+1}\frac{\mathbb{I}_{b_{i,t}^{n_i+1}<0}}{\Delta x_i}+\sum_{i=1}^I\lambda_{5,t}^{n_i+1}\frac{\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2}\right\} \\
& +\sum_{j=1}^J\sum_{i=1}^I\left(\lambda_{6,j,t}^n\frac{\partial b_{i,t}^n}{\partial u_{j,t}^n}\frac{\mathbb{I}_{b_t^n<0}-\mathbb{I}_{b_t^n>0}}{\Delta x_i}+\lambda_{6,j,t}^{n_i-1}\frac{\partial b_{i,t}^{n_i-1}}{\partial u_{j,t}^{n_i-1}}\frac{\mathbb{I}_{b_t^{n_i-1}>0}}{\Delta x_i}-\lambda_{6,j,t}^{n_i+1}\frac{\partial b_{i,t}^{n_i+1}}{\partial u_{j,t}^{n_i+1}}\frac{\mathbb{I}_{b_t^{n_i+1}<0}}{\Delta x_i}\right) \\
& -\rho\lambda_{5,t}^n-\left(\frac{\lambda_{5,t}^n-\lambda_{5,t-1}^n}{\Delta t}-\frac{\beta^{-1}-1}{\Delta t}\lambda_{5,t-1}^n\right) \qquad (127)
\end{aligned}
$$

Again these, two expressions are identical up to the last time index in the last line (λ_5^n), and thus the difference is $\left\|\varrho\left(\lambda_{5,t}-\lambda_{5,t-1}\right)\right\|$.

Next, consider the two discretized optimality conditions with respect to μ_t^n (111) and (125). After some rearranging they are given by

$$
\begin{aligned}
\mu_t\left(x\right):\quad 0= \quad & -\lambda_{4,t}f_{4,t}^n \qquad (128) \\
& +\sum_{i=1}^I b_{i,t}^n\left[\mathbb{I}_{b_{i,t}^n>0}\frac{\lambda_{7,t}^{n_i+1}-\lambda_{7,t}^n}{\Delta x_i}+\mathbb{I}_{b_{i,t}^n<0}\frac{\lambda_{7,t}^n-\lambda_{7,t}^{n_i-1}}{\Delta x_i}\right]+\frac{1}{2}\sum_{i=1}^I\left(\sigma_i^2\right)^n\frac{\lambda_{7,t}^{n_i+1}+\lambda_{7,t}^{n_i-1}-2\lambda_{7,t}^n}{2\left(\Delta x_i\right)^2} \\
& +\frac{\lambda_{7,t}^n-\lambda_{7,t-1}^n}{\Delta t}-\varrho\lambda_{7,t}^n
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \mu_t^n}:0 \quad = \quad & -\lambda_{4,t}f_{4,t}^n \qquad (129) \\
& +\lambda_{7,t}^n\left\{-\sum_{i=1}^I\left[\left(\mathbb{I}_{b_{i,t}^n>0}-\mathbb{I}_{b_{i,t}^n<0}\right)\frac{b_{i,t}^n}{\Delta x_i}\right]-\sum_{i=1}^I\frac{-2\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2}\right\} \\
& -\sum_{i=1}^I\left[\lambda_{7,t}^{n_i-1}\frac{\mathbb{I}_{b_{i,t}^n<0}b_{i,t}^n}{\Delta x_i}\right]+\sum_{i=1}^I\lambda_{7,t}^{n_i-1}\frac{\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2} \\
& +\sum_{i=1}^I\left[\lambda_{7,t}^{n_i+1}\frac{\mathbb{I}_{b_{i,t}^n>0}b_{i,t}^n}{\Delta x_i}\right]+\sum_{i=1}^I\lambda_{7,t}^{n_i+1}\frac{\left(\sigma_i^2\right)^n}{2\left(\Delta x_i\right)^2} \\
& +\frac{\lambda_{7,t}^n-\lambda_{7,t-1}^n}{\Delta t}-\frac{\beta^{-1}-1}{\Delta t}\lambda_{7,t-1}^n,
\end{aligned}
$$

which again differ in $\|\varrho(\lambda_{7,t}-\lambda_{7,t-1})\|$.

Finally, consider the two discretized optimality conditions with respect to $u_{l,t}^{n}(x)$, (112) and (126). After some rearranging they are given by

$$\begin{aligned} u_{l,t}(x): \quad 0 = & \quad -\lambda_{4,t}\frac{\partial f_4}{\partial u_{l,t}}\mu_t^n \\ + & \quad \sum_{j=1}^{J}\lambda_{6,l,t}^n\left(\frac{\partial^2 f_{5,t}^n}{\partial u_{j,t}^n \partial u_{l,t}^n}+\sum_{i=1}^{I}\frac{\partial^2 b_{i,t}^n}{\partial u_{j,t}^n \partial u_{l,t}^n}\left[\mathbb{I}_{b_{i,t}^n>0}\frac{v_t^{n_i+1}-v_t^n}{\Delta x_i}+\mathbb{I}_{b_{i,t}^n<0}\frac{v_t^n-v_t^{n1}}{\Delta x_i}\right]\right) \\ - & \quad \sum_{i=1}^{I}\left[\mathbb{I}_{b_{i,t}^n>0}\frac{\lambda_{7,t}^{n_i+1}-\lambda_{7,t}^n}{\Delta x_i}+\mathbb{I}_{b_{i,t}^n<0}\frac{\lambda_{7,t}^n-\lambda_{7,t}^{n_i-1}}{\Delta x_i}\right]\frac{\partial b_{i,t}^n}{\partial u_{l,t}}\mu_t^n \end{aligned} \tag{130}$$

$$\begin{aligned} \frac{\partial L}{\partial u_{l,t}^n}:0 \quad = \quad & -\lambda_{4,t}\frac{\partial f_{4,t}^n}{\partial u_{l,t}^n}\mu_t^n \\ & +\sum_j \lambda_{6,t}^n\left\{\frac{\partial^2 f_{5,t}^n}{\partial u_{j,t}^n \partial u_{l,t}^n}+\sum_{i=1}^{I}\frac{\partial^2 b_{i,t}^n}{\partial u_{j,t}^n \partial u_{l,t}^n}\left[\mathbb{I}_{b_{i,t}^n>0}\frac{v_t^{n_i+1}-v_t^n}{\Delta x_i}+\mathbb{I}_{b_{i,t}^n<0}\frac{v_t^n-v_t^{n_i-1}}{\Delta x_i}\right]\right\} \\ & +\left[\sum_{i=1}^{I}\left(\lambda_{7,t}^n-\lambda_{7,t}^{n_i-1}\right)\left[\mathbb{I}_{b_{i,t}^n<0}\frac{1}{\Delta x_i}\right]+\sum_{i=1}^{I}\left(\lambda_{7,t}^{n_i+1}-\lambda_{7,t}^n\right)\left[\mathbb{I}_{b_{i,t}^n>0}\frac{1}{\Delta x_i}\right]\right]\frac{\partial b_{i,t}^n}{\partial u_{l,t}}\mu_t^n, \end{aligned}$$

which are identical.To summarize, whether one discretize the optimality conditions of the planner and then discretizes them, or one discretizes the planner's problem and then derives the optimality conditions, one arrives to a set of optimality conditions that coincide in everything but the timing of the multiplier in the term $\varrho\lambda_t$. Provided that multipliers experience no jumps, the difference between the two approaches goes to 0 as $\Delta t \to 0$. Note that this issue has nothing to do with heterogeneity.

D. Solving the Nuño and Thomas model using Dynare

Here we apply the "discretize-optimize" methodology outlined in Section 3 to the heterogeneous-agent model introduced in Nuño and Thomas (2016). This is a model *à la* Aiyagari-Bewley-Huggett with non-state-contingent long-term nominal debt contracts. Finding the optimal policy in this problem requires that the central bank takes into account not only the dynamics of the state distribution (given by the KF equation) but also the HJB equation. Figure 6 displays the time-0 optimal policy (inflation) in this case, compared to the one obtained through the "optimize-discretize" methodology employed in Nuño and Thomas (2016). Optimal inflation coincides in both cases, up to a numerical error that is reduced as we increase the number of grid points and we reduce the time step.

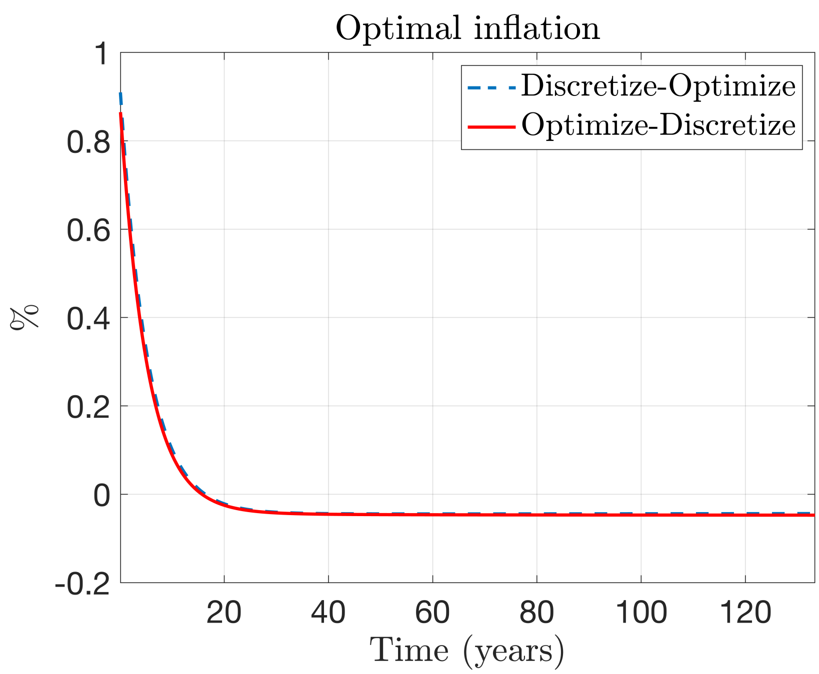

Figure 6: Time-0 optimal monetary policy using the two approaches.

Notes: The figure shows the optimal path of inflation in the Nuño and Thomas (2016) model using the "discretize-optimize" and "optimize-discretize" methods.