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### COVID Crisis: Inflation and Deflation Pressures

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# COVID-19: Inflation and Deflation Pressures

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## Abstract

Pandemics can lead to rich inflation dynamics with strong inflationary as well as deflationary forces. Initially, deflationary pressures play a dominating role because idiosyncratic risk is elevated until recovery arrives. The inflation dynamics are highly dependent on government policies. Government lending programs allow the contact-intensive, distressed sector to become more indebted, which prevents deflation if the pandemic is short-lived but amplifies deflation if the pandemic lasts longer. Redistribution moderates the deflationary forces and can lead to excessive inflation in the long-run.

## 1 Introduction

The COVID-19 pandemic has brought dramatic economic changes around the world. Many industries have been forced to shutdown or restructure operations and governments have run large budget deficits to fund extensive support programs. Historical experience suggests that these changes are likely to lead to subtle inflationary and deflationary dynamics, which are highly dependent on policy decisions. In this paper, we develop a model to decompose the various short run and long run inflation and deflation pressures that are emerging as governments respond to the pandemic.

We consider a model with two sectors, one of which is unexpectedly shutdown for a random duration. Agents in both sectors produce using idiosyncratically risky capital. Capital volatility ends up increasing the longer the pandemic lasts. There are three key frictions. First, idiosyncratic capital risk is not insurable. This implies that “safe” assets, such as government debt, provide a “self-insurance” benefit and so trade at a premium. Second, capital is costly to adjust and only tradable within a sector. This implies that the shutdown

sector must borrow or convert capital in consumption goods in order to smooth consumption. Finally, the agents face a borrowing constraint.

We derive five main results. First, we show that the initial pandemic shock has an ambiguous impact on bond demand and so its impact on the price level is ambiguous. This is because there are counteracting forces affecting bond demand when the pandemic occurs. Aggregate productivity decreases, which lowers aggregate net-worth and so bond demand. However, the volatility outlook for the economy also increases and so agents undertake more precautionary saving, which increases bond demand.

Second, without any government fiscal policy, there is a strong deflationary force as long as the pandemic lasts. Agents in the shutdown sector smooth consumption by selling bonds and converting capital. This decreases aggregate capital stock, net-worth and bond demand, which generates inflation. However, over time, there is also a reallocation of capital away from the shutdown sector. This increases idiosyncratic risk, which increases bond demand and generates deflation. In our numerical examples, the later force dominates and we get sustained deflation during the pandemic. Once the pandemic ends, the forces reverse because agents start to rebuild capital and idiosyncratic risk starts to decrease. This leads to sustained inflation during the recovery. In this sense, our model generates an “inflation whipsaw”, where a deflationary force during the pandemic is quickly followed by an inflationary force once the pandemic ends.

Third, if the government introduces a lending program, then agents are less afraid hitting the borrowing constraint later on and hence they are more aggressive in their consumption smoothing. They borrow more rather than convert capital goods into consumption goods. This is because they anticipate a large increase capital prices when the recovery arrives. If the pandemic is short, then this ensures a fast recovery because capital doesn’t have to be rebuilt. However, if the pandemic lasts longer than expected, then the shutdown sector become highly indebted and agents rapidly destroy capital. We can think about this as the agents’ “gamble on recovery” not paying off. Under this policy, inflationary pressures dominate early in the pandemic but they are overpowered by deflationary forces if the recovery takes longer than expected. In this sense, introducing a government lending program makes the inflation and deflation dynamics more sensitive to the length of the pandemic.

Fourth, we show that intratemporal redistribution across sectors is inflationary. Unlike in many other papers, this does not occur because the sectors have a different marginal propensity to consume out of wealth. Instead,

it occurs because the sectors have a different willingness to hold capital. This means that fiscal policy makers face a trade-off. If they don't address inequality, then they get a strong deflationary force during the crisis and a slow recovery in output once the crisis ends. Alternatively, if they redistribute wealth during the pandemic, then they get a faster recovery but also a strong inflationary force during the crisis and recovery phase.

Finally, we consider intertemporal redistribution. Ricardian equivalence holds if the government introduces a lending program to eliminate the borrowing constraint and funds transfers through lump-sum taxation. However, under other policies, Ricardian equivalence breaks. If the government uses lump-sum taxation but does not introduce a lending program, then  $B$  is indifferent about the timing of taxes but sector  $A$  would prefer that the government raises taxes once their borrowing constraint is no longer binding. If the government raises taxes proportional to wealth, then the transfer scheme provides additional insurance against idiosyncratic risk. Ultimately, this decreases bond demand and so generates more inflation.

## 1.1 Related Literature

In our model, bond (or money) demand comes from precautionary saving to self-insure against non-contractible shocks. This generates the key feature of our model that changes to the price and quantity of risk affect the portfolio choice between bonds (or money) and capital. The emphasis on treating money demand as a portfolio choice problem is part of a long tradition (e.g. [Tobin \(1965\)](#)) and also builds on more recent papers, such as [Brunnermeier and Sannikov \(2016b\)](#), [Di Tella \(2019\)](#), and [Szoke \(2019\)](#), which emphasise how portfolio reallocation into money can impact investment. For tractability, we work with a reduced form model with exogenous market incompleteness that prevents agent from insuring against idiosyncratic capital risk. This uses the framework developed in [Bewley \(1980, 1983\)](#), [Angeletos \(2007\)](#), and [Brunnermeier and Sannikov \(2016a\)](#). Although we are not interested in the micro-foundations of money or debt demand in this paper, we would get a qualitatively similar link between risk and money demand in models with a richer foundation. For example, prominent models of liquidity premia typically focus on self-insurance against idiosyncratic shocks to the marginal utility of consumption, the marginal productivity of investment or the timing of supplier payments (e.g. [Diamond and Dybvig \(1983\)](#), [Holmström and Tirole \(1998\)](#), [Lagos and Wright \(2005\)](#), and [Kiyotaki and Moore \(2019\)](#)). Alternatively, we

would get a similar channel if we used a “safe asset” model or inserted stochastic death rates into an OLG model. The contribution of our paper is to examine how changes to precautionary saving during a period of high volatility and inequality impact inflation and investment.

There is also a long literature studying the link between fiscal policy and inflation (e.g. Sargent and Wallace (1981), Sargent (1982), Woodford (1995), Sims (1994), Brunnermeier et al. (2020)). Our main contribution is to integrate this literature with precautionary saving demand for money and a pandemic style shock.

Finally, there have been many recent papers on the COVID crisis (e.g. Guerrieri et al. (2020), Kaplan et al. (2020), Eichenbaum et al. (2020), e Castro (2020), as well as many others). Our main contribution is focus on aggregate risk.

## 2 Model

### 2.1 Environment

We consider a continuous time, infinite horizon economy populated by a continuum of agents, who are interpreted as entrepreneurs and indexed by  $i \in [0, 2]$ . The agents can be in one of two sectors, indexed by  $I \in \{A, B\}$ . Each sector has its own type of capital, which is used to produce a different type of intermediate good. There is also a final good, which is used for consumption and investment.

Agent  $i$  in sector  $I$  has preferences described by the expected utility function:

$$\mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} u(c_t^{Ii}) dt \right],$$

where  $\mathbb{E}_t$  is the expectation taken with respect to information at time  $t$ ,  $c_t^{Ii}$  denotes the agent’s consumption of the final good at time  $t$  and  $u(c) = \log(c)$ .

Agent  $i$  in sector  $I$  operates a technology that produces intermediate good  $I$  according to the production function:

$$y_t^{Ii} dt = a_t^I k_t^{Ii} dt,$$

where  $a_t^I$  denotes the productivity in sector  $I$ ,  $y_t^{Ii}$  denotes the flow output of good  $I$  produced by agent  $i$ , and  $k_t^{Ii}$  denotes the capital owned by agent  $i$  in

sector  $I$ . Absent capital trading, capital stock evolves according to:

$$dk_t^{Ii} = (\Phi(\iota_t^{Ii}) - \delta)k_t^{Ii} dt + \tilde{\sigma}_t^I k_t^{Ii} d\tilde{Z}_t^{Ii},$$

where  $\iota_t^{Ii}$  is the flow investment rate per unit capital,  $\Phi(\iota) = (1/\phi) \log(1 + \phi\iota)$  is the flow capital created by  $\iota$  (net of adjustment costs),  $\tilde{Z}_t^{Ii}$  denotes an idiosyncratic Brownian motion process and  $\tilde{\sigma}_t$  denotes the volatility of the idiosyncratic process. We allow  $\tilde{\sigma}_t$  to vary with time to consider the possibility that idiosyncratic risk could change as the economic environment changes. In numerical examples, we will focus on the case that  $\tilde{\sigma}_t = \tilde{\sigma}(\kappa_t) = \tilde{\sigma} + \tilde{\sigma}^\kappa(\kappa_t - \kappa^{SS})$ , where  $\kappa_t = K_t^A/K_t$  is the ratio of total capital in sector A to total capital in the economy and  $\kappa^{SS}$  is steady state value of  $\kappa_t$ . We interpret this equation as saying that if the capital allocation is more distorted, then idiosyncratic risk increases.

A representative final goods firm uses the input goods to produce the final consumption good according to the aggregation function:

$$Y_t = \left[ \alpha^A (Y_t^A)^{\frac{\epsilon-1}{\epsilon}} + \alpha^B (Y_t^B)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}.$$

where  $Y_t^I$  denotes total inputs goods from sector  $I$  and  $\alpha^I$  reflects the relative importance of good  $I$  in the CES aggregator.

The only aggregate shock in the economy is the arrival of a pandemic, which is modelled in the following way. Before the pandemic occurs, productivity in both sectors is constant at  $a_t^A = a_t^B = \bar{a}$ . At time  $t_0$ , the pandemic begins and productivity  $a_t^A$  immediately decreases from  $\bar{a}$  to  $\underline{a}$ , while productivity  $a_t^B$  stays at  $\bar{a}$ . Productivity  $a_t^A$  then remains at  $\underline{a}$  for the duration of the pandemic. We interpret the decrease in  $a_t^A$  as the shutdown of production in the non-essential services sector during the pandemic. The pandemic ends according to a Poisson jump process, denoted by  $J_t$ , with arrival rate  $\lambda_t$ . After the pandemic ends,  $a_t$  reverts immediately to  $\bar{a}$ . In summary, if we let  $t_1$  denote the time at which the pandemic ends, then

$$a_t^A = \begin{cases} \bar{a}, & \text{for } t < t_0 \\ \underline{a} & \text{for } t_0 \leq t < t_1 \\ \bar{a}, & \text{for } t \geq t_1 \end{cases} \quad a_t^B = \bar{a}$$

The initial pandemic shock is unexpected but agents in the economy know the

probability with which the pandemic ends.

The only type of financial asset is an infinitesimally short term bond. We normalise the interest rate on bonds to  $i^M$  units of bonds. Bonds can be issued by the government and by private agents. However, private agents face a constraint that their share of wealth in bonds, denoted by  $\theta_t^M$ , cannot be less than  $\underline{\theta}$ . We let  $M_t$  denote the positive net supply of bonds at time  $t$  (i.e. the bonds issued by the government). Although we call the government liabilities bonds, they could also be interpreted as interest paying reserves.

There are competitive markets for the final consumption good, the intermediate goods and bonds. The market for capital is competitive but segmented. Agents in sector  $I$  can only trade capital with other agents in sector  $I$ . We denote prices in the following way. Let the price of the final consumption good be the numeraire. Let  $p_t^I$  denote the price of good  $I \in \{A, B\}$ . Let  $q_t^I$  denote the price of capital  $I \in \{A, B\}$ . Let  $1/P_t$  denote the price of bonds so that  $P_t$  denotes the price of goods in term of bonds. We guess (and verify) that there exist  $\mu_t^{qI}, \mu_t^P, \nu_t^{qI}$  and  $\nu_t^P$  such that the asset prices evolve according to:

$$\begin{aligned} dq_t^I &= \mu_t^{qI} q_t^I dt + \nu_t^{qI} q_t^I dJ_t \\ dP_t &= \mu_t^P P_t dt + \nu_t^P P_t dJ_t \end{aligned}$$

## 2.2 Agent Problems

We now specify the government budget constraint and the agent problems. We start by defining some additional notation that will be useful throughout this section. Let  $m_t^{Ii}$  and  $n_t^{Ii} := q_t^I k_t^{Ii} + (1/P_t)m_t^{Ii}$  denote the bond holdings and net-worth of agent  $i$  in sector  $I$ . Let  $K_t^I, M_t^I, N_t^I, Y_t^I, I_t^I$ , and  $C_t^I$  denote the total capital, bonds, net-worth, output, investment and consumption in sector  $I$  at time  $t$ .

### 2.2.1 Government

The government sets an exogenous path of transfers. We allow the transfers to depend upon the sector,  $I$ , but not upon the idiosyncratic  $\tilde{Z}_t^i$  process or the recovery shock,  $dJ_t$ . Let  $T_t^{Ii}$  denote the transfer to agent  $i$  in sector  $I$ . We restrict transfers to be proportional to agent net-worth. This implies that:

$$T_t^{Ii} dt = \tau_t^I n_t^{Ii} dt$$

for some transfer rate  $\tau_t^I$  that is independent of  $i$ . We further restrict  $\tau_t^I$  to be of the form  $\tau_t^I = \tau^I(\eta_t)$ , where  $\eta_t := N_t^A/N_t$  is sector  $A$ 's share of total wealth<sup>1</sup>. Observe that  $T_t^{Ii} > 0$  represents a transfer to agent  $i$  in sector  $I$  and  $T_t^{Ii} < 0$  represents a tax on agent  $i$  in sector  $I$ . The total transfer to sector  $I$  is then denoted by  $T_t^I = \int T_t^{Ii} di = \tau_t^I N_t^I$ .

The government also sets a path of bond supply, which we denote by  $dM_t = (\mu_t^M - i^M)M_t dt$ . The budget constraint of the government becomes:

$$(T_t^A + T_t^B)dt = \frac{1}{P_t}dM_t + d\left(\frac{1}{P_t}\right)dM_t = \frac{(\mu_t^M - i^M)M_t}{P_t}dt \quad (2.1)$$

### 2.2.2 Final Goods Firm

Given intermediate input goods prices,  $\{p_t^I\}_{I \in \{A,B\}}$ , at each  $t$ , the final goods producer chooses intermediate inputs,  $\{Y_t^I\}_{I \in \{A,B\}}$ , to solve the static problem:

$$\begin{aligned} \max_{\{Y_t^I\}_{I \in \{A,B\}}} \{Y_t - p_t^A Y_t^A - p_t^B Y_t^B\} \quad s.t. \\ Y_t = \left[ \alpha^A (Y_t^A)^{\frac{\epsilon-1}{\epsilon}} + \alpha^B (Y_t^B)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} \end{aligned} \quad (2.2)$$

The first order conditions give the intermediate good demand functions:

$$\begin{aligned} Y_t^A &= \left( \frac{\alpha^A}{p_t^A} \right)^\epsilon Y_t \\ Y_t^B &= \left( \frac{\alpha^B}{p_t^B} \right)^\epsilon Y_t \end{aligned}$$

### 2.2.3 Entrepreneur

Given price and transfer processes, agent  $i$  in sector  $I$  solves the optimization problem:

$$\begin{aligned} \max_{c, \nu, \theta} \left\{ \mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} u(c_t^{Ii}) dt \right] \right\} \quad s.t. \\ dn_t^{Ii} = \left[ \mu_t^{rMI} + (\mu_t^{rkIi} - \mu_t^{rMI})\theta_t^{kIi} - \frac{c_t^{Ii}}{n_t^{Ii}} \right] n_t^{Ii} dt \\ + \tilde{\sigma}_t \theta_t^{kIi} n_t^{Ii} d\tilde{Z}_t + \left[ \nu_t^{1/P} + (\nu_t^{qkI} - \nu_t^{1/P})\theta_t^{kIi} \right] n_t^{Ii} dJ_t \end{aligned}$$

where  $\theta_t^{kIi} := q_t^I k_t^{Ii} / n_t^{Ii}$  is the share of wealth that agent  $i$  in sector  $I$  allocates to capital,  $\theta_t^{rMI} := (1/P_t)m_t^{Ii} / n_t^{Ii} = 1 - \theta_t^{kIi}$  is the share of wealth that agent

<sup>1</sup>This restriction will ensure that the problem is recursive



$i$  in sector  $I$  allocates to bonds,  $\nu_i^{1/P}(1/P_t)$  is the jump in  $1/P_t$  when the recovery occurs,  $\mu_t^{rkIi}$  is the expected return from holding capital and  $\mu_t^{rMI}$  is the expected return from holding bonds. The expected returns (after transfers) are given by the expressions:

$$\begin{aligned}\mu_t^{rkIi} &:= \frac{p_t^I a_t^I - l_t^{Ii}}{q_t^I} + \Phi(l_t^{Ii}) - \delta + \mu_t^{qI} + \tau_t^I \\ \mu_t^{rMI} &:= i^M - \mu_t^P + \tau_t^I\end{aligned}$$

## 2.3 Equilibrium

We now define and characterise a recursive equilibrium for this economy. We start by defining the state variables. The idiosyncratic state variable for agent  $i$  in sector  $I$  is their net worth  $n_t^{Ii}$ . The aggregate state variables for the economy are<sup>2</sup>:  $X_t := \{K_t, M_t, \kappa_t, \eta_t, a_t\}$ , where recall that:

$$\kappa_t := \frac{K_t^A}{K_t}, \quad \eta_t := \frac{N_t^A}{N_t}, \quad a_t := a_t^A$$

Where convenient, we will also use the additional notation that  $\eta_t^I := N_t^I/N_t$  and  $\kappa_t^I := K_t^I/K_t$ .

### 2.3.1 Entrepreneur HJBE

Let  $V^{Ii}(n^{Ii}, X)$  denote the value function for agent  $i$  in sector  $I$  with idiosyncratic wealth  $n^{Ii}$  when the aggregate state is  $X$ . Then,  $V^{Ii}(n^{Ii}, X)$  solves the HJBE:

$$\begin{aligned}& \rho V^{Ii}(n^{Ii}, X) \\ &= \max_{c^{Ii}, \nu^{Ii}, \theta^{kIi}} \left\{ u(c^{Ii}) + \partial_n V^{Ii}(n^{Ii}, X) \left( \mu^{rMI} + (\mu^{rkIi} - \mu^{rMI})\theta^{kIi} - \frac{c^{Ii}}{n^{Ii}} \right) n^{Ii} \right. \\ & \quad + \partial_X V^{Ii}(n^{Ii}, X) \mu^X(X) + \frac{1}{2} \partial_n^2 V^{Ii}(n^{Ii}, X) \tilde{\sigma}(\kappa)^2 (\theta^{kIi})^2 (n^{Ii})^2 \\ & \quad + \lambda \left( V^{Ii} \left( \left( 1 + \nu^{1/P} + (\nu^{qkI} - \nu^{1/P})\theta^{kIi} \right) n^{Ii}, (1 + \nu^X(X))X \right) \right. \\ & \quad \left. \left. - V^{Ii}(n^{Ii}, X) \right) \right\} \quad s.t. \quad \theta^{kIi} \leq 1 - \underline{\theta}\end{aligned}\tag{2.3}$$

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<sup>2</sup>Observe that, even though we have a continuum of agents, we will be able to aggregate agents within each sector and so the wealth and capital shares of sector A will be sufficient state variables for determining equilibrium prices.

where  $\mu^X(X)X$  is the drift in the aggregate state variable and  $\nu^X(X)X$  is the jump in  $X$  when the recovery occurs.

Standard arguments show that the value function takes the form:

$$V^{Ii}(n^{Ii}, X) = \frac{1}{\rho} \log(n^{Ii}) + \Gamma^I(X)$$

for some function  $\Gamma^I(X)$  independent of  $i$ . It follows that the first order conditions for the HJBE become:

$$\begin{aligned} c^{Ii} &= \rho n^{Ii} \\ \iota^{Ii} &= \frac{1}{\phi}(q^I - 1) =: \iota^I \\ \mu^{rkI} - \mu^{rMI} &= \bar{\sigma}(\kappa_t)^2 \theta^{kI} - \frac{\lambda(\nu^{qI} - \nu^{1/P})}{1 + \nu^{nI}} + \rho \psi^I \end{aligned} \quad (2.4)$$

where  $\psi^I$  is the Lagrange multiplier on the constraint  $\theta^{kI} = 1 - \theta^{mI} \geq 1 - \underline{\theta}$  and where, under the optimal choices, the expected return on capital and investment adjustment cost are given by:

$$\begin{aligned} \mu^{rkI} &:= \frac{p^I a^I - \iota_t^I}{q^I} + \Phi(\iota^I) - \delta + \mu^{qI} + \tau^I \\ \Phi(\iota^I) &= \frac{1}{\phi} \log(1 + \phi \iota^I) = \frac{1}{\phi} \log(q^I) \end{aligned}$$

We can immediately see that all agents in a given sector  $I$  choose the same investment rate per unit of capital,  $\iota^I$ , and the same portfolios:  $\theta_t^{kI}$  and  $\theta_t^{mI} = 1 - \theta_t^{kI}$ . We can observe that the aggregate component of the value function,  $\Gamma^I(X)$ , does not end up appearing in the first order conditions and so we do not solve explicitly for  $\Gamma^I(X)$ .

### 2.3.2 Equilibrium Definition

**Definition 1.** A (recursive) equilibrium is a collection of price functions,  $\{p^I(X), q^{kI}(X), P(X), \}_{I \in \{A, B\}}$ , laws of motion for the (aggregate) state variables, as characterised by the drifts,  $\{\mu^K(X), \mu^M(X), \mu^\eta(X), \mu^\kappa(X), \mu^a(X)\}$ , and jump exposures,  $\{\nu^K(X), \nu^M(X), \nu^\eta(X), \nu^\kappa(X), \nu^a(X)\}$ , government policy functions,  $\{\tau^I(\eta)\}_{I \in \{A, B\}}$ , and entrepreneur value functions,  $\{V^{Ii}(n, X)\}_{i \in [0, 2], I \in \{A, B\}}$ , and policy functions,  $\{c^{Ii}(n), \iota^{Ii}(n, X), \theta^{Ii}(X)\}_{i \in [0, 2], I \in \{A, B\}}$  such that:

- Given price functions and laws of motion for the aggregate state

variables, the value function and entrepreneur policy functions solve the entrepreneur HJBE (2.3),

- The final good producer solves their optimization problem (2.2),
- The government budget constraint (2.1) is satisfied,
- The market clearing conditions are satisfied:

$$\begin{aligned}
\int c_t^i di + \int i_t^i di &= Y_t && \dots \text{Final consumption good,} \\
\int y_t^{Ii} di &= Y_t^I && \dots \text{Intermediate good } I \in \{A, B\}, \\
\int k_t^{Ii} di &= K_t^I && \dots \text{Capital in sector } I \in \{A, B\}, \\
\int m_t^i di &= M_t && \dots \text{Bonds.}
\end{aligned}$$

### 2.3.3 Sector Aggregation

Since agent consumption rules are linear in wealth and agent investment and portfolio allocations are independent of wealth, we can solve for prices using the sector level aggregate variables rather than the full distribution. From the first order conditions and production functions we have:

$$\begin{aligned}
Y_t^I &= a_t^I K_t^I \\
I_t^I &= \iota_t^I K_t^I \\
C_t^I &= \rho N_t^I
\end{aligned}$$

and the laws of motion for  $K_t^I$  and  $N_t^I$  are given by:

$$\begin{aligned}
dK_t^I &= (\Phi(\iota_t^I) - \delta) K_t^I dt \\
dN_t^I &= (\mu_t^{rMI} + (\mu_t^{rkI} - \mu_t^{rMI}) \theta_t^{kI} - \rho) N_t^I dt \\
&\quad + (\nu_t^{1/P} + (\nu_t^{qkI} - \nu_t^{1/P}) \theta_t^{kI}) N_t^I dJ_t
\end{aligned} \tag{2.5}$$

For convenience, we will denote the drift and volatility of  $N_t^I$  by  $\mu_t^{NI} N_t$  and  $\nu_t^{NI} N_t^I$  respectively.

### 2.3.4 Equilibrium Characterization

In appendix B, we derive a set equations that characterise the equilibrium. In this section, we state the main results. The aggregate production function is:

$$Y_t = \mathcal{A}(\kappa_t, a_t)K_t$$

where the “aggregate productivity” is given by:

$$\mathcal{A}(\kappa_t, a_t) = \left( \alpha^A (a_t \kappa_t)^{\frac{\epsilon-1}{\epsilon}} + \alpha^B (\bar{a}(1 - \kappa_t))^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

Although we cannot derive closed form expressions for the asset prices, we show in proposition 1 that we can derive expressions for the asset prices in terms of the share of wealth that agents hold in money,  $\vartheta_t$ , and the share of capital wealth in sector  $A$ ,  $\varphi_t$ . These shares are defined by:

$$\begin{aligned} \vartheta_t &:= \frac{q_t^M (M_t^A + M_t^B)}{N_t} \\ \varphi_t &:= \frac{q_t^A K_t^A}{q_t^A K_t^A + q_t^B K_t^B} \end{aligned}$$

**Proposition 1.** *In equilibrium, the total net-worth of the private sector is:*

$$N_t = \left( \frac{1 + \phi \mathcal{A}(\kappa_t, a_t)}{1 - \vartheta_t + \rho \phi} \right) K_t$$

and the asset prices for capital and bonds are given by:

$$\begin{aligned} q_t^A &= \frac{\varphi_t (1 - \vartheta_t) N_t}{\kappa_t K_t} \\ q_t^B &= \frac{(1 - \varphi_t)(1 - \vartheta_t) N_t}{(1 - \kappa_t) K_t} \\ \frac{1}{P_t} &= \vartheta_t \frac{N_t}{M_t} \end{aligned} \tag{2.6}$$

*Proof.* See appendix B.2. □

We can also derive the laws of motion for the aggregate state variables in terms of the asset prices (and so in terms of  $\varphi_t$  and  $\vartheta_t$ ). This is done in proposition 2.

**Proposition 2.** *The laws of motion for  $K_t$ ,  $\kappa_t$ , and  $\eta_t$  are:*

$$dK_t = (\kappa_t \Phi(\iota_t^A) + (1 - \kappa_t) \Phi(\iota_t^B) - \delta) K_t dt \quad (2.7)$$

$$d\kappa_t = (\Phi(\iota_t^A) - \Phi(\iota_t^B)) \kappa_t (1 - \kappa_t) dt \quad (2.8)$$

$$d\eta_t = (1 - \eta_t) \eta_t \left[ (\mu_t^{NA} - \mu_t^{NB}) dt + \left( \frac{\nu_t^{NA} - \nu_t^{NB}}{1 + \eta_t \nu_t^{NA} + (1 - \eta_t) \nu_t^{NB}} \right) dJ_t \right] \quad (2.9)$$

*Proof.* See appendix B.2.  $\square$

We can use proposition 2 to understand how the state variables evolve. First, consider  $K_t$  and  $\kappa_t$ . From equations (2.7) and (2.8), we can see that the drift of  $K_t$  is determined by the total net investment in both sectors and the drift in  $\kappa_t$  is determined by difference between investment rates across the sectors. Recall that, from the first order conditions,  $\Phi(\iota_t^I) = \log(q^I)$  and so the laws of motion can be expressed as:

$$\begin{aligned} dK_t &= (\log((q_t^A)^{\kappa_t} (q_t^B)^{1-\kappa_t}) - \delta) K_t dt \\ d\kappa_t &= \log\left(\frac{q_t^A}{q_t^B}\right) \kappa_t (1 - \kappa_t) dt \end{aligned}$$

It follows that the drift of  $K_t$  is determined by the  $\kappa_t$  weighted composite capital price and the drift of  $\kappa_t$  is determined by the ratio of capital prices. Neither  $K_t$  nor  $\kappa_t$  changes when the recovery jump occurs because agents cannot trade capital across sectors or frictionlessly destroy capital.

Now, consider the law of motion for  $\eta_t$ . By combining equations (2.9) and (2.5), we can see that the drift of  $\eta_t$  is determined by the sectorial differences in government transfers and earned risk premia:

$$\mu^\eta \eta = (1 - \eta_t) \eta_t (\tau_t^A - \tau_t^B + (\mu_t^{rkA} - \mu_t^{rMA}) \theta_t^{kA} + (\mu_t^{rkB} - \mu_t^{rMB}) \theta_t^{kB})$$

From equation (2.5), we can see that  $\eta_t$  increases when the recovery jump occurs if wealth increase by more in sector  $A$  than in sector  $B$ . This will be the case if  $q_t^A$  increases by more than  $q_t^B$ .

We close the equilibrium characterisation by deriving laws of motion for  $\vartheta_t$  and  $\varphi_t$ . These are given in proposition 3.

**Proposition 3.** *The laws of motion for  $\vartheta_t$  and  $\varphi_t$  are given by:*

$$\begin{aligned} d\varphi_t &= \mu_t^\varphi \varphi_t dt + \nu_t^\varphi \varphi_t dJ_t \\ d\theta_t &= \mu_t^\theta \theta_t dt + \nu_t^\theta \theta_t dJ_t \end{aligned}$$

where

$$\begin{aligned} \mu_t^\varphi = & \left( \tilde{\sigma}_t^2 (1 - \varphi_t) \left( \frac{\varphi_t}{\eta_t} - \frac{1 - \varphi_t}{1 - \eta_t} \right) - \lambda \left( \frac{\nu_t^{qA} - \nu_t^Q}{1 + \nu_t^{NA}} - \frac{\nu_t^{qB} - \nu_t^Q}{1 + \nu_t^{NB}} \right) \right. \\ & \left. - \frac{1}{q_t^K} \left( \frac{p_t^A a_t^A \kappa_t}{\varphi_t} - \frac{p_t^B a_t^B (1 - \kappa_t)}{1 - \varphi_t} + \frac{1}{\phi} \frac{\kappa_t - \varphi_t}{\varphi_t (1 - \varphi_t)} \right) + \psi^A - \psi^B \right) (1 - \varphi_t) \end{aligned}$$

$$\begin{aligned} \nu_t^\varphi &= \frac{\nu_t^{qA} - \nu_t^{qK}}{1 + \nu_t^{qK}} \\ \mu_t^\vartheta &= \rho - (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 \left( \frac{\varphi_t^2}{\eta} + \frac{(1 - \varphi_t)^2}{1 - \eta_t} \right) \\ &+ \lambda (1 - \vartheta_t) \left( \frac{(\nu_t^{qA} - \nu_t^Q) \varphi_t}{1 + \nu_t^{NA}} + \frac{(\nu_t^{qB} - \nu_t^Q) (1 - \varphi_t)}{1 + \nu_t^{NB}} \right) \\ &+ (1 - \vartheta_t) (\mu_t^M - i^M) - (1 - \vartheta_t) (\varphi_t \psi^A + (1 - \varphi_t) \psi^B) \\ \nu_t^\vartheta &= - (1 - \vartheta_t) \left( \frac{\nu_t^{qK} - \nu_t^Q}{\vartheta_t (1 + \nu_t^Q) + (1 - \vartheta_t) (1 + \nu_t^{qK})} \right) \end{aligned}$$

and where  $\nu_t^Q Q_t$  and  $\nu_t^{qK} q_t^K$  are the jumps in  $Q_t := 1/P_t$  and  $q_t^K = \kappa_t q_t^A + (1 - \kappa_t) q_t^B$  respectively when the recover occurs.

*Proof.* See appendix B.2.  $\square$

We are particularly interested in interpreting the equation for the equilibrium share of wealth in bonds,  $\vartheta_t$ , because it is such a prominent component of bond demand. Since it is a backwards equation, it will be helpful to express it in integral form (after discounting by  $e^{-\rho t}$ ):

$$\begin{aligned} \vartheta_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} \vartheta_s \left( (1 - \vartheta_s)^2 \tilde{\sigma}_s^2 \left( \frac{\varphi_s^2}{\eta_s} + \frac{(1 - \varphi_s)^2}{1 - \eta_s} \right) \right. \right. \\ \left. \left. + (1 - \vartheta_s) (i^M - \mu_s^M) + (1 - \vartheta_s) (\varphi_s \psi_s^A + (1 - \varphi_s) \psi_s^B) \right. \right. \\ \left. \left. - \lambda (1 - \vartheta_s) (\varphi_s (\nu_s^{qA} - \nu_s^{qM}) \Delta_s^A + (1 - \varphi_s) (\nu_s^{qB} - \nu_s^{qM}) \Delta_s^B) \right) ds \right] \quad (2.10) \end{aligned}$$

where  $\Delta_s^A$  and  $\Delta_s^B$  are given by:

$$\begin{aligned}\Delta_s^A &= \frac{1}{1 + \nu_s^{NA}} - \frac{1}{1 + \nu_s^N} \\ \Delta_s^B &= \frac{1}{1 + \nu_s^{NB}} - \frac{1}{1 + \nu_s^N}\end{aligned}$$

Equation (2.10) illustrates the forces that determine  $\vartheta_t$ . First, an increase in the future path of idiosyncratic risk will lead to an increase in  $\vartheta_t$ . This is because higher idiosyncratic risk leads to greater precautionary saving, which increases the portfolio share in bonds. Second,  $\vartheta_t$  is larger when  $\varphi_t/\eta_t + (1 - \varphi_t)^2/(1 - \eta_t)$  increases. This is because the distortion of idiosyncratic risk is higher and so idiosyncratic risk insurance is more valuable. Third,  $\vartheta_t$  is higher when  $i^M - \mu_s^M$  is higher. This is because the government is printing less money (net of interest) and so the distortion that comes from redistributing seignorage revenue proportional to net-worth rather than money holdings is lower. Finally,  $\vartheta_t$  is higher when the weighted average distortion in recovery shock exposure, as measured by  $\varphi_s(\nu_s^{qA} - \nu_s^{qM})\Delta_s^A + (1 - \varphi_s)(\nu_s^{qB} - \nu_s^{qM})\Delta_s^B$ , is lower. This term arises because agents are unable to trade claims on the aggregate shock. If such claims existed, then agents would choose  $\nu_s^{NA} = \nu_s^{NB} = \nu_s^N$  and so we would have  $\Delta_t^A = \Delta_t^B = 0$ .

## 2.4 Inflation Dynamics

We can use the bond pricing equation (2.6) to decompose the forces that drive the price level. If we rearrange the equation, then we get:

$$P_t = \frac{M_t}{\vartheta_t N_t} = \frac{M_t}{\vartheta_t \left( \frac{1 + \phi \mathcal{A}(\kappa_t, a_t)}{1 - \vartheta_t + \rho \phi} \right) K_t}$$

The numerator is exogenous bond supply. The denominator is real bond demand. So, as we would expect, the price level is increasing in bond supply and decreasing in bond demand. We can also understand the forces that change bond demand. An increase in productivity,  $\mathcal{A}_t$ , or aggregate capital,  $K_t$ , will increase bond demand because they increase agent net-worth. An increase in  $\vartheta_t$  will increase bond demand both because it increases agent net-worth and because it increases the agent portfolio allocation in bonds.

We can see this more precisely by using Itô's lemma to get that the law of

motion for  $dP_t$ :

$$dP_t = \left( \mu_t^M - \frac{1 + \phi\rho}{1 - \vartheta_t + \phi\rho} \mu_t^\vartheta - \frac{\phi\mathcal{A}_t}{1 + \phi\mathcal{A}_t} \mu_t^A - \mu_t^K \right) dt + \frac{1}{1 + \nu_t^\vartheta} \left( \frac{1 - (1 + \nu_t^\vartheta)\vartheta_t + \rho\phi}{1 - \vartheta_t + \rho\phi} \right) \left( \frac{1 + \phi\mathcal{A}_t}{1 + \phi(1 + \nu_t^A)\mathcal{A}_t} \right) dJ_t \quad (2.11)$$

## 2.5 Asset Pricing Interpretation

The stochastic discount factor for agents in sector  $I$  is given by:

$$\xi_t^I := e^{-\rho t} \partial_n V^I(n_t, X) = e^{-\rho t} \frac{1}{\rho n_t}.$$

Using Ito's lemma, we have that:

$$d\xi_t^I = -\mu_t^{\xi I} \xi_t^I dt - \sigma_t^{\xi I} \xi_t^I d\tilde{Z}_t^{Ii} - \nu_t^{\xi I} \xi_t^I dJ_t$$

where the risk free rate, price of idiosyncratic risk and price of aggregate risk are given by:

$$\begin{aligned} \mu_t^{\xi I} &= \left( \mu^Q - \lambda \left( \frac{\nu_t^{qI} - \nu_t^Q}{1 + \nu_t^{NI}} \right) (1 - \vartheta_t) \frac{\varphi_t^I}{\eta_t^I} + \psi_t^I \right) =: -r_t^I \\ \sigma_t^{\xi I} &= \tilde{\sigma}_t (1 - \vartheta_t) \frac{\varphi_t^I}{\eta_t^I} \\ \nu_t^{\xi I} &= \left( \frac{\nu_t^{NI}}{1 + \nu_t^{NI}} \right) \end{aligned}$$

## 2.6 Steady State

We can derive closed form expressions for the steady state (after scaling by aggregate capital and money supply) before and after the pandemic. The drifts of the state variables become:

$$\begin{aligned} 0 = \mu_t^\kappa &= (\Phi(\iota_t^A) - \Phi(\iota_t^B)) \kappa_t (1 - \kappa_t) = (1 - \kappa_t) \log \left( \frac{q_t^A}{q_t^B} \right) \\ 0 = \mu_t^\eta &= (1 - \eta_t) \eta_t (\mu_t^{NA} - \mu_t^{NB}) = (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 \left( \frac{\varphi_t^2}{\eta_t} + \frac{(1 - \varphi_t)^2}{\eta_t} \right) (1 - \eta_t) \eta_t \end{aligned}$$



which imply that:

$$q_t^A = q_t^B$$

$$\varphi^{SS} = \kappa^{SS} = \eta^{SS} = \frac{1}{2}$$

The drifts of  $\varphi$  and  $\vartheta$  then become:

$$0 = \mu_t^\varphi = \kappa_t(1 - \varphi_t)p_t^A \bar{a} - (1 - \kappa_t)\varphi_t p_t^B \bar{a}$$

$$0 = \rho - \tilde{\sigma}_t^2(1 - \vartheta_t)^2 + (1 - \vartheta_t)(\mu^M - i^M)$$

which imply that:

$$p_t^{A,SS} = p_t^{B,SS}$$

$$1 - \vartheta^{SS} = \frac{\sqrt{\rho + (1 - \vartheta^{SS})(\mu^M - i^M)}}{\tilde{\sigma}(\kappa^{SS})}$$

### 3 Simulations

In this section, we simulate the model under different government policies. We discuss how to solve the numerical solution algorithm in appendix C.

#### 3.1 Baseline Model

We start by considering the baseline model without any government policy. In this case, there is no government lending program or transfer scheme. The parameters used in the numerical simulations are outlined in figure 1. We set  $\lambda = 1$ , which implies that agents expect the pandemic to last one year. We set  $\underline{\vartheta} = 0$ , which restricts all agent borrowing. We set  $s = 2.0$  so that goods  $A$  and  $B$  are imperfect substitutes.

Parameter	Value	Parameter	Value
$\rho$	0.0155	$\tilde{\sigma}^\kappa$	1.0
$\underline{\vartheta}$	0.0	$\phi$	2.0
$a^H$	0.285	$s$	2.0
$a^L$	0.0	$\lambda$	1.0
$\delta$	0.1	$\mu^M$	0.0
$\tilde{\sigma}$	0.125	$i^M$	0.02

Table 1: Parameters for Baseline Model:

Figure 1 plots the time series for simulations with these parameters. We start the economy in the steady state, then introduce the pandemic at time  $t = 0$  and simulate the economy for different possible recovery dates. The solid blue lines show time paths if there is no recovery. The red dashed lines show the time paths if the recovery occurs after 1 year (the expected length of the pandemic). Finally, the yellow dashed lines show the time paths if the recovery occurs after 4 years.

First, we can observe what happens when the pandemic occurs. Productivity in sector  $A$  decreases to zero. This leads to a decrease in  $\mathcal{A}_t$  because  $A$ -type and  $B$ -type goods are not perfect substitutes. It also leads to an increase in  $\vartheta_t$  because agents anticipate higher future idiosyncratic volatility. These two changes have offsetting impacts on bond demand and so, ultimately, the initial change in the price level is ambiguous. Under our parameters, the overall effect is an increase in the price level. We can also see that the capital price decrease by more in sector  $A$  than sector  $B$ . This implies that sector  $A$  has become relatively poorer and so  $\eta_t$  decreases.

Next, we can observe the dynamics during the pandemic. Sector  $A$  sells money and destroys capital in order to smooth consumption. This leads to a decrease in  $\eta_t$  and  $\kappa_t$ . The time path for  $\kappa_t$  is slightly concave because agents destroy more capital once they hit their borrowing constraint. However, the concavity is mild because agents anticipate hitting their borrowing constraint and so sell more capital earlier. As  $\kappa_t$  decreases, the idiosyncratic risk,  $\tilde{\sigma}_t$  increases. This leads agents to substitute their portfolio towards safe assets, which increases  $\vartheta_t$  and decreases the price level,  $P_t$ . The decrease in  $\kappa_t$  also leads to a gradual increase in aggregate productivity because capital is slowly being reallocated to the more productive sector. From the solid blue line, we can see that if the recovery never occurs, then  $P_t$  keeps decreasing and we have sustained deflation. From the dashed red and yellow lines, we can see that, if the recovery occurs, then the trends reverse: productivity immediately increases,  $\vartheta_t$  immediately decreases,  $\eta_t$  and  $\kappa_t$  start to recover, and idiosyncratic risk starts to fall. Ultimately, this leads to an increase in the price level,  $P_t$ . In this sense, our model generates an “inflation whipsaw”, where deflation forces during the pandemic are quickly followed by inflation forces once the pandemic ends.

Figure 2 uses equation (2.11) to decompose the contributions to inflation for the case when no recovery occurs. The red and yellow lines show the contributions from capital growth ( $-\mu^K$ ) and productivity growth ( $-\phi\mathcal{A}/(1 + \phi\mathcal{A})\mu^A$ ) respectively. Negative capital growth during the pandemic is inflationary because it decreases agent net-worth and so bond demand.

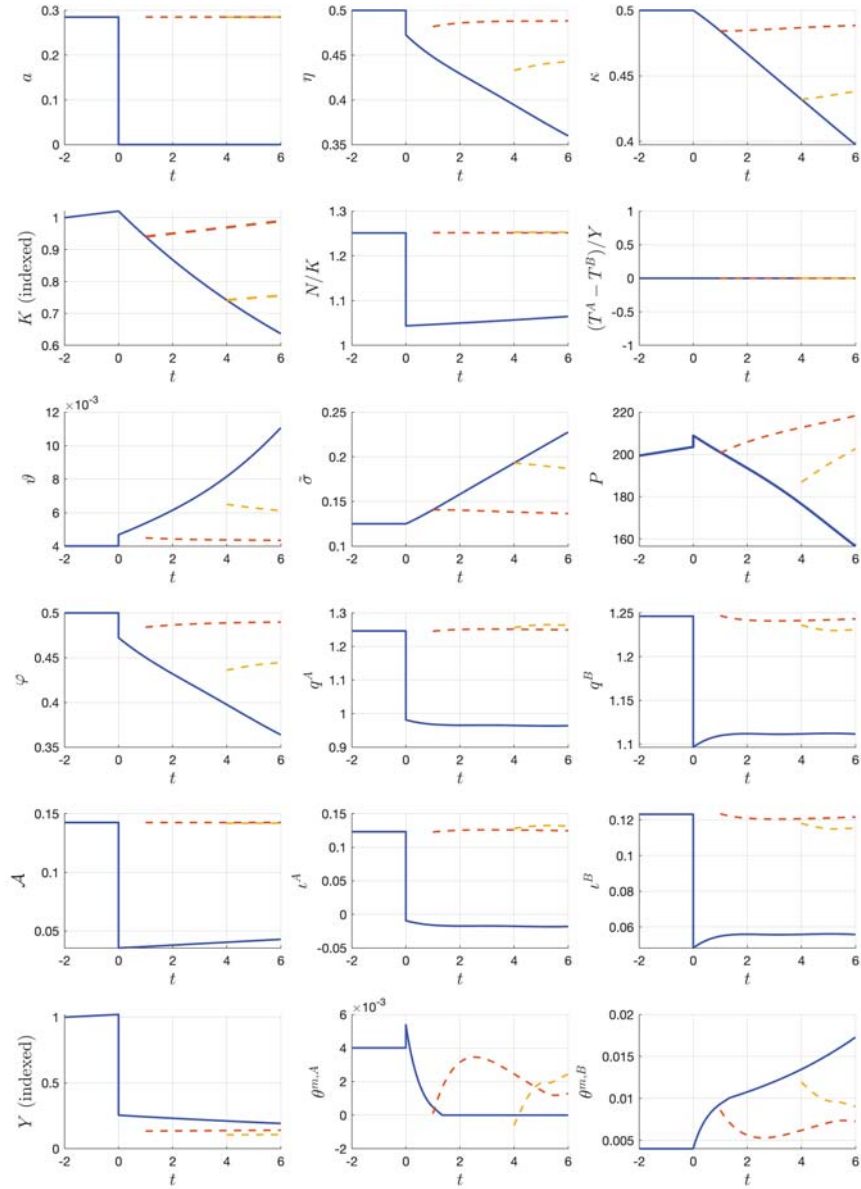


Figure 1: *Time paths with no transfers.* The solid blue lines show the time paths if there is no recovery. The red dashed line shows the time paths if the recovery occurs after 1 year (the expected length of the pandemic). The yellow dashed line shows the time paths if the recovery occurs after 4 years.

By contrast, positive productivity growth during the pandemic is slightly deflationary because it increases agent net-worth and so bond demand. The blue line shows the contribution from portfolio reallocation  $-(1+\phi\rho)/(1-\vartheta+\phi\rho)\mu^\vartheta$ . As can be seen, agent reallocation from capital to bonds is the main deflationary force in the model. The dashed black line shows total inflation, which is dominated by the deflationary forces during the pandemic.

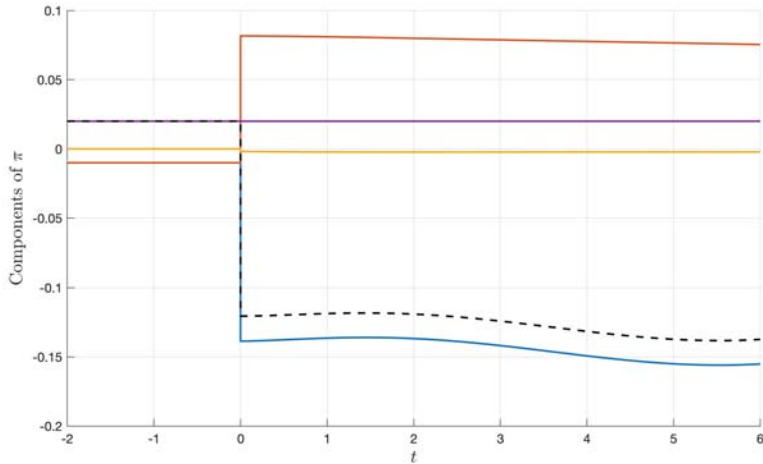


Figure 2: *Decomposition of inflation.* The black dashed line shows inflation. The red line shows the contribution from capital growth:  $-\mu^K$ . The purple line shows the contribution from money supply growth:  $\mu^M + i^M$ . The yellow line shows the contribution from productivity growth:  $-\phi\mathcal{A}/(1+\phi\mathcal{A})\mu^A$ . The blue line shows the contribution from portfolio reallocation:  $-(1+\phi\rho)/(1-\vartheta+\phi\rho)\mu^\vartheta$ .

### 3.2 Government Loans

We now consider a model in which the government “eliminates” the borrowing constraint by offering a lending program. In effect, this means that the government acts as an intermediary between sectors.

The time paths with and without the government lending program are shown in figure 3 and the inflation decomposition is shown in figure 4. As can be seen, once the government lending program is introduced, the time path for  $\kappa_t$  becomes significantly more concave. This is because agents are now more able to “gamble on recovery”. Initially, agents borrow heavily in order to smooth consumption without destroying capital and so  $\kappa_t$  decreases very little. However, if the pandemic lasts longer than expected, then sector  $A$  becomes

highly indebted and aggressively sells capital. Ultimately, this leads to a larger decline in  $\kappa$  and so lower output.

These dynamics have strong implications for inflation. Introducing the government loan program is initially inflationary because the lack of capital destruction prevents idiosyncratic risk from increasing. However, if the pandemic lasts longer than expected, then idiosyncratic risk ends up higher and so the deflationary forces are stronger. In this sense, introducing a government loan program makes the economy more sensitive to the length of the recovery.

### 3.3 Government Transfers

In the previous simulations, the government did not use taxes and transfers to reduce inequality during the pandemic. We now consider the implications of government redistribution. We start by considering an intratemporal tax and transfer policy satisfying:

$$\tau_t^A - \tau_t^B = \tau_0 \left( \frac{\max\{\bar{\eta} - \eta, 0\}}{\eta_t - \underline{\eta}} - \frac{\max\{\bar{\eta} + \eta - 1, 0\}}{1 - \eta_t - \underline{\eta}} \right)$$

This implies that the government starts taxing sector  $B$  ( $A$ ) and subsidizing sector  $A$  ( $B$ ) once  $\eta$  falls below  $\bar{\eta}$  ( $1 - \eta$  increases above  $1 - \bar{\eta}$ ) and prevents  $\eta$  ( $1 - \eta$ ) from falling below (increasing above)  $\underline{\eta}$  ( $1 - \underline{\eta}$ ). The additional parameters used in this section are shown in table 2.

Parameter	Value
$\tau_0$	0.1
$\bar{\eta}$	0.45
$\underline{\eta}$	0.42

Table 2: Additional Parameters for Tax and Transfer Policy:

The time paths with and without the intratemporal government redistribution scheme are shown in figure 5 and the inflation. As expected, the introduction of taxes and transfers moderate the decrease in both  $\eta_t$  and  $\kappa_t$ . This dampens the increase in volatility, which dampens the increase in bond demand and ultimately leads to inflation during the pandemic. In this sense, the policy makers faces a trade-off. If they don't address inequality, then there is deflation during the crisis. However, if they attempt to address inequality, then they get inflation. Furthermore, we get this result even though both sectors have the same marginal propensity to consume out of wealth. Instead,

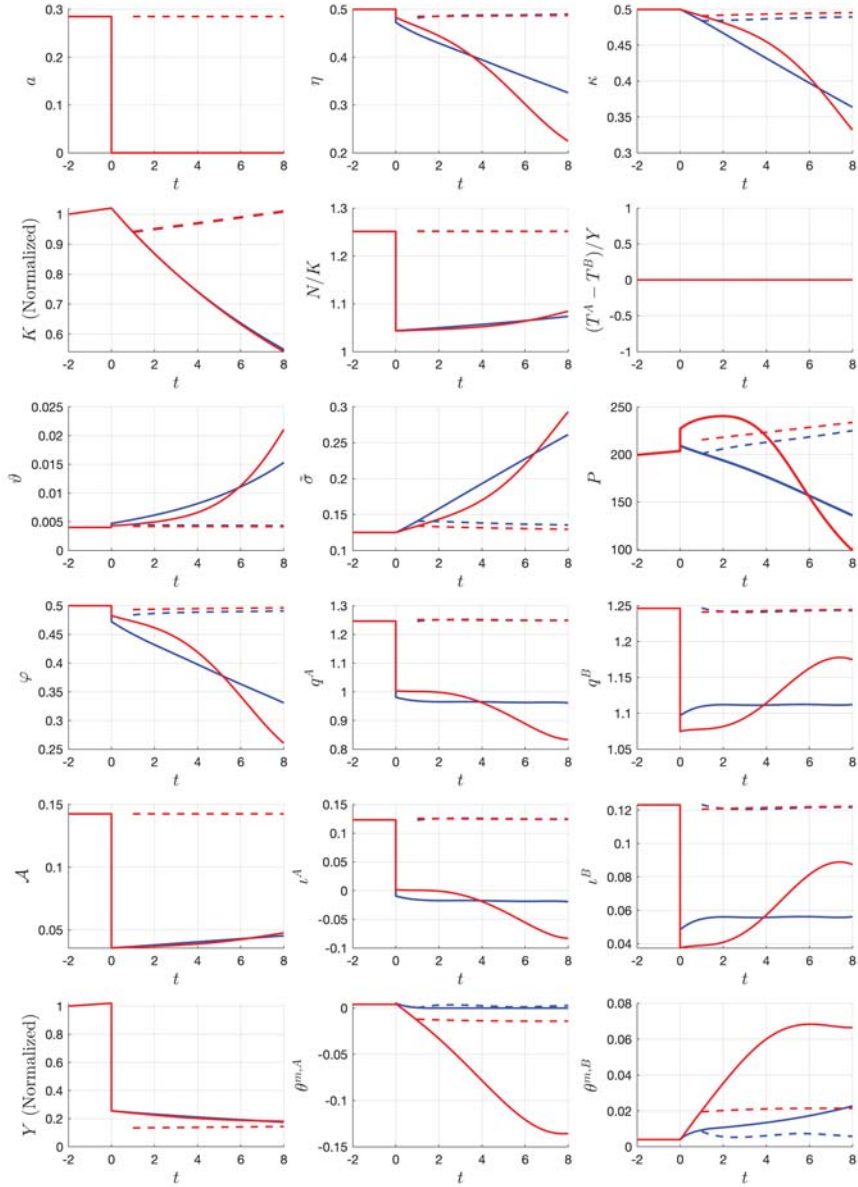


Figure 3: *Time paths with and without government lending.* The blue lines show the time paths for the baseline economy without any government policy. The red lines show the times paths when there is a government lending program. The solid lines show time paths when there is no recovery. The dashed lines show time paths when there is a recovery after one year.

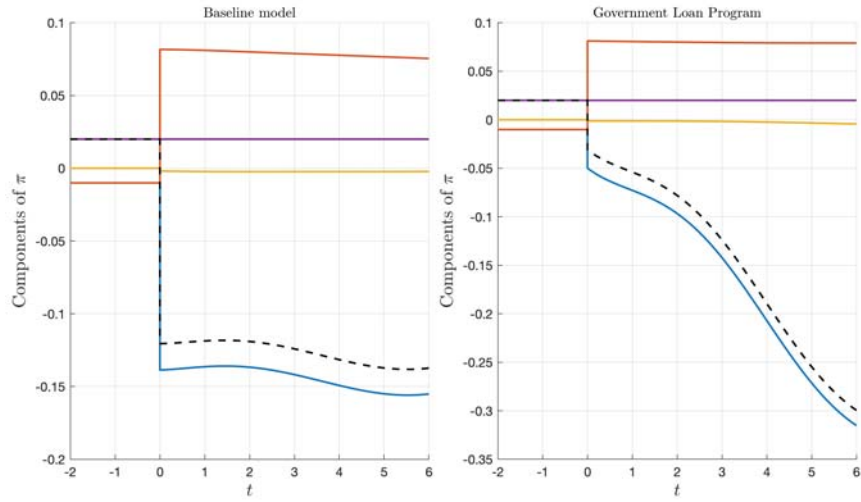


Figure 4: *Decomposition of inflation.* The left plot show the decomposition for the baseline economy without policy. The right plot shows the decomposition for the economy with a government lending program. On both plots, the red lines shows the contribution from capital growth:  $-\mu^K$ . The purple lines shows the contribution from money supply growth:  $\mu^M + i^M$ . The yellow lines show the contribution from productivity growth:  $-\phi\mathcal{A}/(1 + \phi\mathcal{A})\mu^{\mathcal{A}}$ . The blue lines shows the contribution from portfolio reallocation:  $-(1 + \phi\rho)/(1 - \vartheta + \phi\rho)\mu^\vartheta$ .

the redistribution drives inflation because the different sectors have different willingness to hold capital.

*Discussion on Intertemporal Transfers and Ricardian Equivalence:* So far, we have only discussed intratemporal transfers. It worth considering whether agents are indifferent about the timing of government taxes. If the government introduces a lending program and raises lump sum taxes, then Ricardian equivalence holds and there is no difference between intratemporal and intertemporal redistribution. If the government does not introduce a lending program, then sector  $B$  is indifferent about the timing of taxes but sector  $A$  would prefer that the government raises taxes once their borrowing constraint is no longer binding. Finally, if the government raises taxes proportionally to wealth, then the tax and transfer scheme provides partial insurance against idiosyncratic risk. Ultimately, this would decrease bond demand and so generate more inflation.

## 4 Conclusion

We explored the short term and long term inflation and deflation forces arising from different fiscal responses to the COVID-19 pandemic. We started by showing that the arrival of the pandemic has an ambiguous impact on bond demand and so an ambiguous impact on the price level. Next, we showed that, without any government policy, there is a strong deflationary force during the pandemic because idiosyncratic risk increases until the recovery arrives. We then considered the impact of introducing two different government policies: state loans and wealth redistribution. The first policy relaxes private sector borrowing constraints. Agents in the shutdown sector respond by borrowing to smooth consumption rather than destroying capital because they anticipate an increase capital prices when the recovery arrives. This “gambling on recovery” prevents deflation early in the pandemic but leads to higher deflation later on if agents lose their bet and the pandemic lasts longer than expected. Under the second policy, the government raises taxes to finance intratemporal redistribution. In this case, the deflationary forces are dampened because inequality and idiosyncratic risk increase less.

So far, the paper has only considered fiscal policy. The next step is to consider how optimal monetary policy interacts with fiscal policy.



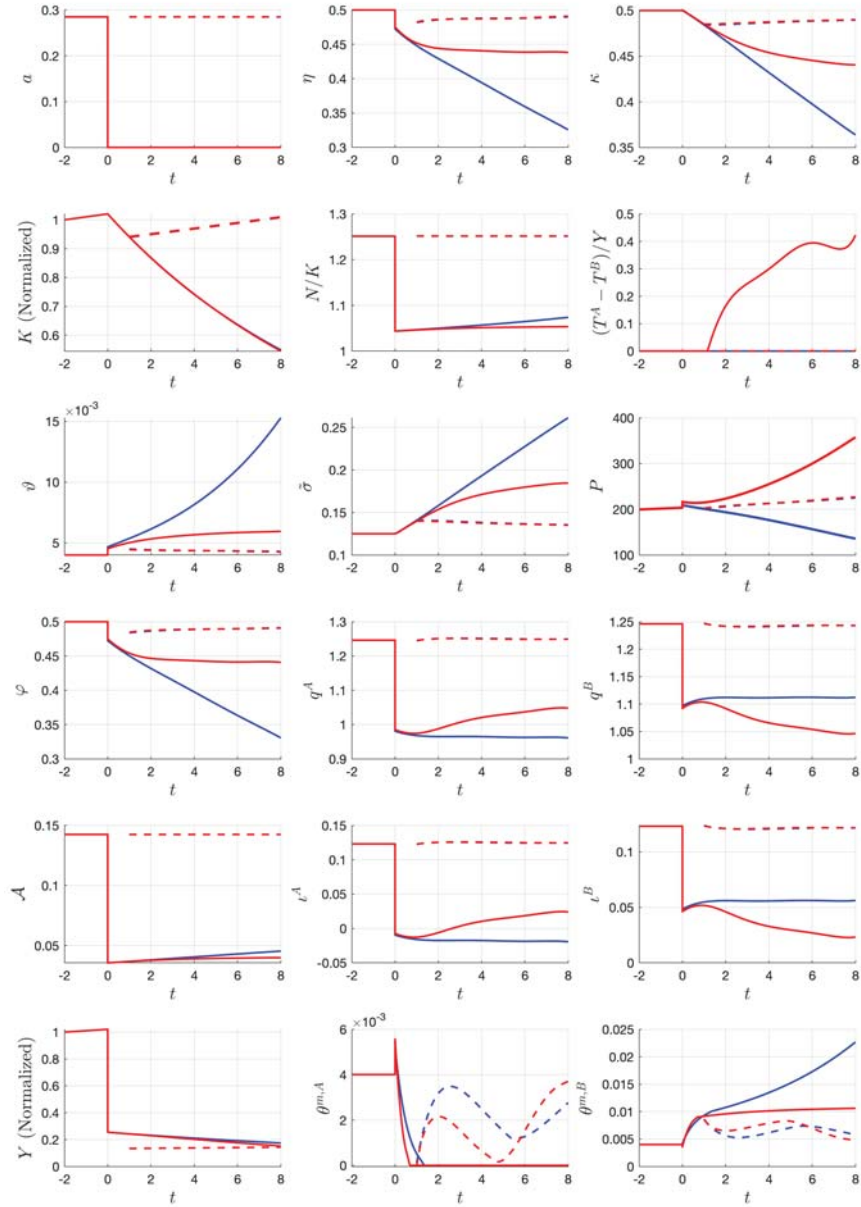


Figure 5: *Time paths with and without redistribution.* The blue lines show the time paths for the baseline economy without any government policy. The red lines show the times paths when the government imposes intratemporal transfers. The solid lines show time paths when there is no recovery. The dashed lines show time paths when there is a recovery after one year.

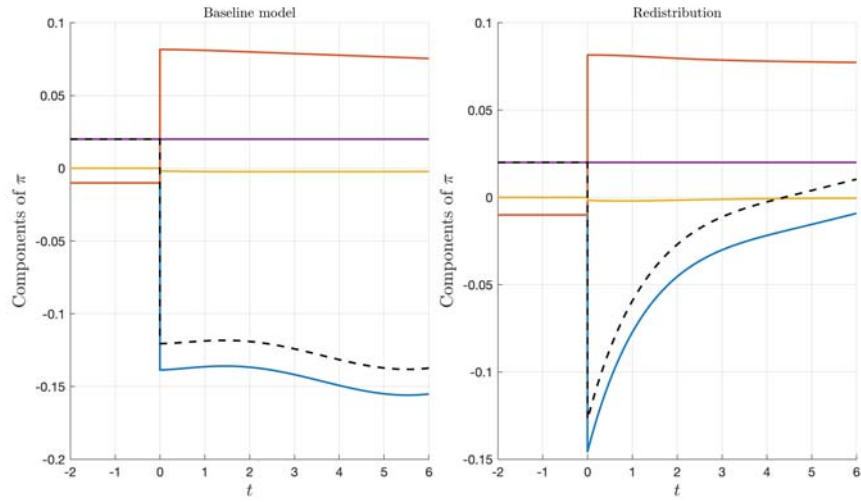


Figure 6: *Decomposition of inflation*. The left plot show the decomposition for the baseline economy without policy. The right plot shows the decomposition when the government imposes intratemporal transfers. On both plots, the red lines shows the contribution from capital growth:  $-\mu^K$ . The purple lines shows the contribution from money supply growth:  $\mu^M + i^M$ . The yellow lines show the contribution from productivity growth:  $-\phi\mathcal{A}/(1 + \phi\mathcal{A})\mu^{\mathcal{A}}$ . The blue lines shows the contribution from portfolio reallocation:  $-(1 + \phi\rho)/(1 - \vartheta + \phi\rho)\mu^\vartheta$ .

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## A Historical Motivation

There are many historical examples where governments have run large budget deficits during times of emergency. Figure 7 shows periods of war, the budget surplus as a % of GDP, the nominal interest rate on short term debt and the inflation rate in the United Kingdom (UK) from 1680 to 2018. Figure 8 shows the same historical time series for the United States of America (USA) from 1860 to 2018. The figures illustrate a number of historical correlations. First, fiscal expansions are typically correlated with inflation. The notable exceptions are the great depression and the recent financial crisis. However, we must also keep in mind that there are important differences between these historical examples and the current COVID-19 pandemic. One important difference is that during a war, there is high government demand for goods and services. By contrast, during the current pandemic, the government has forcibly shut down much human activity in order to decrease aggregate demand. Second, aggressive fiscal contraction is required to restore the price level after a sustained period of inflation. We see this with the attempts to return to the gold standard following the first and second world wars. Third, nominal interest rates typically rise immediately following wars rather than during wars, even when the war is funded with large budget deficits.

Figure 9 shows the historical time series for Germany and the UK and Germany during 1913 to 1925. This provides an instructive case study on how much government policy can impact inflation dynamics. [Balderston \(1989\)](#) gives a nice account about different in strategy. The UK following strategy employed more than 100 years earlier, after the Napoleonic wars, and returned to the pre-WWI gold standard imposing fiscal austerity. This resulted in deflation and slow growth and was heavily criticized including by JM Keynes. Germany's Weimar Republic was politically much less stable. In the fall of 1920 it became clear that Germany could not go for a fiscal consolidation route as the UK implemented its plan. It ended up using the money printing press. Many Central European countries experienced a similar dramatic development as Germany in the 1920.

The historical evidence suggests that the large budget deficits currently being run by countries around the world could generate subtle short term and long term inflation dynamics, and that these dynamics could vary significantly with government policy decisions. In the remainder of the paper, we construct and analyse a structural model in order to help understand which forces are relevant for the current situation.

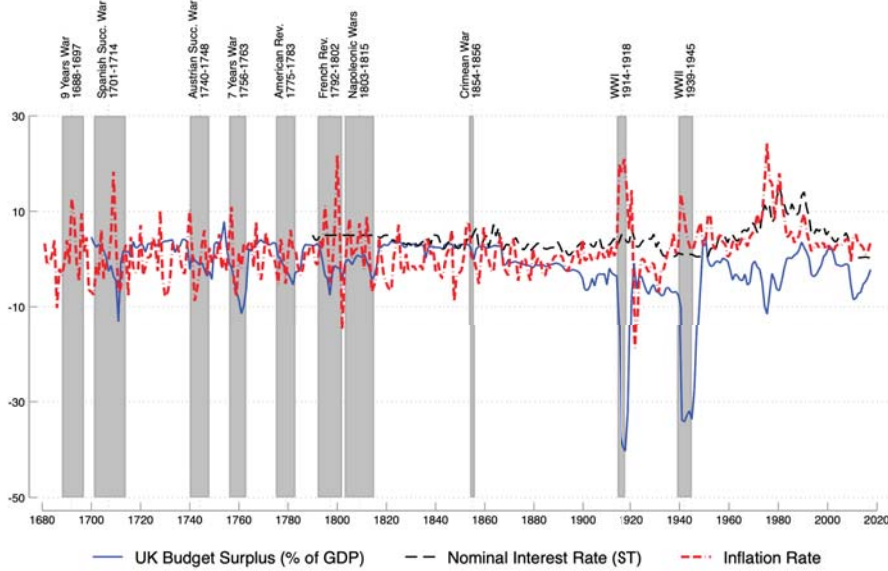


Figure 7: *UK wars, budget surpluses, short term nominal interest rates and inflation.* The budget surplus data comes from [Chantrill \(2020\)](#). The interest rate and inflation data come from [Officer \(2020\)](#). The short term nominal interest rate is a composite of interest rates on bonds with duration less than one year.

## B Derivations and Proofs for Section 2.3 (For Online Appendix Only)

In this appendix, we derive equations that characterise the equilibrium. In section B.1, we provide additional working. In section B.2, we prove propositions 1, 2, and 3 from the section 2.3 in the main text.

### B.1 Preliminary Working

*Additional Notation:* It will be convenient to define additional notation for this appendix. Define the variables:

$$Q_t := \frac{1}{P_t}$$

$$q_t^M := \frac{Q_t M_t}{K_t}$$

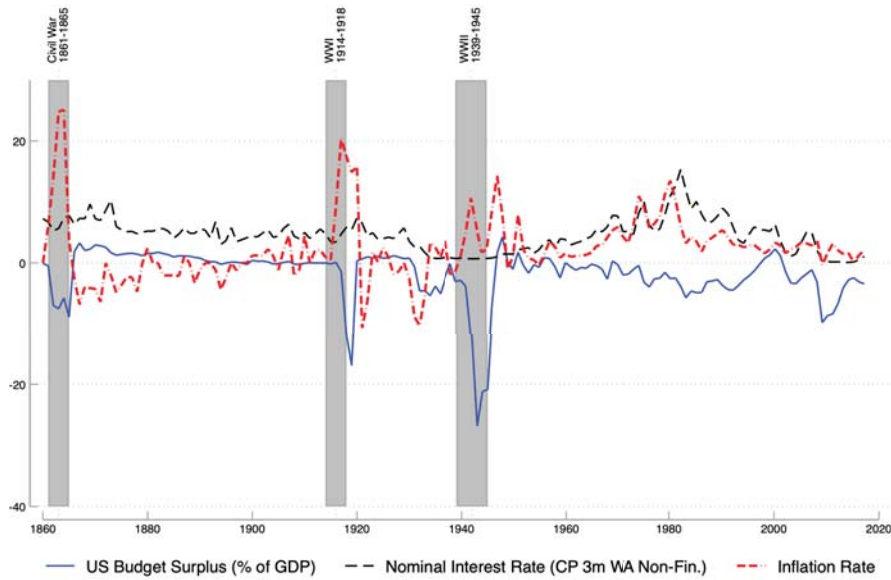


Figure 8: *US Wars, Budget Surpluses, Short Term Nominal Interest Rates and Inflation*. The budget surplus data and nominal interest rate data comes from [FRED \(2020\)](#). The nominal interest rate is a composite of yields on non-financial bonds with duration less than three months. The inflation and GDP data come from [Officer \(2020\)](#).

The variable  $Q_t$  is the price of a bond in terms of units of the final good. The variable  $q_t^M$  is the price of a bond scaled by  $M_t/K_t$ . It is convenient to define this scaled price because it will turn out that  $q_t^K$  nor  $q_t^M$  are both functions of  $\eta_t$  and  $\kappa_t$  but not  $K_t$  or  $M_t$ . That is, the price of bonds is stationary after scaling by  $M_t/K_t$ . Using this notation, we can simplify the share of wealth that agents

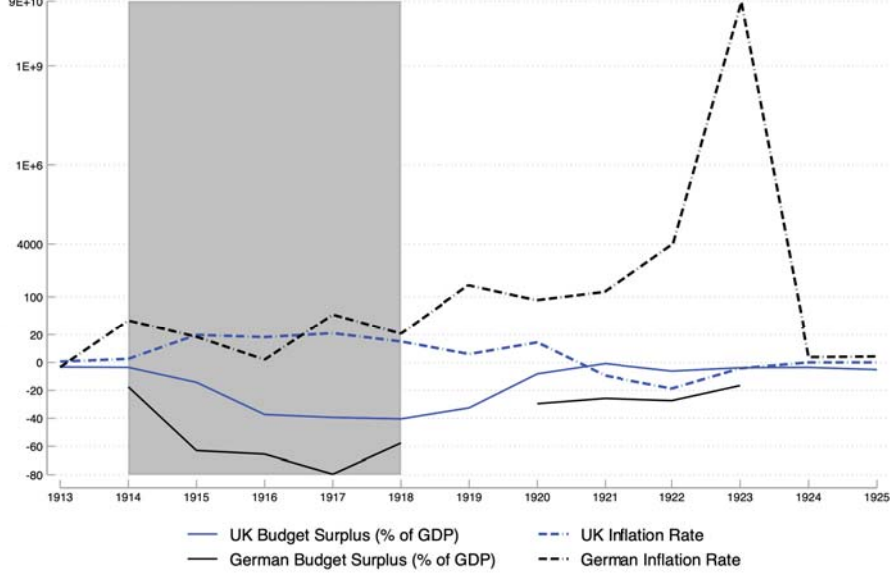


Figure 9: *UK and Germany post World War I.* The UK budget surplus data comes from Chantrill (2020) and the UK inflation data come from Officer (2020). The German data comes from Young (1925).

hold in money,  $\vartheta_t$ , as:

$$\begin{aligned}
\vartheta_t &= \frac{q_t^M (M_t^A + M_t^B)}{N_t} \\
&= \frac{q_t^M (M_t^A + M_t^B)}{q_t^A K_t^A + Q_t M_t^A + q_t^B K_t^B + Q_t M_t^B} \\
&= \frac{q_t^M M_t}{q_t^K K_t + Q_t M_t} \\
&= \frac{Q_t^M K_t}{q_t^K K_t + q_t^M K_t} \\
&= \frac{q_t^M}{q_t^K + q_t^M} \tag{B.1}
\end{aligned}$$

Denote the weighted average variables by:

$$\begin{aligned}
q_t^K &:= \kappa_t q_t^A + (1 - \kappa_t) q_t^B \\
\iota_t &:= \kappa_t \iota_t^A + (1 - \kappa_t) \iota_t^B \\
\Phi_t &:= \kappa_t \Phi(\iota_t^A) + (1 - \kappa_t) \Phi(\iota_t^B)
\end{aligned}$$



Using this notation and the first order conditions, we have that:

$$\begin{aligned}\iota_t &= \kappa_t \frac{1}{\phi} (q_t^A - 1) + (1 - \kappa_t) \frac{1}{\phi} (q_t^B - 1) = \frac{1}{\phi} (q_t - 1) \\ \Phi_t &= \kappa_t \Phi(\iota_t^A) + (1 - \kappa_t) \Phi(\iota_t^B) = \frac{1}{\phi} \log \left( (q_t^A)^{\kappa_t} (q_t^B)^{1 - \kappa_t} \right)\end{aligned}\tag{B.2}$$

and that the share of capital wealth in sector  $A$ ,  $\varphi_t$ , is given by:

$$\varphi_t = \frac{\kappa_t q_t^A}{q_t^K}$$

Finally, where convenient, we will also use the additional notation that  $\eta_t^I := N_t^I / N_t$  and  $\kappa_t^I := K_t^I / K_t$ .

*Sector Aggregation:* Since agent consumption rules are linear in wealth and agent investment and portfolio allocations are independent of wealth, we can solve for prices using the sector level aggregate variables rather than the full distribution.<sup>3</sup> After imposing the first order conditions, for sector  $I$ , the sector aggregate variables become:

$$\begin{aligned}M_t^I &= \int m_t^{Ii} di \\ K_t^I &= \int k_t^{Ii} di = \kappa_t^I K_t \\ N_t^I &= \int n_t^{Ii} di = q^{kI} K_t^I + q^M M_t^I \\ Y_t^I &= \int y_t^{Ii} di = a_t^I K_t^I = a_t^I \kappa_t^I K_t \\ I_t^I &= \int \iota_t^{Ii} k_t^{Ii} di = \iota_t^I K_t^I = \iota_t^I \kappa_t^I K_t \\ C_t^I &= \int c_t^{Ii} di = \rho N_t^I = \rho \eta_t^I N_t\end{aligned}$$

Capital stock in sector I evolves according to:

$$dK_t^I = \int dk_t^{Ii} di = (\Phi(\iota_t^I) - \delta) K_t^I dt =: \mu_t^{KI} K_t^I dt$$

---

<sup>3</sup>We defer the details of constructing a representative agent in each sector to the appendix.

Net worth in sector  $I$  evolves according to:

$$\begin{aligned}
dN_t^I &= \int \left( \left[ \mu_t^{rMI} + (\mu_t^{rkIi} - \mu_t^{rMI})\theta_t^{kIi} - \frac{C_t^{Ii}}{n_t^{Ii}} \right] n_t^{Ii} dt \right. \\
&\quad \left. + \tilde{\sigma}_t \theta_t^{kIi} n_t^{Ii} d\tilde{Z}_t^i + \left[ \nu_t^{qM} + (\nu_t^{qkI} - \nu_t^{qM})\theta_t^{kIi} \right] n_t^{Ii} dJ_t \right) di \\
&= (\mu_t^{rMI} + (\mu_t^{rkIi} - \mu_t^{rMI})\theta_t^{kI} - \rho) N_t^I dt \\
&\quad + (\nu_t^{qM} + (\nu_t^{qkI} - \nu_t^{qM})\theta_t^{kI}) N_t^I dJ_t \\
&=: \mu_t^{NI} N_t^I + \nu_t^{NI} N_t^I dJ_t
\end{aligned}$$

*Aggregate Production:* Using the sector aggregation results, the aggregate production function then becomes:

$$\begin{aligned}
Y_t &= \left( \alpha^A (Y_t^A)^{\frac{s-1}{s}} + \alpha^B (Y_t^B)^{\frac{s-1}{s}} \right)^{\frac{s}{s-1}} \\
&= \left( \alpha^A (a_t \kappa_t)^{\frac{s-1}{s}} + \alpha^B (\bar{a}(1 - \kappa_t))^{\frac{s-1}{s}} \right)^{\frac{s}{s-1}} K_t \\
&= \mathcal{A}(\kappa_t, a_t) K_t
\end{aligned}$$

where:

$$\mathcal{A}(\kappa_t, a_t) = \left( \alpha^A (a_t \kappa_t)^{\frac{\epsilon-1}{\epsilon}} + \alpha^B (\bar{a}(1 - \kappa_t))^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

*Market Clearing Conditions:* We now integrate the FOCs into the market clearing conditions. The intermediate goods market clearing condition for sector  $I$  is given by:

$$\begin{aligned}
\int y_t^{Ii} di &= Y_t^I \\
\Rightarrow a_t^I K_t^I &= \left( \frac{\alpha^I}{p_t^I} \right)^\epsilon Y_t^I \\
\Rightarrow p_t^I &= \nu_t^I \left( \frac{\mathcal{A}(\kappa_t, a_t)}{a_t^I \kappa_t^I} \right)^{\frac{1}{\epsilon}}
\end{aligned}$$

So, more explicitly, we have that:

$$p_t^A = \alpha^A \left( \frac{\mathcal{A}(\kappa_t, a_t)}{a_t \kappa_t} \right)^{\frac{1}{\epsilon}}$$

$$p_t^B = \alpha^B \left( \frac{\mathcal{A}(\kappa_t, a_t)}{\bar{a}(1 - \kappa_t)} \right)^{\frac{1}{\epsilon}}$$

The final goods market clearing condition becomes:

$$Y_t = \int c_t^i di + \int i_t^i di$$

$$\Rightarrow Y_t = \rho N_t + \iota_t^A K_t^A + \iota_t^B K_t^B$$

$$\Rightarrow \mathcal{A}(\kappa_t, a_t) K_t = \rho(q_t^A K_t^A + q_t^B K_t^B + Q_t M_t) + \iota_t^A K_t^A + \iota_t^B K_t^B$$

$$\Rightarrow \mathcal{A}(\kappa_t, a_t) = \rho(q_t^A \kappa_t + q_t^B (1 - \kappa_t) + q_t^M) + \iota_t^A \kappa_t + \iota_t^B (1 - \kappa_t)$$

$$\Rightarrow \mathcal{A}(\kappa_t, a_t) = \rho(q_t^K + q_t^M) + \iota_t \quad (\text{B.3})$$

The capital market clearing condition for sector I becomes:

$$\int k_t^{Ii} di = K_t^I$$

$$\Rightarrow \theta_t^{kI} \frac{N_t^I}{q_t^I} = K_t^I$$

$$\Rightarrow \theta_t^{kI} = (1 - \vartheta_t) \frac{q_t^I \kappa_t^I}{q_t^K \eta_t^I}$$

So, more explicitly, we have that:

$$\theta_t^{kA} = (1 - \vartheta_t) \frac{q_t^A \kappa_t}{q_t \eta_t} \quad (\text{B.4})$$

$$\theta_t^{kB} = (1 - \vartheta_t) \frac{q_t^B (1 - \kappa_t)}{q_t (1 - \eta_t)} \quad (\text{B.5})$$

Substituting equations (B.4) and (B.5) into equation (2.4) gives that:

$$\mu_t^{rkA} - \mu_t^{rMA} = \tilde{\sigma}(\kappa_t)^2 (1 - \vartheta_t) \frac{q_t^A \kappa_t}{q_t \eta_t} - \frac{\lambda(\nu^{qA} - \nu^Q)}{1 + \nu^{NA}} + \psi^A$$

$$\mu_t^{rkB} - \mu_t^{rMA} = \tilde{\sigma}(\kappa_t)^2 (1 - \vartheta_t) \frac{q_t^B (1 - \kappa_t)}{q_t (1 - \eta_t)} - \frac{\lambda(\nu^{qB} - \nu^Q)}{1 + \nu^{NB}} + \psi^B$$

Substituting in the drifts, we get that:

$$\begin{aligned}
& \frac{p_t^A a_t^A - \iota_t^A}{q_t^A} + \Phi(\iota_t^A) - \delta + \mu_t^{qA} - i^M - \mu_t^Q \\
& \quad = \tilde{\sigma}(\kappa_t)^2 (1 - \vartheta_t) \frac{q_t^A \kappa_t}{q_t \eta_t} - \frac{\lambda(\nu^{qA} - \nu^Q)}{1 + \nu^{NA}} + \psi^A \\
& \frac{p_t^B a_t^B - \iota_t^B}{q_t^B} + \Phi(\iota_t^B) - \delta + \mu_t^{qB} - i^M - \mu_t^Q \\
& \quad = \tilde{\sigma}(\kappa_t)^2 (1 - \vartheta_t) \frac{q_t^B (1 - \kappa_t)}{q_t (1 - \eta_t)} - \frac{\lambda(\nu^{qB} - \nu^Q)}{1 + \nu^{NB}} + \psi^B
\end{aligned}$$

## B.2 Proofs of Propositions 1, 2, and 3

*Proof of Proposition 1: (Closed form expressions for the asset prices in terms of  $\vartheta_t$  and  $\varphi_t$ ).* Combining equation (B.2) (the aggregated first order condition for  $\iota$ ), equation (B.3) (the good market clearing condition) and equation (B.1) (the definition of  $\vartheta$ ), we have that:

$$\begin{aligned}
\iota_t &= \frac{(1 - \vartheta_t) \mathcal{A}(\kappa_t, a_t) - \rho}{1 - \vartheta_t + \rho\phi} \\
q_t^K &= \frac{(1 - \vartheta_t) (1 + \phi \mathcal{A}(\kappa_t, a_t))}{1 - \vartheta_t + \rho\phi} \\
q_t^M &= \frac{\vartheta_t (1 + \phi \mathcal{A}(\kappa_t, a_t))}{1 - \vartheta_t + \rho\phi}
\end{aligned}$$

We can immediately see that the price of money can be expressed as:

$$\frac{1}{P_t} = Q_t = \frac{q_t^M K_t}{M_t} = \left( \frac{\vartheta_t (1 + \phi \mathcal{A}(\kappa_t, a_t))}{1 - \vartheta_t + \rho\phi} \right) \frac{K_t}{M_t}$$

Finally, the capital prices can be expressed as:

$$\begin{aligned}
q_t^A &= \frac{\varphi_t q_t^K}{\kappa_t} \\
q_t^B &= \frac{(1 - \varphi_t) q_t^K}{1 - \kappa_t}
\end{aligned}$$

The proposition follows.  $\square$

*Proof of proposition 2: (Law of motion for the state variables).* First, observe

that the law of motion for aggregate capital stock,  $K_t$  evolves according to:

$$\begin{aligned}
dK_t &= (\Phi(\iota_t^A) - \delta)K_t^A dt + (\Phi(\iota_t^B) - \delta)K_t^B dt \\
&= (\kappa_t \Phi(\iota_t^A) + (1 - \kappa_t) \Phi(\iota_t^B) - \delta) K_t dt \\
&=: \mu_t^K K_t dt
\end{aligned}$$

Second, observe that sector A's capital share,  $\kappa_t$ , evolves according to:

$$\begin{aligned}
d\kappa_t &= d(K_t^A / K_t) \\
&= \left( \frac{dK_t^A}{K_t^A} - \frac{dK_t^B}{K_t^B} \right) \frac{K_t^A}{K_t} \frac{K_t^B}{K_t} dt \\
&= \left[ ((\Phi(\iota_t^A) - \delta)dt) - ((\Phi(\iota_t^B) - \delta)dt) \right] \frac{K_t^A}{K_t} \frac{K_t^B}{K_t} \\
&= (\Phi(\iota_t^A) - \Phi(\iota_t^B)) \kappa_t (1 - \kappa_t) dt \\
&= \frac{1}{\phi} \log \left( \frac{\varphi_t (1 - \kappa_t)}{(1 - \varphi_t) \kappa_t} \right) \kappa_t (1 - \kappa_t) dt \\
&=: \mu_t^\kappa \kappa_t dt
\end{aligned}$$

Finally, sector A's wealth share,  $\eta_t$ , evolves according to:

$$\begin{aligned}
d\eta_t &= d \left( \frac{N_t^A}{N_t} \right) \\
&= \frac{N_t^A}{N_t} \left( (\mu_t^{NA} - \mu_t^N) dt + \left( \frac{1 + \nu_t^{NA}}{1 + \nu_t^N} - 1 \right) dJ_t \right) \\
&= (1 - \eta_t) \eta_t \left( (\mu_t^{NA} - \mu_t^{NB}) dt + \left( \frac{\nu_t^{NA} - \nu_t^{NB}}{1 + \eta_t \nu_t^{NA} + (1 - \eta_t) \nu_t^{NB}} \right) dJ_t \right) \\
&=: \mu_t^\eta \eta_t dt + \nu_t^\eta \eta_t dJ_t
\end{aligned}$$

Substituting the FOC and market clearing conditions into the drift gives that:

$$\begin{aligned}
\mu_t^\eta \eta_t &= (1 - \eta_t) \eta_t (\mu_t^{NA} - \mu_t^{NB}) \\
&= (1 - \eta_t) \eta_t (\mu_t^{rMA} + (\mu_t^{rkA} - \mu_t^{rMA}) \theta_t^{kA} - \rho \\
&\quad - (\mu_t^{rMB} + (\mu_t^{rkB} - \mu_t^{rMB}) \theta_t^{kB} - \rho)) \\
&= (1 - \eta_t) \eta_t (\tau_t^A - \tau_t^B + (\mu_t^{rkA} - \mu_t^{rMA}) \theta_t^{kA} - (\mu_t^{rkB} - \mu_t^{rMB}) \theta_t^{kB}) \\
&= (1 - \eta_t) \eta_t \left( \tau_t^A - \tau_t^B + \tilde{\sigma}_t^2 (1 - \vartheta_t)^2 \left( \frac{\varphi_t^2}{\eta_t^2} - \frac{(1 - \varphi_t)^2}{(1 - \eta_t)^2} \right) \right. \\
&\quad \left. - \lambda (1 - \theta_t) \left( \left( \frac{\nu_t^{qA} - \nu_t^{qQ}}{1 + \nu_t^{NA}} \right) \frac{\varphi_t}{\eta} - \left( \frac{\nu_t^{qB} - \nu_t^{qQ}}{1 + \nu_t^{NB}} \right) \frac{1 - \varphi_t}{1 - \eta} \right) \right. \\
&\quad \left. + \psi^A - \psi^B \right)
\end{aligned}$$

and gives that the jump is:

$$\nu_t^\eta \eta_t = \frac{(1 - \vartheta_t) \left( (\nu_t^{qA} - \nu_t^{qQ}) \frac{\varphi_t}{\eta_t} + (\nu_t^{qB} - \nu_t^{qQ}) \frac{1 - \varphi_t}{1 - \eta_t} \right)}{1 + \nu_t^{qQ} + (1 - \vartheta_t) \left( (\nu_t^{qA} - \nu_t^{qQ}) \varphi_t + (\nu_t^{qB} - \nu_t^{qQ}) (1 - \varphi_t) \right)}$$

The result follows.  $\square$

We have expressed the asset prices in terms of  $\vartheta_t$  and  $\varphi_t$ . However, we still need to solve for  $\vartheta_t$  and  $\varphi_t$ . This, requires the derivation of expressions for  $\mu_t^\vartheta$ ,  $\mu_t^\varphi$ ,  $\nu_t^\vartheta$  and  $\nu_t^\varphi$ . We do this in the next proposition.

*Proof of Proposition 3: (Laws of Motion for  $\varphi_t$  and  $\vartheta_t$ ).* First, we derive the law of motion for  $\varphi_t$ . We have that:

$$\begin{aligned}
d\varphi_t &= d \left( \frac{\kappa_t q_t^A}{q_t^K} \right) \\
&= (\mu_t^\kappa + \mu_t^{qA} - \mu_t^{qK}) \varphi_t dt + \left( \frac{\nu_t^{qA} - \nu_t^{qK}}{1 + \nu_t^{qK}} \right) \varphi_t dJ_t \\
&=: \mu_t^\varphi \varphi_t dt + \nu_t^\varphi \varphi_t dJ_t
\end{aligned}$$

Substituting the FOC and market clearing conditions into the drift gives that:

$$\begin{aligned}
\mu_t^\varphi \varphi_t &= (\mu_t^\kappa + \mu_t^{qA} - \mu_t^{qK}) \varphi_t \\
&= \left( \mu_t^{qA} + \mu_t^\kappa - \left( \mu_t^{qA} \frac{q_t^A}{q_t} + (1 - \kappa_t) \mu_t^{qB} \frac{q_t^B}{q_t} + \mu_t^\kappa \kappa_t \left( \frac{q_t^A}{q_t} - \frac{q_t^B}{q_t} \right) \right) \right) \varphi_t \\
&= \left( \mu_t^{qA} + \mu_t^\kappa - \left( \varphi_t \mu_t^{qA} + (1 - \varphi_t) \mu_t^{qB} + \mu_t^\kappa \kappa_t \left( \frac{\varphi_t}{\kappa_t} - \frac{1 - \varphi_t}{1 - \kappa_t} \right) \right) \right) \varphi_t \\
&= \left( \mu_t^{qA} - \mu_t^{qB} + \frac{\mu_t^\kappa}{1 - \kappa_t} \right) (1 - \varphi_t) \varphi_t \\
&= \left( \tilde{\sigma}_t^2 (1 - \varphi_t) \left( \frac{\varphi_t}{\eta_t} - \frac{1 - \varphi_t}{1 - \eta_t} \right) - \lambda \left( \frac{\nu_t^{qA} - \nu_t^{qM}}{1 + \nu_t^{NA}} - \frac{\nu_t^{qB} - \nu_t^{qM}}{1 + \nu_t^{NB}} \right) \right. \\
&\quad \left. - \frac{1}{q_t^K} \left( \frac{p_t^A a_t^A \kappa_t}{\varphi_t} - \frac{p_t^B a_t^B (1 - \kappa_t)}{1 - \varphi_t} + \frac{1}{\phi} \frac{\kappa_t - \varphi_t}{\varphi_t (1 - \varphi_t)} \right) \right. \\
&\quad \left. + \psi^A - \psi^B \right) (1 - \varphi_t) \varphi_t
\end{aligned}$$

and the expression for the jump is:

$$\nu_t^\varphi \varphi_t = \left( \frac{\nu_t^{qA} - \nu_t^{qK}}{1 + \nu_t^{qK}} \right) \varphi_t$$

Now, consider the evolution of  $\vartheta_t$ . We have that:

$$\begin{aligned}
d\vartheta_t &= d \left( \frac{q_t^M}{q_t^M + q_t^K} \right) \\
&= \frac{q_t^M q_t^K}{(q_t^M + q_t^K)^2} \left( (\mu_t^{qM} - \mu_t^{qK}) dt + \left( \frac{\nu_t^{qM} - \nu_t^{qK}}{\frac{q_t^M}{q_t^M + q_t^K} (1 + \nu_t^{qM}) + \frac{q_t^K}{q_t^M + q_t^K} (1 + \nu_t^{qK})} \right) dJ_t \right) \\
&= (1 - \vartheta_t) \vartheta_t \left( (\mu_t^{qM} - \mu_t^{qK}) dt + \left( \frac{\nu_t^{qM} - \nu_t^{qK}}{\vartheta_t (1 + \nu_t^{qM}) + (1 - \vartheta_t) (1 + \nu_t^{qK})} \right) dJ_t \right)
\end{aligned}$$

Substituting the FOCs and market clearing conditions into the drift gives:

$$\begin{aligned}
\mu_t^\vartheta \vartheta_t &= (1 - \vartheta_t) \vartheta_t (\mu_t^{qM} - \mu_t^{qK}) \\
&= \vartheta_t \left( \rho - (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 \left( \frac{\varphi_t^2}{\eta} + \frac{(1 - \varphi_t)^2}{1 - \eta_t} \right) \right) \\
&\quad + \lambda (1 - \vartheta_t) \left( \frac{(\nu_t^{qA} - \nu_t^{qM}) \varphi_t}{1 + \nu_t^{NA}} + \frac{(\nu_t^{qB} - \nu_t^{qM})(1 - \varphi_t)}{1 + \nu_t^{NB}} \right) \\
&\quad + (1 - \vartheta_t) (\mu_t^M - i^M) - (1 - \vartheta_t) (\psi^A + \psi^B)
\end{aligned}$$

and

$$\nu_t^\vartheta \vartheta_t = -(1 - \vartheta_t) \vartheta_t \left( \frac{\nu_t^{qK} - \nu_t^{qM}}{\vartheta_t (1 + \nu_t^{qM}) + (1 - \vartheta_t) (1 + \nu_t^{qK})} \right) \quad \square.$$

In the main text, the differential equation for  $\vartheta_t$  is written in integral form. This is done in the following way. If we multiply by  $e^{-\rho t}$ , then we have that

$$\begin{aligned}
d(e^{-\rho t} \vartheta_t) &= e^{-\rho t} (-\rho + \mu_t^\vartheta) \vartheta_t dt + e^{-\rho t} \nu_t^\vartheta \vartheta_t dJ_t \\
\Rightarrow e^{-\rho T} \vartheta_T - e^{-\rho t} \vartheta_t &= \int_t^T e^{-\rho s} (-\rho + \mu_s^\vartheta) \vartheta_s ds + \int_t^T e^{-\rho s} \nu_s^\vartheta \vartheta_s dJ_s
\end{aligned}$$

After taking expectations, rearranging and taking the limit as  $T \rightarrow \infty$ , we get that:

$$\vartheta_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} (\rho - \mu_s^\vartheta - \lambda \nu_s^\vartheta) \vartheta_s ds \right]$$

To help understand the equation, we can expand the expressions for the drift



and jump exposure. This gives that:

$$\begin{aligned}
& \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( (1 - \vartheta_s)^2 \tilde{\sigma}_s^2 \left( \frac{\varphi_s^2}{\eta} + \frac{(1 - \varphi_s)^2}{1 - \eta_s} \right) \right) \right. \\
& - (1 - \vartheta_s)(\mu_s^M - i^M) + (1 - \vartheta_s)(\varphi_s \psi_s^A + (1 - \varphi_s) \psi_s^B) \\
& - \lambda(1 - \vartheta_s) \left( \frac{(\nu_s^{qA} - \nu_s^{qM}) \varphi_s}{1 + \nu_s^{NA}} + \frac{(\nu_s^{qB} - \nu_s^{qM})(1 - \varphi_s)}{1 + \nu_s^{NB}} \right) \\
& \left. + \lambda(1 - \vartheta_s) \left( \frac{\nu_s^{qK} - \nu_s^{qM}}{\vartheta_s(1 + \nu_s^{qM}) + (1 - \vartheta_s)(1 + \nu_s^{qK})} \right) \right] \vartheta_s ds \\
& = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( (1 - \vartheta_s)^2 \tilde{\sigma}_s^2 \left( \frac{\varphi_s^2}{\eta} + \frac{(1 - \varphi_s)^2}{1 - \eta_s} \right) \right) \right. \\
& - (1 - \vartheta_s)(\mu_s^M - i^M) + (1 - \vartheta_s)(\varphi_s \psi_s^A + (1 - \varphi_s) \psi_s^B) \\
& - \lambda(1 - \vartheta_s) \left( \frac{(\nu_s^{qA} - \nu_s^{qM}) \varphi_s}{1 + \nu_s^{qM} + (\nu_s^{qA} - \nu_s^{qM}) \vartheta_s^k A} \right) \\
& - \lambda(1 - \vartheta_s) \left( \frac{(\nu_s^{qB} - \nu_s^{qM})(1 - \varphi_s)}{1 + \nu_s^{qM} + (\nu_s^{qB} - \nu_s^{qM}) \vartheta_s^k B} \right) \\
& \left. + \lambda(1 - \vartheta_s) \left( \frac{\varphi_s \nu_s^{qA} + (1 - \varphi_s) \nu_s^{qB} - \nu_s^{qM}}{1 + \vartheta_s \nu_s^{qM} + (1 - \vartheta_s)(\varphi_s \nu_s^{qA} + (1 - \varphi_s) \nu_s^{qB})} \right) \right] \vartheta_s ds \\
& = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( (1 - \vartheta_s)^2 \tilde{\sigma}_s^2 \left( \frac{\varphi_s^2}{\eta} + \frac{(1 - \varphi_s)^2}{1 - \eta_s} \right) \right) \right. \\
& - (1 - \vartheta_s)(\mu_s^M - i^M) + (1 - \vartheta_s)(\varphi_s \psi_s^A + (1 - \varphi_s) \psi_s^B) \\
& + \lambda(1 - \vartheta_s) \left( \frac{(\nu_s^{qA} - \nu_s^{qM}) \varphi_s}{1 + \vartheta_s \nu_s^{qm} + (1 - \vartheta_s) \nu_s^{qK}} - \frac{(\nu_s^{qA} - \nu_s^{qM}) \varphi_s}{1 + \vartheta_s^m A \nu_s^{qm} + (1 - \vartheta_s^m A) \nu_s^{qA}} \right) \\
& \left. + \lambda(1 - \vartheta_s) \left( \frac{(\nu_s^{qB} - \nu_s^{qM})(1 - \varphi_s)}{1 + \vartheta_s \nu_s^{qm} + (1 - \vartheta_s) \nu_s^{qK}} - \frac{(\nu_s^{qB} - \nu_s^{qM})(1 - \varphi_s)}{1 + \vartheta_s^m B \nu_s^{qm} + (1 - \vartheta_s^m B) \nu_s^{qB}} \right) \right] ds
\end{aligned}$$

where we have used that  $\nu_t^{qK} = \varphi_t \nu_t^{qA} + (1 - \varphi_t) \nu_t^{qB}$  and  $\theta_t^{mI} := 1 - \theta_t^{kI} = 1 - (1 - \vartheta_t) \varphi_t q_t^K / \kappa_t$ . Substituting in the expressions for the jumps in net-worth,  $\nu_t^{NA}$ ,  $\nu_t^{NB}$ , and  $\nu_t^N$ , gives the required expression.

## C Solution Algorithm (For Online Appendix Only)

### C.1 Post Pandemic Economy

We guess (and verify numerically) that outside of the pandemic the functions  $\vartheta_t$  and  $\varphi_t$  take the form:

$$\begin{aligned}\vartheta_t &= \vartheta(\eta_t, \kappa_t) \\ \varphi_t &= \varphi(\eta_t, \kappa_t)\end{aligned}$$

Using Ito's lemma we have that:

$$\begin{aligned}d\vartheta(\eta_t, \kappa_t) &= [\mu_t^\eta \eta_t \partial_\eta \vartheta(\eta_t, \kappa_t) + \mu_t^\kappa \kappa_t \partial_\kappa \vartheta(\eta_t, \kappa_t)] dt \\ d\varphi(\eta_t, \kappa_t) &= [\mu_t^\eta \eta_t \partial_\eta \varphi(\eta_t, \kappa_t) + \mu_t^\kappa \kappa_t \partial_\kappa \varphi(\eta_t, \kappa_t)] dt\end{aligned}$$

So, equating drifts gives:

$$\begin{aligned}\tilde{\mu}^\vartheta(\eta_t, \kappa_t) &= \tilde{\mu}^\eta(\eta_t, \kappa_t) \partial_\eta \vartheta(\eta_t, \kappa_t) + \tilde{\mu}^\kappa(\eta_t, \kappa_t) \partial_\kappa \vartheta(\eta_t, \kappa_t) \\ \tilde{\mu}^\varphi(\eta_t, \kappa_t, \bar{a}) &= \tilde{\mu}^\eta(\eta_t, \kappa_t) \partial_\eta \varphi(\eta_t, \kappa_t) + \tilde{\mu}^\kappa(\eta_t, \kappa_t) \partial_\kappa \varphi(\eta_t, \kappa_t)\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mu}^\vartheta(\eta_t, \kappa_t) &= \left( \rho - (1 - \vartheta(\eta_t, \kappa_t))^2 \tilde{\sigma}(\kappa_t)^2 \left( \frac{\varphi(\eta_t, \kappa_t)^2}{\eta_t} + \frac{(1 - \varphi(\eta_t, \kappa_t))^2}{1 - \eta_t} \right) \right. \\
&\quad - (1 - \vartheta(\eta_t, \kappa_t))(\varphi(\eta_t, \kappa_t)\psi_t^A + (1 - \varphi(\eta_t, \kappa_t))\psi_t^B) \\
&\quad \left. + (1 - \vartheta(\eta_t, \kappa_t))(\mu_t^M - i^M) \right) \vartheta(\eta_t, \kappa_t) \\
\tilde{\mu}^\varphi(\eta_t, \kappa_t, a_t) &= \left( \tilde{\sigma}^2(\kappa_t)(1 - \varphi^0(\eta_t, \kappa_t)) \left( \frac{\varphi(\eta_t, \kappa_t)}{\eta_t} - \frac{1 - \varphi(\eta_t, \kappa_t)}{1 - \eta_t} \right) \right. \\
&\quad - \frac{1}{q_t^K} \left( \frac{p_t^A a_t \kappa_t}{\varphi(\eta_t, \kappa_t)} - \frac{p_t^B \bar{a}(1 - \kappa_t)}{1 - \varphi(\eta_t, \kappa_t)} + \frac{1}{\phi} \frac{\kappa_t - \varphi(\eta_t, \kappa_t)}{\varphi(\eta_t, \kappa_t)(1 - \varphi(\eta_t, \kappa_t))} \right. \\
&\quad \left. \left. + \psi_t^A - \psi_t^B \right) (1 - \varphi^0(\eta_t, \kappa_t)) \varphi^0(\eta_t, \kappa_t) \right. \\
\tilde{\mu}^\eta(\eta_t, \kappa_t) &= \left( \tilde{\sigma}^2(\kappa_t)(1 - \vartheta(\eta_t, \kappa_t))^2 \left( \frac{\varphi(\eta_t, \kappa_t)^2}{\eta_t^2} - \frac{(1 - \varphi(\eta_t, \kappa_t))^2}{(1 - \eta_t)^2} \right) \right. \\
&\quad \left. + \tau^A(\eta_t) - \tau^B(\eta_t) + \psi_t^A - \psi_t^B \right) (1 - \eta_t)\eta_t \\
\tilde{\mu}^\kappa(\eta_t, \kappa_t) &= \frac{1}{\phi} \log \left( \frac{\vartheta(\eta_t, \kappa_t)(1 - \kappa_t)}{(1 - \vartheta(\eta_t, \kappa_t))\kappa_t} \right) (1 - \kappa_t)\kappa_t
\end{aligned}$$

We solve the differential equations using a finite difference method. We discretize the model in the  $\eta$  dimension with the grid points  $\{\eta_1, \dots, \eta_{N_\eta}\}$  and in the  $\kappa$  dimension with the grid points  $\{\kappa_1, \dots, \kappa_{N_\kappa}\}$ . The finite difference approximations to the differential equations are denoted by:

$$\begin{aligned}
\tilde{\mu}_{jk}^\vartheta &= \tilde{\mu}_{jk}^\eta \partial_\eta \vartheta_{jk} + \tilde{\mu}_{jk}^\kappa \partial_\kappa \vartheta_{jk} \\
\tilde{\mu}_{jk}^\varphi &= \tilde{\mu}_{jk}^\eta \partial_\eta \varphi_{jk} + \tilde{\mu}_{jk}^\kappa \partial_\kappa \varphi_{jk}
\end{aligned}$$

The equation is non-linear so we use a semi-implicit Euler method (with an upwind scheme) to iterate the equations backward until we get steady state convergence. This is done with the following steps:

1. Guess initial discretized functions  $\{\vartheta_{j,k}^0\}_{j \leq N_\eta, k \leq N_\kappa}$  and  $\{\varphi_{j,k}^0\}_{j \leq N_\eta, k \leq N_\kappa}$ .
2. At iteration  $n$ , given current guesses for the discretized functions,  $\{\vartheta_{j,k}^n\}_{j \leq N_\eta, k \leq N_\kappa}$  and  $\{\varphi_{j,k}^n\}_{j \leq N_\eta, k \leq N_\kappa}$ , solve the following equations for

$\{\vartheta_{j,k}^{n+1}\}_{j \leq N_\eta, k \leq N_\kappa}$  and  $\{\varphi_{j,k}^{n+1}\}_{j \leq N_\eta, k \leq N_\kappa}$ :

$$\begin{aligned} \frac{\vartheta_{jk}^{n+1} - \vartheta_{jk}^n}{\Delta t} &= [(\tilde{\mu}_{jk}^\eta)^n]^+ \partial_{\eta F} \vartheta_{jk}^{n+1} + [(\tilde{\mu}_{jk}^\eta)^n]^- \partial_{\eta B} \vartheta_{jk}^{n+1} \\ &\quad + [(\tilde{\mu}_{jk}^\kappa)^n]^+ \partial_{\eta F} \vartheta_{jk}^{n+1} + [(\tilde{\mu}_{jk}^\kappa)^n]^- \partial_{\eta B} \vartheta_{jk}^{n+1} - (\tilde{\mu}_{jk}^\vartheta)^n \\ \frac{\varphi_{jk}^{n+1} - \varphi_{jk}^n}{\Delta t} &= [(\tilde{\mu}_{jk}^\eta)^n]^+ \partial_{\eta F} \varphi_{jk}^{n+1} + [(\tilde{\mu}_{jk}^\eta)^n]^- \partial_{\eta B} \varphi_{jk}^{n+1} \\ &\quad + [(\tilde{\mu}_{jk}^\kappa)^n]^+ \partial_{\eta F} \varphi_{jk}^{n+1} + [(\tilde{\mu}_{jk}^\kappa)^n]^- \partial_{\eta B} \varphi_{jk}^{n+1} - (\tilde{\mu}_{jk}^\varphi)^n \end{aligned}$$

where  $\partial_F$  and  $\partial_B$  denote the forward and backward finite difference approximations.

3. Check for convergence. If not, return to step 2.

## C.2 Pandemic Economy

We guess (and verify numerically) that during the pandemic the functions  $\vartheta_t$  and  $\varphi_t$  take the form:

$$\begin{aligned} \vartheta_t &= \hat{\vartheta}(\eta_t, \kappa_t, a_t) \\ \varphi_t &= \hat{\varphi}(\eta_t, \kappa_t, a_t) \end{aligned}$$

where we have used the hat symbol to distinguish the pandemic functions from the post pandemic functions. Using Ito's lemma, we have that:

$$\begin{aligned} d\hat{\vartheta}(\eta_t, \kappa_t) &= (\mu_t^\eta \eta_t \partial_\eta \hat{\vartheta}(\eta_t, \kappa_t) + \mu_t^\kappa \kappa_t \partial_\kappa \hat{\vartheta}(\eta_t, \kappa_t)) dt \\ &\quad + (\vartheta((1 + \nu_t^\eta) \eta_t, \kappa_t, \underline{a}) - \hat{\vartheta}(\eta_t, \kappa_t)) dJ_t \\ d\hat{\varphi}(\eta_t, \kappa_t) &= [\mu_t^\eta \partial_\eta \hat{\varphi}(\eta_t, \kappa_t) + \mu_t^\kappa \partial_\kappa \hat{\varphi}(\eta_t, \kappa_t)] dt \\ &\quad + (\varphi((1 + \nu_t^\eta) \eta_t, \kappa_t, \underline{a}) - \hat{\varphi}(\eta_t, \kappa_t)) dJ_t \end{aligned}$$

where, because the pandemic is unanticipated,  $\vartheta(\eta_t, \kappa_t)$  and  $\varphi^0(\eta_t, \kappa_t)$  can be solved independently of  $\hat{\vartheta}(\eta_t, \kappa_t)$  and  $\hat{\varphi}(\eta_t, \kappa_t)$  using the algorithm from the previous section. Equating the drifts and jump exposures we have that:

$$\begin{aligned} \tilde{\mu}^{\hat{\vartheta}}(\eta_t, \kappa_t) &= \tilde{\mu}^\eta(\eta_t, \kappa_t) \partial_\eta \hat{\vartheta}(\eta_t, \kappa_t) + \tilde{\mu}^\kappa(\eta_t, \kappa_t) \partial_\kappa \hat{\vartheta}(\eta_t, \kappa_t) \\ \nu^{\hat{\vartheta}}(\eta_t, \kappa_t) \hat{\vartheta}(\eta_t, \kappa_t) &= \vartheta((1 + \nu^\eta(\kappa_t, \eta_t)) \eta_t, \kappa_t) - \hat{\vartheta}(\eta_t, \kappa_t) \\ \tilde{\mu}^{\hat{\varphi}}(\eta_t, \kappa_t, \underline{a}) &= \tilde{\mu}^\eta(\eta_t, \kappa_t) \partial_\eta \hat{\varphi}(\eta_t, \kappa_t) + \tilde{\mu}^\kappa(\eta_t, \kappa_t) \partial_\kappa \hat{\varphi}(\eta_t, \kappa_t) \\ \nu^{\hat{\varphi}}(\eta_t, \kappa_t) \hat{\varphi}(\eta_t, \kappa_t) &= \varphi((1 + \nu^\eta(\kappa_t, \eta_t)) \eta_t, \kappa_t) - \hat{\varphi}(\eta_t, \kappa_t) \end{aligned}$$

where now we have that the drifts include the recovery jump:

$$\begin{aligned}
\tilde{\mu}^{\hat{\vartheta}}(\eta_t, \kappa_t) &= \left( \rho - (1 - \hat{\vartheta}(\eta_t, \kappa_t))^2 \tilde{\sigma}^2(\kappa_t) \left( \frac{\hat{\varphi}(\eta_t, \kappa_t)^2}{\eta_t} + \frac{(1 - \hat{\varphi}(\eta_t, \kappa_t))^2}{1 - \eta_t} \right) \right. \\
&\quad + \lambda(1 - \hat{\vartheta}(\eta_t, \kappa_t)) \left( \frac{(\nu_t^{qA} - \nu_t^{qM})\hat{\varphi}(\eta_t, \kappa_t)}{1 + \nu_t^{NA}} \right. \\
&\quad \left. \left. + \frac{(\nu_t^{qB} - \nu_t^{qM})(1 - \hat{\varphi}(\eta_t, \kappa_t))}{1 + \nu_t^{NB}} \right) \right) \\
&\quad + (1 - \hat{\vartheta}(\eta_t, \kappa_t))(\mu_t^M - (\psi^A - \psi^B)) \hat{\vartheta}(\eta_t, \kappa_t) \\
\tilde{\mu}^{\hat{\varphi}}(\eta_t, \kappa_t, a_t) &= \left( \tilde{\sigma}^2(\kappa_t)(1 - \hat{\varphi}(\eta_t, \kappa_t)) \left( \frac{\hat{\varphi}(\eta_t, \kappa_t)}{\eta_t} - \frac{1 - \hat{\varphi}(\eta_t, \kappa_t)}{1 - \eta_t} \right) \right. \\
&\quad - \lambda \left( \frac{\nu_t^{qA} - \nu_t^{qM}}{1 + \nu_t^{NA}} - \frac{\nu_t^{qB} - \nu_t^{qM}}{1 + \nu_t^{NB}} \right) \\
&\quad - \frac{1}{q_t^K} \left( \frac{p_t^A a_t \kappa_t}{\hat{\varphi}(\eta_t, \kappa_t)} - \frac{p_t^B \bar{a}(1 - \kappa_t)}{1 - \hat{\varphi}(\eta_t, \kappa_t)} + \frac{1}{\phi} \frac{\kappa_t - \hat{\varphi}(\eta_t, \kappa_t)}{\hat{\varphi}(\eta_t, \kappa_t)(1 - \hat{\varphi}(\eta_t, \kappa_t))} \right) \\
&\quad \left. + \psi^A - \psi^B \right) (1 - \hat{\varphi}(\eta_t, \kappa_t)) \hat{\varphi}(\eta_t, \kappa_t) \\
\tilde{\mu}^{\eta}(\eta_t, \kappa_t) &= \left( \tilde{\sigma}^2(\kappa_t)(1 - \hat{\vartheta}(\eta_t, \kappa_t))^2 \left( \frac{\hat{\varphi}(\eta_t, \kappa_t)^2}{\eta_t^2} - \frac{(1 - \hat{\varphi}(\eta_t, \kappa_t))^2}{(1 - \eta_t)^2} \right) \right. \\
&\quad - \lambda(1 - \hat{\theta}(\eta_t, \kappa_t)) \left( \left( \frac{\nu_t^{qA} - \nu_t^{qM}}{1 + \nu_t^{NA}} \right) \frac{\hat{\varphi}(\eta_t, \kappa_t)}{\eta_t} \right. \\
&\quad \left. - \left( \frac{\nu_t^{qB} - \nu_t^{qM}}{1 + \nu_t^{NB}} \right) \frac{1 - \hat{\varphi}(\eta_t, \kappa_t)}{1 - \eta_t} \right) \\
&\quad \left. + \tau^A(\eta_t) - \tau^B(\eta_t) + \psi_t^A - \psi_t^B \right) (1 - \eta_t) \eta_t \\
\tilde{\mu}^{\kappa}(\eta_t, \kappa_t) &= \frac{1}{\phi} \log \left( \frac{\hat{\varphi}(\eta_t, \kappa_t)(1 - \kappa_t)}{(1 - \hat{\varphi}(\eta_t, \kappa_t))\kappa_t} \right) (1 - \kappa_t) \kappa_t
\end{aligned}$$

Once again, we solve the differential equations using a finite difference method with a semi-implicit Euler method. We use the same discretization scheme. This is done with the following steps:

1. Guess initial discretized functions  $\{\hat{\vartheta}_{j,k}^0\}_{j \leq N_\eta, k \leq N_\kappa}$  and  $\{\hat{\varphi}_{j,k}^0\}_{j \leq N_\eta, k \leq N_\kappa}$ .
2. At iteration  $n$ , given current guesses for the discretized functions,

$\{\hat{\vartheta}_{j,k}^n\}_{j \leq N_\eta, k \leq N_\kappa}$  and  $\{\hat{\varphi}_{j,k}^n\}_{j \leq N_\eta, k \leq N_\kappa}$ , solve the following equations for  $\{\hat{\vartheta}_{j,k}^{n+1}\}_{j \leq N_\eta, k \leq N_\kappa}$  and  $\{\hat{\varphi}_{j,k}^{n+1}\}_{j \leq N_\eta, k \leq N_\kappa}$ :

$$\begin{aligned} \frac{\hat{\vartheta}_{j,k}^{n+1} - \hat{\vartheta}_{j,k}^n}{\Delta t} &= [(\tilde{\mu}_{j,k}^\eta)^n]^+ \partial_{\eta F} \hat{\vartheta}_{j,k}^{n+1} + [(\tilde{\mu}_{j,k}^\eta)^n]^- \partial_{\eta B} \hat{\vartheta}_{j,k}^{n+1} \\ &\quad + [(\tilde{\mu}_{j,k}^\kappa)^n]^+ \partial_{\eta F} \hat{\vartheta}_{j,k}^{n+1} + [(\tilde{\mu}_{j,k}^\kappa)^n]^- \partial_{\eta B} \hat{\vartheta}_{j,k}^{n+1} - (\tilde{\mu}_{j,k}^{\hat{\vartheta}})^n \\ \frac{\hat{\varphi}_{j,k}^{n+1} - \hat{\varphi}_{j,k}^n}{\Delta t} &= [(\tilde{\mu}_{j,k}^\eta)^n]^+ \partial_{\eta F} \hat{\varphi}_{j,k}^{n+1} + [(\tilde{\mu}_{j,k}^\eta)^n]^- \partial_{\eta B} \hat{\varphi}_{j,k}^{n+1} \\ &\quad + [(\tilde{\mu}_{j,k}^\kappa)^n]^+ \partial_{\eta F} \hat{\varphi}_{j,k}^{n+1} + [(\tilde{\mu}_{j,k}^\kappa)^n]^- \partial_{\eta B} \hat{\varphi}_{j,k}^{n+1} - (\tilde{\mu}_{j,k}^{\hat{\varphi}})^n \\ \nu_{j,k}^{\hat{\vartheta}} &= \frac{\vartheta((1 + \nu_{j,k}^\eta) \eta_j, \kappa_k) - \hat{\vartheta}_{j,k}}{\hat{\vartheta}_{j,k}} \\ \nu_{j,k}^{\hat{\varphi}} &= \frac{\varphi((1 + \nu_{j,k}^\eta) \eta_j, \kappa_k) - \hat{\varphi}_{j,k}}{\hat{\varphi}_{j,k}} \end{aligned}$$

where  $\partial_F$  and  $\partial_B$  denote the forward and backward finite difference approximations.

3. Check for convergence. If not, return to step 2.