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We study a small open economy displaying Pareto-distributed wealth resulting from random death. The government runs a distribution scheme on inheritance. We present the mathematical background that allows to study the dynamics of means. We end up with ordinary differential equations for the mean of age and of individual and government wealth. We also study distributional dynamics analytically. Starting from any distribution of age and wealth, the aggregate distribution converges, both on a transition path towards a steady state and on a transition path towards balanced growth, to an exponential distribution of age and a Pareto-distribution of wealth. The findings are illustrated for different distribution schemes.

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Keywords: analytical dynamics of mean and distribution, age and wealth, government budget, stochastic differential equation, Kolmogorov backward equation

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1 Introduction

[Motivation] Distributional analyses gain in importance in times where wealth or income appear to become increasingly unequally distributed across individuals. An observed distribution that differs at two points in time can be understood in (at least) two ways. First, by comparative static analysis of a stationary distribution or by studying the transition process from the first realization of the distribution to the second. As the former is the dominating approach in the literature and the latter is often performed numerically, we focus on the latter from an analytical perspective for the case of a 'perpetual youth' model.

[Setup] We model the age process of representatives of a dynasty by a stochastic differential equation. Wealth of the dynasty is derived from this age process while individuals are alive. Newborns are endowed with a constant initial wealth level. We assume that capital is the only source of income. This setup implies a stochastic differential equation for the wealth process as well. We study dynasties in a small open economy framework with free international capital flows. We investigate convergence properties towards a steady state and balanced growth path.

We emphasize that our setup reflects the age process of a typical 'perpetual youth' Yaari-Blanchard model (Yaari, 1965, Blanchard, 1985). The wealth process in our setup is an alternative representation of the evolution of wealth in Jones (2014, 2015). An endowment that is identical at birth reminds of Kasa and Lei (2018), among others (see below for a detailed comparison with the literature).

[Contribution] Our contribution is fourfold. The crucial departure from the economic birth-death literature we are aware of lies in a modelling choice. We represent the age process resulting from a birth-death process by a stochastic differential equation (SDE). As a consequence, the wealth process in our economy can also be represented by an SDE. This allows us to apply standard stochastic methods to understand its properties.

Second, building on the SDE, we derive an ordinary differential equation (ODE) that describes how mean age and mean wealth evolve over time. While mean age always converges to a constant, we derive conditions under which mean wealth does so as well and under which it converges to a growth path. We derive an SDE for government wealth as well. The government taxes income (with a positive or negative rate), receives all wealth at death and endows each newborn with a constant initial wealth level. The corresponding ODE tells us under which conditions the debt to GDP ratios approach a steady state or a balanced growth path.

Third, combining conditions at the household level with conditions for government wealth, we obtain equilibrium conditions. The economy can converge to a steady-state economy or to a balanced growth path. The conditions are expressed in terms of the interest rate, time preference rate, death rate and intertemporal elasticity of substitution.

Finally, we obtain results on the dynamics of distributions by steps that differ from the more popular Fokker-Planck equations (FPEs). We rather solve the SDEs and derive distributions from these solutions. We show that our age process converges to an exponential and our wealth process converges to a Pareto distribution in the limit. Starting from any arbitrary initial condition, we characterize transitional dynamics of age and wealth distributions analytically and illustrate them graphically.

[Table of contents] The next section briefly surveys the literature to which we relate our work. Section 3 introduces the model. Section 4 provides the stochastic background of our analysis. Section 5 provides findings on the evolution of expected age, individual wealth and government wealth. Section 6 first derives conditions for a steady state vs. a balanced growth path, illustrates the difference between time paths for expected values and realizations and offers our analytical characterization of transitional dynamics of distributions. The final section concludes.

2 Related literature

Birth-death process. We employ Poisson processes to model birth-death processes as many other papers in the wealth-distribution literature (Cao and Lou, 2017, Gabaix et al., 2016, Kasa and Lei, 2018, Aoki and Nirei, 2017). We share with others (Blanchard, 1985, Benhabib et al., 2016, Gabaix et al., 2016, Kaymak and Poschke, 2016, Kasa and Lei, 2018, Toda, 2014, Benhabib et al., 2011, Aoki and Nirei, 2017, Benhabib et al., 2019, Itskhoki and Moll, 2019) that death and birth rates are identical. This differs from analyses that allow rates of birth and death to differ leading to population growth (Jones, 2014, 2015, Cao and Lou, 2017). Yaari (1965) and d'Albis (2007) consider age-dependent death rates.

Identical initial endowment. Newborns in our model receive a constant initial endowment. This captures the idea of equality of chances with respect to initial wealth.[2] This assumption is also made by Kasa and Lei (2018). Newborns born at the same point in time receive identical endowments also in Jones (2015). This endowment can grow over time, however. The literature employing insurance companies in finite-life models in the tradition of Yaari (1965) and Blanchard (1985) redistribute wealth intragenerationally. Both use insurance companies selling an annuity to a consumer who then receives a frequent payment until the point of death. After that, the insurance company claims the annuity without any further obligation (Yaari, 1965), or keeps the individual's total wealth (Blanchard, 1985). In models in the Blanchard-tradition, all individuals also have an identical initial wealth level (of zero).[3]

Kolmogorov backward equations. Analysis of the mean is facilitated by employing Kolmogorov backward equations. An introduction can be found in Stokey (2008, ch. 3.7). They are also applied in finance papers like Cox and Ross (1976), Aoki (1995), Kawai (2009), or Eberlein and Glau (2014).

Kolmogorov forward / Fokker-Planck equations. Fokker-Planck equations (FPEs) became very popular recently and we share the belief in their usefulness with Benhabib, Bisin and Zhu (2016), Achdou et al. (2020), Jones and Kim (2017), Cao and Luo (2017), Aoki and Nirei (2017), Kaplan et al. (2018), Nuño and Moll (2018) and Itskhoki and Moll (2019).[4,5]

Probability theory. Given the nature of our project, we used textbooks on probability theory. They include Oksendal (1998), Kallenberg (1997), or Privault (2018). In order to understand stochastic integral equations, their solutions, and the infinitesimal generator of Markov processes, Protter (1995) is helpful. Davis (1993) establishes a theorem on the evolution of an expected value as being entirely determined by a generator for certain assumptions which is essential when analyzing the mean.

Poisson processes. Going beyond the analysis of wealth distributions, we emphasize that Poisson processes are ubiquitous in other parts of economics as well. Modelling strategies to which our method of analyzing the mean could be applied include models of R&D (Aghion et al., 2001, Grossman and Helpman, 1991, Helpman et al., 2005, Klette and Kortum, 2004, Aghion and Howitt, 1992). Search and matching models in the tradition of Diamond (1982), Mortensen (1982) and Pissarides (1985) also build on Poisson processes as do some business cycles models (Brunnermeier and Sannikov, 2014, Wälde, 2005, He and Krishnamurthy, 2011,

[2] We ignore other determinants of equality of chances such as cognitive and non-cognitive skills or family background. We also acknowledge a long literature studying alternative redistribution schemes. Recent contributions include Cao and Luo (2017), Benhabib et al. (2016) and Benhabib et al. (2011).

[3] Including an annuity in our model would lead to a different deterministic evolution of wealth over time. It would not change our main points, however. Thus, we share the ideas of Jones (2014, 2015), Toda (2014, p. 329), or Cao and Luo (2017) and omit insurance markets.

[4] Achdou et al. (2014) provide an overview of partial differential equation models in macroeconomics. Ahn et al. (2017) describe numerical methods for continuous time models.

[5] Bayer and Wälde (2015, p. 4) provide a short survey on the use of FPEs in economics prior to these papers. Bayer and Wälde (2010a, sect. 5) showed how to derive FPEs for relatively general cases (using a Bewley-Huggett-Aiyagari model as example).

2013, or Di Tella, 2017) or the trade literature in the tradition of Melitz (2003).

Pareto and double-Pareto distributions. Pareto distributions have become very popular recently (Piketty and Zucman, 2015). They appeared in the analysis of top income changes (Saez and Zucman, 2016), income growth per person, population growth (Jones, 2015), financial deregulation, (corporate) taxes (Cao and Luo, 2017) or bequests and saving rate inequality (Benhabib et al., 2019). Some models derive a 'double Pareto distribution' for wealth. This can be achieved by introducing a diffusion processes in a model with exponentially distributed lifetimes (Reed, 2001, 2003, Toda, 2012, 2014).[6] Our focus is on analytical results for transitional dynamics of distributions. We believe that they can also be applied to (appropriately modified) double-Pareto structures.[7]

3 The model

The model presentation starts from small agents (one individual and one dynasty), passes by a large agent (the government) and ends in equilibrium (the small open economy).

3.1 The individual

Each individual is endowed with a time preference rate $\hat{\rho} > 0$ and has a finite life that ends at a random point in time. This point T is exponentially distributed with parameter δ, denoted death rate. The individual maximizes expected utility $E_t \int_t^T e^{-\hat{\rho}[s-t]} u(c(s))\, ds$, where expectations are formed with respect to $T > t$ given information up to t. Instantaneous utility is $u(c(s))$ and the individual chooses the time path of consumption $c(s)$. It is well-known from Blanchard-Yaari models that this maximization problem is identical to maximizing a deterministic objective function

$$U(t) = \int_t^\infty e^{-\rho[s-t]} u(c(s))\, ds, \tag{1}$$

where discounting takes place at the rate $\rho = \hat{\rho} + \delta$, i.e. adding the death rate δ to the time preference rate. As the objective function shows, the individual cares about own consumption only. They do not value bequests or utility of offsprings. Consumption $c(s)$ is therefore perceived to be deterministic. All bequests in our model will be accidental. We consider a standard, constant relative risk aversion (henceforth CRRA), instantaneous utility function

$$u(c(t)) = \frac{c^{1-\sigma} - 1}{1 - \sigma}. \tag{2}$$

The budget constraint of our individual is deterministic as well and reads

$$\dot{a}(t) = (r - \tau)\, a(t) - c(t). \tag{3}$$

In the absence of labor income, the only source of income is interest r on individual wealth $a(t)$. Furthermore, a wealth tax τ is paid to the state.[8] Consumption reduces wealth accumulation and the price of the consumption good is normalized to one. The time derivative of wealth is denoted by the usual $\dot{a}(t)$.

[6] See also the analyses by Benhabib et al. (2016), Reed (2011) or Toda and Walsh (2015).

[7] So far double-Pareto findings are built on a combination of Brownian motion and exponential age. Toda (2014) writes that "the double Pareto property is robust in the sense that it depends only on multiplicative growth and the geometric age distribution and not on the details of the stochastic process governing growth". Gabaix (1999) conjectures that the power law should hold even if the multiplicative process is time-varying. Hence, obtaining double-Pareto findings employing Poisson processes only seems possible.

[8] The wealth tax τ turns into a capital income tax τ_c when we replace τ by $r\tau_c$. The budget constraint (3) would then read $\dot{a}(t) = (1 - \tau_c)\, ra(t) - c(t)$.

When we solve the individual's maximization problem, optimal consumption is available in closed form (see app. A.1),

$$c(t) = \phi a(t) \text{ with } \phi \equiv \frac{\rho - (1-\sigma)(r-\tau)}{\sigma}. \tag{4}$$

Consumption is a constant share ϕ out of wealth. Deriving this solution also shows that consumption grows at a rate

$$z \equiv \frac{r - \tau - \rho}{\sigma}. \tag{5}$$

As long as the net interest rate $r - \tau$ exceeds the time preference rate ρ, wealth of the individual increases over time.

Now imagine an individual is born at t_B. Age of the individual is then $t - t_B$. Using (4), (3) and endowing the individual with initial wealth $a(t_B)$, wealth is a function of age and follows

$$a(t) = a(t_B) e^{z[t-t_B]}. \tag{6}$$

This finding is well-known from many closed-form solutions or steady-state properties: Wealth also grows at the rate z.

3.2 The dynasty

Turning to a dynasty i, an offspring is born once an individual dies. A dynasty is therefore characterized by a stochastic age process and a stochastic wealth process. We describe both of them by stochastic differential equations driven by Poisson processes. This is the key novelty of our paper from a methodological perspective.

3.2.1 Age

We start by specifying the age process. As emphasized above, our specification is representative of age processes in many papers employing a birth-death framework with constant population size. Our findings obtained below are possible as we model this age process by a stochastic differential equation. It reads

$$dX_i(t) = bdt - X_i(t_-) dQ_i^\delta(t). \tag{7}$$

Age of the currently alive individual of dynasty i at a point in time t is denoted by $X_i(t)$. It increases linearly and deterministically in time with slope b. When age and time are measured in the same units, b equals one. Age drops to zero at random points in time, i.e. when the increment $dQ_i^\delta(t)$ of the Poisson process $(Q(t))_{t\geq 0}$ equals one. Poisson processes $Q_i^\delta(t)$ are independent of each other.

The arrival rate of this Poisson process is the constant death rate introduced above before (1). Age dropping to zero means that an individual that dies is replaced by a newly born offspring of age zero. Population size N therefore remains constant. We denote the initial age of the currently alive individual of dynasty i by x_i.[9]

3.2.2 Wealth

We can now describe the wealth process of a dynasty i. While alive, an individual accumulates wealth according to (6). When death hits according to (7), all wealth of a dynasty goes to

[9] Parameters of the process (7) could differ across dynasties. Concerning the age process, we only allow for differences in initial age in this paper.

the state which, in turn, endows the newborn with some initial endowment $\bar{a}$.[10] This wealth accumulation and redistribution scheme is captured by

$$dA_i\left(X_i\left(t\right)\right) = zA_i\left(X_i\left(t\right)\right)dt + \left[\bar{a} - A_i\left(X_i\left(t_-\right)\right)\right]dQ_i^\delta\left(t\right). \tag{8}$$

Wealth A_i of dynasty i whose currently alive member has age $X_i\left(t\right)$ at time t changes according to a deterministic and a stochastic part. The deterministic part incorporates optimal wealth accumulation (6) at the individual level via the parameter z: as long as the individual is alive, (8) describes a deterministic growth of wealth at the rate of z, just as (6).[11] The stochastic part shows that in the case of death at t, wealth is reduced by the wealth level $A_i\left(X_i\left(t_-\right)\right)$ at t_-, i.e. an instant before death. Wealth is increased by $\bar{a}$ such that the newborn starts with this initial endowment.[12]

3.3 The government

Consider a government that levies a tax on wealth, collects all wealth at the moment of death and endows all newborns with an initial constant amount of wealth. We can express the change in government wealth based on *one dynasty* by the following SDE

$$dG_i\left(A_i\left(t\right)\right) = \tau A_i\left(t\right)dt + \left[A_i\left(t_-\right) - \bar{a}\right]dQ_i^\delta\left(t\right). \tag{9}$$

The first source of income is given by tax revenue $\tau A_i\left(t\right)$. The second source is wealth $A_i\left(t_-\right)$ of individuals being transferred to the state at the moment of death, i.e. when $dQ_i^\delta\left(t\right) = 1$. Government spending consists in endowing the newborn with a constant amount of wealth $\bar{a}$. We do not impose a balanced government budget at each instant (as e.g. Benhabib et al., 2016). We rather allow the government to trade government bonds on the international capital market.[13]

When we denote total government wealth by $G\left(t\right)$, its evolution follows from summing over all dynasties N,

$$dG\left(t\right) = \Sigma_{i=1}^N\left\{\tau A_i\left(t\right)dt + \left[A_i\left(t_-\right) - \bar{a}\right]dQ_i^\delta\left(t\right)\right\}. \tag{10}$$

3.4 The small open economy

We study a small open economy. All of our distributional findings below concerning age and wealth can therefore be understood as findings describing the population of a small open economy. In this small open economy, international capital flows fix the domestic interest rate r. Optimal consumption as described in (4) is therefore determined by the international interest rate (and preferences of households).

Concerning production processes, we could think of AK technologies as in Toda (2014). With an AK technology (used domestically and abroad), domestic output would be indeterminate as it does not matter where capital is allocated. In the case of a neoclassical production

[10] As discussed above, the absence of planned bequests and identical endowments of newborns is a common assumption in the literature (Kasa and Lei, 2018, Jones, 2015 and models in the Blanchard, 1985, tradition).

[11] Even though $A_i\left(X_i\left(t\right)\right)$ precisely describes the deterministic link between wealth and age in models of this type, we will also employ $A_i\left(t\right)$ when appropriate.

[12] An obvious extension reduces the inheritance tax from 100% as in (8) to some $0 < \tau_b < 1$. The wealth constraint would read $dA_i\left(X_i\left(t\right)\right) = zA_i\left(X_i\left(t\right)\right)dt + \left[\bar{a} - \tau_b A_i\left(X_i\left(t_-\right)\right)\right]dQ_i^\delta\left(t\right)$. Extending our analyses for the mean and aggregate equilibrium is straightforward. The analysis of distributional dynamics is an order of magnitude more complex. We return to this issue once we have understood distributional dynamics of wealth following (8).

[13] This allows to study wealth of domestic dynasties and government wealth independently of each other.

function with capital and labour, the wage w would be fixed as well.[14] We could add labour income to the household budget constraint. Closed-form solutions would persist.

The production process in our model is not central to the results, however, as long as the consumption good is homogenous and traded internationally. Domestic production would be constant in the neoclassical case with a constant interest rate r. Households would nevertheless grow richer and experience exponential consumption growth as they accumulate wealth abroad. For all of our results on inequality and output, we either measure output by consumption or by wealth directly.

4 Mathematical background: Describing the mean of a stochastic process

This section discusses principles behind computing means in section 4.1. Section 4.2 looks at a linear stochastic differential equation (SDE) that describes a stochastic process. This section also computes the time derivative of the mean of this stochastic process. We propose two approaches: a "fast and intuitive" approach and one that follows a general rigorous approach from stochastic theory. Both approaches yield the same results.[15]

4.1 Preliminaries

We are interested in a class of real-valued stochastic processes $(X(t))_{t\geq 0}$. This class can be described as solutions of an SDE driven by a Poisson process $(Q(t))_{t\geq 0}$ with intensity $\lambda > 0$. Given suitable functions $f, g : \mathbb{R} \to \mathbb{R}$, the SDE takes the form[16]

$$dX(t) = f(X(t))dt + g(X(t_-))dQ(t), \quad t \geq 0. \tag{11}$$

Intuitively, the dynamics of the solution to (11) is the following: The path $(X(t))$ moves along solution curves of the ordinary differential equation $\dot{x} = f(x)$. Whenever the Poisson process $(Q(t))$ jumps at a certain time, say τ, the process jumps from its position $X(\tau_-)$ immediately before τ to its new position $X(\tau) = X(\tau_-) + g(X(\tau_-))$.

For completeness, let us briefly discuss the general mathematical set-up behind (11): The process is defined on a filtered probability space[17] $\left(\Omega, \mathcal{F}, P, \tilde{\mathcal{F}}\right)$ where Ω is the sample space, $\mathcal{F}$ is a $\sigma-$algebra and P is a probability measure on $\mathcal{F}$ and $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t)_{t\geq 0}$ is a filtration of sub-σ-algebras of $\mathcal{F}$. Strictly speaking, $X : [0, \infty) \times \Omega \to \mathbb{R}$ is then a function of time and "randomness", where $X(t, \omega)$ is the (random) value of the process at time $t \geq 0$ in the sample point ω. We will follow the usual approach and suppress the dependence on ω in the notation, so $X(t)$ denotes the real-valued random variable which describes the state of the process at a fixed time $t \geq 0$. We will write $(X(t)) \equiv (X(t))_{t\geq 0}$ to denote the (random) path of the process and sometimes simply write X to denote the process when the context is clear. We will try to follow the usual notational convention to denote random variables by capital letters and possible (fixed) values by small letters. A Poisson process $(Q(t))_{t\geq 0}$ with intensity $\lambda > 0$ on

[14] It would also imply a constant domestic wage rate. In the presence of unemployment as in Bayer et al. (2019), the domestic capital stock would adjust accordingly.

[15] Readers interested in understanding means can go to section 4.2 immediately. The analysis of the mean can be understood without the rigorous background in section 4.1.

[16] As usual, we understand (11) to be a shorthand notation for the equation $X(t) = X(0) + \int_0^t f(X(s))\,ds + \int_0^t g(X(s_-))\,dQ(s)$ with $t \geq 0$.

[17] We can and will assume that the "usual conditions" are fulfilled, i.e., $\tilde{\mathcal{F}}$ is right-continuous and complete, see e.g., Garcia and Griego (1994, p. 338).

$\left(\Omega, \mathcal{F}, P, \tilde{\mathcal{F}}\right)$ is a process with $Q(0) = 0$ which is constant between jumps of size $+1$ with the property that $Q(t) - Q(s)$ is independent of $\tilde{\mathcal{F}}_s$ and Poisson distributed with mean $\lambda(t-s)$ for any $0 \leq s < t$.

Under suitable assumptions,[18] it is known that (11) has a unique solution for any starting value $X(0)$ (which could itself be random) which is adapted to the filtration $\tilde{\mathcal{F}}$ and has right-continuous paths. Furthermore, if $X(0)$ has finite expectation $E[|X(0)|] < \infty$, we have then $E[|X(t)|] < \infty$ for all $t > 0$ as well. The analogous statement holds for second moments.

The solutions are semi-martingales and also (strong) Markov processes.[19] This allows to use tools both from stochastic analysis and from the theory of Markov processes in order to analyze the behavior of the process $(X(t))$.

For the Markov process viewpoint, we need a family P_x, $x \in \mathbb{R}$ of probability measures on $(\Omega, \mathcal{F})$ where for given $x \in \mathbb{R}$, P_x describes the law when starting from the fixed $x = X(0)$, in particular $P_x(X(0) = x) = 1$. We will write expectations with respect to P_x as E_x such that

$$E_x[h(X(t))] = E[h(X(t)) \mid X(0) = x], \tag{12}$$

where $h : \mathbb{R} \to \mathbb{R}$ is (a suitable) test function.[20] The transition semigroup[21] is $(P_t)_{t\geq 0}$ where P_t is defined via

$$P_t h(x) \equiv E_x[h(X(t))], \quad x \in \mathbb{R}. \tag{13}$$

In an economic spirit, if the test function $h : \mathbb{R} \to \mathbb{R}$, represented utility, $P_t h(x)$ is the expected value at time 0 of the random utility $h(X(t))$ at time t given that we know $X(0) = x$.

The generator $\mathcal{A}$ of a (Feller) transition semigroup $(P_t)_{t\geq 0}$ is defined as

$$\mathcal{A}h\,(x) = \lim_{t\to 0, t>0} \frac{P_t h(x) - h(x)}{t}, \quad x \in \mathbb{R} \tag{14}$$

for functions h in its domain $\mathcal{D}(\mathcal{A})$. By definition, $\mathcal{D}(\mathcal{A})$ consists of all functions $h \in C(\mathbb{R})$ for which the limit on the right hand side of (14) exists (in the "strong" sense of the supremum norm on $C(\mathbb{R})$, i.e., uniformly in x). For a more probabilistic interpretation of (14), we re-write this as $\mathcal{A}h\,(x) = \lim_{t\to 0, t>0} \frac{E_x[h(X(t))] - h(x)}{t}$. Thus, for a very small positive time $t > 0$ and a given starting point x, we have

$$E_x[h(X(t)] \approx h(x) + t\mathcal{A}h\,(x)$$

and, hence, $\mathcal{A}h(x)$ describes approximately[22] how the mean of $h(X(t))$ changes from its initial value $h(x)$ over a very short time interval.

The generator for the solution of (11) looks as follows[23]

$$\mathcal{A}h\,(x) = f(x)h'(x) + \lambda\,[h(x + g(x)) - h(x)] \tag{15}$$

[18] We will either assume that f is Lipschitz continuous, that is there exist $c_f < \infty$ so that $|f(x) - f(y)| \leq c_f|x - y|$ holds for all $x, y \in \mathbb{R}$ and that g is either Lipschitz continuous or bounded (see for example Garcia and Griego, 1994, Theorem 6.2).

[19] In fact, they belong to the class of piece-wise deterministic Markov processes: Between the jump times of $(Q(t))$, $(X(t))$ follows a differentiable curve. Such processes are discussed in much greater detail in Davis (1993). See also Garcia and Griego (1994, p. 362) for the Markov property.

[20] Suitable means that the expectation in (12) is well-defined. This is, for example, the case when h is measurable and bounded or non-negative.

[21] The semigroup property means $P_t P_s = P_{t+s}$, compare e.g. Protter (2004, p. 35). It is known that for our examples $(P_t)_{t\geq 0}$ is a so-called Feller transition semigroup, see e.g. Davis (1993, Thm. (27.6)), that is $P_t : C(\mathbb{R}) \to C(\mathbb{R})$ where $C(\mathbb{R})$ denotes the set of continuous functions which vanish at $\pm\infty$. This is mathematically convenient since it allows to work on the Banach space $C(\mathbb{R})$.

[22] A precise meaning of $\approx$ is here that in fact $E_x[h(X(t)] = h(x) + t\mathcal{A}h\,(x) + o(t)$ as $t \downarrow 0$, where the "error term" $o(t)$ goes to 0 faster than linearly in t.

[23] See, e.g., Garcia and Griego (1994, p. 361–362).

and $\mathcal{D}(\mathcal{A})$ contains all differentiable functions $h \in C(\mathbb{R})$ such that the derivative $h' \in C(\mathbb{R})$.
Let us briefly discuss why (15) holds. For an intuitive approach, consider $h \in \mathcal{D}(\mathcal{A})$, $X(0) = x$, $t > 0$ very small. Then $P_x(Q(t) = 1) = \lambda t + O(t^2)$, $P_x(Q(t) = 0) = 1 - \lambda t + O(t^2)$, $P_x(Q(t) \geq 2) = O(t^2)$. On the event $\{Q(t) = 0\}$ (no jump before time t), we have, by linearizing the ODE, $X(t) \approx x + tf(x)$; on the event $\{Q(t) = 1\}$ we have $X(t) \approx x + g(x)$. Hence

$$\begin{aligned} E_x[h(X(t))] &\approx (1 - \lambda t)h(x + tf(x)) + \lambda t h(x + g(x)) \\ &= h(x + tf(x)) - h(x) + \lambda t\,[h(x + g(x)) - h(x)] - \lambda t\,[h(x + tf(x)) - h(x)] \end{aligned}$$

subtracting $h(x)$ on both sides, dividing by t and letting $t \downarrow 0$ then yields (15) (use the chain rule on $\frac{d}{dt}h(x + tf(x))$ and observe that $\lambda t\big(h(x + tf(x)) - h(x)\big) = O(t^2)$).

A rigorous argument goes as follows: Applying to $(X(t))_{t\geq 0}$ the chain rule for paths of bounded variation,[24] we find

$$\begin{aligned} h(X(t)) &= h(X(0)) + \int_0^t h'(X(s))f(X(s))\,ds + \int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)dQ(s) \\ &= h(X(0)) + \int_0^t h'(X(s))f(X(s))\,ds + \lambda \int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)ds \\ &\quad + \int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)d[Q(s) - \lambda s]. \end{aligned}$$

By martingale properties of compensated Poisson processes (see, e.g. Garcia and Griego, 1994, Thm. 5.3), the process

$$\int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)d[Q(s) - \lambda s], \quad t \geq 0$$

is a martingale, in particular its expectation equals 0. Thus, taking expectations with respect to P_x shows

$$\begin{aligned} E_x[h(X(t))] &= h(x) + E_x\left[\int_0^t h'(X(s))f(X(s))\,ds\right] \\ &\quad + \lambda E_x\left[\int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)ds\right] \\ &= h(x) + \int_0^t E_x\left[h'(X(s))f(X(s))\right]ds \\ &\quad + \lambda \int_0^t E_x\left[\Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)\right]ds, \end{aligned}$$

where we used Fubini's theorem in the second equation. Thus

$$\begin{aligned} \frac{E_x[h(X(t))] - h(x)}{t} &= \frac{1}{t}\int_0^t E_x\left[h'(X(s))f(X(s))\right]ds \\ &\quad + \frac{\lambda}{t}\int_0^t E_x\left[\Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)\right]ds. \end{aligned}$$

Using the fact that $\lim_{s\downarrow 0} X(s) = \lim_{s\downarrow 0} X(s_-) = X(0)$ because paths are right-continuous, this shows (15) by taking $t \downarrow 0$.

This a good place to highlight the difference between Kolmogorov backward equations and Kolmogorov forward equations (aka Fokker-Planck equations). If distributional properties are

[24] See, e.g., Garcia and Griego (1994, p. 344).

to be understood, the forward equation is applied. If the interest lies in the mean, the backward equation can be used (see for instance Kallenberg, 1997, p. 192). For Markov processes the general Kolmogorov backward equation reads (compare Davis, 1993, p.30, equ. 14.11)

$$\frac{d}{dt} E_x \left[h \left(X \left(t \right) \right) \right] = E_x \left[\left(\mathcal{A} h \right) \left(X \left(t \right) \right) \right] \tag{16}$$

for all functions $h \in \mathcal{D}(\mathcal{A})$.[25]

We are particularly interested in computing the mean

$$\mu(x, t) \equiv E_x[X(t)] \tag{17}$$

for processes of the form (11). We will do this in the following section.

4.2 An example

We start by looking at a stochastic process $X(t)$ described by a SDE,

$$dX \left(t \right) = -aX \left(t \right) dt + bdQ \left(t \right) \tag{18}$$

with $X(0) > 0$ and $a, b > 0$. To connect (18) to (11), set $f(x) = -ax$ and $g(x) = b$. This implies that $X(t) \geq 0 \; \forall \; t$ as the deterministic decay is exponential, i.e. $X(t)$ approaches zero asymptotically in the absence of jumps. The arrival rate of $Q(t)$ is given by the constant $\lambda > 0$. The support of $X(t)$ is $\mathbb{R}^*_+$, i.e. neither zero nor infinity are included, $]0, \infty[$. The support is infinitely large as in principle $Q(t)$ can jump very often relative to the speed of a. Figure 1 shows one possible realization of process (18).

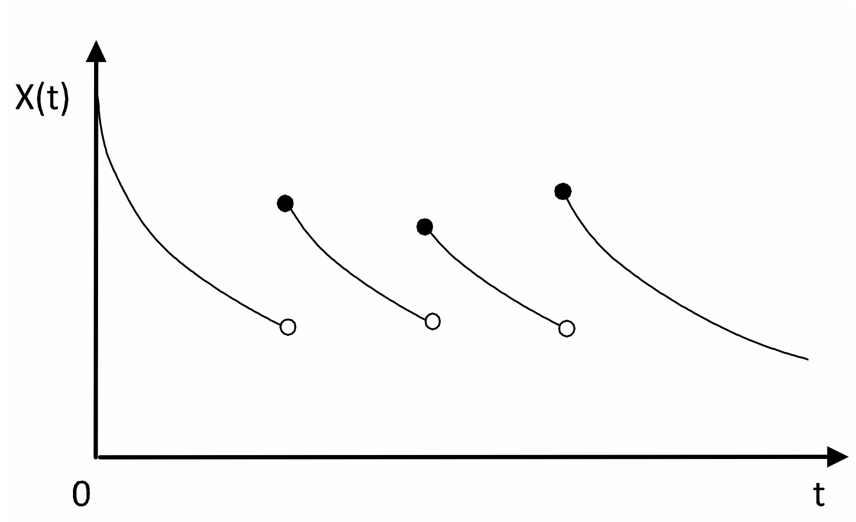

Figure 1 *One possible realization of the stochastic process* $(X(t))_{t \geq 0}$ *in (18)*

4.2.1 The mean (simple approach)

We will now derive the expected value $E_x[X(t)]$ in a rather straightforward way. The linearity of (18) helps in this respect.

[25] As further information, a brief and reader-friendly introduction is Garcia and Griego (1994). Standard texts include Davis (1993), Protter (1995), Privault (2018), Kallenberg(1997) and Liggett (2010). In particular, Liggett (2010, ch. 3) has a very readable introduction to Feller processes.

In a first step, we express the SDE in (18) in its integral version. It reads $X(t) - X(0) = -a\int_0^t X(s)\,ds + b\int_0^t dQ(s)$. When we apply the expectations operator E_x from (12), we get

$$\begin{aligned} E_x[X(t)] - E_x[X(0)] &= -a\int_0^t E_x[X(s)]\,ds + bE_x\left[\int_0^t dQ(s)\right] \\ &= -a\int_0^t E_x[X(s)]\,ds + b\lambda\int_0^t ds. \end{aligned} \tag{19}$$

We can pull the expectations operator inside the integral as the appropriate version of a Fubini theorem holds (see Protter, 2004, p. 207 or Bichteler and Lin, 1995, p. 277, ex. 4.1 for more background). The second equality uses the martingale result of Garcia and Griego (1994, theorem 5.3).

In a second step, we rewrite (19) employing $\mu(x,t)$ from (17) and obtain $\mu(x,t) - \mu(x,0) = -a\int_0^t \mu(x,s)\,ds + b\lambda\int_0^t ds$. Computing the derivative with respect to time t gives

$$\frac{d\mu(x,t)}{dt} \equiv \dot{\mu}(x,t) = -a\mu(x,t) + b\lambda. \tag{20}$$

The Kolmogorov backward equation has thus turned into an ordinary differential equation. It describes the change over time of the expected value of $X(t)$. Expectations are formed from the perspective of the initial point of the process, here 0. The initial condition for $t = 0$ is $\mu(x,0) = x$.

It would then be straightforward to study the properties of this ODE. As one can easily verify, the mean converges to the fixpoint $\mu^* = b\lambda/a$ from above and below.

4.2.2 The mean (generic approach)

We now show how to derive the ODE for the mean in (20) in a way more closely related to section 4.1. The idea consists in using the identity function $h(x) = x$ as a test function and insert it into the corresponding Kolmogorov backward equation.

With $h(x) = x$, $h'(x) = 1$, (16) becomes

$$\frac{d}{dt}E_x[X(t)] = E_x[(\mathcal{A}h)(X(t))] = E_x[-aX(t) + b] = -aE_x[X(t)] + b.$$

Replacing $E_x[X(t)]$ by $\mu(x,t)$ from (17) again, yields (20). Hence, we can either work with the integral version of an SDE and form expectations as in section 4.2 or we use the machinery from section 4.1 to obtain an ODE for the mean of our stochastic process. The second approach also shows why (and when) the first approach works so nicely: We need that for $h(x) = x$, $\mathcal{A}h(x)$ is an affine function of x.

5 Mean dynamics for dynasties, the population and the government

We now apply the results of section 4 to the model from section 3. There is no need to apply our findings to an individual as an individual experiences a deterministic evolution of age and wealth up to death. We therefore study expectations for dynasties, the population and the government.

5.1 Expected age

- A dynasty

Define $\mu(x_i, t)$ as the expected age of the currently alive individual of dynasty i at point in time t given that the initial age at $t = 0$ is known and equals x_i. With definition (17),

$$\mu(x_i, t) = E_x X_i(t) \equiv E[X_i(t) | X_i(0) = x_i]. \tag{21}$$

There are two sources of heterogeneity in age over time. First, individuals differ in initial age x_i. Second, they differ in the realization of death and birth. This is why the mean $\mu(x_i, t)$ has the same functional form for all dynasties. The heterogeneity in initial age is captured by the argument x_i, the heterogeneity in the realization of death and birth is captured by forming the expectation.[26]

Applying (16) to our age process (7) yields a linear ODE describing the evolution of the expected age over time (see app. A.2),

$$\dot{\mu}(x_i, t) = b - \delta\mu(x_i, t). \tag{22}$$

Expected age increases with b and decreases with the death rate δ times the current expected age of an individual. Solving this equation for $t > 0$ yields

$$\mu(x_i, t) = e^{-\delta t}[x_i - \mu^*] + \mu^* \tag{23}$$

where $\mu^* = b/\delta$ is the limit for very large t. Intuitively, when we ask about the expected age of an individual belonging to a dynasty i for very far in the future, current age does not matter. The expected age is larger, the faster the individual grows old (i.e. the larger b) and is smaller, the larger the death rate δ: If individuals are replaced by a newborn with age 0 very frequently, individuals do not have enough time to grow old.

The analytical solution (23) shows that expected age at $t = 0$ is current age and that the impact of initial age on expected age falls over time at the rate of δ. As figure 2 shows, this is true both for initial age above and below the long-run level μ^*.

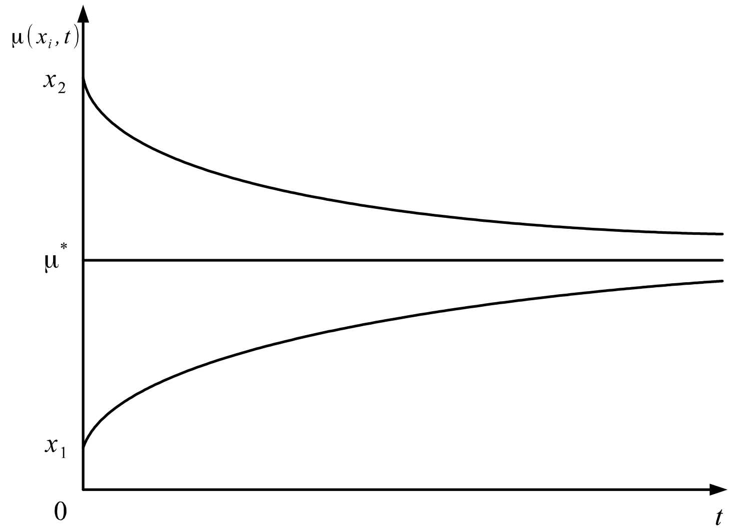

Figure 2 *Evolution of individual mean age over time*

[26] Concerning a generalization briefly discussed in footnote 9, we would write $\mu_i(x_i, t)$ if parameters differed across dynasties. If only death rate differ, one could employ $\mu(x_i, \delta_i, t)$.

Note that a declining expected age in the case of initial age $\mu(x_i, 0)$ above the long-run value μ^*, makes sense. As the term 'perpetual youth' says, expected remaining life-time b/δ is independent of age. Yet, age of an old individual will drop by more when replaced by a newborn compared to age of a young individual. Hence, expected age of any individual above μ^* starts falling as soon as we look into the future.

- The population

Now imagine we study a population that consists of many dynasties. For a given point in time, (individuals representing the) dynasties all have different ages. Average age $\bar{X}(t)$ of the population at some t is

$$\bar{X}(t) \equiv \Sigma_{i=1}^{N} X_i(t)/N. \tag{24}$$

When t lies in the future, $\bar{X}(t)$ is a sum of random variables and thereby a random variable as well. Applying expectations shows that expected average age equals the average of expected individual age, $E\bar{X}(t) = \frac{1}{N}\left(\Sigma_{i=1}^{N}\mu(x_i, t)\right)$.[27] Employing our solution from (23), some simple algebra (available upon request) yields exactly the same structure as for expected individual age

$$E\bar{X}(t) = e^{-\delta t}\left[\Sigma_{i=1}^{N} x_i/N - \mu^*\right] + \mu^*. \tag{25}$$

The only difference consists in the initial condition. Initial age x_i is replaced by average initial age of the population, $\Sigma_{i=1}^{N} x_i/N$. Graphically, average expected age of the population evolves in the same way as displayed in fig. 2 for individual age.

5.2 Expected wealth

- A dynasty

We now return to wealth, our main variable of interest. We define the expected level of dynasty wealth, $\eta(a_i, t)$, as

$$\eta(a_i, t) = E_a(A_i(t)) \equiv E[A_i(t)\,|A_i(0) = a_i]. \tag{26}$$

Following the intuitive description from above, we are at an initial point in time 0, consider an individual with initial age $X_i(0)$ and endow them with initial wealth $A_i(0) = a_i$. The mean $\eta(a_i, t)$ then provides the expected value for individual i with initial wealth a_i at a future point in time t.Following similar steps as for age, we again obtain a linear differential equation (see app. A.3),

$$\dot{\eta}(a_i, t) = \delta\bar{a} + (z - \delta)\,\eta(a_i, t). \tag{27}$$

Expected wealth depends on the death rate δ, on endowment $\bar{a}$ of a newborn and on the growth rate z of individual consumption and wealth from (5). Solving this equation yields

$$\eta(a_i, t) = (a_i - \eta^*)\,e^{(z-\delta)t} + \eta^*, \qquad \text{with } \eta^* \equiv -\frac{\delta}{z-\delta}\bar{a}. \tag{28}$$

The solution shows that expected wealth can rise or fall over time. For the age process, expected age could rise or fall as well. The time path in (23) shows that this depends only on the initial condition: If age is above the long-run value, expected age falls. Otherwise expected age rises. The "convergence parameter" for age is the death rate δ. Here, expected wealth rises or falls independently of the initial condition. The convergence parameter here is $z - \delta = \frac{r-\tau-\rho}{\sigma} - \delta$,

[27] A more precise notation for the expectations operator E would follow (12) and display all initial conditions $\{x_1, x_2, ..., x_N\}$. We stick to this simpler notation here as more involved notation does not promise additional insights.

which can be positive or negative, depending on all those five parameters. When individual wealth increases quickly while alive, i.e. z is large, dynasty wealth will rise as well.

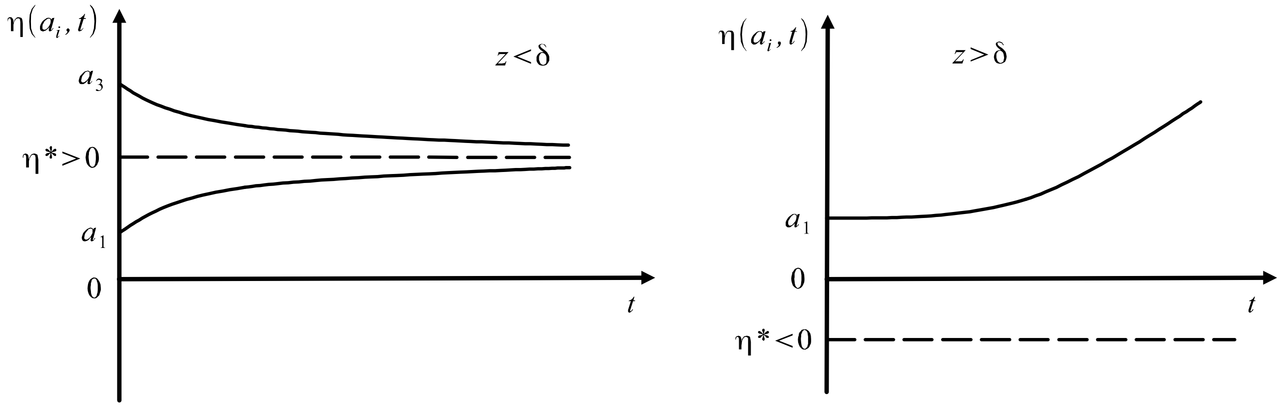

Figure 3 *Expected wealth as a function of parameters z and δ*

When $z < \delta$, the long-run expected wealth level $\eta^* \equiv \frac{\delta}{\delta - z}\bar{a}$ is positive. The initial expected value is a_i which can be of course larger or smaller than η^*. Any initial wealth level converges to η^*, i.e. η^* is a globally stable fix point. This fix point is larger than the initial endowment $\bar{a}$ as long as $\delta / (\delta - z) > 1$ i.e. as long as $z > 0$.

For the empirically more relevant case of $z > \delta$, the stationary level η^* is negative. When the initial condition $a_i = \eta^*$, which would require initial debt, the expected wealth level always stays at η^*. For $a_i < \eta^*$, expected wealth falls to minus infinity, for $a_i > \eta^*$ and thereby for all positive initial wealth levels, expected wealth rises without bound. Hence, η^* is an unstable fix point.

- The population

Average wealth of the population at t is defined as the average over dynasty wealth levels $A_i(t)$,

$$\bar{A}(t) \equiv \Sigma_{i=1}^{N} A_i(t) / N. \tag{29}$$

Once again, the population size N is independent of time and, therefore, constant. As dynasty wealth from (8) is stochastic, we need to form expectations for any point in time $t > 0$ in order to be able to make any model predictions. We obtain

$$E\bar{A}(t) = E\left[\Sigma_{i=1}^{N} A_i(t) / N\right] = \Sigma_{i=1}^{N} E_a [A_i(t)] / N. \tag{30}$$

As we can pull the expectation operator into the sum, we end up with a familiar expression, namely $E_a [A_i(t)]$. As $E_a [A_i(t)] = \eta(a_i, t)$ from (26), the expected population mean equals the mean over expected dynasty means, $E\bar{A}(t) = \Sigma_{i=1}^{N} \eta(a_i, t) / N$. Employing the solution for expected dynasty wealth (28) yields

$$E\bar{A}(t) = \left(\Sigma_{i=1}^{N} a_i / N - \eta^*\right) e^{(z-\delta)t} + \eta^* \tag{31}$$

where η^* is the same expression as defined in (28) for expected dynasty wealth.

Here we need to distinguish the two cases of $z - \delta$ being positive or negative as well. As $z > \delta$ is the empirically relevant case, we focus on this assumption. The long-run average wealth level η^* in our economy is then negative and an unstable fix point. As the initial average wealth $\Sigma_{i=1}^{N} a_i / N$ needs to be positive by empirical relevance, expected average wealth increases at the exponential rate $z - \delta > 0$.

If we assume that population size goes to infinity, i.e. $N \rightarrow \infty$, the variance of average wealth $\bar{A}(t)$ from (29), as a result of the law of large numbers,[28] tends towards 0. Hence, in any practical sense the observation $\bar{A}(t)$ equals the expected value $E\bar{A}(t)$ (plus some small error). We can therefore equate $E\bar{A}(t)$ with empirical data.

5.3 Expected government wealth

The government runs a redistribution scheme based on inheritances. Is this scheme feasible for government wealth? Under which conditions will the government exhibit a balanced budget in the long run? To start answering these questions, we first study the evolution of expected government wealth as an outcome of applying its distribution scheme to one dynasty only. We then aggregate over all dynasties. The full answer will be obtained when we study equilibrium in section 6.1.

- Expected government wealth based on one dynasty

We define expected government wealth following (9) as

$$\gamma(a_i, t) \equiv E\left[G_i\left(A_i\left(X\left(t\right)\right)\right)|A_i\left(X\left(0\right)\right) = a_i\right]. \tag{32}$$

The initial condition a_i is the same as in (26). Following methods from above, the mean $\gamma(a_i, t)$ follows (see app. A.4)

$$\dot{\gamma}(a_i, t, \tau) = -\delta\bar{a} + (\tau + \delta)\,\eta(a_i, t). \tag{33}$$

The dynamics can be easily understood when comparing this ODE with the ODE for expected wealth of a dynasty in (27). Expected wealth of a dynasty rises in δ (via a first channel) as the dynasty receives $\bar{a}$ when an offspring is born. Expected wealth of the government falls in δ as a new offspring is an expenditure for the government. Expected wealth of a dynasty falls in δ (via a second channel) as the household loses expected wealth $\eta(a_i, t)$. By contrast, government wealth rises in this second channel as the government receives this expected wealth. Expected wealth of the household rises at the rate of z, resulting from the optimal consumption decision of the household. Wealth of the government rises at τ as this is the tax rate applied to expected wealth of the dynasty.

While ODEs between the dynasty and government level have very similar interpretations, the solution of (33) looks very different from the solution at the dynasty level. This is not surprising as the right hand side of the government's expected wealth (for this dynasty) contains expected wealth of the dynasty. Hence, (33) is not an autonomous differential equation but needs to be solved by taking the solution of the dynasty budget constraint (28) into account. The solution to (33) reads (see app. A.4.3)

$$\gamma(a_i, t, \tau) = G_{i,0} + ((\tau + \delta)\,\eta^* - \delta\bar{a})\,t + \frac{(\tau + \delta)(a_i - \eta^*)}{z - \delta}\left(e^{(z-\delta)t} - 1\right), \tag{34}$$

where $G_{i,0}$ describes the government wealth at the initial point in time 0 stemming from dynasty i. Given its complexity and differences compared to expected wealth of a dynasty, different interpretations emerge. Equilibrium implications or requirements of (34) will be discussed later.

[28] The summands in $\Sigma_{i=1}^{N} A_i(X_i(t))$ are independent of each other as $A_i(X_i(t))$ is a deterministic function of random age X_i and, given independence of Poisson processes in (7), random variables X_i are independent of each other.

- Expected total government wealth

Similar to (10), expected total wealth of the government is simply the sum over dynasty-specific means, $\Gamma(t,\tau) = \Sigma_{i=1}^{N}\gamma(a_i,t)$. After some steps (see app. A.4.4), we obtain an expression for expected government wealth per capita that reads

$$\frac{\Gamma(t,\tau)}{N} = \frac{G_0}{N} + ((\tau+\delta)\eta^* - \delta\bar{a})\,t + \frac{(\tau+\delta)\left(\bar{A}(0)-\eta^*\right)}{z-\delta}\left(e^{(z-\delta)t}-1\right). \tag{35}$$

Juxtaposing this equation with (34) shows that dynasty initial wealth $G_{i,0}$ is replaced by G_0/N and dynasty initial wealth a_i is replaced by average initial wealth $\bar{A}(0)$ defined as in (29). The interpretation is therefore in perfect analogy to the earlier expression.

We can apply a law of large numbers for per capita government wealth in the same way as we did for average individual wealth $\bar{A}(t)$ from (29). Hence, we can equate expected government wealth per capita $\Gamma(t,\tau)/N$ to observed government wealth in the data.

6 Aggregate and distributional findings

We now characterize equilibrium in our small open economy. Subsequently, we describe distributional properties of age and wealth.

6.1 Steady state and balanced growth path equilibrium

Depending on parameter values, the model ends up in a stationary equilibrium or on a growth path. We say that our economy is in a steady-state equilibrium when both individual variables (e.g. dynasty wealth) and aggregate variables (e.g. government wealth) converge to stationary values. The economy is in a growth equilibrium when individual and aggregate variables converge to a balanced growth path where (most) variables grow at identical rates. Interestingly, distributions can be stationary on a balanced growth path.

6.1.1 Convergence to a general steady-state equilibrium

As the expected wealth analysis, summarized in figure 3, has shown, a partial stationary equilibrium holds if $z < \delta$, i.e. when the rate of wealth growth falls short to the death rate δ. Expected wealth of a dynasty (28) as well as expected average wealth of the population (31) converge to their long-run value η^*, where the latter is described by (28). With constant wealth, consumption and utility are constant as well, where the rate z is treated as exogenous by individuals.

In order to describe the general stationary equilibrium, however, we need to consider the evolution of the government budget as well. For a steady-state equilibrium, we require that government wealth approaches a long-run constant value.

- A stationary government wealth implies an endogenous tax rate

The solution (34) shows that expected government wealth from one dynasty can rise or fall over time. The exponential term clearly shows that one necessary condition for a steady state is $z < \delta$. This makes sure that the final term in (34) approaches a constant. The term linear in time t, by contrast, requires that the term in front of t equals zero at each instant,

$$\tau\eta^* = \delta\left[\bar{a} - \eta^*\right]. \tag{36}$$

This condition needs to be fulfilled in order to describe a steady-state equilibrium as otherwise the state runs a surplus or a deficit in the long run. If η^* is larger in equilibrium than $\bar{a}$ (which holds for $z > 0$ as shown in the discussion of figure 3), this condition suggests a negative tax rate: A positive government income per birth, $\eta^* > \bar{a}$, implies subsidies to capital income, $\tau < 0$.

In order to determine τ, we start from (36) and employ η^* from (28). After some steps (see app. A.4.3), the resulting tax rate reads

$$\tau = \frac{r-\rho}{1-\sigma}. \tag{37}$$

Interestingly, this tax rate is constant over time. As (36) already suggests, a long-run balanced budget only requires that the tax rate obeys long-run values, here, the long-run expected wealth of a dynasty. Short run average wealth of the population or current government wealth do not matter.

As the tax rate has now endogenously been determined, we need to adjust the growth rate z from (5). After some simple steps (see appendix A.5), the revised wealth growth rate reads

$$z = \frac{r-\rho}{\sigma-1}. \tag{38}$$

- Conditions for convergence to a steady-state equilibrium

So far, we obtained two necessary conditions for a steady-state equilibrium. First, steady state requires $z < \delta$ as (i) expected household wealth then approaches a constant and as (ii) the second term of the wealth expression for the government (35) also approaches a constant. Second, steady state requires an endogenous tax rate, the condition for τ in (37). This tax rate makes sure that government wealth approaches a constant in the long run. The endogenous tax rate led to the new expression for z in (38). A steady state for both the household and the government level therefore requires that $z < \delta$ also holds for z from (38).

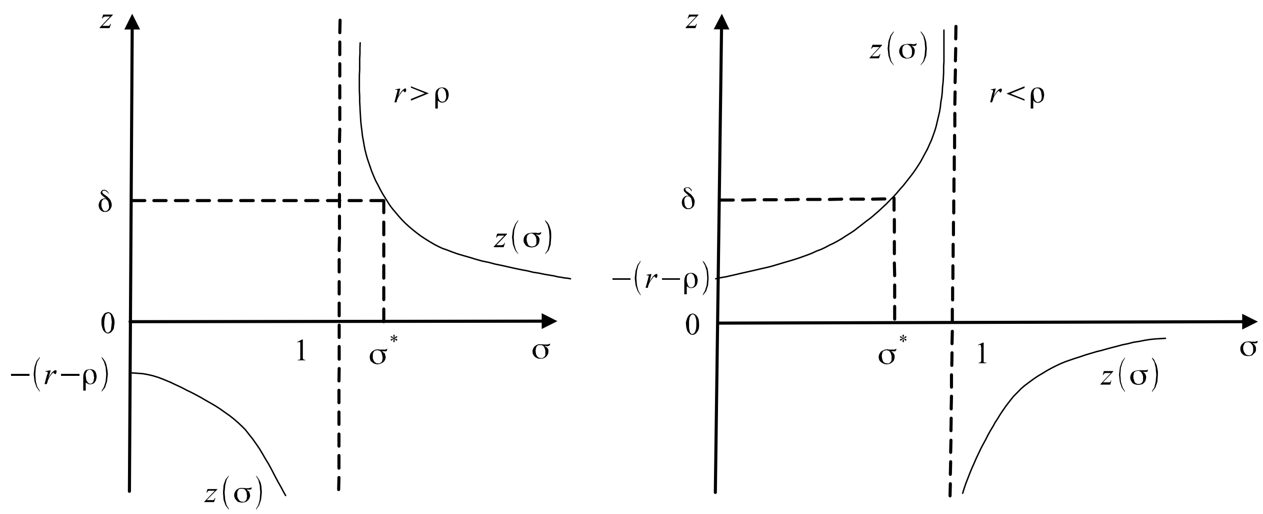

Figure 4 *The steady-state condition for σ for $r > \rho$ (left) and $r < \rho$ (right)*

To understand when $z < \delta$, consider figure 4. It plots z from (38) as a function of σ. The left panel displays the case of $r > \rho$, the right panel of $r < \rho$. There are poles at $\sigma = 1$. We understand when $z < \delta$ by defining a threshold level σ^* that implies $z = \delta$. This threshold level is given by

$$\sigma^* \equiv \frac{r-\rho+\delta}{\delta} \tag{39}$$

and is also shown in both panels for an example of δ.

When $r > \rho$, the threshold level is $\sigma^* > 1$ for all positive δ. There is a steady state (where $z < \delta$ with endogenous τ from (37)) if and only if $\sigma < 1$ or $\sigma > \sigma^*$. For levels inbetween, i.e. for $1 < \sigma < \sigma^*$, the economy is on a growth path. When $r < \rho$ (right panel) and $\delta < -(r-\rho)$ (as *not* drawn in the panel), the threshold level is negative, $\sigma^* < 0$. There is a steady state if and only if $\sigma > 1$. For $\sigma < 1$, the economy is on a growth path. By contrast, when $r < \rho$ and $\delta > -(r-\rho)$ (as drawn in the right panel), the threshold level is between zero and one, $0 < \sigma^* < 1$. There is a steady state if and only if $\sigma < \sigma^*$ or $\sigma > 1$. For $\sigma^* < \sigma < 1$, the economy is on a growth path. These conditions are summarized in the following table.

$r > \rho$		$\sigma < 1$ $\sigma^* < \sigma$	steady state
		$1 < \sigma < \sigma^*$	growth path
$r < \rho$	$\delta < -(r - \rho)$	$\sigma > 1$	steady state
		$\sigma < 1$	growth path
	$\delta > -(r - \rho)$	$\sigma < \sigma^*$ $1 < \sigma$	steady state
		$\sigma^* < \sigma < 1$	growth path

Table 1 *Parameter conditions for steady state and growth path*

- The importance of risk aversion for the equilibrium type

We would like to emphasize the importance of σ in determining the equilibrium type. To the best of our knowledge, risk aversion never played this role in any models of the optimal growth or new growth theory. Risk aversion (in optimal saving rules of the type $\dot{c}/c = (r-\rho)/\sigma$) amplifies the growth rate, but does not have an effect on the sign of the growth rate – whether the economy ends up in a steady state or on a balanced growth path.

This importance of sigma is illustrated in figure 5. The horizontal axis plots the difference between r and ρ, the vertical axis plots risk aversion σ.

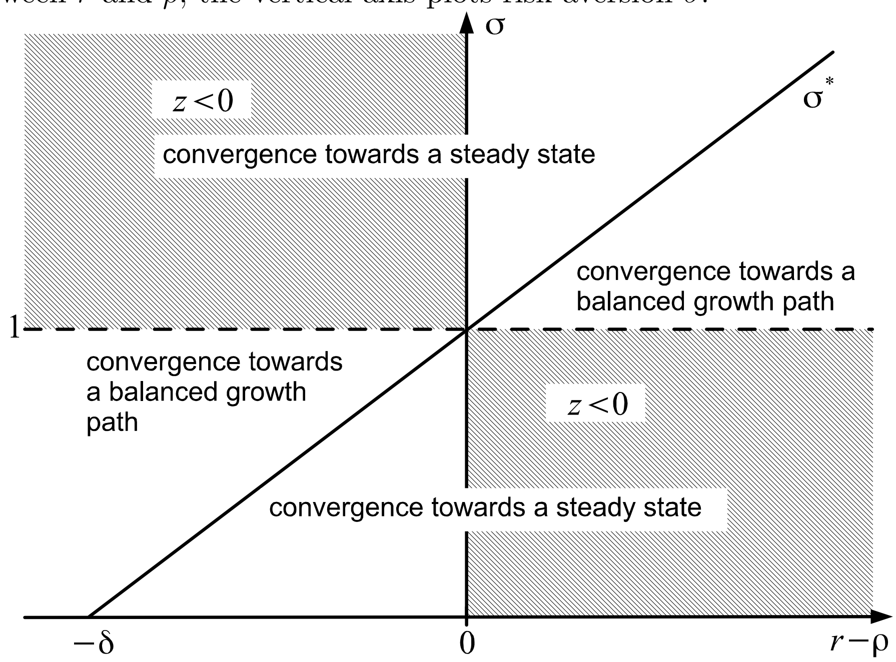

Figure 5 *Steady state and balanced growth path regions*

Consider first the case of a positive difference $r - \rho$. With a risk aversion below 1, the economy ends up in a steady state. As $z < 0$, wealth falls over time. This holds in individual data but not for empirical aggregate averages over the lifetime. This is therefore the empirically less relevant steady state. A risk aversion equal to 1 (see app. A.4.5) or larger than 1 but still below σ^* implies a balanced growth path (provided that (44) holds). When σ rises further, we return to a steady-state economy. For this region of $\sigma > \sigma^*$, z is positive such that expected dynasty wealth $\eta(t, a_i)$ rises and approaches the steady state from below (as illustrated by the trajectory starting at a_1 in the left panel of figure 3).

When $r - \rho$ is negative but still larger than $-\delta$, an increase in σ also moves the economy through three regimes. With risk aversion below σ^*, the economy is in a steady state with positive z. Wealth evolves as just illustrated by $\eta(t, a_1)$ in figure 3. A higher σ brings us to a growth equilibrium and risk aversion above one leads to an empirically non-convincing steady state with $z < 0$.

For $r - \rho < -\delta$, the economy starts (at low σ) in a growth equilibrium. Risk aversion exceeding 1 yields the steady state just described.

Why does σ play this role here? It enters the condition whether $z > \delta$ (growth) or $z < \delta$ (steady state) as a determinant of z. Hence, the reason why σ matters for the equilibrium type is the fact that individual wealth growth and aggregate wealth growth differ. In a model without death, aggregate wealth would grow at z from (5). The sign of this growth rate is independent of σ. In our model with death and birth, the sign of the aggregate wealth growth rate $z - \delta$ is determined by σ. The precise channel through which σ affects aggregate growth or steady state is not z from (5) but z from (38). Hence, it is also the presence of a government budget that we require to be balanced in a steady state that determines the effect of σ on the type of equilibrium.

What is the intuition behind this channel of σ? The parameter should be understood here in its interpretation (of its inverse) as intertemporal elasticity of substitution (and not in terms of risk aversion). We prefer this interpretation as the channel through which σ acts is through its effect on the wealth growth rate z of an individual while alive, i.e. in the absence of any risk.

Generally speaking, the higher $1/\sigma$, the higher the individual (deterministic) growth rate. This is a well-known property and visible here in (5). High needs to be understood in an absolute sense, however. If the numerator in (5) is negative, the growth rate has a high negative number. The sign also plays a crucial role in the expression for the growth rate z from (38) with the tax rate from (37) guaranteeing a balanced government budget. In this version for z, which is relevant for our discussion here, the sign of z depends on the sign of $r - \rho$ and on the level of σ, given the term "-1" in the denominator of (38).

When r exceeds ρ, the growth rate z (38) is negative as long as σ is below 1, i.e. as long as the intertemporal elasticity of substitution (IES) exceeds 1. Hence, $z < \delta$ and the economy is in a steady-state equilibrium. This is the grey lower right area in figure 5. When the IES falls (σ rises further) below 1, the growth rate (38) turns positive. We are in a growth equilibrium. When the IES falls further (σ rises even more), the growth rate z (38) falls. At some point (when σ exceeds σ^*), the growth rate is below the death rate and the economy is back in a steady-state equilibrium.

An interpretation in the same spirit can be given for r falling short of ρ.

- Equilibrium convergence to a steady state

We can now summarize equilibrium dynamics of means in our small open economy. The economy starts with initial wealth levels a_i for dynasties i. Expected dynasty wealth converges to a steady state following (28). Average wealth in our economy follows (31). These paths are illustrated in the left panel of figure 3.

Expected government wealth follows

$$\frac{\Gamma(t,\tau)}{N} = \frac{G_0}{N} + \left(\bar{A}(0) - \eta^*\right)\left(1 - e^{(z-\delta)t}\right). \tag{40}$$

It can rise or fall over time, depending on whether initial wealth $\bar{A}(0)$ of households lies above or below expected wealth η^*. Interestingly, the per-capita government wealth in the steady state, $G_0/N + \bar{A}(0) - \eta^*$, displays two initial conditions. Usually (as e.g. in dynasty wealth (28) or average wealth (31)), initial conditions vanish in the long run. Here, they persist as (initial) government wealth is not directly owned by households and therefore not subject to the death-birth process (as discussed after (9)).

Concerning realized consumption and wealth growth while alive, given the optimal consumption share ϕ from (4) and τ from (37), consumption reads $c(t) = ra(t)$. This illustrates that, if taxes are instantaneously chosen such that the long-term budget is balanced, the optimal, utility-maximizing share of wealth consumed is equal to the gross before-tax interest rate. From the budget constraint (3), wealth therefore falls at the rate of τ. Remember that τ can be positive or negative, depending on parameter values.

- Expected values vs. realizations

It appears useful to be explicit about the difference between an expected evolution and realizations in our model. To this end, figure 6 illustrates expected wealth dynamics vs. realized wealth while alive. Such a distinction helps intuition and is central for relating the model to data. Equilibrium dynamics for expected wealth in a steady-state economy are shown in black in the left panel of figure 6. In addition to figure 3, this panel also shows an example of a realized wealth path in red. The corresponding paths for the growing economy are in the right panel to which we turn later.

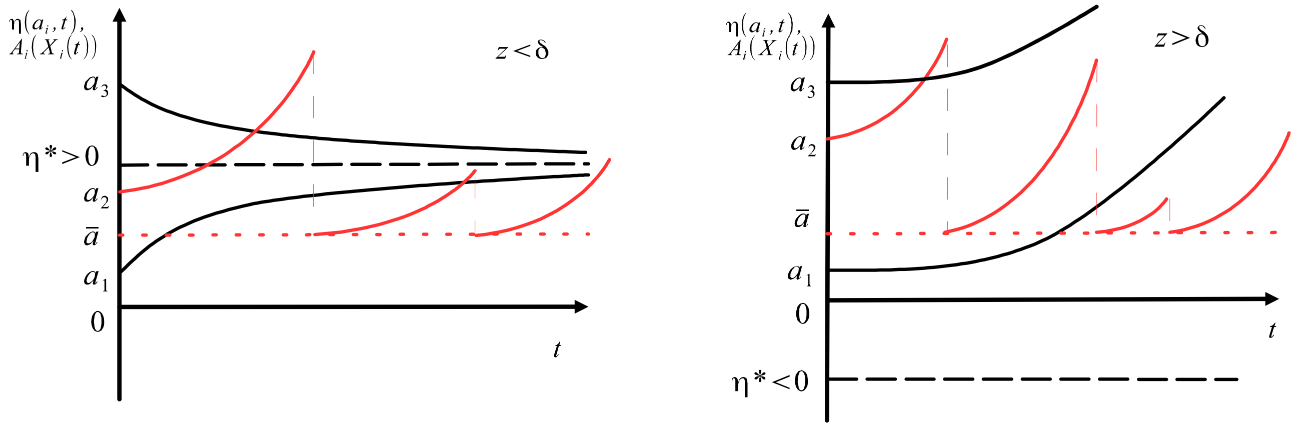

Figure 6 *Realized wealth paths (red) vs. expected wealth paths (black) of a dynasty*

In a steady-state economy with a long-run balanced government budget, the link between expected dynasty wealth η^* and initial endowment $\bar{a}$ from (28) adjusts due to the endogenous tax rate and the implied new accumulation rate z from (38). After some steps (see appendix A.5), expected dynasty wealth reads

$$\eta^* = \frac{(1-\sigma)\delta}{r - \rho + (1-\sigma)\delta}\bar{a}. \tag{41}$$

Obviously, η^* exceeds $\bar{a}$ if $r < \rho$ and falls short of it for $r > \rho$. The left panel shows the case of a $z < \delta$ economy converging towards a steady state with $r < \rho$. When we look at expected dynasty wealth $\eta(a_i, t)$ in black, it approaches η^* irrespective of initial conditions a_1 or a_3. By contrast, when we look at an example of a realized growth path $A_i(t)$ of a dynasty i from (8) in red, it starts at the initial level a_2 and grows at the constant rate z as long as the current representative of the dynasty stays alive. Whenever the individual is replaced by an offspring, wealth jumps to $\bar{a}$. The black curves also represent realized average wealth in the economy as a whole, i.e. $\bar{A}(t)$ from (31).

Empirically, the expected value $\eta(t)$ or $\bar{A}(t)$ should be employed when looking at average wealth data (as some deaths do occur in a large group of individuals). When looking at wealth of one individual, the realized wealth path $a(t)$ should be used as point of reference.

6.1.2 Convergence towards a balanced growth path

Figure 5 shows parameter values for which the economy finds itself on a growth path. On such a path, condition (36) making sure that the government wealth approaches a constant might not be required. A growing economy would not require a constant (expected) government wealth level. It would be enough to think of government wealth (or debt) as staying within a certain range of GDP or overall wealth (think of the Maastricht criteria of the EU).

- Convergence of government wealth to a balanced growth path

In this vein, we divide government wealth (35) per capita by aggregate average wealth (31) and obtain

$$\frac{\Gamma(t,\tau)/N}{E\bar{A}(t)} = \frac{G_0/N + ((\tau+\delta)\eta^* - \delta\bar{a})\,t + \frac{\tau+\delta}{z-\delta}\left(e^{(z-\delta)t} - 1\right)\left(\bar{A}(0) - \eta^*\right)}{\left(\bar{A}(0) - \eta^*\right)e^{(z-\delta)t} + \eta^*}. \tag{42}$$

When we now consider the long run, i.e. $t \to \infty$, both the linear growth expression, $((\tau+\delta)\eta^* - \delta\bar{a})\,t$, and the exponential growth expression, $\left(\bar{A}(0) - \eta^*\right)e^{(z-\delta)t}$, tend towards infinity. As the exponential term grows faster than linear or constant terms, limit arguments yield

$$\lim_{t\to\infty} \frac{\Gamma(t,\tau)/N}{E\bar{A}(t)} = \lim_{t\to\infty} \frac{\frac{G_0/N + ((\tau+\delta)\eta^* - \delta\bar{a})t}{e^{(z-\delta)t}} + \frac{\tau+\delta}{z-\delta}\left[1 - \frac{1}{e^{(z-\delta)t}}\right]\left(\bar{A}(0) - \eta^*\right)}{\bar{A}(0) - \eta^* + \frac{\eta^*}{e^{(z-\delta)t}}} = \frac{\tau+\delta}{z-\delta}. \tag{43}$$

In the long run, government wealth relative to expected average wealth is constant. The government budget is balanced asymptotically even though in absolute terms government wealth features linear and exponential growth. In the long run, government wealth follows the same growth path as dynasty wealth – independently of the tax rate τ.

Even though the government has one additional degree of freedom in a growing economy as compared to an economy that converges to a steady state, the tax is subject to one constraint: It must not be too large such that $z > \delta$ still holds. This is the case as long as the tax does not exceed an upper bound (see app. A.5)

$$z > \delta \Leftrightarrow \tau < \tau^* \equiv r - \rho - \delta\sigma. \tag{44}$$

If it did, individual returns to wealth would fall too much and wealth growth z would become smaller than the death rate. The economy would return to a steady-state equilibrium.[29]

[29] If by accident τ takes on the expression (37), the linear component in (43) is removed. The debt to wealth ratio still converges to the same constant in the limit.

- Equilibrium convergence to a balanced growth path

Equilibrium dynamics in our growing small open economy are as follows. Expected wealth of a dynasty starts from an initial value and grows at a rate $z - \delta$ as described in (28). Population average wealth follows (31). The debt to GDP ratio (42) in our growing economy is potentially non-monotonic over time. It starts at $t = 0$ at $\frac{G_0/N}{\bar{A}(0)}$ and converges to $\frac{\tau+\delta}{z-\delta}$ from (42).

Equilibrium dynamics for expected and realized wealth are shown in the right panel of figure 6. (Remember that (41) only holds for the left panel.) As for the steady-state economy, black curves show expected growth paths for dynasties, $\eta(a_i, t)$, given initial conditions a_1 or a_3. Both grow at the same rate and there is no convergence in expectation. Realized wealth of subsequent representatives of one dynasty starting at initial wealth a_2 are shown by the red curve. Each offspring starts at $\bar{a}$ and experiences higher wealth growth than expected wealth growth. Given that average wealth $\bar{A}(t)$ from (31) is again (as in the steady-state economy) also represented by the black curves, each individual becomes richer over life relative to the population average.

6.2 Distributional dynamics

How do our aggregate equilibrium dynamics square with transitional distributional dynamics of age and wealth? We now present analytical findings on the dynamics of these distributions on the adjustment paths towards a steady state or towards the balanced growth path.[30]

6.2.1 The age distribution

We denote the probability that age X_i of an individual belonging to dynasty i is below a certain age y at time t by

$$\varphi(x_i, B, t) \equiv \Pr(X_i(t) \in B | X_i(0) = x_i) \tag{45}$$

where $B = [0, y] \subset \mathbb{R}$ and x_i is initial age at time 0.

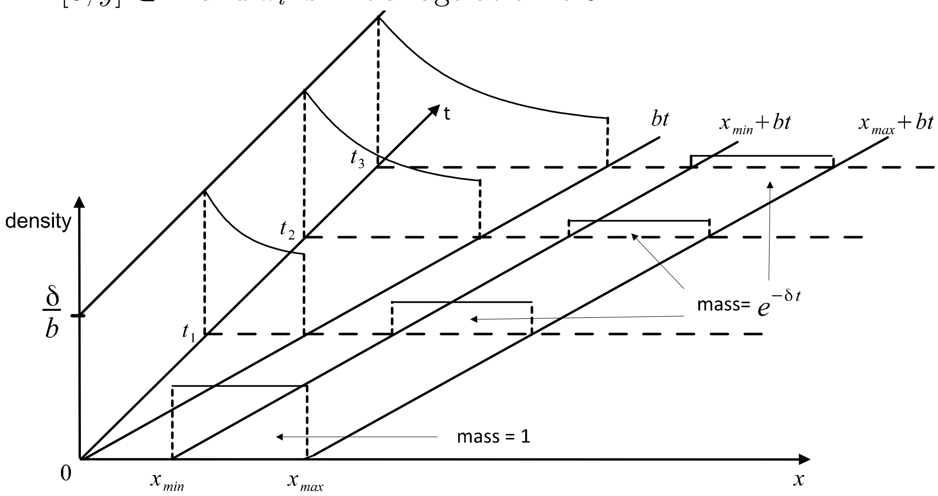

Figure 7 *Age distribution with an initial uniform distribution*

After some steps, we obtain an explicit solution for the age distribution at each point in time. The derivation works in analogy to deriving the wealth distribution in the next section.

[30] We know from more abstract (probability based) work by Bayer et al. (2019) that processes like our age and the related wealth process are characterized by the existence of a unique long-term distribution which is stable. The latter means that for all (meaningful) initial distributions, an initial distribution converges over time to this unique and stable long-term distribution.

Hence, we restrict our presentation to an intuitive explanation here by offering a visualization in figure 7.[31]

As an example, assume an initial uniform distribution of age. The density is horizontal between minimum, $x_{\min}$, and maximum age, $x_{\max}$. It has a probability mass of 1 at $t = 0$. Note that we can allow for any initial distribution.

Over time, some individuals grow older and "stay within" the uniform distribution. Others die and "leave the density" at the rate δ introduced in the age process (7). The boundaries of the uniform distribution increase deterministically over time with slope b, also stemming from (7). The probability mass is given by $e^{-\delta t}$.

When individuals "are reborn" with age 0, probability mass moves from the uniform distribution to the newly emerging distribution. All newborns start at age 0, which thereby constitutes the lower bound of the new (sub-) density. This subdensity features an upper bound bt. This is the age of an individual reborn at $t = 0$ and alive since then. After having moved from the uniform to the new density, individuals continue to be subject to the death-birth process from (7). As the latter is driven by a standard Poisson process, the subdensity between 0 and bt is a truncated exponential density. Its probability mass is $1 - e^{-\delta t}$.

In the very long run, age is exponentially distributed with parameter δ/b. The long-run probability to be of age y or younger is given by the distribution $P(y) = 1 - e^{-\frac{\delta}{b}y}$.

6.2.2 The wealth distribution

We now describe the derivation of the wealth distribution in detail. We first undertake the fundamental analytical steps. Subsequently, we illustrate the dynamics of the wealth density for an initial mass point and for an initial (non-degenerate) distribution.

- Deriving distributional dynamics

Define the probability that realized dynasty wealth $A_i(t)$ from (8) for an initial wealth level of a_i and at a point in time t lies within a certain range or set $B \subset \mathbb{R}$ by

$$\pi(a_i, B, t) \equiv \Pr(A_i(t) \in B | A_i(0) = a_i). \tag{46}$$

We introduce an indicator function $\boldsymbol{I}_A(z) = 1$ if $z \in A$ and zero otherwise.

The essential step in translating this definition into informative expressions consists in solving the SDE (8). Given the framework defined and discussed in section 4.1 and given an initial condition $A_i(0) = a_i \geq 0$, the unique solution (strong and weak solutions coincide in this framework) to (8) reads (see appendix A.6.1)

$$A_i(t) = \boldsymbol{I}_{Q_i^\delta(t)}(0)\, a_i e^{zt} + \left(1 - \boldsymbol{I}_{Q_i^\delta(t)}(0)\right) \bar{a} e^{z(t-T)}, \tag{47}$$

where T marks the most recent point in time before t where a jump of $(Q_i^\delta(s))_{s\geq 0}$ occurred, i.e. where a member of dynasty i deceased for the last time.

The indicator function $\boldsymbol{I}_{Q_i^\delta(t)}(0)$ equals 1 for $Q_i^\delta(t) = 0$ and zero otherwise. The former represents the absence of death: the individual initially representing dynasty i lives on to accumulate wealth based on the initial value of wealth a_i and rate z. The latter describes the opposite, namely an individual being born as a result of the previous individual's death. Wealth initially starts with $\bar{a}$ and then exponentially accumulates over the time span between birth date T and today t at rate z.

[31] A formal derivation, following exactly the same steps as studied in detail below for wealth, is available upon request.

We now specify the set B from (46) as $B = [\bar{a}, x]$. We also assume, to avoid tedious case-by-case analyses, that $a_i > \bar{a}$. We can then rewrite the probability in (46) as $\pi(a_i, x, t) \equiv \Pr(A_i(t) \leq x | A_i(0) = a_i)$. Building on the solution in (47), this probability can be expressed by (see appendix A.6.2)

$$\pi(a_i, x, t) = e^{-\delta t} \boldsymbol{I}_B \left(a_i e^{zt}\right) + \int_B \frac{\delta}{z} \frac{\bar{a}^{\frac{\delta}{z}}}{v^{\frac{\delta}{z}+1}} \boldsymbol{I}_{[\bar{a}, \bar{a}e^{zt}]}(v)\, dv. \tag{48}$$

To understand this expression, consider three ranges for x. Initially, imagine x is small, i.e. $\bar{a} \leq x < \bar{a}e^{zt}$. Then $\boldsymbol{I}_B(a_i e^{zt}) = 0$ and $\boldsymbol{I}_{[\bar{a}, \bar{a}e^{zt}]}(x) = 1$. The probability (48) reads

$$\pi(a_i, x, t) = \int_{\bar{a}}^{x} \frac{\delta}{z} \frac{\bar{a}^{\frac{\delta}{z}}}{v^{\frac{\delta}{z}+1}} dv \text{ for } \bar{a} < x < \bar{a}e^{zt}. \tag{49}$$

In the second range $\bar{a}e^{zt} < x < a_i e^{zt}$, it still holds that $\boldsymbol{I}_B(a_i e^{zt}) = 0$. In addition, the second indicator function is zero, $\boldsymbol{I}_{[\bar{a}, \bar{a}e^{zt}]}(v) = 0$ for all $v > \bar{a}e^{zt}$. Hence, we can replace the general set B by a lower bound $\bar{a}$ and an upper bound $\bar{a}e^{zt}$ such that (48) reads

$$\pi(a_i, x, t) = \int_{\bar{a}}^{\bar{a}e^{zt}} \frac{\delta}{z} \frac{\bar{a}^{\frac{\delta}{z}}}{v^{\frac{\delta}{z}+1}} dv \text{ for } \bar{a}e^{zt} < x < a_i e^{zt}. \tag{50}$$

Note that the integral in (50) is not a function of x but only of time t. For the third range when $x \geq a_i e^{zt}$, the probability that $A_i(t)$ is smaller than x is one, $\pi(a_i, x, t) = 1$.

In simple words, when we are interested in the probability that wealth x is small (the first range), this probability can only come from being reborn. Wealth of an individual that lived as of 0 would be $a_i e^{zt}$ and would be too high. Hence, we only consider the range of wealth from endowment $\bar{a}$ at birth to the wealth level x of interest, as shown in (49). The integrand in (49) is the Pareto density. It follows from the Pareto density in the general expression (48) which in turn is the outcome of a simple parameter substitution (see appendix A.6.2) starting from (47). Why do we see the Pareto density in (49) from an intuitive perspective? First, when we are interested in small wealth levels x, small levels would result from just being reborn. When being young, wealth cannot be much larger than initial endowment $\bar{a}$. Second, we obtain a Pareto distribution in the short run for the same reason that there are Pareto distributions in the long run: There is exponentially distributed age. We can start our analysis of the wealth distribution from any initial age or wealth distribution. As soon as an individual is reborn, however, our age process (7) makes sure that age is back to an exponential distribution. With exponential age distribution, it seems intuitive, that we obtain a (truncated) Pareto distribution for wealth in the transition.

When our wealth level x of interest is a bit larger (second range), we integrate in (50) over the entire range from $\bar{a}$ to $\bar{a}e^{zt}$. When we think about its construction, we integrate over the entire density apart from the probability of not having died. So the integral in (50) equals $1 - e^{-\delta t}$ where $e^{-\delta t}$ is the probability of still being alive at t.

- Illustration for an initial mass point

We have described the distribution function most generally in (48). Figure 8 illustrates this expression for an initial mass point.

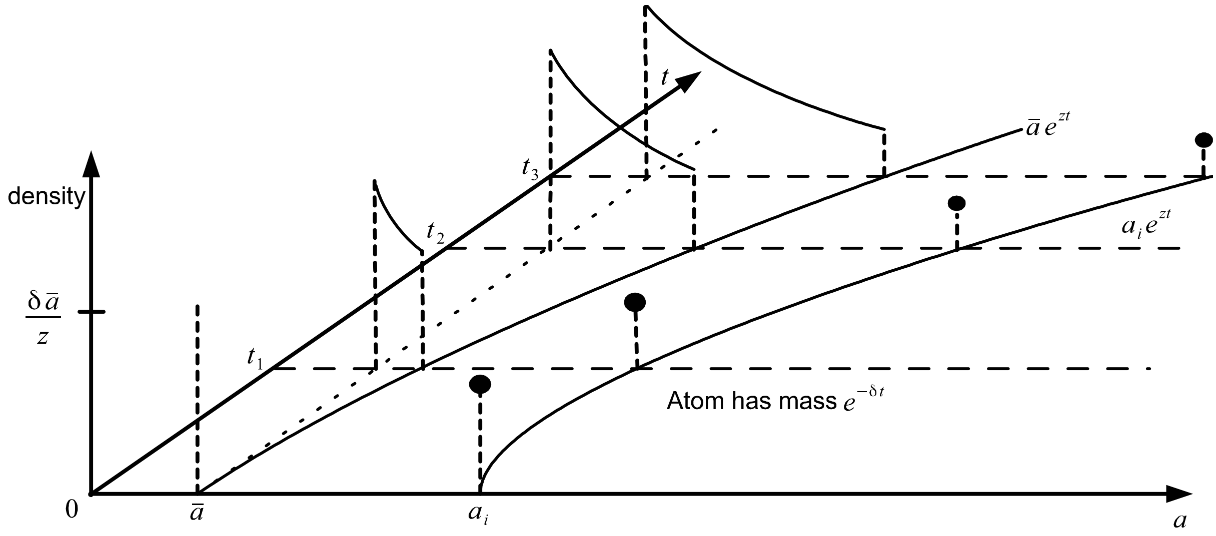

Figure 8 *Dynamics of the wealth density for an initially degenerate distribution*

When we start from an initial condition $a(0) = a_i$, the probability to hold wealth a_i in $t = 0$ equals one. At any point t, the wealth distribution has a probability mass of $e^{-\delta t}$ at $a_i e^{zt}$ where δ is again the death rate from the age process (7). As long as the individuals do not die, they start with a_i and accumulate wealth at the rate of z. The probability to survive until t is given by the probability mass $e^{-\delta t}$.

Now imagine the individual is replaced by an offspring. Wealth jumps to $\bar{a}$. If this jump takes place at $t = 0$, the maximum wealth level that can be reached is $\bar{a}e^{zt}$. Hence, as shown in (49), there is an expanding support $[\bar{a}, \bar{a}e^{zt}]$ within which wealth is (truncated) Pareto distributed with density

$$f(a) = \frac{\delta}{z} \frac{\bar{a}^{\frac{\delta}{z}}}{a^{\frac{\delta}{z}+1}}. \tag{51}$$

As the mass point loses mass over time at rate δ, the truncated Pareto density gains in mass at rate δ. As we assume $a_i > \bar{a}$, the mass point at $a_i e^{zt}$ is always to the right of the upper bound of the Pareto support. In the long run, the mass-point vanishes and the support of the Pareto density is $[\bar{a}, \infty[$.

- Illustration for an initial distribution

Now consider figure 9 where the initial condition is given by an initial distribution. Let the support be given by $[a_i, a^{\max}[$ where $a^{\max}$ could be infinity. We could think of this initial distribution as being Pareto whose parameters follow from a certain tax policy being characterized by $\bar{a}_{\text{old}}$ and a tax rate τ_{old} translating into a z_{old}. This initial distribution has a mass of 1 and reads

$$f(a) = \frac{\delta}{z_{\text{old}}} \frac{\bar{a}_{\text{old}}^{\frac{\delta}{z}}}{a^{\frac{\delta}{z}+1}} \text{ for } a \geq \bar{a}_{\text{old}}.$$

As time goes by, the support of the initial distribution moves to the right and is given by $[\bar{a}_{\text{old}} e^{zt}, a^{\max} e^{zt}[$. Given the death rate δ, the mass of the distribution is $e^{-\delta t}$.

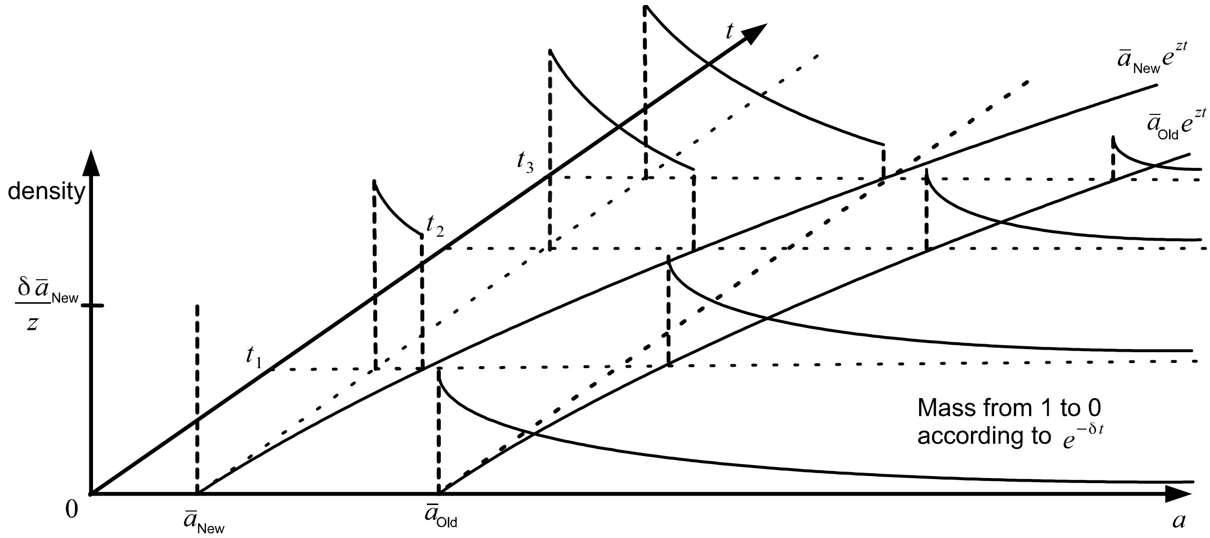

Figure 9 *Dynamics of the wealth density with an initial Pareto distribution of wealth*

For wealth levels $\bar{a} < w < \bar{a}e^{zt}$, the density behaves identically to the case of an initial condition being given by a fixed number. With a new tax policy, the density on the support $\bar{a}_{\text{new}} < a < \bar{a}_{\text{new}}e^{zt}$ is truncated Pareto,

$$f(a) = \frac{\delta}{z_{\text{new}}} \frac{\bar{a}_{\text{new}}^{\frac{\delta}{z}}}{a^{\frac{\delta}{z}+1}} \text{ for } \bar{a}_{\text{new}} < a < \bar{a}_{\text{new}}e^{zt}. \tag{52}$$

In the long run, the complete mass "walked into" the latter density function. This marks the completion of the transition from an old to a new stationary distribution of wealth. As we assume $\bar{a}_{\text{old}} > \bar{a}_{\text{new}}$, the old stationary distribution's lower bound is always to the right of the upper bound. The long-run support of the Pareto density is $[\bar{a}_{\text{new}}, \infty[$.

6.2.3 The link between the distribution and the mean

Let us briefly comment on the link between the distributions and the mean. For our economy converging to a steady state, it is not hard to imagine that mean wealth converging to η^* (left panel in figure 3) goes hand in hand with a density that converges to a stable density (figure 8 or 9). But how does an exploding mean (right panel in figure 3) in our growing economy square with a density that is stationary in the long run?

The answer comes from the property that the Pareto distribution has an undefined mean for $z > \delta$. Computing the mean over a range $[\bar{a}_{\text{new}}, a]$ based on the (truncated) density $f(a)$ from (52), we get (see appendix A.6.3) $\int_{\bar{a}_{\text{new}}}^{a} x f(x)\, dx = \omega \bar{a}_{\text{new}}^{\omega} \left[\frac{a^{1-\omega} - \bar{a}_{\text{new}}^{1-\omega}}{1-\omega} \right]$ where $\omega \equiv \delta / z_{\text{new}}$. This mean approaches infinity when z_{new} approaches δ and a becomes larger and larger. When we look at figure 8, it is clear that for any finite t, we have a finite mean of wealth. Yet, mean wealth grows and approaches infinity as the long-run density is Pareto with $z > \delta$ and a support $[\bar{a}, \infty[$ and therefore an infinite mean. Wealth of the economy grows in expectation, yet the distribution of wealth approaches a stationary distribution.

6.2.4 A note on Fokker-Planck equations

Before concluding, we would like to point out the link to Fokker-Planck equations. Following the usual steps (see the references in the literature section or app. A.7), the FPE for wealth reads

$$\frac{\partial p(A_i, t)}{\partial t} = -(z + \delta)\, p(A_i, t) - z A_i \frac{\partial p(A_i, t)}{\partial A_i}.$$

The analytical solution and its illustration in the above figures lead to three observations. The transition from the original to the new distribution in figure 9 can best be understood by a transfer of probability mass from one distribution to another. The original density is characterized by a uniform loss of density at rate δ across its entire range. This simply means that individuals of each wealth level die at the same rate. It is also characterized by an exponential shift to the right driven by the growth of its lower bound.

The new density is characterized by $\frac{\partial p(A_i,t)}{\partial t} = 0$. The new density gains probability mass by an increasing upper bound, not by an increasing density for any given A_i. Third, the entire density is characterized by non-differentiability (with respect to wealth) at $\bar{a}_{\text{new}} e^{zt}$ and at $\bar{a}_{\text{old}} e^{zt}$.

Approximating this evolution by a numerical solution to the FPE could easily miss these points. We acknowledge that more complex models do not allow for analytical solutions and solving FPEs is the only option to understand model properties. Yet, features of the analytical structures identified here are bound to be present in more complex models as well.

6.2.5 Outlook: Generalizing the tax scheme

Let us briefly return to the issue of extending the SDE on wealth (8) as discussed in footnote 12. The task is considerably more complex than above as can be seen from the solution of the SDE (8) in (47). Our above solution is "simple" as the initial wealth endowment after being reborn is $\bar{a}$, irrespectively of the previous wealth level $A_i(X(t_-))$. Hence, the solution (47) displays one term only in addition to the case of no jump.

When we allow for an inheritance tax lower than 100%, initial wealth after being reborn is a function of wealth of the previous dynasty representative. The generalization of (8) leads to a generalized solution of (47) with a countable but infinite number of terms. These terms consist of multiple integrals. Understanding their property is the objective of future research.

7 Conclusion

This paper studies a small open economy with finitely lived households and an inheritance-based redistribution scheme for wealth. We describe the death-birth process of members of dynasties by a stochastic differential equation. This allows us to describe expected age and wealth of a dynasty by ordinary differential equations. Expected government wealth also follows an ordinary differential equation. By a law of large numbers, per capita wealth and per capita government wealth are deterministic.

The economy approaches either a steady state or a balanced growth path, depending on the interest rate, time preference rate, death rate and risk aversion. Especially the latter is crucial for pinning down equilibrium properties. Requiring a balanced long-run government budget endogenizes the tax rate on wealth for a steady-state economy. When the economy approaches a balanced growth path, the tax rate is a free parameter.

Solving our SDEs for age and dynasty wealth, we can analytically describe the transition of the age and wealth distribution from any initial distribution to its long-run exponential and Pareto distribution, respectively. These transitions are illustrated both for initial degenerate distributions and for initial well-behaved distributions. The effects of a change in fiscal policies are illustrated.

We explain how a balanced growth path at the aggregate level is consistent with a stationary wealth distribution in the long run. The key is the mean of a Pareto distribution that approaches infinity when the shape parameter is smaller than one.

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A Appendix

A.1 Deriving the closed-form solution (4)

We first derive the Keynes-Ramsey rule which reads $\frac{\dot{c}}{c} = \frac{r-\tau-\rho}{\sigma}$. The derivation is standard and is available upon request. We then obtain the closed-form solution by guess and verify.[32]

A.1.1 Guess

Guesses need to be based on the utility function and the Bellman equation of the maximization problem. The value function is defined as $V(a) \equiv \max_{c(t)} U(t)$ subject to the constraint. The Bellman equation reads[33]

$$\rho V(a(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(a(t)) \right\}. \tag{A.1}$$

Computing the expected differential $dV(a)$, given the constraint (3) and using a change-of-variable formula (for a non-technical approach, see Wälde, 2012), yields $dV(a) = V'(a)\{(r-\tau)a - c\}dt$. Since everything is certain in t, we rewrite (A.1) as

$$\rho V(a) = \max_{c} \{u(c) + V'(a)[(r-\tau)a - c]\}. \tag{A.2}$$

The first-order condition then reads

$$u'(c) = V'(a). \tag{A.3}$$

Keeping the expression for instantaneous utility (2) in mind, we start from a guess $J(a)$ for the true value function $V(a)$. The guess, its derivative and a guess for consumption read

$$J(a) = \Gamma_0 a^{1-\sigma} - \Gamma_2, \tag{A.4}$$

$$J'(a) = (1-\sigma)\Gamma_0 a^{-\sigma}, \tag{A.5}$$

$$c = \Gamma_1 a. \tag{A.6}$$

A.1.2 Verification

We have to verify that our guesses satisfy the first-order condition and the Bellman equation. The former, i.e. (A.3), can be rewritten, employing the derivative of our guess (A.5) and the derivative of (2) with respect to c, as $c^{-\sigma} = (1-\sigma)\Gamma_0 a^{-\sigma}$. Rearranging with respect to Γ_0 yields $\Gamma_0 = \frac{1}{1-\sigma}\left(\frac{c}{a}\right)^{-\sigma}$. Replacing c with our guess (A.6) and reallocating leads to

$$\Gamma_1 = [(1-\sigma)\Gamma_0]^{-\frac{1}{\sigma}}, \tag{A.7}$$

which determines Γ_1.

Considering the Bellman equation and inserting our guesses, i.e. (A.4), (A.5) and (A.6), we get

$$\rho\Gamma_0 a^{1-\sigma} - \rho\Gamma_2 = \left(\frac{\Gamma_1^{1-\sigma} a^{1-\sigma} - 1}{1-\sigma}\right) + (1-\sigma)\Gamma_0 a^{-\sigma}[(r-\tau)a - \Gamma_1 a].$$

Replacing Γ_1 and reallocating yields

$$\frac{1}{1-\sigma} - \rho\Gamma_2 = (1-\sigma)^{\frac{-1}{\sigma}} \Gamma_0^{-\frac{1-\sigma}{\sigma}} a^{1-\sigma} + (1-\sigma)\Gamma_0(r-\tau)a^{1-\sigma}$$
$$- (1-\sigma)^{-\frac{1-\sigma}{\sigma}} \Gamma_0^{-\frac{1-\sigma}{\sigma}} a^{1-\sigma} - \rho\Gamma_0 a^{1-\sigma}.$$

[32] An overview of guess and verify approaches is provided by Wälde (2011).

[33] For some formal background, see Sennewald (2007) and the references therein.

This equation holds if, for instance, the left hand side and the right hand side are 0. Hence, the left hand side requires

$$\Gamma_2 = \frac{1}{\rho(1-\sigma)}.$$

For Γ_0 on the right hand side, we get

$$\begin{aligned}
0 &= (1-\sigma)^{\frac{-1}{\sigma}} \Gamma_0^{-\frac{1-\sigma}{\sigma}} a^{1-\sigma} + (1-\sigma)\Gamma_0(r-\tau)a^{1-\sigma} - (1-\sigma)^{-\frac{1-\sigma}{\sigma}} \Gamma_0^{-\frac{1-\sigma}{\sigma}} a^{1-\sigma} - \rho\Gamma_0 a^{1-\sigma} \\
&= \Gamma_0 a^{1-\sigma}\left[(1-\sigma)^{\frac{-1}{\sigma}} \Gamma_0^{-\frac{1}{\sigma}} + (1-\sigma)(r-\tau) - (1-\sigma)^{-\frac{1-\sigma}{\sigma}} \Gamma_0^{-\frac{1}{\sigma}} - \rho\right].
\end{aligned}$$

For an economically meaningful solution, we need

$$(1-\sigma)^{\frac{-1}{\sigma}} \Gamma_0^{-\frac{1}{\sigma}} + (1-\sigma)(r-\tau) - (1-\sigma)^{-\frac{1-\sigma}{\sigma}} \Gamma_0^{-\frac{1}{\sigma}} - \rho = 0.$$

Rewriting yields

$$\Gamma_0 = \frac{1}{1-\sigma}\left[\frac{\sigma}{\rho-(1-\sigma)(r-\tau)}\right]^{\sigma}$$

and therefore, with (A.7),

$$\Gamma_1 = \frac{\rho-(1-\sigma)(r-\tau)}{\sigma}.$$

A.1.3 The closed-form solution

The previous section provides our final solutions for the guesses above

$$\begin{aligned}
J(a) &= \frac{1}{1-\sigma}\left[\frac{\sigma}{\rho-(1-\sigma)(r-\tau)}\right]^{\sigma} a^{1-\sigma} - \frac{1}{\rho[1-\sigma]} \\
J'(a) &= \left[\frac{\sigma}{\rho-(1-\sigma)(r-\tau)}\right]^{\sigma} a^{-\sigma} \\
c &= \left(\frac{\rho-(1-\sigma)(r-\tau)}{\sigma}\right) a.
\end{aligned}$$

This consumption equation is used in the main text.

A.1.4 Deriving a closed-form solution for the logarithmic case ($\sigma = 1$)

- Guess

Keeping the expression for instantaneous utility in mind, we begin again by guessing the value function and the consumption path

$$J(a) = \Gamma_1 \ln(a) - \Gamma_2, \quad J'(a) = \frac{\Gamma_1}{a}, \quad c = \Gamma_3 a.$$

- Verification

We have to verify our guesses satisfying the first-order condition and the Bellman equation. Given our guess, the first-order condition reads

$$\frac{1}{c} = \frac{\Gamma_1}{a}, \quad \frac{1}{\Gamma_3} = \Gamma_1.$$

Considering the Bellman equation and inserting our guesses, we get

$$\rho\Gamma_1 \ln(a) - \rho\Gamma_2 = \ln[\Gamma_3 a] + \frac{\Gamma_1}{a}[(r-\tau)a - \Gamma_3 a] \Leftrightarrow$$
$$\rho\Gamma_1 \ln(a) - \rho\Gamma_2 = \ln(\Gamma_3) + \ln(a) + \Gamma_1[(r-\tau) - \Gamma_3].$$

Reallocating with respect to a and the remaining constant expressions allow us to pin values for Γ_1, Γ_2 and Γ_3 such that both sides of the equation are 0. We obtain

$$\rho\Gamma_1 \ln(a) - \ln(a) = \rho\Gamma_2 + \ln(\Gamma_3) + \Gamma_1[(r-\tau) - \Gamma_3] \Leftrightarrow$$
$$(\rho\Gamma_1 - 1)\ln(a) = \rho\Gamma_2 + \ln(\Gamma_3) + \frac{r-\tau}{\Gamma_3} - 1.$$

In order for the left hand side to be 0, we get $\Gamma_1 = \frac{1}{\rho}$. This leads to $\Gamma_3 = \frac{1}{\Gamma_1} = \rho$. Regarding the right hand side, we get $\Gamma_2 = \frac{1}{\rho} - \frac{\ln(\Gamma_3)}{\rho} - \frac{r-\tau}{\rho\Gamma_3}$. Inserting Γ_3, we get $\Gamma_2 = \frac{1-\ln(\rho)-\frac{r-\tau}{\rho}}{\rho}$.

- Closed-form solution

We obtain

$$J(a) = \frac{1}{\rho}\ln(a) - \frac{1-\ln(\rho)-\frac{r-\tau}{\rho}}{\rho}, \quad J'(a) = \frac{1}{\rho a}, \quad c = \rho a.$$

A.2 Deriving (22) – Evolution of the mean for age

The age process for one individual representing dynasty i is given by (7). In order to determine the evolution of the expected individual age for the given process, we follow the simple approach outlined in 4.2.1. We are allowed to do so since we have shown that both, the generic and the simple approach, will yield the same result.

A.2.1 Deriving the ODE

In a first step, we express the SDE in (7) in its integral version. It reads

$$X_i(t) - X_i(0) = b\int_0^t ds - \int_0^t X_i(s_-)\, dQ_i^\delta(s).$$

When we apply the expectations operator E_x from (21) onto our SDE, we get

$$\begin{aligned} E_x[X_i(t)] - E_x[X_i(0)] &= b\int_0^t ds - E_x \int_0^t X_i(s_-)\, dQ_i^\delta(s) \\ &= b\int_0^t ds - \delta\int_0^t E_x[X_i(s_-)]\, ds. \end{aligned} \tag{A.8}$$

We can pull the expectations operator inside the integral as the necessary properties hold (compare section 4.2). The last line also applies the martingale result of Garcia and Griego (1994, theorem 5.3).

In a second step, we rewrite (A.8) employing $\mu(x_i, t)$ from (21) and obtain

$$\mu(x_i, t) - \mu(x_i, 0) = b\int_0^t ds - \delta\int_0^t \mu(x_i, s)\, ds.$$

Computing the derivative with respect to time t gives

$$\dot{\mu}(x_i, t) = b - \delta\mu(x_i, t), \tag{A.9}$$

which corresponds to (22).

(A.9) describes the timely change of the mean age of an individual representing dynasty i at point in time t. This change is characterized by a constant positive drift b, with which the mean steadily rises. The second term $\delta\mu$ shows the decrease over time. This fraction is supposed to increase as the individual ages further because the older an individual gets, the larger the mean age and ultimately, the stronger the negative impact. As soon as $\delta\mu$ supersedes b, the mean age will decrease over time. In the long run, expected age reads

$$\mu^* \equiv \mu^*(x_i) = \lim_{t\to\infty} \mu(x_i, t) = \frac{b}{\delta}, \tag{A.10}$$

which reflects the unit age increases with b over the arrival of death δ.

A.2.2 Solving the ODE

In order to find a solution for $\mu(x_i, t)$, we guess an expression for which its time derivative yields (22) when inserting the solution candidate. Eventually, the equation that describes the individual expected age is

$$\mu(x_i, t) = e^{-\delta t} x_i + \left(1 - e^{-\delta t}\right)\frac{b}{\delta}, \tag{A.11}$$

which can be viewed as the probability of having reached a certain age, i.e. $e^{-\delta t}$, multiplied by the age at initial point in time, i.e. x_i, and the probability of not having reached a certain age, i.e. $1 - e^{-\delta t}$, multiplied by the expected age in the long run.

A.2.3 Checking the solution

The derivative of (A.11) with respect to t reads

$$\dot{\mu}(x_i, t) = -\delta e^{-\delta t} x_i + b e^{-\delta t} \tag{A.12}$$

$$= -\delta\left[e^{-\delta t} x_i - \frac{b}{\delta} e^{-\delta t} + \frac{b}{\delta}\right] + b$$

$$= -\delta\left[e^{-\delta t} x_i + \left(1 - e^{-\delta t}\right)\frac{b}{\delta}\right] + b \tag{A.13}$$

$$= -\delta\mu(x_i, t) + b, \tag{A.14}$$

where (A.14) is exactly equal to (22), showing that we obtained a perfectly valid solution.

A.3 Deriving (27) – Evolution of the mean of wealth

Given the process introduced in section 3.2.2, we can again apply the simple approach from 4.2.1 and determine the mean and its evolution over time. We employ $A_i(t)$ here, as discussed in footnote 11.

A.3.1 Deriving the ODE

Following the simple approach, we begin with expressing (8) using integrals. We get

$$A_i(t) - A_i(0) = z\int_0^t A_i(s)\, ds - \int_0^t (A_i(s_-) - \bar{a})\, dQ_i^\delta(s).$$

When we apply the expectations operator E_a from (26) onto our SDE, we get

$$\begin{aligned} E_a[A_i(t)] - E_a[A_i(0)] &= zE_a \int_0^t A_i(s)\,ds - E_a \int_0^t (A_i(s_-) - \bar{a})\,dQ_i^\delta(s) \\ &= z \int_0^t E_a[A_i(s)]\,ds - \delta \int_0^t E_a[A_i(s_-)]\,ds + \delta\bar{a} \int_0^t ds. \end{aligned} \tag{A.15}$$

We can pull the expectations operator inside the integral as the necessary properties hold again (compare section 4.2). The last line also applies the martingale result of Garcia and Griego (1994, theorem 5.3). Since $\bar{a}$ is a constant, it does not require expectations to be formed.

In a second step, we rewrite (A.15) employing $\eta(a_i, t)$ from (26) and obtain

$$\eta(a_i, t) - \eta(a_i, 0) = z \int_0^t \eta(a_i, s)\,ds - \delta \int_0^t \eta(a_i, t)\,ds + \delta\bar{a}t.$$

Computing the derivative with respect to time t gives

$$\dot{\eta}(a_i, t) = \delta\bar{a} + (z - \delta)\,\eta(a_i, t). \tag{A.16}$$

This linear ODE describes mean wealth evolving positively with a fraction δ of the starting wealth of a newborn $\bar{a}$ and then either continuing positively or negatively, depending on $z \lesseqgtr \delta$. Setting (A.16) equal to 0 allows to come up with the stationary solution

$$\eta^* \equiv \eta^*(a_i) = -\frac{\delta}{z - \delta}\bar{a} = \frac{1}{1 - \frac{z}{\delta}}\bar{a}. \tag{A.17}$$

Hence, the long-run mean wealth of an individual is a fraction $\frac{1}{1-\frac{z}{\delta}}$ of the wealth newborns are endowed with, i.e. $\bar{a}$. Given $\delta > 0$ by construction, the whole fraction is either negative (in case $\frac{z}{\delta} > 1$), implying the mean of individual wealth to characterize an unstable steady-state value, or the fraction is smaller than 1 (in case $\frac{z}{\delta} < 0$, which occurs when $r < \tau + \phi$) or larger than 1 (in case $\frac{z}{\delta} < 1$) implying a positive mean wealth in the long run.

A.3.2 Solving the ODE

A solution to (A.16) is (see e.g. Wälde, 2012, p.95)

$$\eta(a_i, t) = (a_i - \eta^*)\,e^{(z-\delta)t} + \eta^* \tag{A.18}$$

where we used η^* from (28).

A.3.3 Checking the solution

Computing a time derivative of (A.18) yields

$$\dot{\eta}(a_i, t) = (a_i - \eta^*)\,e^{(z-\delta)t}[z - \delta] = (\eta(a_i, t) - \eta^*)(z - \delta),$$

where the second equality employed (A.18). Rearranging and employing η^* from (28), or (A.17) in the appendix, yields

$$\dot{\eta}(a_i, t) = \eta(a_i, t)[z - \delta] + \delta\bar{a},$$

which is (A.16).

A.4 Deriving (34) – Evolution and determination of expected public wealth

A.4.1 Deriving the ODE for expected public wealth

Even though we could resort to the simple approach again, for illustration purposes we apply the approach from sect. 4.1 to the SDE (9) of government wealth. We calculate the time evolution of expected government wealth and then solve this ODE to obtain an explicit expression for the expected government wealth based on one dynasty.

Calculating the mean starts with setting up the differential of an auxiliary function f. We begin with

$$\begin{aligned} df\left(G_i\left(\zeta\right)\right) &= f'\left(G_i\left(\zeta\right)\right)\tau A_i\left(\zeta\right)d\zeta \\ &\quad + \left\{f\left(G_i\left(\zeta_-\right)+A_i\left(\zeta_-\right)-\bar{a}\right)-f\left(G_i\left(\zeta_-\right)\right)\right\}dQ_i^{\delta}\left(\zeta\right). \end{aligned}$$

If we think of ζ as a future point in time $t+\epsilon$, we can rewrite the just-stated equation in its integral representation

$$\begin{aligned} f\left(G_i\left(t+\epsilon\right)\right) &= f\left(G_i\left(t\right)\right)+\int_t^{t+\epsilon}\tau A_i\left(s\right)f'\left(G_i\left(s\right)\right)ds \\ &\quad +\int_t^{t+\epsilon}f\left(G_i\left(s_-\right)+A_i\left(s_-\right)-\bar{a}\right)-f\left(G_i\left(s_-\right)\right)dQ_i^{\delta}\left(s\right). \end{aligned} \tag{A.19}$$

Forming expectations gives

$$\begin{aligned} E_a\left[f\left(G_i\left(t+\epsilon\right)\right)-f\left(G_i\left(t\right)\right)\right] &= \tau\int_t^{t+\epsilon}E_a\left[A_i\left(s\right)f'\left(G_i\left(s\right)\right)\right]ds \\ &\quad +\delta\int_t^{t+\epsilon}E_a\left[\begin{array}{c} f\left(G_i\left(s_-\right)+A_i\left(s_-\right)-\bar{a}\right) \\ -f\left(G_i\left(s_-\right)\right)\end{array}\right]ds. \end{aligned} \tag{A.20}$$

As G_i entirely depends on dynasty i's wealth, we apply the expectation operator E_a from above. Rewriting leads to

$$\begin{aligned} E_a\left[f\left(G_i\left(t+\epsilon\right)\right)-f\left(G_i\left(t\right)\right)\right] &= \tau\int_t^{t+\epsilon}E_a\left[A_i\left(s\right)f'\left(G_i\left(s\right)\right)\right]ds \\ &\quad +\delta\int_t^{t+\epsilon}E_a\left[\begin{array}{c} f\left(G_i\left(s_-\right)+A_i\left(s_-\right)-\bar{a}\right) \\ -f\left(G_i\left(s_-\right)\right)\end{array}\right]ds. \end{aligned} \tag{A.21}$$

Dividing by ϵ and letting ϵ move towards 0, we obtain

$$\begin{aligned} \lim_{\epsilon\to0}\frac{E_a\left[f\left(G_i\left(t+\epsilon\right)\right)-f\left(G_i\left(t\right)\right)\right]}{\epsilon} &= \tau\lim_{\epsilon\to0}\frac{\int_t^{t+\epsilon}E_a\left[A_i\left(s\right)f\left(G_i\left(s\right)\right)\right]ds}{\epsilon} \\ &\quad +\delta\lim_{\epsilon\to0}\frac{\int_t^{t+\epsilon}E_a\left[\begin{array}{c} f\left(G_i\left(s_-\right)+A_i\left(s_-\right)-\bar{a}\right) \\ -f\left(G_i\left(s_-\right)\right)\end{array}\right]ds}{\epsilon}. \end{aligned} \tag{A.22}$$

Using (26) for E_a, we write

$$\begin{aligned} &\lim_{\epsilon\to0}\frac{E\left[f\left(G_i\left(t+\epsilon\right)\right)|A_i\left(t\right)=a\right]-E\left[f\left(G_i\left(t\right)\right)|A_i\left(t\right)=a\right]}{\epsilon} \\ &=\tau\lim_{\epsilon\to0}\frac{\int_t^{t+\epsilon}E\left[A_i\left(s\right)f'\left(G_i\left(s\right)\right)|A_i\left(t\right)=a\right]ds}{\epsilon} \\ &+\delta\lim_{\epsilon\to0}\frac{\int_t^{t+\epsilon}E\left[f\left(G_i\left(s_-\right)+A_i\left(s_-\right)-\bar{a}\right)-f\left(G_i\left(s_-\right)\right)|A_i\left(t\right)=a\right]ds}{\epsilon}. \end{aligned} \tag{A.23}$$

As the left hand side reflects the infinitesimal generator, we apply its definition (compare section 4.1) and get

$$\mathcal{A}f\left(G_i\left(t\right)\right) = \tau \lim_{\epsilon\to 0} \frac{\int_t^{t+\epsilon} E\left[A_i\left(s\right) f'\left(G_i\left(s\right)\right) | A_i\left(t\right) = a\right] ds}{\epsilon} + \delta \lim_{\epsilon\to 0} \frac{\int_t^{t+\epsilon} E\left[f\left(G_i\left(s_-\right) + A_i\left(s_-\right) - \bar{a}\right) - f\left(G_i\left(s_-\right)\right) | A_i\left(t\right) = a\right] ds}{\epsilon}.$$

In order to take the limit, we have to apply L'Hôspital's rule. If numerator and denominator of both limit expressions move towards 0 or $\pm\infty$, we are allowed to do so. As this is fulfilled, we get

$$\mathcal{A}f\left(G_i\left(t\right)\right) = \tau A_i\left(t\right) f'\left(G_i\left(t\right)\right) + \delta\left(f\left(G_i\left(t\right) + A_i\left(t\right) - \bar{a}\right) - f\left(G_i\left(t\right)\right)\right). \tag{A.24}$$

Specifying $f\left(G_i\left(t\right)\right)$ to be the identity function leads to

$$\mathcal{A}f\left(G_i\left(t\right)\right) = \tau A_i\left(t\right) + \delta\left(A_i\left(t\right) - \bar{a}\right).$$

Knowing that $\frac{d}{dt}E\left[f\left(G_i\left(t\right)\right)\right] = E\left[\left(\mathcal{A}f\right)\left(G_i\left(t\right)\right)\right]$ and using the definition (32) gives

$$\dot{\gamma}\left(a_i, t\right) = E_a\left[\tau A_i\left(t\right) + \delta\left(A_i\left(t\right) - \bar{a}\right)\right], \tag{A.25}$$

which ultimately leads to

$$\dot{\gamma}\left(a_i, t\right) = -\delta\bar{a} + \left(\tau + \delta\right)\eta\left(a_i, t\right). \tag{A.26}$$

This linear ODE describes mean government budget evolving negatively with a fraction δ of the starting wealth of a newborn $\bar{a}$, which describes the expense of the state, and then changes positively according to $\tau + \delta$ multiplied with the expected wealth of a dynasty η.

A.4.2 Solving the ODE

To solve the ODE, we employ the solution for $\eta\left(a_i, t\right)$ from (28), reproduced here for convenience, $\eta\left(a_i, t\right) = \left(a_i - \eta^*\right) e^{(z-\delta)t} + \eta^*$. Plugging this into (A.26) yields

$$\begin{aligned}\dot{\gamma}\left(a_i, t\right) &= -\delta\bar{a} + \left(\tau + \delta\right)\left[\left(a_i - \eta^*\right) e^{(z-\delta)t} + \eta^*\right] \\ &= \left(\tau + \delta\right)\eta^* - \delta\bar{a} + \left(\tau + \delta\right)\left(a_i - \eta^*\right) e^{(z-\delta)t} \\ &\equiv A + Be^{(z-\delta)t}.\end{aligned}$$

The solution to this ODE is

$$\gamma\left(t\right) = \tilde{\gamma} + At + \frac{B}{z-\delta} e^{(z-\delta)t},$$

which can easily be verified by computing the time derivative: $\dot{\gamma}\left(t\right) = A + Be^{(z-\delta)t}$. Hence, with auxiliary parameters being replaced, we have

$$\gamma\left(a_i, t\right) = \tilde{\gamma} + \left(\left(\tau + \delta\right)\eta^* - \delta\bar{a}\right) t + \frac{\left(\tau + \delta\right)\left(a_i - \eta^*\right)}{z-\delta} e^{(z-\delta)t}. \tag{A.27}$$

- The initial value for government wealth

Focusing on the initial value of (A.27), we set $t = 0$ and get

$$\gamma\left(a_i, 0\right) = \tilde{\gamma} + \frac{\left(\tau + \delta\right)\left(a_i - \eta^*\right)}{z-\delta}.$$

When we want $\gamma(a_i, 0)$ to equal a certain initial government wealth level, we can compute $\tilde{\gamma}$. Imagine, we want $\gamma(a_i, 0)$ to equal some government wealth level $G_{i,0}$, where $G_{i,0} \in \mathbb{R}^{\geq 0}$. Then

$$\tilde{\gamma} = G_{i,0} - \frac{(\tau + \delta)(a_i - \eta^*)}{z - \delta}.$$

Thus, the solution (A.27) eventually reads

$$\gamma(a_i, t) = G_{i,0} - \frac{(\tau + \delta)(a_i - \eta^*)}{z - \delta} + ((\tau + \delta)\eta^* - \delta\bar{a})\, t + \frac{(\tau + \delta)(a_i - \eta^*)}{z - \delta} e^{(z-\delta)t}. \tag{A.28}$$

A.4.3 Deriving (37)

In order to achieve a balanced government budget, the government chooses a tax rate τ such that its expected value approaches a constant in the long run. This is what we show first. Additionally, a budget-balancing tax rate shapes the wealth accumulation of the government. This is why, secondly, we concentrate on the expression for expected government wealth from one dynasty, given a budget balancing tax τ.

- The tax rate under a balanced government budget constraint

When we ask about the limit, i.e. $t \to \infty$, we start from (A.28) and we get, given $z < \delta$,

$$\lim_{t\to\infty} \gamma(a_i, t) = G_{i,0} - \frac{(\tau + \delta)(a_i - \eta^*)}{z - \delta} + (\tau\eta^* - \delta[\bar{a} - \eta^*]) \lim_{t\to\infty} t.$$

This limit is plus or minus infinity unless $(\tau + \delta)\eta^* = \delta\bar{a}$. Hence, the latter fixes the tax rate to $\tau = \left(\frac{\bar{a}}{\eta^*} - 1\right)\delta$. Inserting η^* from (28) leads to

$$\tau = \left(\frac{\bar{a}}{-\frac{\delta}{z-\delta}\bar{a}} - 1\right)\delta = \left(\frac{z - \delta}{-\delta} - 1\right)\delta = -z + \delta - \delta = -z. \tag{A.29}$$

Employing z defined in (5), we get

$$\tau = -\frac{r - \tau - \rho}{\sigma} \Leftrightarrow \tau = \frac{r - \rho}{1 - \sigma}. \tag{A.30}$$

- Expected government wealth

Assuming a balanced budget, the solution for mean government wealth (A.28) reads

$$\gamma(a_i, t) = G_{i,0} + \frac{(\tau + \delta)(a_i - \eta^*)}{z - \delta}\left(e^{(z-\delta)t} - 1\right).$$

Inserting τ from (37) in the main text or from (A.30) in the appendix, we get

$$\gamma(a_i, t) = G_{i,0} + (a_i - \eta^*)\left(1 - e^{(z-\delta)t}\right).$$

A.4.4 Aggregate government wealth

Aggregating γ from (34), we write

$$\begin{aligned}\Gamma(t,\tau) &= \Sigma_{i=1}^{N}\gamma(a_i,t)\\ &= \Sigma_{i=1}^{N}\left(G_{i,0}-\frac{(\tau+\delta)(a_i-\eta^*)}{z-\delta}+((\tau+\delta)\eta^*-\delta\bar{a})t+\frac{(\tau+\delta)(a_i-\eta^*)}{z-\delta}e^{(z-\delta)t}\right).\end{aligned}$$

Using $G_0 \equiv \Sigma_{i=1}^{N} G_{i,0}$ as well as $\bar{A}(0) = \Sigma_{i=1}^{N}\frac{a_i}{N}$, where the latter, i.e. $\bar{A}(0)$, stems from (31), we obtain

$$\Gamma(t,\tau) = G_0 + N\left[((\tau+\delta)\eta^*-\delta\bar{a})t+\frac{(\tau+\delta)\left(\bar{A}(0)-\eta^*\right)}{z-\delta}\left(e^{(z-\delta)t}-1\right)\right]. \tag{A.31}$$

The difference between the state's wealth based on one individual dynasty and all dynasties lies in the aggregation of the initial government wealth $G_{i,0}$ as well as N times the linear and exponential evolution of government wealth stemming from one dynasty i. We are allowed to do the latter as the law of motion holds for every dynasty in the same way.

A.4.5 Expected government wealth for $\sigma = 1$

Appendix A.1.4 shows that optimal consumption in the logarithmic case ($\sigma = 1$) reads $c(t) = \rho a(t)$. Given the budget constraint $\dot{a}(t) = (r-\tau)a(t) - c(t)$, wealth evolution now reads $\dot{a}(t) = za(t)$ where

$$z = r - \tau - \rho.$$

We now *assume* that a steady-state equilibrium exists and study its properties. We start by considering the limit for government wealth γ,

$$\lim_{t\to\infty}\gamma(a_i,t) = G_{i,0}-\frac{(\tau+\delta)(a_i-\eta^*)}{z-\delta}+(\tau\eta^*-\delta[\bar{a}-\eta^*])\lim_{t\to\infty}t.$$

We see that the exponential expression drops out due to $z < \delta$. We again require a balanced budget in the long run. We therefore impose

$$\tau\eta^* - \delta[\bar{a}-\eta^*] = 0 \tag{A.32}$$

as in the non-logarithmic case.

We finally want to determine the expression for the tax rate taking the endogeneity of η^* into account. We begin by inserting z into η^* from (28) and the resulting revised η^* into (A.32). We get

$$\begin{aligned}\tau\frac{\delta}{\delta-r+\tau+\rho}\bar{a}-\delta\left(1-\frac{\delta}{\delta-r+\tau+\rho}\right)\bar{a}=0 \Leftrightarrow \frac{(\tau+\delta)\delta}{\delta-r+\tau+\rho}-\delta=0 \Leftrightarrow\\ (\tau+\delta)\delta=(\delta-r+\tau+\rho)\delta \Leftrightarrow 0=(-r+\rho)\delta\end{aligned}$$

where reallocating eventually yields

$$r = \rho. \tag{A.33}$$

This implies that a steady-state equilibrium only exists for this special case. For all other cases, a steady-state equilibrium does not exist. We conclude that the logarithmic case with $r \neq \rho$ implies a growth equilibrium.

A.5 Some simple derivations

- Deriving (38)

We consider (5) and insert (37). This yields

$$z = \frac{r-\tau-\rho}{\sigma} = \frac{r-\frac{r-\rho}{1-\sigma}-\rho}{\sigma} = \frac{(r-\rho)(1-\sigma)-(r-\rho)}{1-\sigma}\frac{1}{\sigma} = \frac{-\sigma(r-\rho)}{1-\sigma}\frac{1}{\sigma} = -\frac{r-\rho}{1-\sigma}.$$

- Deriving (41)

We start from $\eta^* = -\frac{\delta}{z-\delta}\bar{a}$ given in (28). Inserting the updated z from (38), we get

$$\eta^* = -\frac{\delta}{-\frac{r-\rho}{1-\sigma}-\delta}\bar{a} = -\frac{\delta}{\frac{-r+\rho-(1-\sigma)\delta}{1-\sigma}}\bar{a} = -\delta\frac{(1-\sigma)}{-r+\rho-(1-\sigma)\delta}\bar{a} = \frac{\delta(1-\sigma)}{r-\rho+(1-\sigma)\delta}\bar{a}.$$

- Deriving (44)

The growth rate z is described by (5). As we are in the balanced growth path equilibrium, we are interested under which conditions $z > \delta$. We find

$$z > \delta \Leftrightarrow \frac{r-\tau-\rho}{\sigma} > \delta \Leftrightarrow r-\rho-\sigma\delta > \tau \Leftrightarrow \tau < \tau^*,$$

where we defined

$$\tau^* \equiv r-\rho-\sigma\delta.$$

A.6 Dynamics of the wealth distribution

We start from the law of motion for wealth described by the SDE (8), expressed here in its integral version,

$$A_i(t) = a_i + \int_0^t zA_i(s)\,ds + \int_0^t [\bar{a}-A_i(s_-)]\,dQ_i^\delta(s). \tag{A.34}$$

The main part provides the solution of the SDE (8) in (47). We treat this expression in this appendix as a guess and denote it by $\tilde{A}_i(t)$,

$$\tilde{A}_i(t) = \boldsymbol{I}_{Q_i^\delta(t)}(0)\,a_i e^{zt} + \left(1-\boldsymbol{I}_{Q_i^\delta(t)}(0)\right)\bar{a}e^{z(t-T)}. \tag{A.35}$$

The next section proves that this guess satisfies (A.34) and thereby, by definition, is a solution.

A.6.1 Proof for the solution of the SDE

Let us distinguish two cases, $Q_i^\delta(t) = 0$ and $Q_i^\delta(t) > 0$. For $Q_i^\delta(t) = 0$, i.e. the individual has survived from 0 to t, our guess (A.35) reads $\tilde{A}_i(t) = a_i e^{zt}$. The integral version (A.34) reads

$$A_i(t) = a_i + \int_0^t zA_i(s)\,ds.$$

Employing our guess, i.e. replacing $A_i(s)$ with $a_i e^{zs}$, we get

$$A_i(t) = a_i + \int_0^t za_i e^{zs}ds = a_i + a_i\left[e^{zt}-1\right] = a_i e^{zt} = \tilde{A}_i(t).$$

Hence, our guess fulfills (A.34) for $Q_i^\delta(t) = 0$.

In the case of $Q_i^\delta(t) > 0$, we assume jump times T_1 to $T_{Q_i^\delta(t)}$ and set $T_0 = 0$. Employing these points in time, our integral equation (A.34) can be written as[34]

$$A_i(t) = a_i + \sum_{j=1}^{Q_i^\delta(t)} \int_{T_{j-1}}^{T_j} zA_i(s)\,ds + \int_{T_{Q_i^\delta(t)}}^{t} zA_i(s)\,ds + \sum_{j=1}^{Q_i^\delta(t)} \left[\bar{a} - A_i(T_j)\right]. \tag{A.36}$$

To understand this, look at the second term on the right hand side first. The integral $\int_{T_{j-1}}^{T_j} zA_i(s)\,ds$ "sums up" capital income $zA_i(s)$ between birth at T_{j-1} and death at T_j (where T_j is birth of the offspring). The second term therefore represents the sum of all capital income between $T_0 = 0$ and the point in time of the last jump, $T_{Q_i^\delta(t)}$. Employing our guess, i.e. replacing $A_i(s)$ by $a_i e^{zs}$ or by $\bar{a}e^{z[s-T_{j-1}]}$, we can rewrite the second term of (A.36) as

$$\sum_{j=1}^{Q_i^\delta(t)} \int_{T_{j-1}}^{T_j} zA_i(s)\,ds = \int_0^{T_1} za_i e^{zs} ds + \sum_{j=2}^{Q_i^\delta(t)} \int_{T_{j-1}}^{T_j} z\bar{a}e^{z[s-T_{j-1}]} ds. \tag{A.37}$$

The third term of (A.36) simply integrates over capital income between the last jump time $T_{Q_i^\delta(t)}$ and today t. Employing our guess, this term can be replaced by

$$\int_{T_{Q_i^\delta(t)}}^{t} zA_i(s)\,ds = \int_{T_{Q_i^\delta(t)}}^{t} z\bar{a}e^{z\left[s-T_{Q_i^\delta(t)}\right]} ds. \tag{A.38}$$

The $A_i(T_j)$-part of the fourth term of (A.36) can be rewritten, using again our guess for $j = 1$ or $j > 1$, as

$$A_i(T_j) = \left\{ \begin{array}{c} a_i e^{zT_1} \\ \bar{a}e^{z[T_j - T_{j-1}]} \end{array} \right\} \text{ for } j \left\{ \begin{array}{c} = \\ > \end{array} \right\} 1. \tag{A.39}$$

Inserting (A.37), (A.38) and (A.39) into (A.36), we obtain

$$\begin{aligned} A_i(t) = a_i &+ \int_0^{T_1} za_i e^{zs} ds + \sum_{j=2}^{Q_i^\delta(t)} \int_{T_{j-1}}^{T_j} z\bar{a}e^{z[s-T_{j-1}]} ds + \int_{T_{Q_i^\delta(t)}}^{t} z\bar{a}e^{z\left[s-T_{Q_i^\delta(t)}\right]} ds \\ &+ \left(\bar{a} - a_i e^{zT_1}\right) + \sum_{j=2}^{Q_i^\delta(t)} \left[\bar{a} - \bar{a}e^{z[T_j - T_{j-1}]}\right]. \end{aligned}$$

Calculating the integral expressions leads to

$$\begin{aligned} A_i(t) &= a_i + a_i \left[e^{zs}\right]_0^{T_1} + \sum_{j=2}^{Q_i^\delta(t)} \bar{a}\left[e^{z[s-T_{j-1}]}\right]_{T_{j-1}}^{T_j} + \bar{a}\left[e^{z\left[s-T_{Q_i^\delta(t)}\right]}\right]_{T_{Q_i^\delta(t)}}^{t} \\ &\quad + \bar{a} - a_i e^{zT_1} + \sum_{j=2}^{Q_i^\delta(t)} \left[\bar{a} - \bar{a}e^{z[T_j - T_{j-1}]}\right] \\ &= a_i e^{zT_1} + \sum_{j=2}^{Q_i^\delta(t)} \bar{a}\left[e^{z[T_j - T_{j-1}]} - 1\right] + \bar{a}\left[e^{z\left[t-T_{Q_i^\delta(t)}\right]} - 1\right] \\ &\quad + \bar{a} - a_i e^{zT_1} + \sum_{j=2}^{Q_i^\delta(t)} \bar{a}\left[1 - e^{z[T_j - T_{j-1}]}\right] \end{aligned}$$

[34] Technically speaking, we should make a distinction in this expression between T_j and $T_{j(-)}$, an instant before T_j. As this leads to cumbersome notation and does not increase readability, we always write T_j.

$$= \bar{a}\left[e^{z\left[t-T_{Q_i^\delta(t)}\right]}-1\right]+\bar{a}=\bar{a}e^{z\left[t-T_{Q_i^\delta(t)}\right]}.$$

As our guess (A.35) for $Q_i^\delta(t)>0$ and therefore $\boldsymbol{I}_{Q_i^\delta(t)}(0)=0$ reads

$$\tilde{A}_i(t)=\bar{a}e^{z[t-T]}$$

and T in (A.35) is the point in time of the last jump (i.e. it is $T_{Q_i^\delta(t)}$), our guess fullfills (A.34) for $Q_i^\delta(t)>0$ as well. Hence, (A.35) and thereby (47) is a solution to (A.34) and thereby (8).

A.6.2 The probability distribution of $A_i(t)$ – deriving (48)

- The distribution function and an illustration

We denote and define the probability distribution of $A_i(t)$ from (47) by

$$\pi(a_i,B,t)\equiv\Pr(A_i(t)\in B|A_i(0)=a_i).$$

We obtain a more informative expression for this density when starting from the solution of the SDE in (47), which we reproduce here for reference,

$$A_i(t)=\boldsymbol{I}_{Q_i^\delta(t)=0}(0)a_ie^{zt}+\Big(1-\boldsymbol{I}_{Q_i^\delta(t)=0}(0)\Big)\bar{a}e^{z[t-T]}. \tag{A.40}$$

In this equation, $T<t$ is the most recent point in time before t where a jump of $\left(Q_i^\delta(s)\right)_{s\geq 0}$ occurred (we implicitly define $T=0$ when $Q_i^\delta(t)=0$, i.e. in case that no such jump occurred at all). For fixed $t\geq 0$ we have $P\big(Q_i^\delta(t)=0\big)=e^{-\delta t}$. For $Q_i^\delta(t)>0$, by translation invariance and reflection symmetry of the law of a Poisson process, the distribution of $T-t$ is exponential with rate δ conditioned on being smaller than t. Thus, using the formula of total probability, we see from (A.40) that, for any (measurable) set $B\subset\mathbb{R}$, the distribution reads

$$\begin{aligned}\pi(a_i,B,t)&=P\left(Q_i^\delta(t)=0\right)P\left(A_i(t)\in B|Q_i^\delta(t)=0\right)\\&\quad+P\left(Q_i^\delta(t)>0\right)P\left(A_i(t)\in B\,|Q_i^\delta(t)>0\right)\\&=e^{-\delta t}\boldsymbol{I}_B(a_ie^{zt})+(1-e^{-\delta t})\int_0^t\frac{\delta e^{-\delta u}}{1-e^{-\delta t}}\boldsymbol{I}_B(\bar{a}e^{zu})\,du\\&=e^{-\delta t}\boldsymbol{I}_B(a_ie^{zt})+\int_0^t\delta e^{-\delta u}\boldsymbol{I}_B(\bar{a}e^{zu})\,du.\end{aligned} \tag{A.41}$$

To understand this expression, consider some set $B=\left\{A_i(t)|\,w^{\min}<A_i(t)<w\right\}$. Imagine that $a_ie^{zt}>w$. Then the indicator function $\boldsymbol{I}_B(a_ie^{zt})$ equals 0 in the first term of (A.41). In words, when t is so large that in the case of no jump wealth would have grown above w, then the event 'no jump' cannot imply $w^{\min}<A_i(t)<w$. As the event 'no jump' has a probability of $e^{-\delta t}$, this probability does not count towards $(A_i(t)\in B|A_i(0)=a_i)$. The probability of B can only result from the event 'at least one jump'.

When there was at least one jump, the most recent one occurred at some u between 0 and t. Since the last jump, accumulated wealth amounts to $\bar{a}e^{zu}$ as shown in the argument of the indicator function of the integral. Now we define u_w by $\bar{a}e^{zu_w}=w$, i.e. the length in time since the last jump, such that wealth is just not above w and therefore still in B, is u_w. Define further $u_w^{\min}$ by $\bar{a}e^{zu_w^{\min}}=w^{\min}$, i.e. $u_w^{\min}$ is the smallest length in time such that wealth just reaches B. Then, for this example, the probability (A.41) reads

$$\Pr\left(w^{\min}<A_i(t)<w|A_i(0)=a_i\right)=\int_{u_w^{\min}}^{u_w}\delta e^{-\delta u}du.$$

The probability that wealth $A_i(t)$ lies between $w^{\min}$ and w is the "sum" (the integral) over all the probabilities that the last jump occurred between $u_w^{\min}$ and u_w.

Consider a final example where $B = \mathbb{R}$. Then the probability for $A_i(t) \in B$ must obviously equal one. Then $\boldsymbol{I}_B(a_i e^{zt}) = \boldsymbol{I}_B(\bar{a}e^{zu}) = 1$ and the probability reads

$$\pi(a_i, B, t) = e^{-\delta t} + \int_0^t \delta e^{-\delta u} du = e^{-\delta t} - \left[e^{-\delta u}\right]_0^t = 1. \tag{A.42}$$

- Rewriting the distribution function

We now rewrite the distribution function (A.41) such that the link to the Pareto distribution becomes aparent. Define the wealth level v such that $v = \bar{a}e^{zu}$. This implies $u = \frac{1}{z}(\ln v - \ln \bar{a})$ and $du = \frac{1}{zv}dv$. Then, our distribution function reads

$$\begin{aligned}\pi(a_i, B, t) &= e^{-\delta t}\boldsymbol{I}_B\left(a_i e^{zt}\right) + \int_{\bar{a}}^{\bar{a}e^{zt}} \delta e^{-\frac{\delta}{z}(\ln v - \ln \bar{a})}\boldsymbol{I}_B(v)\frac{1}{zv}dv \\ &= e^{-\delta t}\boldsymbol{I}_B\left(a_i e^{zt}\right) + \int_{\bar{a}}^{\bar{a}e^{zt}} \frac{\delta}{z}\frac{\bar{a}^{\frac{\delta}{z}}}{v^{\frac{\delta}{z}+1}}\boldsymbol{I}_B(v)\,dv \\ &= e^{-\delta t}\boldsymbol{I}_B\left(a_i e^{zt}\right) + \int_B \frac{\delta}{z}\frac{\bar{a}^{\frac{\delta}{z}}}{v^{\frac{\delta}{z}+1}}\boldsymbol{I}_{\{\bar{a},\bar{a}e^{zt}\}}(v)\,dv,\end{aligned} \tag{A.43}$$

where the first equality simply exploits the link between time u and wealth v and the second equality rearranges. The third equality swaps lower and upper bound for integration and the condition for the indicator function. This is equation (48) in the main text.

- The stationary distribution

Taking the limit for $t \to \infty$ of (A.43) yields the wealth distribution in the long run. Considering the set $B = \{A_i(t)|\, \bar{a} < A_i(t) < w\}$, we obtain

$$\begin{aligned}\lim_{t\to\infty} \pi(a_i, B, t) &= 0 + \int_{\bar{a}}^{\infty} \frac{\delta}{z}\frac{\bar{a}^{\frac{\delta}{z}}}{v^{\frac{\delta}{z}+1}}\boldsymbol{I}_B(v)\,dv = \int_{\bar{a}}^{w} \frac{\delta}{z}\frac{\bar{a}^{\frac{\delta}{z}}}{v^{\frac{\delta}{z}+1}}dv + \int_{w}^{\infty} \frac{\delta}{z}\frac{\bar{a}^{\frac{\delta}{z}}}{v^{\frac{\delta}{z}+1}}\boldsymbol{I}_B(v)\,dv \\ &= \frac{\delta}{z}\bar{a}^{\frac{\delta}{z}}\frac{\left[v^{-\frac{\delta}{z}}\right]_{\bar{a}}^{w}}{-\frac{\delta}{z}} + 0 = -\bar{a}^{\frac{\delta}{z}}\left[w^{-\frac{\delta}{z}} - \bar{a}^{-\frac{\delta}{z}}\right] = 1 - \left(\frac{\bar{a}}{w}\right)^{\frac{\delta}{z}}.\end{aligned}$$

A.6.3 The mean of the truncated Pareto distribution

Computing the mean over range $[\bar{a}_{\text{new}}, a]$ employing the truncated Pareto density from (52) yields

$$\begin{aligned}\int_{\bar{a}_{\text{new}}}^{a} xf(x)\,dx &= \frac{\delta}{z_{\text{new}}}\bar{a}_{\text{new}}^{\frac{\delta}{z}}\int_{\bar{a}_{\text{new}}}^{a} x^{-\frac{\delta}{z_{\text{new}}}}dx = \frac{\delta}{z_{\text{new}}}\bar{a}_{\text{new}}^{\frac{\delta}{z}}\left[\frac{x^{1-\frac{\delta}{z_{\text{new}}}}}{1-\frac{\delta}{z_{\text{new}}}}\right]_{\bar{a}_{\text{new}}}^{a} \\ &= \frac{\delta}{z_{\text{new}}}\bar{a}_{\text{new}}^{\frac{\delta}{z_{\text{new}}}}\left(\frac{a^{1-\frac{\delta}{z_{\text{new}}}} - \bar{a}_{\text{new}}^{1-\frac{\delta}{z_{\text{new}}}}}{1-\frac{\delta}{z}}\right) = \omega\bar{a}_{\text{new}}^{\omega}\left[\frac{a^{1-\omega} - \bar{a}_{\text{new}}^{1-\omega}}{1-\omega}\right],\end{aligned}$$

where the last equality employed

$$\omega \equiv \frac{\delta}{z_{\text{new}}}.$$

When we study the limit for $\omega \to 1$, we see that $\frac{a^{1-\omega}-\bar{a}_{\text{new}}^{1-\omega}}{1-\omega}$ approaches zero in numerator and denominator. Employing L'Hôspital's rule, we get

$$\lim_{\omega\to 1} \frac{a^{1-\omega} - \bar{a}_{\text{new}}^{1-\omega}}{1-\omega} = \lim_{\omega\to 1} \frac{e^{\ln(a)(1-\omega)}\left(-\ln(a)\right) - e^{\ln(\bar{a}_{\text{new}})(1-\omega)}\left(-\ln\left(\bar{a}_{\text{new}}\right)\right)}{-1}$$
$$= \lim_{\omega\to 1} e^{\ln(a)(1-\omega)} \ln(a) - \lim_{\omega\to 1} e^{\ln(\bar{a}_{\text{new}})(1-\omega)} \ln\left(\bar{a}_{\text{new}}\right)$$
$$= \ln(a) - \ln\left(\bar{a}_{\text{new}}\right).$$

This obviously approaches infinity when a approaches infinity.

A.7 The FPE for the wealth distribution

We start from the law of motion for wealth in (8). To simplify notation, we abbreviate $A_i(t)$ by A_i here and start from

$$dA_i = zA_i dt + [\bar{a} - A_i] dQ_i^{\delta}(t). \tag{A.44}$$

Being in t, consider the probability that some realization A_i of wealth is smaller than some a at a future point in time $t+h$ for a small $h > 0$. We denote this probability by $P(a, t+h)$. Most generally speaking, a can be larger or smaller than $\bar{a}$. As wealth is larger than $\bar{a}$ after rebirth, we focus on $a > \bar{a} > 0$.[35] We can write

$$P(a, t+h) = e^{-\delta h}\left[P(a,t) - \int_{(1-zh)a}^{a} p(y,t)\, dy\right] + \left(1 - e^{-\delta h}\right)\left[P(a,t) + 1 - P(a,t)\right]. \tag{A.45}$$

When there is no jump over the period of lenght h, for which the probability is $e^{-\delta h}$, the probability mass in $t+h$ below a is given by the current mass $P(a,t)$ minus the outflow from the range $a - zah = (1-zh)a$ to a. This is the first term on the right-hand side. As h is very small, there is either no jump or one jump. In the case of a jump, occuring with probability $1 - e^{-\delta h}$, the mass in $t+h$ is the current mass $P(a,t)$ increased by everything above a, i.e. $1 - P(a,t)$. The resulting probability is 1. Simply speaking, when there is a jump, all wealth drops to $\bar{a}$. As we assumed $a > \bar{a} > 0$, $P(a, t+h) = 1$ conditional on a jump between t and $t+h$.

Approximating the exponential functions $e^{-\delta h}$ by $1 - \delta h$ yields

$$P(a, t+h) = (1 - \delta h)\left[P(a,t) - \int_{(1-zh)a}^{a(t)} p(y,t)\, dy\right] + \delta h.$$

Subtracting $P(a,t)$ and rearranging, we get

$$P(a, t+h) - P(a,t) = (\delta h - 1)\int_{(1-zh)a}^{a} p(y,t)\, dy - \delta h P(a,t) + \delta h.$$

Dividing by h yields

$$\frac{P(a,t+h) - P(a,t)}{h} = (\delta h - 1)\frac{\int_{(1-zh)a}^{a} p(y,t)\, dy}{h} - \delta P(a,t) + \delta.$$

Letting h become small, we get

$$\frac{\partial P(a,t)}{\partial t} = -zap(a,t) + \delta\left(1 - P(a,t)\right). \tag{A.49}$$

[35] When assuming $\bar{a} > a > 0$, the derivation is slightly different. The final Fokker-Planck equation is the same as for $a > \bar{a} > 0$. The derivation is available upon request.

Note that the last step employed L'Hospital's rule stating, given well-known conditions,

$$\lim_{h\to 0}\frac{f(x)}{g(x)}=\lim_{h\to 0}\frac{f'(x)}{g'(x)}.$$

Setting $f(x)\equiv\int_{(1-zh)a}^{a}p(y,t)\,dy$ and $g(x)\equiv h$, we get

$$\lim_{h\to 0}\frac{f'(x)}{g'(x)}=\frac{zap(a,t)}{1}=zap(a,t),$$

where we used the Leibniz rule to determine the derivative of $f(x)$.

Now compute the derivative of (A.49) with respect to a and get

$$\frac{\partial p(a,t)}{\partial t}=-zp(a,t)-za\frac{\partial p(a,t)}{\partial a}-\delta p(a,t).$$

Hence, the Fokker-Planck equation describing the evolution of the density of wealth over time reads

$$\frac{\partial p(a,t)}{\partial t}=-(z+\delta)p(a,t)-za\frac{\partial p(a,t)}{\partial a}.$$