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Cross-Dynastic Intergenerational Altruism

Frikk Nesje

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Frikk Nesje[1]

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Abstract

I study whether saving behavior reveals socially relevant intertemporal preferences. To this end, I decompose the present generation's preference for the next into its dynastic and cross-dynastic components in a model of saving. If people are concerned about the next generation as such, then they might assign welfare weights on other dynasties. With such cross-dynastic intergenerational altruism, saving for one's descendants benefits present members of other dynasties. These preference externalities imply that socially relevant intertemporal preferences cannot be inferred from saving behavior. Numerically, I show that even "small" preferences for the next generation as such can lower the efficient discount rate by 20% to 40%, as compared to Nordhaus' calibration.

Keywords: Intergenerational altruism, social discounting, time-inconsistency, declining discount rates, generalized consumption Euler equations, interdependent utility, isolation paradox.

JEL Classification: D64, D71, H43, Q01, Q54.

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[1]Department of Economics, University of Copenhagen, Øster Farimagsgade 5, Building 26, 1353 Copenhagen K, Denmark. E-mail: frikk.nesje@econ.ku.dk.

1 Introduction

Intergenerational altruism plays a role in many economic situations, including bequests from parents to their descendants (Ray, 1987; Galperti and Strulovici, 2017). While studies of such situations typically model intergenerational altruism as links between parents and their own descendants (following Barro, 1974), surveys (Cropper et al., 1991, 1992, 1994; Johanneson and Johansson, 1997; Frederick, 2003) and experiments (Chermak and Krause, 2002; Fischer et al., 2004; Hauser et al., 2014; Fehr-Duda and Fehr, 2016; Molina et al., 2018) reveal a broader concern for future generations. This means that more involved networks of altruistic links should be analyzed.

It is natural to consider parents caring about their own descendants, and to recursively extend this preference assuming that thoughtful parents know that their descendants care about their descendants (Barro, 1974). This leads to altruistic links for future generations inside each family, in line with the typical models of intergenerational altruism (for generalizations, see Phelps and Pollak, 1968; Sáez-Marti and Weibull, 2005; Galperti and Strulovici, 2017). However, these parents may also care about the descendants of other parents for reasons such as sustainability. If such preferences also describe how their own descendants and the descendants of other parents care about the future, thoughtful parents take future generations across all families into account, leading to an infinite chain of concerns.

I formalize such broader concern for future generations recursively and study the long term implications. The particular application is to distribution of resources between generations, respecting the sacrifice that the present generation is willing to make for future generations (Goulder and Williams, 2012; Kelleher, 2017). Characterizing such distributions relates to the quantification of the discount rate, and is of high policy relevance (Stern, 2007; Nordhaus, 2007; Drupp et al., 2018). Critically, the distribution depends on the relative weight assigned to the next generation by the present generation. These weights in turn, result from intergenerational altruism and are essential for how to elicit how the present generation trades off its own consumption against the interests of future generations. Climate policy, for example, must balance the mitigation costs incurred by the present generation against the benefits from a stable climate that accrue for future generations (Kolstad et al., 2014).

Economists frequently impute the weight on future generations from returns in the market on either corporate capital, equities or bonds, depending on project maturity and risk profile (Arrow et al., 1995; Gollier, 2012). Hence, the discount rate is implied by saving behavior (e.g., Nordhaus, 2008). However, the result that the weight on future generations can be quantified through saving behavior is obtained in the traditional model of altruism only with regard to own descendants. Indeed, in that model, the return to saving reflects the importance of future utility for the present generation.

It is a conclusion of the present analysis that calibration from saving behavior might not reveal the sacrifice that the present generation is willing to make if altruism exists for the descendants of others. With such preferences, saving for one's descendants benefits others in the present generation, thus giving rise to preference externalities. Hence, the return to saving no longer reflects the importance of future utility for the present generation. This means that even "small" preferences for the next generation as such, when combined, may imply a "large" preference that is not captured by the market. This discrepancy is the focus of the present paper, and it leads to a number of departures from the traditional model – including a lowering of the discount rate.

The analysis is based on a stationary infinite horizon model. Generations are non-overlapping and they live for one period only. There is a finite number of dynasties, interpreted as parallel families or social groups. The welfare of the present generation depends on their own utility and the welfare of the next generation across dynasties. The recursive formulation allows for a novel decomposition of intergenerational altruism into its dynastic and cross-dynastic components. Dynastic intergenerational altruism gives the own dynasty welfare weights (Barro, 1974), while cross-dynastic intergenerational altruism gives the welfare weights on all other dynasties.

The question whether altruism for the next generation is reflected by the market is posed by considering a game of saving for one's own immediate descendants. The game is a tractable model in which cross-dynastic intergenerational altruism can be studied analytically. The analysis shows the existence of a stationary Markov-perfect equilibrium in linear strategies with an inefficiently low saving rate. The equilibrium saving rate in this equilibrium increases in intergenerational altruism, both within and between dynasties. Keeping the total level of intergenerational altruism fixed, the equilibrium saving rate decreases in the number of dynasties. Assuming that the altruistic weight on each of the other dynasties goes to zero in the limiting case when the number of dynasties goes to infinity, the saving rate reduces to the rate without cross-dynastic intergenerational altruism. This means that the wedge between the observed saving rate and the efficient saving rate is maximized when the number of dynasties goes to infinity.

In contrast, dynasties choose the efficient saving rate if they cooperate, thus capturing in any generation their total altruism for the next generation. The analysis shows that the efficient saving rate increases in intergenerational altruism, both within and between dynasties. The wedge between the efficient and equilibrium saving rates measures the externality problem. This wedge is positive if there is cross-dynastic intergenerational altruism. Moreover, this result is qualitatively robust even with intragenerational altruism, as long as the weight on the utility of the present generation, as compared to the weight on the utility of the next generation, is higher for the own dynasty than the other dynasties. There is strong empirical support for a smaller weight on the other dynasties in this generation than the own dynasty (Bernhard et al., 2006 and references therein; Schelling, 1995).

The wedge between the efficient and equilibrium saving rates can also be established by deriving discount functions. This derivation shows that the external effect of present saving becomes less important over time and vanishes only in the limit. Cross-dynastic intergenerational altruism thus leads to different discount functions in equilibrium as compared to those obtained under under efficiency. In general, the discount rates in equilibrium and under efficiency converge only in the limit as the distance between the time periods compared and the time of evaluation goes to infinity. This means that a dynasty's discount rate is smaller for long term projects, leading to a time-inconsistency problem unless the dynasties cooperate.

The wedge further offers a means for adjusting the discount rate. Accounting for cross-dynastic intergenerational altruism beyond what is reflected by saving behavior translates into an increase in the relative weight on future generations. Nordhaus (2008) offers an influential market-based calibration. Respecting the distribution that would arise following the preference of the present generation (thereby retaining Nordhaus' setting but abstracting away from crowding out of saving), cross-dynastic intergenerational altruism of only 10% and 20% beyond the level of intergenerational altruism inferred from saving behavior imply utility discount rates of 1.2% and 0.9%, compared to the Nordhaus rate of 1.5%. The immediate implication for policy is thus that discount rates inferred from saving behavior should be lowered. The extent of this adjustment depends on the degree of cross-dynastic intergenerational altruism. Even if cross-dynastic intergenerational altruism cannot be inferred from saving behavior, it nevertheless plays an important normative role. Based on the numerical example, the weight on utility 100 years from now increases with 34% and 81%, respectively.

The paper proceeds as follows. Section 2 presents an informal motivating example clarifying how the preference externalities are generated. Section 3 presents the model. Section 4 derives the main results in the context of a wedge between the equilibrium and efficient saving rates and explains the contributions of the paper in this regard. In short, cross-dynastic intergenerational altruism gives a new preference-based justification for why saving behavior may not reveal socially relevant intertemporal preferences (without giving the future more weight than what follows from the interests of the present generation, as in Caplin and Leahy, 2004). Section 5 establishes how the main results relate to time-inconsistency. Since the external effects of present saving weakens over time, cross-dynastic intergenerational altruism serves as a new microfoundation for declining discount rates in equilibrium (different from Phelps and Pollak, 1968; Sáez-Marti and Weibull, 2005; Galperti and Strulovici, 2017). Furthermore, the preference formulation permits the saving rates to be derived from generalized consumption Euler equations (Hiraguchi, 2014; Iverson and Karp, 2021; Laibson, 1998). Section 6 establishes how the main results relate to interdependent utility and explains the contributions of the paper in this regard. In short, preference externalities arise in a stationary infinite horizon setting as long as the relative weight on the utility of the present and next generations is strictly larger for the own dynasty than the other dynasties (generalizing and extending Sen 1961, 1967; Marglin, 1963). Section 7 concludes the paper.

The Appendix contains proofs of theorems. The Online Appendix contains additional proofs of results, an interpretation of the model if descendants can move or marry someone from other dynasties (building on Bernheim and Bagwell, 1988; Laitner, 1991; Zhang, 1994; Myles, 1997), and a numerical exercise illustrating the policy implications (building on Nordhaus, 2008).

2 Motivating example

Define $\alpha \in (0, 1)$ as any generation's altruism for the next generation. Generation 0 thus assigns wight α to the next generation so that $W_0 = (1 - \alpha)u_0 + \alpha W_1$, where W, u and subscript refer to welfare, utility and generation. However, any future generation t will do this in turn: $W_t = (1-\alpha)u_t + \alpha W_{t+1}$. This leads to the following relative weights on $(u_0, u_1, u_2, \dots)$ from the perspective of generation 0: $1 - \alpha,\ (1 - \alpha)\alpha,\ (1 - \alpha)\alpha^2,\ \dots$, which is proportional to (and in line with Samuelson, 1937):

$$1,\ \alpha,\ \alpha^2,\ \dots . \tag{1}$$

This preference is stationary so that time-consistency follows from time-invariance.

Suppose now that there are two dynasties, where the present generation of any dynasty assigns total weight α to the two dynasties in the next generation. This weight consists of two parts, α_D and α_C, where α_D is dynastic intergenerational altruism and α_C is cross-dynastic intergenerational altruism. As extreme cases, any dynasty might care only for its own descendants: $\alpha_D = \alpha$ (Barro, 1974) or equally for all descendants: $\alpha_D = \alpha_C = \alpha/2$. It is natural to assume that $\alpha_D \geq \alpha_C \geq 0$ (e.g., Myles, 1997); that is, a dynasty cares weakly more for its own descendants.

Consider the network of preference links when $\alpha_C > 0$. Figure 1 illustrates the preference of generation 0 in dynasty 1, with positive weights on utilities in both dynasties. The weights on the utilities of generation 2 follow from accounting for the total number of dynastic and cross-dynastic altruistic links forward in time. To illustrate, $\alpha_D^2 + \alpha_C^2$ in Figure 1 follows since generation 0 in dynasty 1

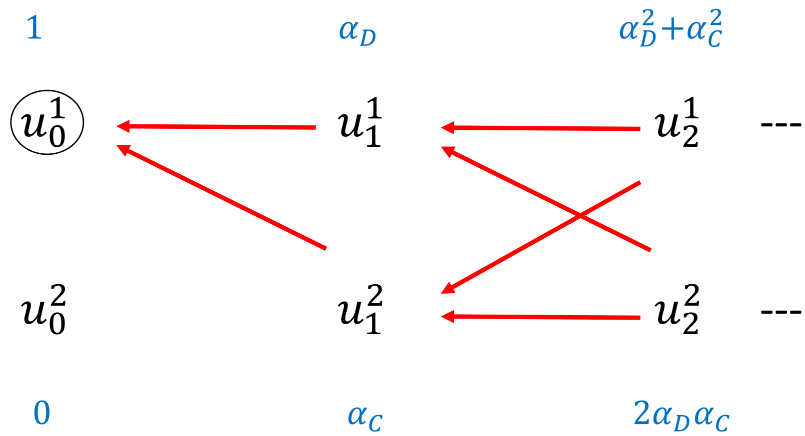


Figure 1: Resulting discount functions with two dynasties (sequences). Welfare implication of incremental utility backward in time (direction of arrows). Subscript refers to generation, superscript to dynasty.

cares dynastically for generation 1, which, again, cares dynastically for generation 2, and since generation 0 in dynasty 1 cares cross-dynastically for generation 1, which, again, cares cross-dynastically for generation 2.

Consider the game where each generation saves for its immedate descendents. In a stationary Markov-perfect equilibrium with linear strategies, generation 0 in dynasty 1 only considers the sequence of weights on within-dynasty utilities as cross-dynastic transfers are not allowed. This gives rise to preference externalities as saving for one's own descendants benefits the present member of the other dynasty. Furthermore, the two first within-dynasty per-period discount factors, α_D and $(\alpha_D^2+\alpha_C^2)/\alpha_D = \alpha_D+\alpha_C^2/\alpha_D$, follow from α_D and $(\alpha_D^2+\alpha_C^2)/\alpha_D$, implying that $\alpha_D + \alpha_C = \alpha \geq \alpha_D + \alpha_C^2/\alpha_D > \alpha_D$. These preferences are not stationary (Koopmans, 1960) since the discount factors depend on the distance between the time periods compared and the time of evaluation. The non-stationarity means that the external effect, being the difference between α and α_D and between α and $\alpha_D + \alpha_C^2/\alpha_D$, becomes less important as this distance increases. It leads to time-inconsistent preferences if preferences are time-invariant (Strotz, 1955–1956; Halevy, 2015).

In contrast, efficient saving captures any generation's total altruism for the next generation. Efficiency thus recovers sequence (1) as the discount factors for the utilities of future generations.

3 Model

Society is divided into $N \geq 2$ equally populated dynasties indexed by $i = 1, 2, \ldots$. Time $t \in \mathbb{N}$ is discrete and countably infinite, where $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are the natural numbers without and with 0. Generations, also indexed by t, are non-overlapping and live for one period only.

Denote the consumption, saving and wealth of the present generation in dynasty i by $c_t^i, k_t^i, x_t^i \in \mathbb{R}_+$, where $\mathbb{R}$ and $\mathbb{R}_+$ are the real numbers without and with the non-negativity restriction. A consumption stream ${}_0c^i = (c_0^i, c_1^i, \ldots) \geq 0$ is feasible given an initial level of wealth $x^i \geq 0$ if there exists a wealth stream ${}_0x^i = (x_0^i, x_1^i, \ldots) \geq 0$ such that $x_0^i = x^i$ and

$$\begin{aligned} x_t^i &= c_t^i + k_t^i \text{ for all } t \in \mathbb{N}_0, \text{ and} \\ x_t^i &= A k_{t-1}^i \text{ for all } t \in \mathbb{N}. \end{aligned}$$

The action taken by each dynasty i is to save $k_t^i \geq 0$ for its own immediate descendants. The residual, c_t^i, is consumed. Hence, cross-dynastic transfers are ruled out by assumption. Wealth is determined by the saving of the previous generation in the same dynasty, k_{t-1}^i, multiplied by a gross productivity parameter, $A \geq 1$. Such a technology is referred to as the AK model, which is a tractable model in which cross-dynastic intergenerational altruism can be studied analytically.

Let

$$X_\tau(x^i) = \{{}_0x^i : 0 \leq x_t^i \leq A x_{t-1}^i \text{ for all } t \in \{1, 2, \ldots, \tau\}\} \tag{2}$$

denote the set of feasible wealth streams until time $\tau \in \mathbb{N}$. Write $X(x^i) = X_\infty(x^i)$. Hence, $X(x^i)$ denotes the set of feasible wealth streams. Furthermore, define $x_t = (x_t^1, x_t^2, \ldots, x_t^N)$ as the distribution of wealth at time $t \in \mathbb{N}_0$.

Define

$$\boldsymbol{c}({}_0x^i) = (x_0^i - x_1^i/A,\ x_1^i - x_2^i/A,\ \ldots)$$

as the consumption stream associated with ${}_0x^i$ and let

$$C(x^i) = \{{}_0c^i :\ \text{there is } {}_0x^i \in X(x^i) \text{ s.t. } {}_0c^i = \boldsymbol{c}({}_0x^i)\}$$

denote the set of feasible consumption streams.

Map consumption $c_t^i \geq 0$ into utility by the utility function $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ defined by:

$$u(c_t^i) = \begin{cases} \ln c_t^i \text{ if } c_t^i > 0, \\ -\infty \text{ if } c_t^i = 0. \end{cases}$$

Hence, the present generation in dynasty i has a logarithmic utility function. Write $\boldsymbol{u}({}_0c^i) = (u(c_0^i), u(c_1^i), \dots)$ and let

$$U(x^i) = \{{}_0u^i : \text{ there is } {}_0c^i \in C(x^i) \text{ s.t. } {}_0u^i = \boldsymbol{u}({}_0c^i)\}$$

denote the set of feasible utility streams. Write $\mathcal{U} = \bigcup_{x^i \in \mathbb{R}_+} U(x^i)$. Furthermore, define $u_t = (u_t^1, u_t^2, \dots, u_t^N)$ as the distribution of utility levels at time $t \in \mathbb{N}_0$ and ${}_tu = (u_t, u_{t+1}, \dots)$ as the utility levels from time t on.

The present generation of dynasty i cares about immediate descendants in all dynasties. Intergenerational altruism is divided into its dynastic, α_D, and cross-dynastic, α_C, components. The following assumption regarding the network of altruistic links will be useful:

Assumption 1 *Altruism parameters have the following restrictions:* $1 > \alpha_D + \alpha_C > 0$ *and* $\alpha_D \geq \alpha_C/(N-1) \geq 0$.

The restrictions embody the extreme cases: $\alpha_D > \alpha_C = 0$ (Barro, 1974), weight only on own immediate descendants, and $\alpha_D = \alpha_C/(N-1) > 0$ (e.g., Myles, 1997), equal weight on the immediate descendants of all dynasties.

The preference of each dynasty is represented by a welfare function $W^i(\cdot)$ which assigns welfare $W^i({}_tu)$ at time t as a function of utility levels ${}_tu$ from time t on. Let $W^{-i}(\cdot)$ denote the vector of welfare functions in other dynasties. Assume that there exists an aggregator function $V : (\mathbb{R} \cup \{-\infty\})^{N+1} \to \mathbb{R} \cup \{-\infty\}$ defined by:

$$\begin{aligned} V(u_t^i, W^i({}_{t+1}u), W^{-i}({}_{t+1}u)) = {} & (1 - \alpha_D - \alpha_C)u_t^i \\ & + \alpha_D W^i({}_{t+1}u) + \frac{\alpha_C}{N-1} \sum_{j \neq i} W^j({}_{t+1}u), \end{aligned} \tag{3}$$

where u_t^i is the utility of the present generation in dynasty i. Assume furthermore that $V(u_t, W^i({}_{t+1}u), W^{-i}({}_{t+1}u)) = -\infty$ if $u_t^i = -\infty$.

The aggregator function, V, implicitly determines the welfare function:

$$W^i({}_t u) = V(u_t, W^i({}_{t+1}u), W^{-i}({}_{t+1}u)).$$

It means that intergenerational altruism is constant, non-paternalistic and sensitive only for the next generation, in the sense that the welfare of the present generation in dynasty i is derived from its own utility and the welfare of immediate descendants in the different dynasties (adapting the terminology of Ray, 1987).

3.1 Equilibrium concept

The strategic problem is how to best respond to the present saving of other dynasties and the future saving of all dynasties. The set of feasible histories at time τ depends on the initial wealth x and is given by $h_\tau(x) = X_\tau(x^1) \times \cdots \times X_\tau(x^i) \times \cdots \times X_\tau(x^N)$, where, as defined in (2), $X_\tau(x^i)$ denotes the set of feasible wealth streams until time τ for a single dynasty i. Write $h_0(x) = x$. When deciding how much to save, the dynasties see the entire history, h_τ. Write the union of histories as $\mathcal{H}(x) = \bigcup_{\tau \in \mathbb{N}_0} h_\tau(x)$. A strategy $k^{i,\sigma} : \mathcal{H}(x) \to \mathbb{R}_+$ maps the union of histories into present saving. A strategy is defined to be unimprovable if there exists no history after which welfare can be increased by changing saving only for this history. A profile of such strategies is a subgame-perfect equilibrium (SPE) if and only if, for any i and for any history h_τ, it is unimprovable. This follows because the game is continuous at infinity.

The analysis will be restricted to Markovian strategies (Maskin and Tirole, 2001). Define a Markovian strategy $k^{i,\mu} : \mathbb{R}^N_+ \to \mathbb{R}_+$ as a function from present wealth x_t to present saving, where x_t contains all payoff-relevant information at time t (the last entry into h_τ). The strategy is also stationary as it is independent of calendar time.

Let x and x_{+1} denote the present and next period wealth levels. Write optimal behavior in the form of a value function from dynamic programming. In particular, a value function $U^i : \mathbb{R}^N_+ \to \mathbb{R} \cup \{-\infty\}$ defined over wealth levels satisfies

$$U^i(x) = \max_{k^i \in [0, x^i]} V(u^i, U^i_{+1}, U^{-i}_{+1})$$

$$= \max_{k^i \in [0, x^i]} \left\{ (1 - \alpha_D - \alpha_C) u^i + \alpha_D U^i_{+1} + \frac{\alpha_C}{N-1} \sum_{j \neq i} U^j_{+1} \right\}, \tag{4}$$

where $u^i = u(x^i - k^i)$ is defined as the utility of the present generation in dynasty i, $U^i = U^i(A(x^i - k^i),\ x^{-i}_{+1})$ the induced welfare of the immediate descendants of the same dynasty and $U^j = U^j(x^{-i}_{+1},\ A(x^i - k^i))$ the induced welfare of the immediate descendants of another dynasty.

A Markovian strategy is unimprovable if it satisfies $k^{i,\mu}(x) = \text{argmax}_{k^i} U^i(x)$ for all i and wealth x. As above, a profile of such strategies is an SPE if, and only if, it is unimprovable. A stationary Markovian strategy profile that is an SPE is a stationary Markov-perfect equilibrium (MPE).

4 Main results

4.1 Equilibrium

In a stationary MPE with linear strategies, only present wealth in the own dynasty matters when deciding how much to save for immediate descendants. This follows since if there are no cross-dynastic transfers and no links between dynasties that depend on wealth levels outside the dynasty, only utilities in the own dynasty can be affected. Hence, the saving of one dynasty is independent of the wealth levels of other dynasties. This insight is key when establishing the following theorem:

Theorem 1 *Under Assumption 1, there exists a stationary MPE where all dynasties use the linear strategy:*

$$k^{i,\mu}(x_t) = s x^i_t \tag{5}$$

for all i and x_t, where the constant saving rate, s, is given by

$$s = \alpha_D + \frac{\alpha_C^2}{(N-1)(1 - \alpha_D - \alpha_C) + \alpha_C}. \tag{6}$$

This is the limit of the unique finite time horizon SPE when time goes to infinity.

The existence of a stationary MPE where all dynasties use a linear strategy follows by applying the unimprovability property and is proven in the Appendix.

It is also proven in the Online Appendix that there exists a unique SPE in the finite horizon game for any horizon and that the equilibrium specified in Theorem 1 is the limit of this SPE when the horizon goes to infinity. In particular, the equilibrium strategies used in these finite horizon games go to the linear strategy with s given by (6) when the horizon goes to infinity.

The following corollary describes the properties of the equilibrium saving rate, s:

Corollary 1 *Under Assumption 1, the equilibrium saving rate, s, has the following properties:*

(i) $s = \alpha_D$ *if* $\alpha_C = 0$.

(ii) s *increases in* α_D.

(iii) s *increases in* α_C.

(iv) s *decreases in* N *if* $\alpha_C > 0$.

(v) $s \to \alpha_D$ *if* $N \to \infty$.

This follows from expression (6) and is proven in the Online Appendix A.

Without cross-dynastic intergenerational altruism, the saving rate reduces to α_D. This follows as the cross-dynastic intergenerational altruism of the descendants in the other dynasties in the next generation has (almost) no concern for dynasty i. The saving rate decreases in relation to the number of dynasties, as it increases the externality problem. Since the altruistic weight on other dynasties goes to zero in the limiting case, when the number of dynasties goes to infinity, the saving rate reduces to α_D. This means that cross-dynastic intergenerational altruism does not affect the equilibrium saving rate when the number of dynasties is infinitely large.

4.2 Efficiency

Recall that the equilibrium saving rate is inefficient due to the preference externalities. Interpret the efficient saving rate as the saving rate that would emerge if all dynasties bargain efficiently regarding its level based on the assumption of cooperation also in the future. This means that the present representatives of all dynasties come together with the aim of realizing a trajectory that is Pareto efficient for the present generation, where their preferences also include the preference for the future.

As reported in the following theorem, there exists a stationary saving rate that, if used also in the future, results in a trajectory which is Pareto efficient for the present generation in terms of the altruistic welfare of the dynasties (see Milgrom, 1993 and Hausman, 2011 for perspectives on preference satisfaction in behavioral welfare analysis):

Theorem 2 *Under Assumption 1, saving according to*

$$k_t^i = s^* x_t^i \tag{7}$$

for all i and x_t, where the constant saving rate, s^*, is given by

$$s^* = \alpha_D + \alpha_C, \tag{8}$$

implies a trajectory that is Pareto efficient for the present generation in terms of their altruistic welfare, given that the rule is used in the future.

It is proven in the Appendix that if one dynasty maximizes its own welfare provided that the saving rates in all dynasties are set as equal to each other, then the resulting saving rate s^* is given by (8). Since this is the result even when dynasties are asymmetric and regardless of which dynasty is responsible for the maximization, saving according to (7) leads to a trajectory that is Pareto efficient for the present generation in terms of their altruistic welfare.

The following corollary describes the properties of the efficient saving rate, s^*:

Corollary 2 *Under Assumption 1, the efficient saving rate, s^*, has the following properties:*

(i) $s^* = \alpha_D$ *if* $\alpha_C = 0$.

(ii) s^* *increases in* α_D.

(iii) s^* increases in α_C.

This follows from expression (8).

Define by

$$s^* - s = \alpha_C - \frac{\alpha_C^2}{(N-1)(1-\alpha_D-\alpha_C)+\alpha_C} \tag{9}$$

the wedge between the efficient and equilibrium saving rates. The following corollary describes the wedge, $s^* - s$:

Corollary 3 *Under Assumption 1, the equilibrium saving rate, s, is inefficient if $\alpha_C > 0$.*

This follows from expression (9) and is proven in the Online Appendix A.

The efficient saving rate, s^*, increases in intergenerational altruism. It reduces to α_D without cross-dynastic intergenerational altruism. With cross-dynastic intergenerational altruism, the efficient saving rate, s^*, is always larger than the equilibrium saving rate, s. It follows from Corollaries 1 and 2 that this wedge increases to α_C in the limiting case when the number of dynasties goes to infinity.

The present generation's preference for future generations is reflected by s^* and can only be inferred from saving behavior when there is no cross-dynastic intergenerational altruism so that $s^* - s = 0$. Efficient saving therefore translates into an increase in the relative weight on all future generations when accounting for cross-dynastic intergenerational altruism. The policy implication could be a lowering of discount rates inferred from saving behavior in the market, even if there is limited cross-dynastic intergenerational altruism.

This critique of dynamic revealed preference theory (Arrow and Kurz, 1970) differs from Caplin and Leahy (2004). By considering the set of efficient distributions of altruistic welfare, Caplin and Leahy (2004) show conditions under which it would be efficient to give the future more weight than assigned by the present through its altruistic preferences (see Millner and Heal, 2021 for a perspective). In contrast, in the present paper, the future is not given more weight than what is already included in the altruistic preferences of the present.

5 Time-inconsistency

It will be useful to derive a non-recursive formulation of the welfare function W^i. The following theorem establishes the non-recursive formulation:

Theorem 3 *Under Assumption 1, welfare can be written non-recursively:*

$$W^i({}_tu) = (1-\alpha_D-\alpha_C)\Big(\sum_{\tau=0}^{\infty}\Delta_\tau u^i_{t+\tau} + \sum_{j\neq i}\sum_{\tau=0}^{\infty}\Gamma_\tau u^j_{t+\tau}\Big), \tag{10}$$

with discount functions

$$\Delta_\tau = \frac{1}{N}\big((\alpha_D+\alpha_C)^\tau + (N-1)(\alpha_D - \frac{\alpha_C}{N-1})^\tau\big), \tag{11}$$

$$\Gamma_\tau = \frac{1}{N}\big((\alpha_D+\alpha_C)^\tau - (\alpha_D - \frac{\alpha_C}{N-1})^\tau\big). \tag{12}$$

It is proven in the Appendix that the welfare function (10) follows by repeated substitution of W^i and W^j's into V from (3). The discount functions (11) and (12) are proven by means of induction.

The discount functions (11) and (12) give the weights that the present generation of dynasty i puts on the utility of generation τ in the same dynasty and each of the other dynasties. They imply the following weights on the first two generations:

$$\Delta_0 = 1, \quad \Delta_1 = \alpha_D,$$
$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{\alpha_C}{N-1}.$$

Figure 1 illustrates these weights for $N = 2$. More generally, $\Delta_\tau \geq \Gamma_\tau$ for all $\tau \in \mathbb{N}$.

The following observation will be helpful when interpreting the term structure of the discount factor. The total weight on all other dynasties, $(N-1)\Gamma_{\tau+1}$, is important for the construction of $\Delta_{\tau+2}$ (in the proof of Theorem 3). Using expressions (11) and (12), the link between Δ_τ and $\Delta_{\tau+2}$ via $(N-1)\Gamma_{\tau+1}$ can be written as

$$\frac{\alpha_C}{N-1}(N-1)\frac{\alpha_C}{N-1} = \frac{\alpha_C^2}{N-1}. \tag{13}$$

Intuitively, although dynasty i gives weight α_C to the other dynasties, the other dynasties give weight $\alpha_C/(N-1)$ to dynasty i. This weight goes to zero as the number of dynasties goes to infinity.

5.1 Declining discount rates

To illustrate that cross-dynastic intergenerational altruism implies declining discount rates *in equilibrium*, consider the non-recursive formulation of the welfare function (10). Importantly, note that since the behavior of one dynasty does not depend on the utilities of other dynasties (based on Theorem 1), only the first summation is relevant for time-inconsistency.

Assume for the moment that $\alpha_C = 0$. Then, $\Delta_\tau = \alpha_D^\tau$ and $\Gamma_\tau = 0$ for all $\tau \in \mathbb{N}_0$. Inserting in (10) gives the dynastic intergenerational altruism welfare function:

$$(1 - \alpha_D) \sum_{\tau=0}^{\infty} \alpha_D^\tau u_{t+\tau}^i.$$

Since $\Delta_\tau / \Delta_{\tau-1} = \alpha_D$ for all $\tau \in \mathbb{N}$, all generations weight within-dynasty utility similarly. This implies a geometric discount function (i.e., constant discount rates). Hence, the preference of each dynasty is time-consistent.

This is no longer the case with cross-dynastic intergenerational altruism. The following propositions generalize the claim related to time-inconsistency in Section 2:

Proposition 1 *Under Assumption 1, the preference of each dynasty is non-stationary and thus time-inconsistent if* $\alpha_C > 0$.

This follows from expressions (11) and (12) and is proven in the Online Appendix A.

Proposition 2 *Under Assumption 1,*

(i) $\Delta_\tau / \Delta_{\tau-1}$ *converges to* $\alpha_D + \alpha_C$ *only in the limit, as time goes to infinity, if* $\alpha_D > \alpha_C/(N-1) > 0$.

(ii) $\Delta_\tau / \Delta_{\tau-1}$ *converges to* $\alpha_D + \alpha_C$ *in the next time period if* $\alpha_D = \alpha_C/(N-1)$.

This follows from the proof of Proposition 1.

There are two cases: If $\alpha_D > \alpha_C/(N-1)$, then $\Delta_\tau/\Delta_{\tau-1}$ increases from α_D and converges only in the limit to $\alpha_D + \alpha_C$ so that all generations weight within-dynasty utility differently. This is a discount function with declining discount rates. If $\alpha_D = \alpha_C/(N-1)$, $\Delta_\tau/\Delta_{\tau-1}$ increases from α_D and converges to $\alpha_D+\alpha_C$ in the next time period so that only subsequent generations weight within-dynasty utility differently. This implies a "quasi-hyperbolic" discount function. In both cases, the preference of each dynasty is time-inconsistent.

This observation differs from Phelps and Pollak (1968), Sáez-Marti and Weibull (2005) and, more recently, Galperti and Strulovici (2017) since time-inconsistency in these papers follows from intergenerational altruism being sensitive beyond the next generation of the same dynasty. Here, time-inconsistency is due to altruism for the next generation as such. A notable exception is when the number of dynasties is finite but goes to infinity. In line with expression (13), the weight each other dynasty gives to a dynasty goes to zero as the number of dynasties goes to infinity. This leads to geometric discounting of the own dynasty only in the limit.

In contrast, cross-dynastic intergenerational altruism implies constant discount rates *under efficiency* (Theorem 2). This can be seen from the discount functions (11) and (12), where $(\Delta_\tau+(N-1)\Gamma_\tau)/(\Delta_{\tau-1}+(N-1)\Gamma_{\tau-1}) = \alpha_D+\alpha_C$ for all $\tau \in \mathbb{N}$. From the discussion above, it is clear that $\Delta_\tau/\Delta_{\tau-1}$ increases from α_D and approaches $\alpha_D + \alpha_C$.

An indication of why efficient and equilibrium discounting agree in the limit if $\alpha_D > \alpha_C/(N-1)$ can be obtained from the discount functions as time goes to infinity. In a version of the model in the Online Appendix B, it follows that the external effect of present saving becomes less important over time and vanishes only in the limit. This establishes that a dynasty's discount rate is smaller for the long term. More precisely, I establish that $\lim_{\tau\to\infty} \Delta_\tau/(\Delta_\tau + (N-1)\Gamma_\tau) = 1/N$. Hence, each dynasty's present value of a gain at time t converges to $1/N$ of the social value of this benefit when t approaches infinity.

5.2 Generalized consumption Euler equations

An alternative starting point for deriving the stationary saving rate is Laibson (1998). Laibson studies the extent of undersaving by a "quasi-hyperbolic" discounter that is sophisticated in the sense that he takes into account the fact that his preference is time-inconsistent. Krusell et al. (2002) integrate Laibson's insight into standard discrete-time macroeconomic models. (For continuous-time formulations, see, for instance, Karp, 2007 and Ekeland and Lazrak, 2010.) Hiraguchi (2014) and Iverson and Karp (2021), who are closer to my contribution, generalize Krusell et al. (2002) to arbitrary term structures of the discount rates.

The equilibrium saving rate, s from expression (6), can be derived from the generalized consumption Euler equation of Hiraguchi (2014) and Iverson and Karp (2021):

$$s = \frac{\sum_{\tau=1}^{\infty} \Delta_\tau}{\sum_{\tau=0}^{\infty} \Delta_\tau}, \tag{14}$$

but for a distinct reason. The following proposition establishes this insight:

Proposition 3 *Under Assumption 1, the equilibrium saving rate s from expression (6) follows from the Hiraguchi-Iverson-Karp solution for s (14).*

This follows from expression (11) and is proven in the Online Appendix A.

Hiraguchi (2014) and Iverson and Karp (2021) assume a term structure of the discount rates when deriving the saving rate. Here, the relation is an outcome of the game of saving. This follows as the behavior of one dynasty does not depend on the utilities of other dynasties (from Theorem 1). It is as if society consists of N parallel dynasties with declining discount rates according to expression (11).

6 Interdependent utility

It will be useful to illustrate the qualitative robustness of the main results by considering two formulations of *intra*generational altruism replacing the aggregator function, V, from expression (3). Recent additions to this literature focus on static interdependent utility (Bourlès et al., 2017) as well as dynamic interdependent utility without considering saving behavior (Millner, 2020). My paper addresses the consequences in terms of saving.

6.1 Paternalistic intragenerational altruism

Suppose that the present generation of dynasty i also cares about the utility of contemporaries in other dynasties. Intragenerational altruism is divided into its dynastic, α_A, and cross-dynastic, α_B, components. As argued in the Introduction, there is strong support for $\alpha_A > \alpha_B/(N-1)$.

The following additional assumption regarding the network of altruistic links will be useful:

Assumption 2 *Altruism parameters have the following restrictions:* $\alpha_A + \alpha_B = 1$ *and* $\alpha_A \geq \alpha_B/(N-1) \geq 0$.

These restrictions embody the extreme cases: $\alpha_A > \alpha_B = 0$ (Section 3), weight only on own dynasty contemporaries, and $\alpha_D = \alpha_B/(N-1) > 0$, equal weight on all contemporaries.

The preference of each dynasty is represented by the welfare function $W^i(\cdot)$. Let u_t^{-i} and $W^{-i}(\cdot)$ denote the vectors of utilities and welfare functions in other dynasties. Assume that there exists an aggregator function $V : (\mathbb{R} \cup \{-\infty\})^{2N} \to \mathbb{R} \cup \{-\infty\}$ that implicitly determines the welfare function defined by:

$$V(u_t^i, u_t^{-i}, W^i({}_{t+1}u), W^{-i}({}_{t+1}u)) = (1-\alpha_D-\alpha_C)\big(\alpha_A u_t^i + \frac{\alpha_B}{N-1}\sum_{j\neq i} u_t^j\big) + \alpha_D W^i({}_{t+1}u) + \frac{\alpha_C}{N-1}\sum_{j\neq i} W^j({}_{t+1}u), \tag{15}$$

where u^j is the utility of the present generation of another dynasty. Assume, furthermore, that $V(u_t^i, u_t^{-i}, W^i({}_{t+1}u), W^{-i}({}_{t+1}u)) = -\infty$ if $u_t^i = -\infty$ or if $\alpha_B > 0$, $u_t^j = -\infty$.

The following proposition establishes the non-recursive formulation of the welfare function:

Proposition 4 *Under Assumptions 1 and 2, welfare can be written non-recursively:*

$$W^i({}_t u) = (1-\alpha_D-\alpha_C)\Big(\sum_{\tau=0}^{\infty}\Delta_\tau u_{t+\tau}^i + \sum_{j\neq i}\sum_{\tau=0}^{\infty}\Gamma_\tau u_{t+\tau}^j\Big), \tag{16}$$

with discount functions

$$\Delta_\tau = \frac{1}{N}\Big(\alpha_A\big((\alpha_D+\alpha_C)^\tau + (N-1)(\alpha_D - \frac{\alpha_C}{N-1})^\tau\big) + \alpha_B\big((\alpha_D+\alpha_C)^\tau - (\alpha_D - \frac{\alpha_C}{N-1})^\tau\big)\Big), \tag{17}$$

$$\Gamma_\tau = \frac{1}{N}\Big(\frac{\alpha_B}{N-1}\big((\alpha_D+\alpha_C)^\tau + (N-1)(\alpha_D - \frac{\alpha_C}{N-1})^\tau\big) + (\alpha_A + \frac{(N-2)\alpha_B}{N-1})\big((\alpha_D+\alpha_C)^\tau - (\alpha_D - \frac{\alpha_C}{N-1})^\tau\big)\Big). \tag{18}$$

This follows from an application of the proof of Theorem 3 and is proven in the Online Appendix A.

The discount functions (17) and (18) give the weights that the present generation of dynasty i puts on the utility of generation τ in the same dynasty and each of the other dynasties. They imply the following weights on the first two generations:

$$\Delta_0 = \alpha_A, \qquad \Delta_1 = \alpha_D\alpha_A + \alpha_C\frac{\alpha_B}{N-1},$$

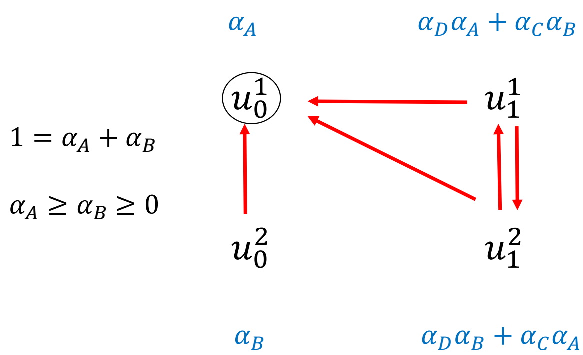
$$\Gamma_0 = \frac{\alpha_B}{N-1}, \quad \Gamma_1 = \frac{\alpha_C}{N-1}\alpha_A + \Big(\alpha_D + \frac{(N-2)\alpha_C}{N-1}\Big)\frac{\alpha_B}{N-1}.$$

Figure 2a illustrates these weights for $N = 2$. To see this, consider α_A and α_B. This follows directly from Assumption 2 as the weights that the present generation of dynasty i puts on itself and contemporaries in the other dynasty. The weight on the next generation in the same dynasty is $\alpha_D\alpha_A + \alpha_C\alpha_B$ and follows as dynasty i cares dynastically and cross-dynastically. By Assumption 2, the dynastic link is weighted by the share put on the own dynasty utility and the cross-dynastic link by the share put on the other dynasty utility. The weight $\alpha_D\alpha_B + \alpha_C\alpha_A$ follows by symmetry. More generally, $\Delta_\tau \geq \Gamma_\tau$ for all $\tau \in \mathbb{N}_0$.

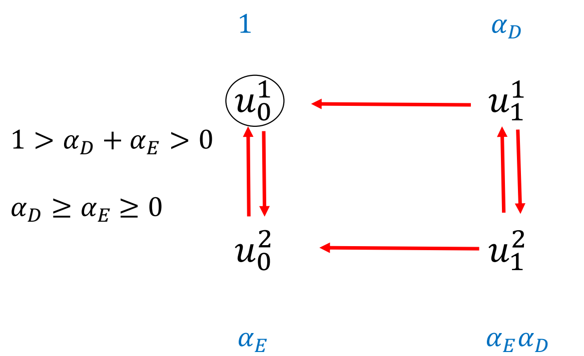
The following propositions generalize Propositions 1 and 2:

Proposition 5 *Under Assumptions 1 and 2, the preference of each dynasty is non-stationary and thus time-inconsistent if* $\alpha_A > \alpha_B/(N-1)$ *and* $\alpha_C > 0$.

This follows from expressions (17) and (18) and is proven in the Online Appendix A.



(a) Paternalistic cross-dynastic intragenerational altruism.



(b) Non-paternalistic cross-dynastic intragenerational altruism.

Figure 2: Resulting discount functions with two dynasties (sequences) for alternative preference formulations. Welfare implication of incremental utility backward in time (direction of arrows). Subscript refers to generation, superscript to dynasty.

Proposition 6 *Under Assumptions 1 and 2,*

(i) $\Delta_\tau/\Delta_{\tau-1}$ *converges to* $\alpha_D + \alpha_C$ *only in the limit, as time goes to infinity, if* $\alpha_A > \alpha_B/(N-1)$ *and* $\alpha_D > \alpha_C/(N-1) > 0$*.*

(ii) $\Delta_\tau/\Delta_{\tau-1}$ *converges to* $\alpha_D + \alpha_C$ *in the next time period if* $\alpha_A > \alpha_B/(N-1)$ *and* $\alpha_D = \alpha_C/(N-1)$*.*

This follows from the proof of Proposition 5.

Assuming $\alpha_A > \alpha_B/(N-1)$, there are two cases: If $\alpha_D > \alpha_C/(N-1)$, then $\Delta_\tau/\Delta_{\tau-1}$ increases from $\alpha_D + \alpha_C\alpha_B/((N-1)\alpha_A)$ and converges only in the limit to $\alpha_D + \alpha_C$ so that all generations weight within-dynasty utility differently. This is a discount function with declining discount rates. If $\alpha_D = \alpha_C/(N-1)$, $\Delta_\tau/\Delta_{\tau-1}$ increases from $\alpha_D + \alpha_C\alpha_B/((N-1)\alpha_A)$ and converges to $\alpha_D + \alpha_C$ in the next time period so that only subsequent generations weight within-dynasty utility differently. This implies a "quasi-hyperbolic" discount function. In both cases, the preference of each dynasty is time-inconsistent.

The following corollary describes the equilibrium and efficient saving rates:

Corollary 4 *Under Assumptions 1 and 2, the equilibrium and efficient saving rates can be written:*

$$s = \alpha_D + \alpha_C \frac{\alpha_A\alpha_C + \alpha_B(1-\alpha_D)}{\alpha_A((N-1)(1-\alpha_D-\alpha_C)+\alpha_C)+\alpha_B\alpha_C}, \tag{19}$$

$$s^* = \alpha_D + \alpha_C. \tag{20}$$

This follows from expressions (17) and (18) and is proven in the Online Appendix A.

It follows from Assumptions 1 and 2 that $s^* - s \geq 0$. Furthermore, $s^* - s > 0$ if $\alpha_A > \alpha_B/(N-1)$ and $\alpha_C > 0$. This means that all results of the main text hold qualitatively even with paternalistic cross-dynastic intragenerational altruism as long as the weight on the other dynasties in this generation is smaller than the weight on the own dynasty in this generation.

Somewhat surprisingly for the stationary infinite horizon setting, it reduces to the following condition for $s^* > s$:

$$\frac{\Delta_0}{\Delta_1} > \frac{\Gamma_0}{\Gamma_1},$$

that the relative weight on the utility of the present and next generations is strictly larger for the own dynasty than for the other dynasties. Equilibrium saving is thus inefficient due to the discrepancy between the dynastic and cross-dynastic discount functions, but only in the first two generations. This intuition follows from Assumptions 1 and 2. If there is a discrepancy between the relative discounting of the first two generations, there is also a discrepancy in any two generations.

6.2 Non-paternalistic intragenerational altruism

Suppose that the present generation of dynasty i only cares cross-dynastically for the welfare of other dynasties in the present generation rather than the next generation. While there is less support for such preferences, it illustrates the limit of the analysis.

The following alternative assumption on the network of altruistic links will be useful:

Assumption 3 *Altruism parameters have the following restrictions:* $1 > \alpha_D + \alpha_E > 0$ *and* $\alpha_D \geq \alpha_E/(N-1) \geq 0$.

The restrictions embody the extreme cases: $\alpha_D > \alpha_E = 0$ (Barro, 1974), weight only on own immediate descendants, and $\alpha_D = \alpha_C/(N-1) > 0$, equal weight on immediate descendants in the own dynasty and contemporaries in the other dynasties.

The preference of each dynasty is represented by the welfare function $W^i(\cdot)$. Assume that there exists an aggregator function $V : (\mathbb{R} \cup \{-\infty\})^{N+1} \to \mathbb{R} \cup \{-\infty\}$ that implicitly determines the welfare function defined by:

$$V(u_t^i, W^i({}_{t+1}u), W^{-i}({}_tu)) = (1 - \alpha_D - \alpha_E)u_t^i + \alpha_D W^i({}_{t+1}u) + \frac{\alpha_E}{N-1} \sum_{j \neq i} W^j({}_tu). \tag{21}$$

Assume, furthermore, that $V(u_t^i, W^i({}_{t+1}u), W^{-i}({}_tu)) = -\infty$ if $u_t^i = -\infty$.

The following proposition establishes the non-recursive formulation of the welfare function:

Proposition 7 *Under Assumption 3, welfare can be written non-recursively:*

$$W^i({}_tu) = (1 - \alpha_D - \alpha_E)\Big(\sum_{\tau=0}^{\infty} \Delta_\tau u_{t+\tau}^i + \sum_{j \neq i} \sum_{\tau=0}^{\infty} \Gamma_\tau u_{t+\tau}^j \Big), \tag{22}$$

with discount functions

$$\Delta_\tau = \alpha_D^\tau, \tag{23}$$

$$\Gamma_\tau = \frac{\alpha_E}{N-1}\alpha_D^\tau, \tag{24}$$

when Δ_0 *is normalized to* 1.

This follows from an application of the proof of Theorem 3 and is proven in the Online Appendix A.

The discount functions (23) and (24) give the weights that the present generation of dynasty i puts on the utility of generation τ in the same dynasty and each of the other dynasties. They imply the following weights on the first two generations:

$$\Delta_0 = 1, \qquad \Delta_1 = \alpha_D,$$
$$\Gamma_0 = \frac{\alpha_E}{N-1}, \quad \Gamma_1 = \frac{\alpha_E}{N-1}\alpha_D.$$

Figure 2b illustrates these weights for $N = 2$. To see this, consider the weight that the present generation of dynasty i puts on itself. Since cross-dynastic intra-generational altruism is reciprocal, this weight is $(1+\alpha_E+\alpha_E^2+\dots) = 1/(1-\alpha_E)$. Contemporaries in the other dynasty are additionally weighted cross-dynastically, $\alpha_E(1+\alpha_E+\alpha_E^2+\dots) = \alpha_E/(1-\alpha_E)$. Both dynasties care dynastically about the next generation so that the resulting weights are $\alpha_D/(1-\alpha_E)$ for dynasty i and $\alpha_E\alpha_D/(1-\alpha_E)$ for dynasty j. Multiply through by $1-\alpha_E$ to ensure $\Delta_0 = 1$. More generally, $\Delta_\tau > \Gamma_\tau$ for all $\tau \in \mathbb{N}_0$.

The following corollary describes the equilibrium and efficient saving rates:

Corollary 5 *Under Assumption 3, the equilibrium and efficient saving rates can be written:*

$$s = \alpha_D, \tag{25}$$

$$s^* = \alpha_D. \tag{26}$$

This follows from expressions (23) and (24).

Hence, $s^* - s = 0$ for all α_E. This means that cross-dynastic intragenerational altruism alone is not sufficient for deriving the main results. Cross-dynastic altruism needs to be sensitive to the welfare of future generations.

This reduces to the following condition, implying $s^* = s$:

$$\frac{\Delta_0}{\Delta_1} = \frac{\Gamma_0}{\Gamma_1},$$

that the relative weights on the utility of the present and next generations are equal for the own dynasty and the other dynasties. Equilibrium saving is efficient due to the similarity between the dynastic and cross-dynastic discount functions.

6.3 The "isolation paradox"

Sen (1961, 1967) and Marglin (1963) develop a model of dynamic interdependent utility and saving (for a recent addition to this literature, see Robson and Szentes, 2014). In the terminology of this paper, they study a two-period model in which present members of dynasties are altruistic toward their own descendants and descendants in other dynasties. Each dynasty decides how much to save for its own immediate descendants. As in the present paper, the equilibrium saving rate is inefficiently low.

Sen (1961) names it the "isolation paradox" since each dynasty would agree collectively to save more, although no dynasty is willing to do so in "isolation" (borrowing the explanation of Newbery, 1990). Attempting to solve this problem, Sen (1967) considers a bargain between all dynasties aiming to realize a trajectory that is Pareto efficient for the present generation in terms of their altruistic welfare. The efficient saving rate, s^*, can be interpreted as the saving rate that would emerge if all dynasties bargain over how much to save for immediate descendants. Thus, the interpretation resembles that of the "isolation paradox" literature. Yet, the precise condition for the "isolation paradox" to arise remains criticized (e.g., Lind, 1964).

In Sen's two-period model of within-dynasty saving, the equilibrium saving rate, s, is inefficient if the relative weight on the utility of the present and next generations is strictly larger for the own dynasty than for the other dynasties.

Using my notation, that is

$$\frac{\Delta_0}{\Delta_1} > \frac{\Gamma_0}{\Gamma_1}. \tag{27}$$

It follows from the discussion above that this condition is equal to the condition for $s^* > s$. Sen's (1967) condition thereby generalizes to a stationary infinite horizon setting:

Remark 1 *Under Assumptions 1 and 2 or Assumption 3, the condition for the "isolation paradox" to arise in Sen's two-period model, given by expression (27), is equal in the stationary infinite horizon model.*

Hence, only the utility weights in the first two generations are relevant for determining whether equilibrium saving is inefficient. This intuition follows from Assumptions 1 and 2 or Assumption 3. If there is a discrepancy between the relative discounting of the first two generations, there is also a discrepancy in any two generations.

Accounting for cross-dynastic intergenerational altruism also exposes a limitation to, as well as extends, Sen's (1967) "isolation paradox." In Sen's two-period model, cross-dynastic intergenerational altruism cannot affect the decision of how much to save. This is not the case in the model used in this paper, except in the limiting case, when the number of dynasties goes to infinity:

Remark 2 *Under Assumption 1, α_C affects the decision on how much to save, except in the limit as $N \to \infty$. In Sen's two-period model, this is not the case for any N.*

7 Concluding remarks

In this paper, I ask whether the trade-off between present utility and future welfare can be inferred from saving behavior. In answering this question, I study a setting with cross-dynastic intergenerational altruism. Cross-dynastic intergenerational altruism refers to the welfare weight on the next generation in other dynasties. This can be motivated by a concern for sustainability. Crucially, saving for one's own descendants benefits present members of other dynasties. This gives rise to preference externalities since the other dynasties also care cross-dynastically. The analysis shows that intergenerational altruism may not be inferred from saving behavior as long as the relative weight on the utility of the present and next generations is strictly larger for the own dynasty than the other dynasties. Another finding is that the external effect of present saving decreases over time. This implies that the utility discount rate consistent with saving behavior decreases. In general, this discount rate converges to the efficient level only in the limit, as time goes to infinity.

Yet, the utility discount rate in public guidelines is typically informed by saving behavior (OECD, 2018). The main results presented a wedge between the efficient and equilibrium saving rates, measuring the preference externalities due to cross-dynastic intergenerational altruism. This implies a shift of the relative weights forward in time by accounting for cross-dynastic intergenerational altruism to correct the externality problem (see Online Appendix C for an illustration). The immediate implication for policy guidelines is that discount rates implied from saving behavior should be lowered, even if there is limited cross-dynastic intergenerational altruism. This adjustment is of particular importance for long term problems such as climate change.

The analysis has clarified the conceptual basis for the above claims in a model of within-dynasty saving. In the Introduction, I argued that the condition on preference parameters for preference externalities to emerge is likely to hold in practice. However, it should be noted that not all transfers to future generations are carried out in the form of dynastic saving. I leave it as further work to additionally consider transfers to the immediate descendants of all dynasties and whether such transfers may crowd out transfers to one's own immediate descendants (as pointed out for the "isolation paradox" by Newbery, 1990).

I also leave it as further work to study how to integrate into the analysis of the present paper another externality problem, namely the global externality that greenhouse gas emissions lead to. With cross-dynastic intergenerational altruism, cooperation between the present members of all dynasties must not only account for the traditional emissions externalities and technological externalities (Harstad, 2020) but also, if possible, preference externalities. Cross-dynastic intergenerational altruism may therefore provide an additional reason for cooperation on climate change.

Appendix

This section contains proofs of theorems. Additional proofs of results are presented in the Online Appendix A.

Proof of Theorem 1 – Existence

The proof of existence is an application of the unimprovability property. Assume that all generations in dynasties $j \neq i$ and all future generations in dynasty i use the linear strategy (5). This gives a marginal propensity to consume of $1 - s$.

Write:

$$\begin{aligned} y &\equiv U^i(x_t), \\ z^j &\equiv U^j(x_t), \\ u &\equiv \ln((1-s)x_t^i), \\ v^j &\equiv \ln((1-s)x_t^j). \end{aligned}$$

By observing that the gross growth rate is sA, it follows from (4):

$$\begin{aligned} y &= (1-\alpha_D-\alpha_C)u + \alpha_D\left(y+\ln(sA)\right) + \frac{\alpha_C}{N-1}\sum_{\ell}\left(z^\ell+\ln(sA)\right), \\ z^j &= (1-\alpha_D-\alpha_C)v^j + \alpha_D\left(z^j+\ln(sA)\right) \\ &\qquad + \frac{\alpha_C}{N-1}\Big((y+\ln(sA)) + \sum_{\ell\neq j}\left(z^\ell+\ln(sA)\right)\Big), \end{aligned}$$

for all j. Solving the set of these equations yields:

$$\begin{aligned} y &= \frac{((N-1)(1-\alpha_D)-(N-2)\alpha_C)u + \alpha_C\sum_\ell v^\ell}{(N-1)(1-\alpha_D)+\alpha_C} \\ &+ \frac{\alpha_D+\alpha_C}{1-\alpha_D-\alpha_C}\ln(sA), \end{aligned} \tag{28}$$

$$\begin{aligned} z^j &= \frac{((N-1)(1-\alpha_D)-(N-2)\alpha_C)v^j + \alpha_C(u+\sum_{\ell\neq j} v^\ell)}{(N-1)(1-\alpha_D)+\alpha_C} \\ &+ \frac{\alpha_D+\alpha_C}{1-\alpha_D-\alpha_C}\ln(sA) \end{aligned} \tag{29}$$

for all j.

Insert for (28) to (29) in (4). The problem is to show that $k_t^i = sx_t^i$ maximizes

$$(1-\alpha_D-\alpha_C)\ln(x_t^i-k_t^i)+\alpha_D\frac{((N-1)(1-\alpha_D)-(N-2)\alpha_C)\ln((1-s)Ak_t^i)}{(N-1)(1-\alpha_D)+\alpha_C}$$
$$+\alpha_C\frac{\alpha_C\ln((1-s)Ak_t^i)}{(N-1)(1-\alpha_D)+\alpha_C}.$$

The first derivative is:

$$-\frac{1-\alpha_D-\alpha_C}{x_t^i-k_t^i}+\frac{\alpha_D((N-1)(1-\alpha_D)-(N-2)\alpha_C)+\alpha_C^2}{(N-1)(1-\alpha_D)+\alpha_C}\frac{1}{k_t^i},$$

which yields the first-order condition:

$$\frac{1-\alpha_D-\alpha_C}{x_t^i-k_t^i}=\frac{\alpha_D((N-1)(1-\alpha_D)-(N-2)\alpha_C)+\alpha_C^2}{(N-1)(1-\alpha_D)+\alpha_C}\frac{1}{k_t^i}.$$

Therefore:

$$\frac{k_t^i}{x_t^i}=\alpha_D+\frac{\alpha_C^2}{(N-1)(1-\alpha_D-\alpha_C)+\alpha_C}=s,$$

which gives $k_t^i = sx_t^i$.

The second derivative is:

$$-\frac{1-\alpha_D-\alpha_C}{(x_t^i-k_t^i)^2}-\frac{\alpha_D((N-1)(1-\alpha_D)-(N-2)\alpha_C)+\alpha_C^2}{(N-1)(1-\alpha_D)+\alpha_C}\frac{1}{(k_t^i)^2}$$

and is strictly negative for $k_t^i \in (0, x_t^i)$. This verifies that the problem is concave. $k_t^i = sx_t^i$ thus maximizes the problem. There is no profitable deviation for the present generation in dynasty i when all generations of dynasties j and all future generations in dynasty i use the linear strategy (5). ■

Proof of Theorem 2

Based on the assumption of cooperation in the future, I can replace s by s^* (from expression (8)) in y, z^j, u and v^j from the proof of Theorem 1. Denote the new expressions by y^*, z^{*j}, u^* and v^{*j}.

Let one dynasty maximize its own welfare provided that the saving rates in all dynasties are set as equal to each other. Let b denote the saving rate across dynasties. The problem is to show that $b = s^*$ maximizes

$$(1-\alpha_D-\alpha_C)\ln((1-b)x_t^i)$$

$$
\begin{aligned}
&+\alpha_D\Big(\frac{((N-1)(1-\alpha_D)-(N-2)\alpha_C)\ln((1-s^*)Abx_t^i)}{(N-1)(1-\alpha_D)+\alpha_C} \\
&\qquad\qquad +\frac{\alpha_C\sum_\ell \ln((1-s^*)Abx_t^\ell)}{(N-1)(1-\alpha_D)+\alpha_C}\Big) \\
&+\frac{\alpha_C}{N-1}\sum_{j\neq i}\Big(\frac{((N-1)(1-\alpha_D)-(N-2)\alpha_C)\ln((1-s^*)Abx_t^j)}{(N-1)(1-\alpha_D)+\alpha_C} \\
&\qquad\qquad +\frac{\alpha_C(\ln((1-s^*)Abx_t^i)+\sum_{\ell\neq j}\ln((1-s^*)Abx_t^\ell))}{(N-1)(1-\alpha_D)+\alpha_C}\Big).
\end{aligned}
$$

The first derivative with respect to b is:

$$
-\frac{1-\alpha_D-\alpha_C}{1-b} + (\alpha_D+\alpha_C)\frac{(N-1)(1-\alpha_D)-(N-2)\alpha_C+(N-1)\alpha_C}{(N-1)(1-\alpha_D)+\alpha_C}\frac{1}{b}.
$$

This yields the first-order condition:

$$
\frac{1-\alpha_D-\alpha_C}{1-b} = \frac{\alpha_D+\alpha_C}{b}.
$$

To illustrate this, note that the terms multiplied by $1/b$ can be written as:

$$
(\alpha_D+\alpha_C)\frac{(N-1)(1-\alpha_D)+\alpha_C}{(N-1)(1-\alpha_D)+\alpha_C}.
$$

Therefore:

$$
b = \alpha_D+\alpha_C = s^*,
$$

which gives $k_t^i = s^* x_t^i$ for all i.

The second derivative with respect to b is:

$$
-\frac{1-\alpha_D-\alpha_C}{(1-b)^2} - \frac{\alpha_D+\alpha_C}{b^2}
$$

and is strictly negative for $b \in (0,1)$. This verifies that the problem is concave. $k_t^i = s^* x_t^i$ for all i thus maximizes the problem. In fact, this is the result even when dynasties are asymmetric and regardless of which dynasty is responsible for the maximization. Hence, saving according to (7) implies a trajectory that is Pareto efficient for the present generation in terms of their altruistic welfare, given that the rule is used in the future. ■

Proof of Theorem 3

The welfare function (10) follows by repeated substitution of W^i and W^j's into V from (3). Discount functions (11) and (12) are proven by means of induction.

The base case: Discount functions (11) and (12) hold for $\tau = 0$ since $\Delta_0 = 1$ and $\Gamma_0 = 0$.

The step case: Suppose that discount functions (11) and (12) hold for $\tau - 1$. Then,

$$\begin{aligned}
\Delta_\tau &= \alpha_D \Delta_{\tau-1} + \frac{\alpha_C}{N-1}(N-1)\Gamma_{\tau-1} \\
&= \alpha_D \Delta_{\tau-1} + \alpha_C \Gamma_{\tau-1} \\
&= \frac{1}{N}\big(\alpha_D(\alpha_D + \alpha_C)^{\tau-1} + \alpha_D(N-1)(\alpha_D - \frac{\alpha_C}{N-1})^{\tau-1} \\
&\qquad + \alpha_C(\alpha_D + \alpha_C)^{\tau-1} - \alpha_C(\alpha_D - \frac{\alpha_C}{N-1})^{\tau-1}\big) \\
&= \frac{1}{N}\big((\alpha_D + \alpha_C)^\tau + (N-1)(\alpha_D - \frac{\alpha_C}{N-1})^\tau\big),
\end{aligned}$$

by inserting for $\Delta_{\tau-1}$ and $\Gamma_{\tau-1}$. And,

$$\begin{aligned}
\Gamma_\tau &= \frac{\alpha_C}{N-1}\Delta_{\tau-1} + (\alpha_D + \frac{(N-2)\alpha_C}{N-1})\Gamma_{\tau-1} \\
&= \frac{1}{N}\big(\frac{\alpha_C}{N-1}(\alpha_D + \alpha_C)^{\tau-1} + \frac{\alpha_C}{N-1}(N-1)(\alpha_D - \frac{\alpha_C}{N-1})^{\tau-1} \\
&\qquad + (\alpha_D + \frac{(N-2)\alpha_C}{N-1})(\alpha_D + \alpha_C)^{\tau-1} - (\alpha_D + \frac{(N-2)\alpha_C}{N-1})(\alpha_D - \frac{\alpha_C}{N-1})^{\tau-1}\big) \\
&= \frac{1}{N}\big((\alpha_D + \alpha_C)^\tau - (\alpha_D - \frac{\alpha_C}{N-1})^\tau\big),
\end{aligned}$$

by inserting for $\Delta_{\tau-1}$ and $\Gamma_{\tau-1}$. This proves that discount functions (11) and (12) hold for all $\tau \in \mathbb{N}_0$.

It follows from (11) and (12) that $\Delta_\tau + (N-1)\Gamma_\tau = (\alpha_D + \alpha_C)^\tau$. This ensures that W^i is well-defined on $\mathcal{U}^N$. ■

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Online Appendix A

This section contains additional proofs of results.

Proof of Theorem 1 – Uniqueness

The following proves that (5) is the unique MPE in the finite horizon game.

Let the remaining horizon be H. Write

$$U^i((h_{H+1}, x_H)) = \max_{k^i} F(x, k^i, H), \tag{30}$$

$$k^i((h_{H+1}, x_H)) = \frac{\sum_{\tau=1}^{H} \Delta_\tau}{\sum_{\tau=0}^{H} \Delta_\tau} x^i = \operatorname*{argmax}_{k^i} F(x, k^i, H) := s_H x^i, \tag{31}$$

where based on a finite horizon version of (10):

$$F(x, k^i, H) = (1 - \alpha_D - \alpha_C)\Big(\ln(x^i - k^i) + \sum_{\tau=1}^{H} \Delta_\tau \ln(k^i) + C_H \Big), \tag{32}$$

with constant

$$C_H = \sum_{j \neq i} \sum_{\tau=1}^{H} \Gamma_\tau \ln(k^j) + \sum_{\tau=1}^{H} (\alpha_D + \alpha_C)^\tau \ln\Big((1 - s_{H-\tau}) \prod_{\ell=1}^{\tau-1} s_{H-\ell}\ A^\tau \Big),$$

depending on the present saving of other dynasties and growth terms implied by future play. The value function (30) and strategy (31) are proven by means of induction.

The base case: Expressions (30) and (31) hold for $H = 0$ due to the convention $\sum_{\tau=1}^{0} \Delta_\tau = 0$.

The step case: The problem for dynasty i with remaining horizon H is to maximize (32) with respect to k^i. The first derivative is:

$$-\frac{1}{x^i - k^i} + \frac{\sum_{\tau=1}^{H} \Delta_\tau}{k^i},$$

which yields the first-order condition:

$$\frac{1}{x^i - k^i} = \frac{\sum_{\tau=1}^{H} \Delta_\tau}{k^i}.$$

Therefore:

$$k^i = \frac{\sum_{\tau=1}^{H} \Delta_\tau}{\sum_{\tau=0}^{H} \Delta_\tau} x^i,$$

which gives $k^i = s_H x^i$. The second derivative is:

$$-\frac{1}{(x^i - k^i)^2} - \frac{\sum_{\tau=1}^{H} \Delta_\tau}{(k^i)^2}$$

and strictly negative for $k^i \in (0, x^i)$. This verifies that the problem is concave. The solution $k^i = s_H x^i$ satisfies the strategy (31) as well as the value function (30) due to the independence of the k^j's.

The above proof establishes uniqueness in a finite horizon game. From expression (14), it is clear that

$$\lim_{H \to \infty} s_H x^i = s x^i.$$

Hence, it is shown that there exists a unique SPE in the finite horizon game for any horizon. The equilibrium strategies used in these finite horizon games go to the linear strategy with s given by (6) when the horizon goes to infinity. ■

Proof of Corollary 1

Statements are proven one by one:

(i) follows by inserting for $\alpha_C = 0$ in expression (6).

(ii) follows by taking the first derivative of s with respect to α_D:

$$1 + \frac{\alpha_C^2 (N-1)}{\big((N-1)(1-\alpha_D-\alpha_C)+\alpha_C\big)^2} > 0,$$

since $1 > \alpha_D + \alpha_C$.

(iii) follows by taking the first derivative of s with respect to α_C:

$$\frac{2\alpha_C}{(N-1)(1-\alpha_D-\alpha_C)+\alpha_C} + \frac{\alpha_C^2 (N-2)}{\big((N-1)(1-\alpha_D-\alpha_C)+\alpha_C\big)^2} > 0.$$

(iv) follows by taking the first derivative of s with respect to N:

$$-\alpha_C^2 \frac{(1-\alpha_D-\alpha_C)}{\big((N-1)(1-\alpha_D-\alpha_C)+\alpha_C\big)^2} < 0.$$

(v) follows by taking the following limit:

$$\lim_{N\to\infty} \alpha_D + \frac{\alpha_C^2}{(N-1)(1-\alpha_D-\alpha_C)+\alpha_C} = \alpha_D + 0 = \alpha_D.$$

This completes the proof. ■

Proof of Corollary 3

Assume $\alpha_C > 0$. Compare the equilibrium saving rate, s from (6), with the efficient saving rate, s^*:

$$\alpha_C > \frac{\alpha_C^2}{(N-1)(1-\alpha_D-\alpha_C)+\alpha_C},$$

since $(N-1)(1-\alpha_D-\alpha_C) > 0$. This verifies that the equilibrium saving rate is inefficiently low for all $N > 1$. ■

Proof of Proposition 1

Write the relative utility weight of two subsequent generations

$$\frac{\Delta_\tau}{\Delta_{\tau-1}} = \frac{\alpha_D\Delta_{\tau-1} + \alpha_C\Gamma_{\tau-1}}{\Delta_{\tau-1}} = \alpha_D + \alpha_C\frac{\Gamma_{\tau-1}}{\Delta_{\tau-1}}, \tag{33}$$

by inserting from (11). Combine expressions (11) and (12),

$$\frac{\Gamma_{\tau-1}}{\Delta_{\tau-1}} = \frac{(\alpha_D+\alpha_C)^{\tau-1} - (\alpha_D - \frac{\alpha_C}{N-1})^{\tau-1}}{(\alpha_D+\alpha_C)^{\tau-1} + (N-1)(\alpha_D - \frac{\alpha_C}{N-1})^{\tau-1}}. \tag{34}$$

There are two cases:

Case 1: Assume $\alpha_D > \alpha_C/(N-1)$. The fraction $\Delta_\tau/\Delta_{\tau-1}$ in (33) increases from α_D and converges only in the limit to $\alpha_D + \alpha_C$. This follows directly from (34): $\Gamma_0/\Delta_0 = 0$, Γ_τ/Δ_τ increases in τ (since the nominator increases in τ and the denominator decreases) and $\lim_{\tau\to\infty}\Gamma_\tau/\Delta_\tau = 1$. This means that all generations weight within-dynasty utility differently. Hence, the preference of each dynasty is time-inconsistent

Case 2: Assume $\alpha_D = \alpha_C/(N-1)$. The fraction $\Delta_\tau/\Delta_{\tau-1}$ in (33) increases from α_D and converges to $\alpha_D + \alpha_C$ in the next time period. This follows directly from (34): $\Gamma_0/\Delta_0 = 0$ and $\Gamma_\tau/\Delta_\tau = 1$ for all $\tau \in \mathbb{N}$. This means that subsequent generations weight within-dynasty utility differently. Hence, the preference of each dynasty is time-inconsistent. ■

Proof of Proposition 3

Define the geometric series

$$\sum_{\tau=0}^{\infty}(\alpha_D+\alpha_C)^{\tau}=\frac{1}{1-\alpha_D-\alpha_C}, \tag{35}$$

$$(N-1)\sum_{\tau=0}^{\infty}(\alpha_D-\frac{\alpha_C}{N-1})^{\tau}=\frac{N-1}{1-\alpha_D+\frac{\alpha_C}{N-1}}. \tag{36}$$

Hence, according to (11), it follows from (35) and (36) that

$$\sum_{\tau=0}^{\infty}\Delta_{\tau}=\frac{1}{N}\Big(\frac{1}{1-\alpha_D-\alpha_C}+\frac{N-1}{1-\alpha_D+\frac{\alpha_C}{N-1}}\Big),$$

which, by rewriting (14), implies

$$s=\frac{\sum_{\tau=0}^{\infty}\Delta_{\tau}-1}{\sum_{\tau=0}^{\infty}\Delta_{\tau}}=\alpha_D+\frac{\alpha_C^2}{(N-1)(1-\alpha_D-\alpha_C)+\alpha_C}.$$

This is identical to expression (6), the equilibrium saving rate. ■

Proof of Proposition 4

The welfare function (16) follows by repeated substitution of W^i and W^j's into V from (15). Discount functions (17) and (18) are proven by means of induction.

The base case: Discount functions (17) and (18) hold for $\tau=0$ since $\Delta_0=\alpha_A$ and $\Gamma_0=\alpha_B/(N-1)$.

The step case: Suppose that discount functions (17) and (18) hold for $\tau-1$. Then,

$$\begin{aligned}\Delta_{\tau}&=\alpha_D\Delta_{\tau-1}+\alpha_C\Gamma_{\tau-1}\\&=\frac{1}{N}\Big(\alpha_A\big((\alpha_D+\alpha_C)^{\tau}+(N-1)(\alpha_D-\frac{\alpha_C}{N-1})^{\tau}\big)\\&\qquad+\alpha_B\big((\alpha_D+\alpha_C)^{\tau}-(\alpha_D-\frac{\alpha_C}{N-1})^{\tau}\big)\Big),\end{aligned}$$

by inserting for $\Delta_{\tau-1}$ and $\Gamma_{\tau-1}$ (and noting the similarity to Theorem 3). And,

$$\Gamma_{\tau}=\frac{\alpha_C}{N-1}\Delta_{\tau-1}+\Big(\alpha_D+\frac{(N-2)\alpha_C}{N-1}\Big)\Gamma_{\tau-1}$$

$$= \frac{1}{N}\Big(\frac{\alpha_B}{N-1}\big((\alpha_D+\alpha_C)^\tau + (N-1)(\alpha_D - \frac{\alpha_C}{N-1})^\tau\big)$$
$$+ (\alpha_A + \frac{(N-2)\alpha_B}{N-1})\big((\alpha_D+\alpha_C)^\tau - (\alpha_D - \frac{\alpha_C}{N-1})^\tau\big)\Big),$$

by inserting for $\Delta_{\tau-1}$ and $\Gamma_{\tau-1}$ (and noting the similarity to Theorem 3). This proves that discount functions (17) and (18) hold for all $\tau \in \mathbb{N}_0$.

It follows from (17) and (18) that $\Delta_\tau + \Gamma_\tau = (\alpha_D + \alpha_C)^\tau$. This ensures that W^i is well-defined on $\mathcal{U}^N$. ■

Proof of Proposition 5

Write the relative utility weight of two subsequent generations

$$\frac{\Delta_\tau}{\Delta_{\tau-1}} = \frac{\alpha_D\Delta_{\tau-1} + \alpha_C\Gamma_{\tau-1}}{\Delta_{\tau-1}} = \alpha_D + \alpha_C\frac{\Gamma_{\tau-1}}{\Delta_{\tau-1}}, \tag{37}$$

by inserting from (17). Combine expressions (17) and (18),

$$\frac{\Gamma_{\tau-1}}{\Delta_{\tau-1}} = \frac{\frac{\alpha_B}{N-1}f + \big(\alpha_A + \frac{(N-2)\alpha_B}{N-1}\big)g}{\alpha_A f + \alpha_B g}, \tag{38}$$

where

$$f \equiv (\alpha_D+\alpha_C)^{\tau-1} - (\alpha_D - \frac{\alpha_C}{N-1})^{\tau-1},$$
$$g \equiv (\alpha_D+\alpha_C)^{\tau-1} + (N-1)(\alpha_D - \frac{\alpha_C}{N-1})^{\tau-1}.$$

Assuming $\alpha_A > \alpha_B/(N-1)$, there are two cases:

Case 1: Assume $\alpha_D > \alpha_C/(N-1)$. The fraction $\Delta_\tau/\Delta_{\tau-1}$ in (37) increases from $\alpha_D + \alpha_C\alpha_B/((N-1)\alpha_A)$ and converges only in the limit to $\alpha_D + \alpha_C$. This follows directly from (38): $\Gamma_0/\Delta_0 = \alpha_B/(N-1)\alpha_A$, Γ_τ/Δ_τ increases in τ (since the nominator increases in τ and the denominator decreases) and $\lim_{\tau\to\infty}\Gamma_\tau/\Delta_\tau = 1$. This means that all generations weight within-dynasty utility differently. Hence, the preference of each dynasty is time-inconsistent

Case 2: Assume $\alpha_D = \alpha_C/(N-1)$. The fraction $\Delta_\tau/\Delta_{\tau-1}$ in (37) increases from $\alpha_D + \alpha_C\alpha_B/((N-1)\alpha_A)$ and converges to $\alpha_D + \alpha_C$ in the next time period. This follows directly from (38): $\Gamma_0/\Delta_0 = \alpha_B/(N-1)\alpha_A$ and $\Gamma_\tau/\Delta_\tau = 1$ for all $\tau \in \mathbb{N}$. This means that subsequent generations weight within-dynasty utility differently. Hence, the preference of each dynasty is time-inconsistent. ■

Proof of Corollary 4

For discount function (17), it follows from (35) and (36) that

$$\sum_{\tau=0}^{\infty} \Delta_\tau = \frac{1}{N}\Big(\frac{1}{1-\alpha_D-\alpha_C} + \Big(\alpha_A - \frac{\alpha_B}{N-1}\Big)\frac{N-1}{1-\alpha_D+\frac{\alpha_C}{N-1}}\Big),$$

which, according to the Hiraguchi-Iverson-Karp solution, implies

$$s = \frac{\sum_{\tau=0}^{\infty}\Delta_\tau - \alpha_A}{\sum_{\tau=0}^{\infty}\Delta_\tau} = \alpha_D + \alpha_C \frac{\alpha_A\alpha_C + \alpha_B(1-\alpha_D)}{\alpha_A((N-1)(1-\alpha_D-\alpha_C)+\alpha_C)+\alpha_B\alpha_C}.$$

This is identical to expression (19), the equilibrium saving rate. The efficient saving rate (20) follows immediately from (17) and (18). ■

Proof of Proposition 7

The welfare function (22) follows by repeated substitution of W^i and W^j's into V from (21). Discount functions (23) and (24) are proven by means of induction.

The base case: Discount functions (23) and (24) hold for $\tau = 0$ since $\Delta_0 = 1$ and $\Gamma_0 = \alpha_E$ under the condition that Δ_0 is normalized to 1.

The step case: Suppose that discount functions (23) and (24) hold for $\tau - 1$. Then,

$$\Delta_\tau = \alpha_D \Delta_{\tau-1} = \alpha_D^\tau,$$

by inserting for $\Delta_{\tau-1}$. And,

$$\Gamma_\tau = \frac{\alpha_E}{N-1}\alpha_D\Delta_{\tau-1} = \frac{\alpha_E}{N-1}\alpha_D^\tau,$$

by inserting for $\Delta_{\tau-1}$. This proves that discount functions (23) and (24) hold for all $\tau \in \mathbb{N}_0$.

It follows from (23) and (24) that $\Delta_\tau + \Gamma_\tau = (1+\alpha_E)\alpha_D^\tau$. This ensures that W^i is well-defined on $\mathcal{U}^N$. ■

Online Appendix B

This section offers interpretations of the model if descendants move or marry someone from other dynasties.

The "dynastic family"

In response to Barro's (1974) formulation of intergenerational altruism, Bernheim and Bagwell (1988) consider the case in which each generation consists of a large number of individuals and that links between dynasties imply that individuals belong to different dynasties. A limitation in their analysis is that these links are hypothesized and not modeled. Laitner (1991) and Zhang (1994) formulate links between two dynasties through marital connections but focus on cross-sectional neutrality of policies and assortative mating, respectively. Myles (1997) states a more general preference but is silent with regard to its implications for the discount function.

I present a new interpretation of the discount function. Define for now α_C as the relative probability of immediate descendants ending up in other dynasties (e.g., through mating). (Consult Proposition 8 in the next subsection for a statistical interpretation of the discount functions.) It follows that discount functions (11) and (12) are Markov chains assigning the relative probabilities that descendants end up in different dynasties:

Remark 3 *Under Assumption 1, the fraction $\Delta_\tau/(\Delta_\tau + (N-1)\Gamma_\tau)$ assigns the probability that the descendants of the present generation of a dynasty are in the same dynasty τ generations from now.*

Note that

$$\frac{\Delta_\tau}{\Delta_\tau + (N-1)\Gamma_\tau} = \frac{1}{N}\frac{(\alpha_D + \alpha_C)^\tau + (N-1)(\alpha_D - \frac{\alpha_C}{N-1})^\tau}{(\alpha_D + \alpha_C)^\tau},$$

by inserting from expressions (11) and (12). Observe that $\lim_{\tau\to\infty} \Delta_\tau/(\Delta_\tau + (N-1)\Gamma_\tau) = 1/N$, implying convergence to a uniform distribution if $\alpha_D > \alpha_C/(N-1)$. In fact, the uniform distribution follows as the external effect of present saving becomes less important over time and vanishes only in the limit.

Statistical interpretation

Consider discount functions (11) and (12) for $N = 2$. The following proposition reinterprets these discount functions:

Proposition 8 *Assume $N = 2$. Under Assumption 1, discount functions (11) and (12) can be written:*

$$\Delta_\tau = \sum_{\substack{q \ even \\ 0 \le q \le \tau}} \binom{\tau}{\tau - q} \alpha_D^{\tau-q} \alpha_C^q, \tag{39}$$

$$\Gamma_\tau = \sum_{\substack{q \ odd \\ 0 \le q \le \tau}} \binom{\tau}{\tau - q} \alpha_D^{\tau-q} \alpha_C^q.$$

Proof. The right-hand side of (39) can be simplified. Perform the following rescaling of parameters: $\tilde{\alpha_D} = \alpha_D/(\alpha_D + \alpha_C)$ and $\tilde{\alpha_C} = \alpha_C/(\alpha_D + \alpha_C)$. Since $\tilde{\alpha_D} + \tilde{\alpha_C} = 1$, I can work with sums of binomial distributions. Write the sum over q even and q odd distributions as:

$$\begin{aligned} \sum_{q=0}^{\tau} \binom{\tau}{\tau - q} \alpha_D^{\tau-q} \alpha_C^q &= (\alpha_D + \alpha_C) \sum_{q=0}^{\tau} \binom{\tau}{\tau - q} \tilde{\alpha_D}^{\tau-q} \tilde{\alpha_C}^q \\ &= (\alpha_D + \alpha_C)^\tau, \end{aligned} \tag{40}$$

where the last line follows since the summation is now the total cumulative probability distribution of a binomial distribution and is equal to 1. The difference between q even and q odd distributions can be expressed as:

$$\begin{aligned} &\sum_{\substack{q \text{ even} \\ 0 \le q \le \tau}} \binom{\tau}{\tau - q} \alpha_D^{\tau-q} \alpha_C^q - \sum_{\substack{q \text{ odd} \\ 0 \le q \le \tau}} \binom{\tau}{\tau - q} \alpha_D^{\tau-q} \alpha_C^q \\ &= \sum_{\substack{q \text{ even} \\ 0 \le q \le \tau}} (-1)^q \binom{\tau}{\tau - q} \alpha_D^{\tau-q} \alpha_C^q + \sum_{\substack{q \text{ odd} \\ 0 \le q \le \tau}} (-1)^q \binom{\tau}{\tau - q} \alpha_D^{\tau-q} \alpha_C^q \\ &= \sum_{q=0}^{\tau} \binom{\tau}{\tau - q} \alpha_D^{\tau-q} (-\alpha_C^q) = (\alpha_D - \alpha_C)^\tau, \end{aligned} \tag{41}$$

using the definitions of $\tilde{\alpha_D}$ and $\tilde{\alpha_C}$.

Using the insights from expressions (40) and (41), expression (39) can be written:

$$\Delta_\tau \ = \ \frac{1}{2} (\underbrace{(\alpha_D + \alpha_C)^\tau}_{q \text{ even} + q \text{ odd}} + \underbrace{(\alpha_D - \alpha_C)^\tau}_{q \text{ even - } q \text{ odd}}),$$

which is identical to (11) for $N = 2$.

For completeness, define Γ_τ as:

$$\begin{aligned}\Gamma_\tau &= (\alpha_D + \alpha_C)^\tau - \Delta_\tau \\ &= (\alpha_D + \alpha_C)^\tau - \frac{1}{2}\big((\alpha_D + \alpha_C)^\tau + (\alpha_D - \alpha_C)^\tau\big) \\ &= \frac{1}{2}\big((\alpha_D + \alpha_C)^\tau - (\alpha_D - \alpha_C)^\tau\big),\end{aligned}$$

which is identical to (12) for $N = 2$. ■

From the point of view of the present generation of dynasty i, even time periods allow more cross-dynastic altruistic intergenerational links forward in time compared to the preceding odd time period. This asymmetry is clear when extending Figure 1 forward in time. The expression within the summation in (39) resembles a binomial distribution, with the exception that $\alpha_D + \alpha_C < 1$.

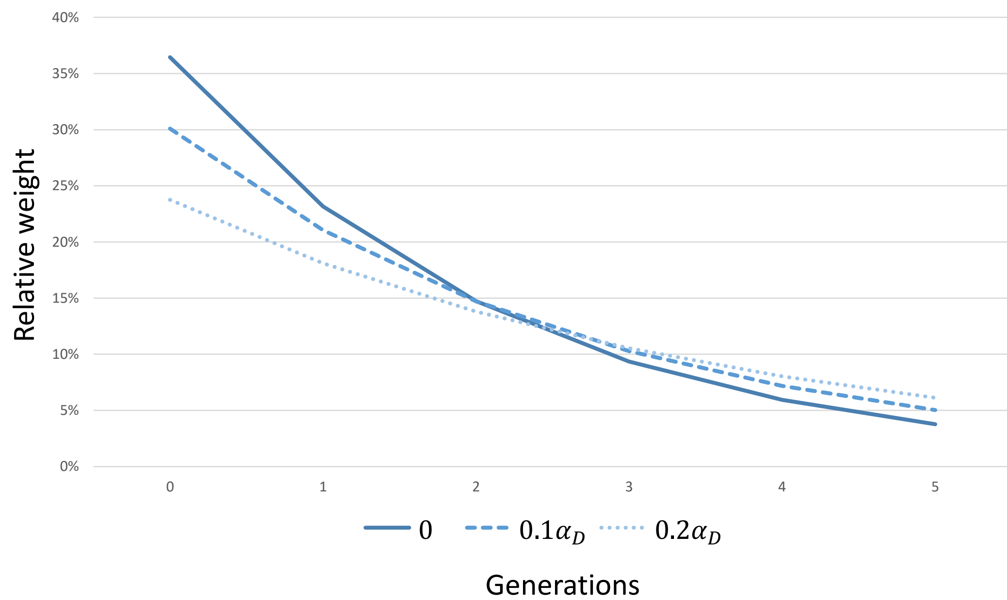
Online Appendix C

This section illustrates an immediate implication for policy guidelines when accounting for cross-dynastic intergenerational altruism.

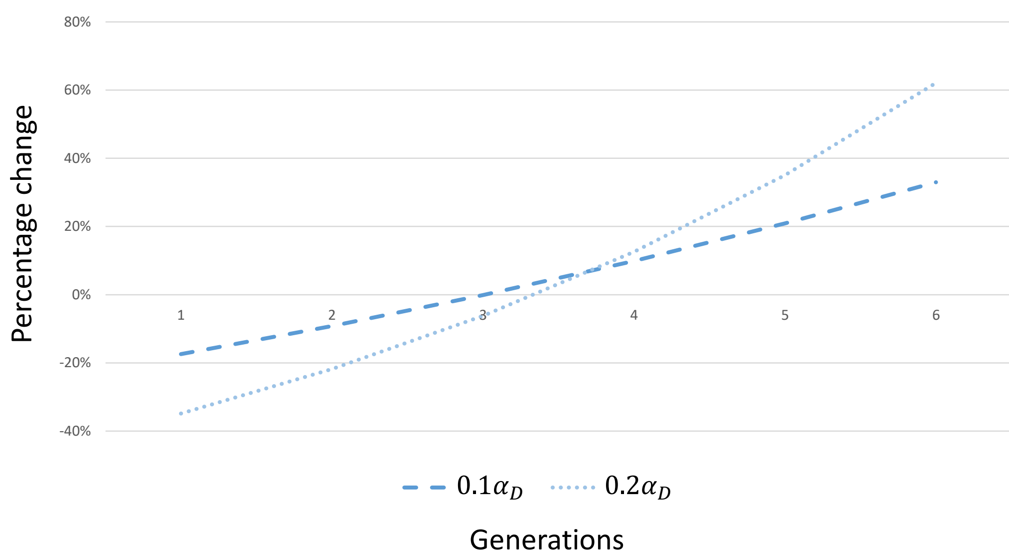
Policy implications

The utility discount rate in public guidelines is typically informed by saving behavior. To illustrate the consequence of the adoption of such discount rates, assume that a generation represents 30 years. Assume furthermore that the number of dynasties goes to infinity, $N \to \infty$. Nordhaus (2008) offers an influential market-based calibration. According to Nordhaus, the *relative* weight on future generations can be expressed as $s \to \alpha_D = 0.985^{30} \approx 64\%$. Corollary 3 gave the following wedge between the efficient and equilibrium saving rates: $s^* - s \to \alpha_C$. This measures the preference externalities due to cross-dynastic intergenerational altruism.

Figure 3a exemplifies the shift of relative weights forward in time by accounting for cross-dynastic intergenerational altruism ($s^* - s \to \alpha_C$, with $\alpha_C = 0, 0.1\alpha_D$



(a) Resulting relative weights on each generation.



(b) Percentage change in the relative weights.

Figure 3: Implications for the relative weights by changes in the wedge between the efficient and equilibrium saving, $s^* - s$. Assume that a generation represents 30 years and that $N \to \infty$. From Nordhaus (2008): $\alpha_D = 0.985^{30}$. The wedge, which is a measure of preference externalities, is given by $s^* - s \to \alpha_C$. Consider cases $\alpha_C = 0$, $0.1\alpha_D$ and $0.2\alpha_D$.

and $0.2\alpha_D$, respectively), thereby correcting the externality problem. From the restriction that α_C is less than or equal to $(N-1)\alpha_D$, it is clear that I consider very low α_C among the weights that satisfy this restriction. Accounting for cross-dynastic intergenerational altruism implies relative weights on future generations of 64%, 70% and 76%, leading to discount rates below the rate inferred from saving behavior (1.2% and 0.9%, compared to the Nordhaus rate of 1.5%). Figure 3b illustrates the percentage change in these weights compared to the Nordhaus calibration, clarifying that even accounting for limited levels of cross-dynastic intergenerational altruism is important. The weight on future generations increases by 10% and 20%, respectively. The immediate implication for policy guidelines is that discount rates implied from saving behavior should be lowered.