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Abstract

Prettner (2019) studies the implications of automation for economic growth and the labor share in a variant of the Solow-Swan model. The aggregate production function allows for two types of capital, traditional and automation capital. Traditional capital and labor are imperfect substitutes whereas automation capital and labor are perfect substitutes. In this paper, we point to a flaw in Prettner's analysis that invalidates his main analytical and computational findings. In contrast to Prettner, we argue that both kinds of capital are perfect substitutes as stores of value, and, therefore, must earn the same rate of return in equilibrium. Our computational analysis shows that the model dramatically overestimates the actual decline in the US labor share over the last 50 years.

JEL-Codes: O110, O330, O410.

Keywords: automation, declining labor share, capital accumulation, long-run growth.

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1 Introduction

In a recent paper, Prettner (2019) studies the implications of automation for economic growth and the labor share in a variant of the one-sector neoclassical growth model of Solow (1956) and Swan (1956). Following Steigum (2011), Prettner modifies the aggregate neoclassical production function and allows for two types of capital that interact differently with the third input, labor. On the one hand, there is *traditional capital*, e. g., machines or structures, on the other hand, there is *automation capital* like robots (Prettner (2019), p. 2). According to Prettner's interpretation traditional capital and labor are imperfect substitutes whereas automation capital and labor are *perfect* substitutes.

In this paper, we point to a flaw in Prettner's analysis that invalidates his main analytical findings as well as the results of his simulations. Prettner starts from the assumption that the accumulation of each type of capital is governed by a separate differential equation relating the change in the respective capital stock to its respective net investment. In both differential equations, the investment rate of the respective type of capital is exogenous and time-invariant. Our main analytical result shows that the assumption of exogenous and time-invariant investment rates are inconsistent with the properly characterized dynamic general equilibrium of the model. In contrast, we show that these investment rates are endogenous and change along the transition. As a consequence, the model's dynamical system, its asymptotic equilibrium growth rate, and its calibration results differ from those proposed by Prettner.

The intuition behind our findings is the following. Households own the stocks of traditional and automation capital. The households' willingness to hold strictly positive amounts of each stock rests on their respective rates of return. Since traditional and automation capital are perfect substitutes as stores of value, households are only willing to hold both stocks if their net rates of return, i. e., the respective rental rate minus the rate of depreciation, are the same. Hence, in equilibrium a no-arbitrage condition must assure the equality of these rates of return.

On the production side, the stocks of traditional and automation capital must both be hired. This requires the equilibrium factor prices to adjust so that profit-maximizing firms are willing act accordingly. However, in equilibrium, the rental rates of both types of capital are tied by the above-mentioned no-arbitrage condition and cannot vary independently. This has two consequences. First, the initial levels of traditional and automation capital have to be such that the marginal products of both stocks coincide. Second, the evolution of these stocks has to maintain the equality of the two marginal products at any moment in time. We show that the latter requirement dictates the equilibrium breakup of total gross investment into additions to traditional and automation capital. It implies that the fraction of gross investment that goes to either type of capital is endogenous and varies over time. As a consequence, Prettner's claim (iii) of a growth maximizing share of savings diverted to automation is ill-posed (Prettner (2019), p. 1294).

Our simulations shows that the obtained evolutions of per-capita assets, traditional and automation capital, output, and their growth rates hinge critically on whether the no-arbitrage condition in the market for capital is taken into account or not. However, we also find that neither Prettner's original simulations nor those obtained when traditional and automation capital earn the same rate of return replicate the recent evolution of the US labor share in a satisfactory way. This finding suggests that either the assumption of a perfect substitutability between automation capital and labor and/or the savings hypothesis with constant savings rates may have to be given up to derive simulations that are better in line with the data.

The remainder of this paper is organized as follows. Section 2 presents the model. To detect Prettner's flaw it proves useful to follow, e. g., Acemoglu (2009), and to interpret the setup as a dynamic general equilibrium model. This is the purpose of Section 3. Section 3.1 defines and explains the dynamic competitive equilibrium of the economy. Section 3.2 derives and interprets the equilibrium factor prices. Section 4 studies the transitional dynamics of the equilibrium and its asymptotic steady state. Section 4.1 characterizes the dynamical system of the equilibrium. Section 4.2 analyses the implied evolution of the investment shares in traditional and automation capital and of the labor share. Section 5 contains computational exercises that compare the implications of the corrected model to those obtained in Prettner (2019). Section 6 concludes. All proves are relegated to Section 7, the Appendix.

2 The Model of Prettner (2019)

Consider a competitive economy in continuous time, $t \in [0, \infty)$. There is a single final good that can be consumed or invested. If invested it may serve as traditional capital or as automation capital. Without affecting our qualitative results, both types of capital depreciate at the same instantaneous rate, $\delta > 0$. Households consume, save, and supply labor.

At all t, there are markets for the final good, both types of capital, and labor. All prices are expressed in units of contemporaneous output.

Production

The production of the final good, $Y(t)$, requires three inputs, labor, $L(t)$, traditional capital, $K(t)$, and automation capital, $P(t)$. The aggregate production function takes the form

$$Y(t) = K(t)^{\alpha} [L(t) + P(t)]^{1-\alpha}, \tag{2.1}$$

i. e., $L(t)$ and $P(t)$ are perfect substitutes.[1]

[1]The defining property of perfect substitutes is that the marginal rate of substitution between factors of

Let $R_K(t)$, $R_P(t)$, and $w(t)$ denote, respectively, the rental rate of traditional capital, of automation capital, and the real wage. Then, the optimal plan of a competitive representative firm maximizes profits, $\Pi(t) = Y(t) - R_K(t)K(t) - R_P(t)P(t) - w(t)L(t)$. This delivers the following first-order Kuhn-Tucker conditions[2]

$$\frac{\partial\Pi(t)}{\partial K(t)} = \alpha\left(\frac{K(t)}{L(t)+P(t)}\right)^{\alpha-1} - R_K(t) \leq 0, \text{ with "<" only if } K(t) = 0, \tag{2.2a}$$

$$\frac{\partial\Pi(t)}{\partial P(t)} = (1-\alpha)\left(\frac{K(t)}{L(t)+P(t)}\right)^{\alpha} - R_P(t) \leq 0, \text{ with "<" only if } P(t) = 0, \tag{2.2b}$$

$$\frac{\partial\Pi(t)}{\partial L(t)} = (1-\alpha)\left(\frac{K(t)}{L(t)+P(t)}\right)^{\alpha} - w(t) \leq 0, \text{ with "<" only if } L(t) = 0. \tag{2.2c}$$

Households

At all t there are $L(t) > 0$ households each endowed with one unit of labor that is inelastically supplied to the labor market. The number of households grows at an exogenous instantaneous rate n.

Let $\mathcal{A}(t)$ denote household assets at t. The economy has two assets, the stock of traditional and the stock of automation capital. Hence, $\mathcal{A}(t) = K(t) + P(t)$ for all t. The households' willingness to hold traditional and automation capital rests on their respective rates of return. As both assets are perfect substitutes as stores of value and depreciate at the same rate, an equilibrium involving $K(t) > 0$ and $P(t) > 0$ must satisfy the no-arbitrage condition

$$R_K(t) = R_P(t). \tag{2.3}$$

Under competitive factor pricing and constant returns to scale household income is $Y(t)$. The household sector consumes, $C(t)$, and saves, $S(t)$, hence, $Y(t) = C(t) + S(t)$. Following the textbook version of the Solow-Swan savings hypothesis, the household sector saves a constant fraction of its income, i. e., $S(t) = sY(t)$, where $s \in (0, 1)$ is the exogenous savings rate.

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production is constant. Here, the latter rate is equal to unity for labor and automation capital. Observe that a more general definition of perfect substitutes would allow for $A > 0$ and $B > 0$ so that the term in brackets becomes $AL(t) + BP(t)$. In this case the marginal rate of substitution is equal to A/B. All qualitative results of our analysis hold true for this generalization.

[2]A comprehensive list of these conditions would also involve statements of the kind $\partial\Pi(t)/\partial K(t) \geq 0$, with ">" only if $K(t) = \infty$, and mutatis mutandis, for the remaining two factors. However, as the supply of all three factors of production is finite and ">" implies an unbounded demand such a constellation is incompatible with the usual equilibrium market clearing conditions that require demand not to exceed supply.

3 Dynamic Competitive Equilibrium

To focus our analysis on the setting discussed in Prettner (2019) we restrict attention to "interior" equilibria, i. e., equilibrium configurations where the representative firm's demand for all three factors of production is strictly positive at all t. Henceforth, we refer to such a configuration as a dynamic competitive equilibrium or simply as an equilibrium.

3.1 Definition

Given initial values $L(0) > 0$, $K(0) > 0$, and $P(0) > 0$, and the evolution of the population, $\dot{L}(t) = nL(t)$, a dynamic competitive equilibrium is a sequence

$$\{Y(t), K(t), P(t), C(t), S(t), R_K(t), R_P(t), w(t)\}_{t=0}^{\infty} \tag{3.1}$$

such that for all $t \geq 0$

(E1) The representative firm maximizes profits.

(E2) Households are willing to hold both types of capital and save a constant fraction s of their income.

(E3) The market for the final good clears, i. e., $C(t) + I(t) = Y(t)$, where $I(t)$ is gross investment.

(E4) The capital market clears, i. e., the demands for traditional and automation capital are equal to the respective supplies.

(E5) The labor market clears.

Equilibrium condition (E1) requires (2.2a) - (2.2c) to hold as equalities. This is ensured for condition (2.2a) since the marginal product of traditional capital satisfies the Inada conditions. However, $\lim_{P(t)\to 0} \partial Y(t)/\partial P(t)$ and $\lim_{L(t)\to 0} \partial Y(t)/\partial L(t)$ are both finite. Therefore, conditions (2.2b) and (2.2c) leave room for corner solutions if $R_P(t)$, respectively, $w(t)$ were sufficiently high. Below, we derive a condition that assures that the equilibrium is interior.

Since automation capital and labor are perfect substitutes, conditions (2.2b) and (2.2c) imply

$$R_P(t) = w(t) = (1-\alpha)\left(\frac{K(t)}{L(t)+P(t)}\right)^{\alpha}. \tag{3.2}$$

The intuition is straightforward. Given $K(t)$ the isoquants of (2.1) in $(P(t), L(t))$ - space are linear with slope equal to the marginal rate of substitution equal to unity. Then, for an interior solution to be profit-maximizing $R_P(t)/w(t) = 1$ must hold. This has three implications. First, since the marginal product of automation capital is the same as the one for labor, both factor prices evolve in line with this marginal product, i. e., they increase in the complementary factor $K(t)$ and fall due to a diminishing marginal product. Second, at equal factor prices, the representative firm is indifferent as to what combination of inputs, $(L(t), P(t))$, to hire. Accordingly, in equilibrium it will hire the respective supplied quantities and find this profit-maximizing. Third, the equilibrium labor share, $LS(t)$, satisfies

$$LS(t) = \frac{w(t)L(t)}{Y(t)} = (1-\alpha)\left(\frac{L(t)}{L(t)+P(t)}\right). \tag{3.3}$$

Since the aggregate production function (2.1) is of the Cobb-Douglas type with constant returns to scale the income share that accrues to $K(t)$ is α. Hence, the share of the remaining factors is $1-\alpha$. The question is then how this share is divided between automation capital and labor. Since both factors of production are perfect substitutes each earns simply a fraction of $1-\alpha$ equal to its weight in $P(t)+L(t)$. As a consequence, the higher $P(t)$, the lower the labor share and the higher the share of automation capital.

(E2) requires the no-arbitrage condition (2.3) to hold. In conjunction with (2.2a) and (2.2b) this implies for all t that the marginal product of $K(t)$ and $P(t)$ has to coincide, i. e.,

$$P(t) = \frac{1-\alpha}{\alpha}K(t) - L(t). \tag{3.4}$$

Again, the intuition is straightforward. Given $(L(t), P(t))$, if $K(t)$ is small then the marginal product of traditional capital is high and the marginal product of automation capital is low. As $K(t)$ increases, the marginal product of traditional capital falls whereas the one of automation capital increases. Therefore, $K(t)$ has to be sufficiently large relative to $L(t)+P(t)$ to guarantee the equality of the two marginal products.

For three reasons, condition (3.4) is important. First, it imposes a constraint on the evolution of $P(t)$ and $K(t)$. In particular, at all t it must hold that

$$\dot{P}(t) = \frac{1-\alpha}{\alpha}\dot{K}(t) - \dot{L}(t). \tag{3.5}$$

Hence, since $\dot{L}(t)$ is exogenous, Prettner's stipulation of two independent laws of capital accumulation (see equation (3) in (Prettner (2019))), one for $K(t)$ and another one for $P(t)$, is inconsistent with the equilibrium of his model.[3]

[3] To develop this point further, observe that equation (3) in Prettner (2019) stipulates the following accumulation equations

$$\dot{K}(t) = s_K I(t) - \delta K(t) \quad \text{and} \quad \dot{P}(t) = (1-s_K)\,I(t) - \delta P(t),$$

where $s_K \in (0,1)$ is the exogenous investment share of traditional capital and, accordingly, $1-s_K$ is the investment share of automation capital. This specification is consistent with (3.5) if and only if

$$(1-s_K)\,I(t) - \delta P(t) = \frac{1-\alpha}{\alpha}\left(s_K I(t) - \delta K(t)\right) - nL(t)$$

Second, if $P(t) > 0$ then (3.4) requires for all t that

$$K(t) > \frac{\alpha}{1-\alpha} L(t). \tag{3.6}$$

Accordingly, not all conceivable evolutions of $K(t)$ will be consistent with an equilibrium.

Third, using (3.4) in the aggregate production function (2.1) reveals that

$$Y(t) = \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} K(t). \tag{3.7}$$

Hence, the no-arbitrage condition (2.3), i. e., the requirement that the relationship between $P(t)$ and $K(t)$ must be linear to maintain the equality of the marginal product of both types of capital, delivers a reduced form of the aggregate production function which is of the AK type. As in standard AK-models, this opens up the possibility of sustained long-run growth.

(E2) and (E3) imply that the market for the final good clears if $I(t) = S(t) = sY(t)$. Since $\mathcal{A}(t) = K(t) + P(t)$, the economy's aggregate capital stock evolves according to

$$\dot{\mathcal{A}}(t) = sY(t) - \delta A(t), \tag{3.8}$$

given $\mathcal{A}(0) = K(0) + P(0) > 0$ and (3.6) for $t = 0$.

(E4) and (E5) reflect the assumption that equilibrium factor prices adjust so that the representative firm is willing to hire the supplied factors of production. At all t, the break up of the entire capital stock in the supply of traditional and automation capital is determined by (3.4).

3.2 Equilibrium Factor Prices

Using (3.4) in (2.2a) - (2.2c) delivers

$$R_K(t) = R_P(t) = w(t) = (1-\alpha)^{1-\alpha}\alpha^{\alpha}. \tag{3.9}$$

Hence, the equilibrium has the remarkable property that all three equilibrium factor prices coincide and are time-invariant. This is the direct consequence of the assumed "double perfect substitutability:"

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or

$$\frac{I(t)}{\alpha}\left(s_K - \alpha\right) = \delta\left(\frac{1-\alpha}{\alpha}K(t) - P(t)\right) + nL(t).$$

In light of (3.4) this boils down to

$$\frac{I(t)}{\alpha}\left(s_K - \alpha\right) = \left(\delta + n\right) L(t).$$

Hence, for all t it must hold that $s_K > \alpha$ and $g_I(t) = n$.

i) as automation capital and labor are perfect substitutes their respective factor price must coincide since an equilibrium with $P(t) > 0$ and $L(t) > 0$ requires the representative firm to be indifferent between the hiring of either factor;

ii) since traditional and automation capital are perfect substitutes as stores of value, an equilibrium with $K(t) > 0$ and $P(t) > 0$ requires that households are indifferent between holding either type of capital.

4 Transitional Dynamics and Steady State

To describe the economy's evolution in terms of per-capita variables let

$$k(t) \equiv \frac{K(t)}{L(t)}, \quad p(t) \equiv \frac{P(t)}{L(t)}, \quad \text{and} \quad a(t) \equiv \frac{\mathcal{A}(t)}{L(t)}$$

denote traditional capital, automation capital, and total assets in per capita terms. Then, (3.4) becomes

$$p(t) = \frac{1-\alpha}{\alpha} k(t) - 1, \tag{4.1}$$

and (3.7) delivers the equilibrium output per capita as

$$y(t) = \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} k(t). \tag{4.2}$$

4.1 Dynamical System

The transitional dynamics of the dynamic competitive equilibrium can be analyzed through the lens of the state variable $k(t)$ and its evolution. To develop its equation of motion observe that (4.1) implies

$$a(t) = k(t) + p(t) = \frac{k(t)}{\alpha} - 1 \quad \text{and} \quad \dot{a}(t) = \dot{k}(t) + \dot{p}(t) = \frac{\dot{k}(t)}{\alpha}. \tag{4.3}$$

Moreover, (3.8) delivers the evolution of per-capita assets as

$$\dot{a}(t) = sy(t) - (\delta + n)\, a(t). \tag{4.4}$$

Combining (4.2) - (4.4), and denoting

$$g = s(1-\alpha)^{1-\alpha}\alpha^{\alpha} - (\delta + n) \tag{4.5}$$

gives the equilibrium differential equation for the evolution of $k(t)$ as

$$\dot{k}(t) = gk(t) + \alpha(\delta + n). \tag{4.6}$$

Finally, (3.6) delivers

$$k(t) > \frac{\alpha}{1-\alpha} \equiv \bar{k}. \tag{4.7}$$

Then, the following proposition holds.

Proposition 1 *(Dynamical System of the Equilibrium)*

Consider a dynamic competitive equilibrium involving $P(t) > 0$ for all t. If the initial conditions $K(0) > 0$, $P(0) > 0$, and $L(0) > 0$ satisfy (3.4) then such an equilibrium exists if and only if

$$s(1-\alpha)^{1-\alpha}\alpha^{\alpha} > \alpha(\delta+n).$$

Moreover, the following cases may arise:

1. *If $s(1-\alpha)^{1-\alpha}\alpha^{\alpha} \geq \delta+n$, then $g_k(t)$ falls monotonically with $\lim_{t\to\infty} g_k(t) = g \geq 0$.*
2. *If $\delta+n > s(1-\alpha)^{1-\alpha}\alpha^{\alpha} > \alpha(\delta+n)$, then*

$$\lim_{t\to\infty} k(t) = \frac{\alpha(\delta+n)}{\delta+n-s(1-\alpha)^{1-\alpha}\alpha^{\alpha}} = k^* > \bar{k}$$

and $\lim_{t\to\infty} g_k(t) = 0$.

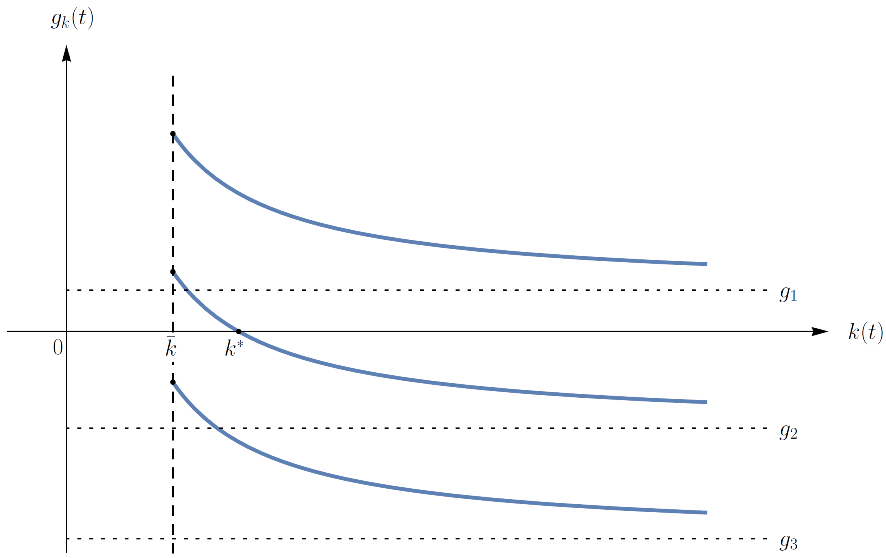
Proposition 1 gives a comprehensive description of the dynamic competitive equilibrium in which the representative firm demands $P(t) > 0$ for all t. This requires the initial conditions to satisfy (3.4), and $k(t) > \bar{k}$ must hold for all t. Figure 4.1 illustrates the following discussion of Proposition 1.

In Case 1, the asymptotic growth rate of traditional capital and all remaining per-capita variables is given by g of (4.5). Moreover, the model parameters are such that g is positive. This case is the one to be compared to Prettner's analysis. Observe that our expression for g differs from the one of Prettner. Indeed, in his equation 4, Prettner (2019) claims that the asymptotic growth rate is given by $g = s(1-s_K)^{1-\alpha}s_K^{\alpha} - (\delta+n)$ where $s_K \in (0,1)$ is the exogenous, time-invariant fraction of gross investment in the accumulation of traditional capital.

In Case 2, parameters are such that $0 > g > -(1-\alpha)(\delta+n)$, i. e., g is negative, but not too strongly so. Here, for any admissible set of initial conditions the economy monotonically converges to some $k^* > \bar{k}$, and the asymptotic growth rate of per-capita variables is zero.

Finally, consider the constellation where $\alpha(\delta+n) \geq s(1-\alpha)^{1-\alpha}\alpha^{\alpha}$. Then, $-(1-\alpha)(\delta+n) \geq g$, and no dynamic competitive equilibrium involving $P(t) > 0$ for all t exists. The intuition is straightforward. The economy may start with initial conditions satisfying $k(0) > \bar{k}$. However, over time $k(t)$ continuously falls. Moreover, either in finite time or asymptotically it holds that $k(t) = \bar{k}$. Then, the representative firm will no longer demand automation capital and the configuration ceases to be interior.

Figure 4.1: The Dynamical System of the Equilibrium.



Note: In Case 1, $g = g_1 > 0$, $g_k(t) > 0$, and $\lim_{t\to\infty} g_k(t) = g_1 > 0$. In Case 2, $g = g_2 < 0$ and $g_k(t) > 0$ holds for some $k(t) > \bar{k}$. Hence, for any $k(0) > \bar{k}$ we have $\lim_{t\to\infty} k(t) = k^*$. If $\alpha(\delta + n) \geq s(1-\alpha)^{1-\alpha}\alpha^{\alpha}$ then $g = g_3 \ll 0$ and $g_k(t) < 0$ for all $k(t) > \bar{k}$.

4.2 Endogenous Investment Rates and Labor Share Dynamics

The following corollary shows that the dynamic competitive equilibrium of Proposition 1 requires that the fraction of gross investment in the accumulation of traditional capital varies over time. Let us denote this fraction by $s_K(t)$. Then, the evolution of traditional and automation capital per capita may be written as

$$\dot{k}(t) = s_K(t)sy(t) - (\delta + n)k(t) \quad \text{and} \quad \dot{p}(t) = (1 - s_K(t))\, sy(t) - (\delta + n)p(t). \tag{4.8}$$

Corollary 1 *(Time-varying Fraction of Gross Investment in Traditional Capital)*

Consider Case 1 and 2 of Proposition 1. Then, for all $t \geq 0$

$$s_K(t) = \alpha\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\frac{\delta+n}{sk(t)}\right),$$

where

$$0 < s_K(t) < \alpha\left(1 + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\frac{\delta+n}{s}\right).$$

Moreover, in Case 1 it holds that $\lim_{t\to\infty} s_K(t) = \alpha$. *In Case 2 we have*

$$\lim_{t\to\infty} s_K(t) = \alpha - \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\frac{g}{s} > 0.$$

Accordingly, Prettner's assumption that $s_K(t) = s_K$ is inconsistent with the dynamic competitive equilibrium. To grasp the intuition use (4.2) in (4.8) to express $g_k(t)$ as

$$g_k(t) = s_K(t)s\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} - (\delta + n). \tag{4.9}$$

Hence, whenever $g_k(t)$ falls $s_K(t)$ must fall as well.

Next, we turn to the evolution of the equilibrium labor share. A direct implication of (3.9) and (4.2) is that

$$LS(t) = \frac{\alpha}{k(t)}. \tag{4.10}$$

The following corollary holds.

Corollary 2 *(Evolution of the Equilibrium Labor Share)*

For Case 1 of Proposition 1 and $g > 0$ *it holds that*

$$\lim_{t\to\infty} LS(t) = 0.$$

For Case 2 of Proposition 1 it holds that

$$\lim_{t\to\infty} LS(t) = \frac{\alpha}{k^*} > 0.$$

Hence, in Case 1, i. e., when the economy's asymptotic growth rate is strictly positive, then the labor share declines over time and vanishes asymptotically. In Case 2, the process of capital accumulation grinds to a halt allowing for the labor share to remain asymptotically strictly positive. Accordingly, the assumption that automation capital and labor are perfect substitutes is not sufficient for a vanishing labor share.

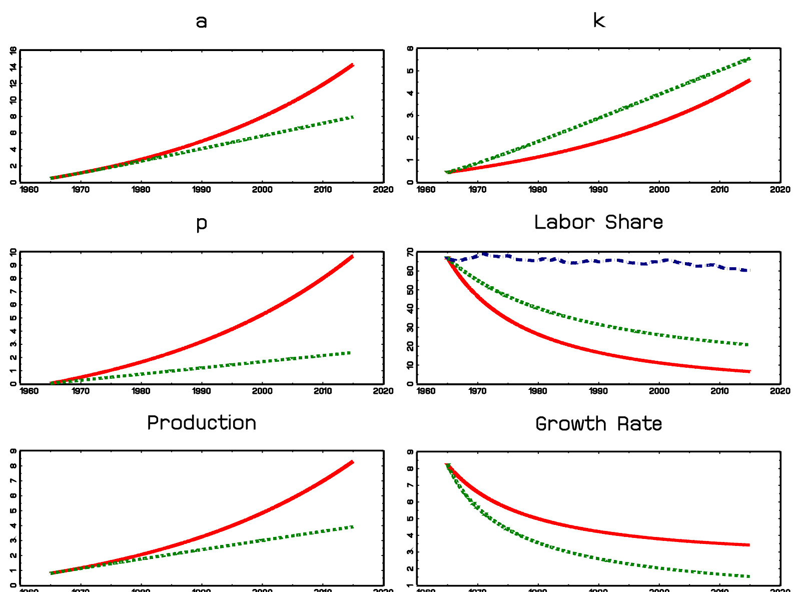
5 Computational Exercises

Between 1965 and 2015 the US labor share declined from 67% to 60% (see, AMECO database of the European Commission). Does a simulation of the dynamic competitive equilibrium of Proposition 1 help us understand this decline? How do Prettner's simulation results change if the no-arbitrage condition (2.3) is taken into account? To address these questions we first simulate the US economy for the time span 1965-2015 for Case 1 of Proposition 1. Here, traditional and automation capital earn the same rate of return, and the allocation of both types of capital is endogenous. Second, we replicate the simulation exercise of Prettner (2019) using his constant and exogenous savings rates for traditional and automation capital that violate the no-arbitrage condition (2.3).

To compute the transition from 1965 to 2015, we calibrate the model as follows. Following Prettner (2019) we set the output elasticity of traditional capital equal to $\alpha = 0.3$. Capital depreciates at an annual rate of $\delta = 7.0\%$ (see, e. g., Trabandt and Uhlig (2011)). According to United Nations (2015) the average annual population growth rate during 1965-2015 amounts to $n = 0.96\%$. Moreover, we set $s = 0.2$. This implies an asymptotic growth rate of per-capita variables of $g = 2.9\%$ and comes close to the actual average annual growth rate of the US economy over the considered time span. Notice that we choose a much lower savings rate than Prettner who picks a value of 0.3. However, with our choice the calibration comes much closer to the US gross private domestic investment share in GDP, which amounted to 17.7% during 1965-2015.[4]

To calibrate the initial values $a(0)$, $k(0)$, and $p(0)$ at time $t = 0$ (corresponding to the year 1965) we use equation (3.3) to express initial automation capital as $p(0) = (1 - \alpha)/LS(0) - 1$. The calibrated value for $p(0)$ obtains with the empirical value of the labor share in 1965, $LS(0) = 67\%$. Next, we use the no-arbitrage condition (3.4) to calibrate $k(0) = (1 - \alpha)/\alpha\,(1 + p(0))$, and, thus, $a(0) = k(0) + p(0)$. In the computation of Prettner's model we use his value for the traditional capital investment share equal to $s_m = 0.7$.

Figure 5.1: Transitional Dynamics with and without the No-arbitrage Condition (2.3)



Note: The solid red lines show the evolutions for Claim 1 of Proposition 1, the broken green lines those of Prettner's simulation. The dashed blue line in the middle right panel represents the actual evolution of the US labor share.

[4]U.S. Bureau of Economic Analysis, Shares of gross domestic product: Gross private domestic investment [A006RE1Q156NBEA], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/A006RE1Q156NBEA, June 1, 2019.

Figure 5.1 depicts the transition dynamics of per-capita assets, $a(t)$, traditional capital, $k(t)$, automation capital $p(t)$, the labor share $LS(t)$, output per worker $y(t)$, and the growth rate of per-capita variables, $g(t)$. The solid red lines show the evolutions for Claim 1 of Proposition 1, the broken green lines those of Prettner's simulation.[5] Evidently, the calibrated evolutions hinge critically on the underlying model specification. In the variant based on Case 1 of Proposition 1, households save a higher fraction of their savings in the form of automation capital (see the evolution of $p(t)$ in the medium left panel) and a lower fraction in the form of traditional capital, $k(t)$. This allows for a higher return on capital so that per-capita assets $a(t)$ grow faster than in the model of Prettner. As a consequence, per-capita output and its growth rate are also higher under the assumptions of Proposition 1 than in Prettner's model.

The medium right panel displays the dynamics of the labor share in the two models. In addition, we also present the time series of the US labor share during 1965-2015 (blue line).[6] Evidently, both models fail to replicate the behavior of the labor share during this period in a satisfactory way. While the labor shares of the two models coincide with the empirical labor share in 1965 due to our calibration strategy, they fall, respectively, to 6.5% and 20.7% by 2015, whereas the actual labor share amounted to roughly 60%.

6 Concluding Remark

This paper identifies a flaw in the analysis of Prettner (2019). We show that the marginal product of both types of capital in the production of the final good must be the same since traditional and automation capital are perfect substitutes as stores of value. This implies a linear constraint on the amounts of traditional and automation capital that is consistent with the competitive equilibrium. As a consequence, and in contrast to Prettner's modelling, the accumulation of traditional and automation capital is not independent. Rather, it has to assure this consistency.[7] A correct analysis of Prettner's model leads to the theoretical results of the present paper. Moreover, we show in our computational analysis that, in its present specification, the model dramatically overestimates the actual decline in the US labor share over the last 50 years.

[5]The transition dynamics are computed as the solution of a simple initial-value problem using the standard fourth-order Runge-Khutta method. The Gauss computer code is available from the authors upon request.

[6]Annual values of the empirical labor share time series have been interpolated to in-between data points using cubic spline interpolation.

[7]Our concern with Prettner's analysis does not hinge on the assumption that both types of capital are perfects substitutes in production but on the fact that they are perfect substitutes as stores of value. Hence, Prettner's modelling strategy with two independent laws of motions for traditional and automation capital would still be flawed if we allowed for both types of capital to be imperfect substitutes in production.

7 Appendix: Proofs

7.1 Proof of Proposition 1

If the initial conditions $K(0) > 0$, $P(0) > 0$, and $L(0) > 0$ satisfy (3.4) then, according to (4.7), $k(0) > \bar{k}$. If, in addition, $s(1-\alpha)^{1-\alpha}\alpha^{\alpha} > \alpha(\delta + n)$ then the two cases mentioned in the proposition may arise.

Case 1 has $g \geq 0$. Hence, from (4.6), for finite t it holds that $g_k(t) > 0$ and $\lim_{t\to\infty} g_k(t) = g \geq 0$.

In Case 2, $g < 0$, however, there is $k(t) \in (\bar{k}, k^*)$ such that $g_k(t) > 0$. Hence, for any $k(0) > \bar{k}$ it holds that $\lim_{t\to\infty} k(t) = k^*$ and $\lim_{t\to\infty} g_k(t) = 0$.

In contrast, if $\alpha(\delta + n) \geq s(1-\alpha)^{1-\alpha}\alpha^{\alpha}$, then $g < 0$, and $g_k(t) < 0$ for all $k(t) > \bar{k}$. Hence, for any $k(0) > \bar{k}$, $k(t)$ declines. If $\alpha(\delta + n) = s(1-\alpha)^{1-\alpha}\alpha^{\alpha}$ then $g_k = 0$ at $k(t) = \bar{k}$. In this case, $k = \bar{k}$ is a steady state with $P(t) = 0$. Hence, $\lim_{t\to\infty} k(t) = \bar{k}$ and the demand for automation capital vanishes asymptotically. If $\alpha(\delta + n) > s(1-\alpha)^{1-\alpha}\alpha^{\alpha}$ then $g_k < 0$ at $k(t) = \bar{k}$. Then, $k(t) = \bar{k}$ is reached in finite time. When $k(t) = \bar{k}$, the marginal product of traditional capital is equal to the marginal product of automation capital if and only if $P(t) = 0$. Hence, the demand for automation capital is zero and the equilibrium configuration is no longer interior. Accordingly, for this parameter constellation an equilibrium with $P(t) > 0$ for all t fails to exist. ■

7.2 Proof of Corollary 1

Equation (4.1) implies for all t that

$$\dot{p}(t) = \frac{1-\alpha}{\alpha}\dot{k}(t).$$

Hence, in conjunction with (4.8) it must hold that

$$(1 - s_K(t))\, sy(t) - (\delta + n)\left(\frac{1-\alpha}{\alpha}k(t) - 1\right) = \frac{1-\alpha}{\alpha}\left(s_K(t)sy(t) - (\delta + n)k(t)\right).$$

Solving for $s_K(t)$ delivers

$$s_K(t) = \alpha\left(1 + \frac{\delta + n}{sy(t)}\right).$$

With (4.2) the expression for $s_K(t)$ stated in the corollary follows. Since $s_K(t)$ declines in $k(t)$ and $k(t) > \bar{k}$ it must hold that

$$s_K(t) < \alpha\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\frac{\delta + n}{s\bar{k}}\right).$$

In Case 1 of Proposition 1 $\lim_{t\to\infty} k(t) = \infty$. Hence, $\lim_{t\to\infty} s_K(t) = \alpha$. In Case 2 of Proposition 1, $\lim_{t\to\infty} k(t) = k^*$. Hence

$$\lim_{t\to\infty} k(t) = \alpha\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\frac{\delta + n}{sk^*}\right).$$

■

7.3 Proof of Corollary 2

Immediate from (4.10) and Proposition 1. ■

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