CES Working Paper Series

THE ALGEBRA OF GOVERNMENT DEBT

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Working Paper No. 121

1996

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I am indebted to Sylvia Staudinger for helpful discussions and for calculating the figures of Tables 1-10. I would like to express my appreciation for the stimulating atmosphere at *CES*.

CES Working Paper No. 121
December 1996

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Abstract

A fiscal policy which achieves a reduction of the debt to GDP ratio below its initial level is labeled as fiscal retrenchment policy. In this paper we investigate four types of fiscal retrenchment policies. Type 1 describes a policy where the government chooses a constant primary budget surplus to GDP ratio, type 2 a policy where the government chooses a constant overall budget deficit to GDP ratio to achieve the reference value of the debt to GDP ratio. Following type 3, the Strong Gros-Rule, the government reduces the debt to GDP ratio every year by 5% of the discrepancy observed at the beginning of the fiscal retrenchment program and the target value. Finally, a type 4 policy (the Weak Gros-Rule) is investigated where the debt to GDP ratio declines every year by 5% of the difference between the one-period lagged debt ratio and the reference value.

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1 Introduction

The previous decade was a time of large fiscal deficits and increasing government debt-GDP ratios in many industrial countries. It was also a decade of growing concern that the excessive debt accumulation is not sustainable. The Maastricht Treaty requires participants in the European Monetary Union to achieve (or to at least show a satisfactory progress towards achieving) an upper limit for its fiscal deficit of 3% of GDP and a limit of 60% for the debt-GDP ratio, numbers which dominate the current public discussion. However, by the end of 1995, the average deficit of the EU member countries was 80% of GDP and the average deficit remained above 5% of GDP. Ten member states have to reduce their current debt-GDP ratio to satisfy the 60% target of GDP. Although the current public discussion of the Maastricht fiscal guidelines has shifted to the fiscal deficit criterion, we focus entirely on the question of imposing the debt-GDP ratio of 60% on countries with an excess debt-GDP ratio.

A fiscal policy which achieves a reduction in the debt-GDP ratio below its initial level is labeled a fiscal retrenchment policy. In this paper we investigate four types of fiscal retrenchment policies which differ in their instruments but have the common feature that they aim at a reduction of the future debt-GDP ratio. In all four cases we assume the Maastricht target of 60%.

a) *Fiscal Retrenchment Policy — Type 1*:

The government chooses a *constant* primary budget surplus to achieve the 60% debt target in T periods.

b) *Fiscal Retrenchment Policy — Type 2*:

The government chooses a *constant* overall budget deficit to achieve this as well.

The solutions of the difference equations in section 2 show that the required primary surplus (fiscal retrenchment type 1) and the overall budget deficit (fiscal retrenchment type 2) depend on the initial value of the debt to GDP ratio, the interest rate, the rate of growth of the economy and on the fiscal adjustment horizon T. For various (empirically relevant) combinations of these parameters we calculate the numerical solutions and present these results in tables 1 to 5 for the type 1 policy and in tables 6-10 for the type 2 policy. These tables provide a brief numerical orientation about the requirements of a fiscal retrenchment program given alternative economic scenarios. It does not say that these fiscal adjustment programs are feasible from a political point of view. It enables policy makers only to compare the fiscal adjustment program in terms of the overall deficit and in terms of the primary surplus and to make an assessment whether the stance of fiscal policy can be classified as "business as usual", "tight", or "very tight".

c) *Fiscal Retrenchment Policy — Type 3 (Strong Gros Rule):*

A different type of fiscal retrenchment has recently been suggested by D. Gros. It aims at reducing the debt to GDP ratio every year by one-twentieth of the difference between the debt ratio at the *beginning* of the retrenchment policy and the Maastricht target ratio of 60%. This rule implies that the debt to GDP ratio will be reduced in $T = 20$ periods from the initial level to the 60% ratio target. Contrary to the fiscal retrenchment policies of type 1 and 2 neither the primary surplus to GDP ratio nor the overall deficit to GDP ratio are constant during the time of adjustment. The primary surplus ratio has its maximum at the beginning of this policy and falls slowly with the corresponding overall deficit ratio.

d) *Fiscal Retrenchment Policy — Type 4 (Weak Gros Rule):*

The Gros rule can be specified in a different and more sophisticated way. In this case the government reduces the debt-GDP ratio by one-twentieth of the discrepancy between the *one-period lagged* debt to GDP ratio and the 60% target value. Under this rule the 60% target value is only reached for $t \to \infty$. Again, the primary surplus ratio is not constant during the adjustment process. The same result holds for the overall deficit ratio except for a single case.

Both rules were applied to the situation of Belgium and it is shown that the strong version of the Gros rule, if implemented, implies a "very tight" fiscal policy. The "pace" of fiscal retrenchment of the strong Gros rule exceeds the "pace" of fiscal retrenchment of type 1, type 2, and type 4 (weak Gros rule).

The paper is organized as follows: In Section 2 we analyze the basic difference equations describing the dynamics of the debt to GDP ratio. We distinguish two cases: The government i) keeps the overall deficit to GDP ratio constant, and ii) adopts a policy of keeping the primary surplus to GDP ratio constant. In section 3 we focus on fiscal retrenchment policy and discuss the four types sketched above: a policy of a constant primary surplus ratio (type 1), a policy of a constant overall deficit ratio (type 2), the strong Gros rule (type 3) and the weak Gros rule (type 4). The subsection on the Gros rules was written with the collaboration of Franz X. Hof.

2 Fiscal Policy Regimes and Public Debt

In this section we summarize some basic algebra of government debt accumulation. For convenience we will make the simplifying assumption that the interest rate r and the rate of growth of the economy θ are constant. Ignoring seigniorage the stock of public debt B_t evolves according to the difference equation

$$B_t = B_{t-1} + D_t \tag{1}$$

where the *overall* deficit D_t is defined by

$$D_t = rB_{t-1} + D_t^p \tag{2}$$

The overall deficit consists of two components, the interest payments on the outstanding debt rB_{t-1} and the *primary* deficit D_t^p given by

$$D_t^p = G_t - T_t \tag{3}$$

where G_t and T_t denote government expenditures on goods and services and net taxes (taxes minus transfers), respectively. The flow budget constraint of the government can therefore be rewritten as

$$B_t = (1 + r)B_{t-1} + D_t^p \tag{4}$$

For this analysis it is useful to rewrite these equations in terms of ratios to GDP. Using the definitions

$$b_t = \frac{B_t}{Y_t}, \qquad d_t = \frac{D_t}{Y_t}, \qquad d_t^p = \frac{D_t^p}{Y_t} \tag{5}$$

where Y denotes GDP, and taking into account that by assumption $(Y_t - Y_{t-1})/Y_{t-1} = \theta$ for all t, the difference equations (1) and (4) become

$$b_t = \left(\frac{1}{1+\theta}\right) b_{t-1} + d_t \tag{6}$$

and

$$b_t = \left(\frac{1+r}{1+\theta}\right) b_{t-1} + d_t^p, \tag{7}$$

respectively. Moreover, from (2), (3), and (5) follows that

$$d_t = \left(\frac{r}{1+\theta}\right) b_{t-1} + d_t^p \tag{8}$$

Rewriting (7) as

$$b_t - b_{t-1} = \left(\frac{r-\theta}{1+\theta}\right) b_{t-1} + d_t^p \tag{9}$$

produces the following important economic information: The change in the debt to GDP ratio b_t is explained by its inheritance of the past, the accumulated debt times the growth corrected interest rate $(r-\theta)/(1+\theta)$ and by the primary deficit to GDP ratio d_t^p. If the interest rate r exceeds the rate of growth of the economy θ, a primary budget surplus (i.e. $d_t^p < 0$) is needed to maintain a constant debt to GDP ratio.

In what follows we analyze the dynamics of the debt to GDP ratio under two fiscal rules: The government either follows a fiscal policy to keep the overall deficit to GDP ratio d_t constant, or adopts the target to keep the primary deficit to GDP ratio d_t^p constant.

2.1 Constant Deficit to GDP Ratio

This fiscal regime was originally investigated by E. Domar (1944). The government borrows each year a constant fraction of GDP to finance its overall deficit and maintains this policy indefinitely. It never repays the debt but continuously rolls it over. Denoting this constant overall deficit to GDP ratio by d and substituting $d_t = d$ into (6) we obtain:

$$b_t = \left(\frac{1}{1+\theta}\right) b_{t-1} + d \tag{10}$$

The solution of (10) is given by:

$$b_t = \left(\frac{1+\theta}{\theta}\right) d + \left[b_o - \left(\frac{1+\theta}{\theta}\right) d\right] \left(\frac{1}{1+\theta}\right)^t \tag{11}$$

If $\theta > 0$, we have

$$\lim_{t \to \infty} b_t = \left(\frac{1+\theta}{\theta} \right) d \tag{12}$$

If the government annually borrows a constant fraction of the GDP, d, to finance the budget deficit and maintains this policy indefinitely, the debt to GDP ratio converges eventually to a constant, namely $\theta^{-1}(1+\theta)\,d$. This was the famous but somewhat misleading message of the Domar model.

2.2 Constant Primary Deficit to GDP Ratio

Now we investigate the economic consequences of an alternative fiscal policy. The government adopts the target to keep the primary deficit to GDP ratio constant. Denoting this constant ratio by d^p and substituting $d_t^p = d^p$ into (7) we obtain:

$$b_t = \left(\frac{1+r}{1+\theta} \right) b_{t-1} + d^p \tag{13}$$

The solution of (13) is given by

$$b_t = -\left(\frac{1+\theta}{r-\theta} \right) d^p + \left(b_0 + \left(\frac{1+\theta}{r-\theta} \right) d^p \right) \left(\frac{1+r}{1+\theta} \right)^t \tag{14}$$

Central to the discussion of this solution is to distinguish two regimes: $r < \theta$ (regime A) and $r > \theta$ (regime B).

Regime A: If $r < \theta$ holds, we have

$$\lim_{t \to \infty} b_t = \left(\frac{1+\theta}{\theta - r} \right) d^p \tag{15}$$

Consider the following experiment: The government annually issues new debt and always refinances it, i.e. the interest charges and the outstanding debt are indefinitely rolled over. Since the interest rate remains forever below the rate of growth of the economy, the debt to GDP ratio converges eventually to a constant, namely $(\theta - r)^{-1}(1+\theta)\,d^p$.

Regime B: If $r > \theta$ holds, we have to distinguish two cases. Denoting the primary surplus to GDP ratio by s_t^p (i.e. $s_t^p = -d_t^p$) we obtain the following results:

B1) If $s^p \neq (1+\theta)^{-1}(r-\theta)\,b_0$, then $\lim_{t \to \infty} b_t$ does not exist.

B2) If $s^p = (1+\theta)^{-1}(r-\theta)\,b_0$, then $b_t = b_0, \forall t > 0$.

From B2) follows that there exists a unique level of s^p which causes b_t to remain at its initial level b_0. It is given by

$$s^p|_{b_t = b_0} = \left(\frac{r-\theta}{1+\theta} \right) b_0 \tag{16}$$

(16) describes the permanent required primary surplus to GDP ratio which ensures that $b_t = b_0$.

In the rest of this section we restrict attention to regime B where $r > \theta$ holds. Denoting the value of s^p which ensures that the No-Ponzi Game Condition (NPGC) is satisfied by $s^p|_{NPG}$ it can be shown that:

$$s^p|_{NPG} = s^p|_{b_t=b_0} \tag{17}$$

Premultiplying both sides of equation (14) by the discount factor $[(1+r)/(1+\theta)]^{-t}$ and taking into account that $s^p = -d^p$ yields:

$$\left(\frac{1+r}{1+\theta}\right)^{-t} b_t = \left(\frac{1+r}{1+\theta}\right)^{-t} \left(\frac{1+\theta}{r-\theta}\right) s^p + b_0 - \left(\frac{1+\theta}{r-\theta}\right) s^p \tag{18}$$

The NPGC requires that the discounted value of the debt to GDP ratio goes to zero as $t \to \infty$, i.e.

$$\lim_{t\to\infty} \left(\frac{1+r}{1+\theta}\right)^{-t} b_t = 0 \tag{19}$$

From (18) follows that

$$\lim_{t\to\infty} \left(\frac{1+r}{1+\theta}\right)^{-t} b_t = b_0 - \left(\frac{1+\theta}{r-\theta}\right) s^p \tag{20}$$

Obviously, the NPGC holds if the right-hand side of (20) is zero. Hence, we have

$$s^p|_{NPG} = \left(\frac{r-\theta}{1+\theta}\right) b_0 \tag{21}$$

According to (21) the NPGC (i.e. the intertemporal budget "solvency" condition) implies that the constant primary surplus to GDP ratio s^p be equal to the outstanding debt ratio times the growth corrected long run interest rate, $(1+\theta)^{-1}(r-\theta)$. Comparing (16) and (21), it is obvious that (17) holds.

A fiscal policy which achieves a constant primary surplus to GDP ratio $s^p = s^p|_{NPG}$ is a "sustainable" fiscal policy as proposed by Blanchard et al. (1990) and Buiter et al. (1993). Note that the NPGC imposes only a mild form of solvency restriction on current fiscal policy. The resulting $s^p|_{NPG}$ is the minimum amount of debt servicing (as a fraction of GDP) required to prevent an increase in the debt to GDP ratio, when the interest rate exceeds the rate of growth of GDP.

3 Fiscal Retrenchment Policies

Now we address the main problem of this paper. A fiscal policy that adopts the target to *reduce* the debt to GDP ratio is more ambitious than a sustainable policy and shall be labeled a fiscal retrenchment policy.

3.1 Fiscal Retrenchment Policy — Type 1

Let us assume that the government starts with an initial debt ratio b_0 and intends to choose that *constant* level of the *primary* surplus to GDP ratio

s^p which yields $b_T = \hat{b}$, where $\hat{b}$ denotes the government's target level. This policy will be labeled Fiscal Retrenchment Policy – Type 1. Substituting $t = T$ and $b_T = \hat{b}$ into (18) and solving for s^p we obtain

$$s^p|_{b_T=\hat{b}} = \left[1 - \left(\frac{1+r}{1+\theta}\right)^{-T}\right]^{-1} \left(\frac{r-\theta}{1+\theta}\right) \left[b_0 - \left(\frac{1+r}{1+\theta}\right)^{-T} \hat{b}\right] \qquad (22)$$

As one can easily see the value of the required constant primary surplus to GDP ratio depends on the initial debt ratio b_0, the target level $\hat{b}$, the length of the time horizon T, the interest rate r, and the growth rate of GDP θ, i.e.

$$s^p|_{b_T=\hat{b}} = s^p|_{b_T=\hat{b}}\,(b_0, \hat{b}, T, r, \theta) \qquad (23)$$

It can be shown that ($r > \theta$ and $b_0 > \hat{b}$) is sufficient for

$$\frac{\partial\, s^p|_{b_T=\hat{b}}}{\partial b_0}, \frac{\partial\, s^p|_{b_T=\hat{b}}}{\partial r} > 0; \qquad \frac{\partial\, s^p|_{b_T=\hat{b}}}{\partial \hat{b}}, \frac{\partial\, s^p|_{b_T=\hat{b}}}{\partial T}, \frac{\partial\, s^p|_{b_T=\hat{b}}}{\partial \theta} < 0 \qquad (24)$$

Equation (22) allows a quick numerical analysis of the requirements of a fiscal retrenchment policy. The numerical solutions of (22) are shown in tables 1 – 5. They denote the necessary primary surplus for various combinations of the parameters b_0, r and θ. Each table is prepared for a given adjustment period, namely for $T = 2$, $T = 5$, $T = 10$, $T = 20$, and $T = 25$ years. The debt to GDP ratio target which should be achieved in T years is 60% in all cases to be consistent with the Maastricht guidelines. The first column b_0 shows alternative initial values of the debt to GDP ratio. The following columns indicate alternative values of $(r - \theta)$. Since countries can not control (generally) the interest rate and the rate of growth of the economy the instrument variable to reduce b_t is the primary surplus. A higher value of the primary surplus can be achieved either by increasing the tax ratio τ or by decreasing the expenditure ratio g. Whether a reduction in government spending is a more successful way to increase the primary surplus permanently or a tax increase, is a controversial issue in recent studies. For example, Alesina and Perotti (1995) found in empirical studies that a cut in government spending is an efficient way to reduce the debt to GDP ratio, while Hughes Hallett and McAdam (1995) argue, on the contrary, that a tax increase in combination with fiscal discipline is a more efficient measure. In this paper we treat the s^p as *one* variable and do not separate it further into a spending and a government revenue component.

Given the fact that an upper limit τ_u exists on feasible government revenues and a lower limit on government expenditure g_l which is politically acceptable, it is obvious that an upper limit on the primary surplus must exist. This maximal value of s^p is vague and difficult to determine. Recent historical experience suggest that a primary surplus up to 1% might be classified as a "business as usual" fiscal policy. A primary surplus up to 3% is a "tight" fiscal policy implying already a painful fiscal adjustment. If the required primary surplus is between 3% and 6%, a massive fiscal retrenchment is necessary and the fiscal policy is "very tight". Budget surpluses above 6% are blatantly unrealistic in a democratic political environment in which the government has to look forward to the next election year. This classification might serve as a benchmark for dividing the numbers in table 1 to 5 into those which have an economic meaning and those which are simply the outcome of numerical calculations.

Primary budget surplus to GDP ratios under fiscal retrenchment policy type 1

b_0 \ $r-\theta$	1	2	3	4	5
140	41.14	42.30	43.49	44.70	45.94
135	38.60	39.73	40.88	42.05	43.25
125	33.53	34.58	35.66	36.76	37.88
120	31.00	32.01	33.05	34.11	35.19
100	20.85	21.72	22.61	23.52	24.45
95	18.32	19.15	20.00	20.87	21.76
90	15.78	16.58	17.39	18.23	19.07
85	13.25	14.01	14.79	15.58	16.39
80	10.71	11.44	12.18	12.93	13.70
75	8.18	8.86	9.57	10.28	11.01
70	5.64	6.29	6.96	7.64	8.33
65	3.10	3.72	4.35	4.99	5.64

Table 1: $T = 2$, $\hat{b} = 60\%$, $r = 6.5\%$

b_0 \ $r-\theta$	1	2	3	4	5
140	17.03	18.08	19.16	20.26	21.40
135	16.00	17.02	18.07	19.14	20.24
125	13.94	14.90	15.89	16.90	17.94
120	12.91	13.85	14.80	15.78	16.79
100	8.80	9.61	10.45	11.30	12.18
95	7.77	8.56	9.36	10.18	11.02
90	6.74	7.50	8.27	9.06	9.87
85	5.71	6.44	7.18	7.94	8.72
80	4.68	5.38	6.09	6.82	7.57
75	3.65	4.32	5.01	5.70	6.41
70	2.63	3.26	3.92	4.58	5.26
65	1.60	2.21	2.83	3.46	4.11

Table 2: $T = 5$, $\hat{b} = 60\%$, $r = 6.5\%$

$r-\theta$ / $b_\circ$	1	2	3	4	5
140	8.99	10.01	11.07	12.16	13.28
135	8.47	9.46	10.49	11.54	12.63
125	7.41	8.35	9.32	10.32	11.34
120	6.89	7.80	8.74	9.70	10.70
100	4.78	5.58	6.40	7.25	8.12
95	4.25	5.03	5.82	6.64	7.47
90	3.73	4.47	5.24	6.02	6.83
85	3.20	3.92	4.65	5.41	6.18
80	2.67	3.36	4.07	4.80	5.54
75	2.15	2.81	3.49	4.18	4.89
70	1.62	2.26	2.91	3.57	4.25
65	1.10	1.70	2.32	2.95	3.60

Table 3: $T = 10$, $\hat{b} = 60\%$, $r = 6.5\%$

$r-\theta$ / $b_\circ$	1	2	3	4	5
140	4.98	6.00	7.07	8.18	9.33
135	4.70	5.70	6.73	7.81	8.94
125	4.15	5.09	6.07	7.08	8.14
120	3.88	4.79	5.73	6.72	7.74
100	2.77	3.57	4.40	5.26	6.15
95	2.50	3.27	4.07	4.89	5.75
90	2.22	2.97	3.74	4.53	5.35
85	1.95	2.66	3.40	4.17	4.95
80	1.67	2.36	3.07	3.8	4.55
75	1.40	2.06	2.74	3.44	4.15
70	1.12	1.75	2.41	3.07	3.75
65	0.84	1.45	2.07	2.71	3.35

Table 4: $T = 20$, $\hat{b} = 60\%$, $r = 6.5\%$

$r-\theta$ / b_0	1	2	3	4	5
140	4.18	5.20	6.28	7.41	8.59
135	3.95	4.95	6.00	7.09	8.24
125	3.50	4.44	5.43	6.46	7.53
120	3.28	4.19	5.15	6.14	7.18
100	2.37	3.18	4.01	4.88	5.77
95	2.15	2.92	3.73	4.56	5.42
90	1.92	2.67	3.44	4.24	5.07
85	1.70	2.42	3.16	3.93	4.72
80	1.47	2.16	2.87	3.61	4.36
75	1.25	1.91	2.59	3.29	4.01
70	1.02	1.66	2.31	2.97	3.66
65	0.79	1.40	2.02	2.66	3.31

Table 5: $T = 25, \quad \hat{b} = 60\%, \quad r = 6.5\%$

3.2 Fiscal Retrenchment Policy — Type 2

Although the tables with the primary surplus show clearly the fiscal adjustment needed to consolidate the government budget, a major part of the public discussion of the Maastricht guidelines and the literature emphasize the stabilization of the overall deficit ratio d_t. Assume that the government starts with an initial debt ratio b_0 and intends to choose that *constant* level of the overall deficit to GDP ratio d which yields $b_T = \hat{b}$. This policy is labeled Fiscal Retrenchment Policy - Type 2. Substituting $t = T$ and $b_T = \hat{b}$ into (11) and solving for d we obtain

$$d|_{b_T=\hat{b}} = \left(\frac{\theta}{1+\theta}\right)\left[(1+\theta)^T - 1\right]^{-1}\left[(1+\theta)^T\hat{b} - b_0\right] \tag{25}$$

This equation can be interpreted in a similar manner as equation (22). The overall deficit ratio $d|_{b_T=\hat{b}}$ indicates the constant overall deficit ratio needed to achieve the reduction of the initial debt to GDP ratio b_0 to the target value $\hat{b}$ in T periods. From inspection of (25) follows that the sign of $d|_{b_T=\hat{b}}$ depends on the expression $\left[(1+\theta)^T\hat{b} - b_0\right]$. If this expression is positive, equation (25) indicates an overall deficit, in the opposite case it indicates an overall surplus.[1] The switch in the signs of the numbers in the matrices can be seen in all tables 6 to 10. The reason is the following: A reduction in the debt-GDP ratio requires a primary surplus in all cases (see tables 1 - 5). However, depending on the initial value b_0 the interest payments on

[1] In the numerical analysis in tables 6 - 10 the positive numbers denote overall surpluses, negative numbers overall deficits.

the existing debt $(1+\theta)^{-1} r b_{t-1}$ can exceed the primary surplus (we have an overall deficit) or falls short of the primary surplus; in this case an overall surplus develops.

Overall budget surplus to GDP ratios under fiscal retrenchment policy type 2

b_o \ θ	5.5	4.5	3.5	2.5	1.5
140	33.77	34.85	35.95	37.08	38.23
135	31.47	32.51	33.58	34.67	35.78
125	26.85	27.83	28.83	29.85	30.90
120	24.55	25.49	26.46	27.44	28.45
100	15.32	16.13	16.96	17.81	18.67
95	13.02	13.79	14.59	15.40	16.23
90	10.71	11.45	12.21	13.00	13.78
85	8.41	9.12	9.84	10.58	11.34
80	6.10	6.78	7.47	8.17	8.89
75	3.79	4.44	5.09	5.76	6.45
70	1.48	2.10	2.72	3.35	4.00
65	−0.82	−0.24	0.35	0.95	1.59

Table 6: $T = 2$, $\hat{b} = 60\%$

b_o \ θ	5.5	4.5	3.5	2.5	1.5
140	10.46	11.41	12.39	13.39	14.41
135	9.61	10.53	11.48	12.46	13.46
125	7.91	8.79	9.68	10.60	11.54
120	7.06	7.90	8.78	9.67	10.58
100	3.67	4.40	5.18	5.96	6.76
95	2.82	3.54	4.28	5.03	5.81
90	1.97	2.66	3.38	4.10	4.85
85	1.12	1.79	2.48	3.18	3.89
80	0.27	0.91	1.57	2.25	2.94
75	−0.58	0.01	0.67	1.32	1.98
70	−1.43	−0.83	−0.23	0.39	1.03
65	−2.27	−1.71	−1.13	−0.54	0.07

Table 7: $T = 5$, $\hat{b} = 60\%$

θ / $b_\circ$	5.5	4.5	3.5	2.5	1.5
140	2.76	3.64	4.56	5.50	6.48
135	2.39	3.26	4.15	5.07	6.02
125	1.66	2.48	3.32	4.20	5.10
120	1.29	2.09	2.91	3.76	4.64
100	−0.18	0.53	1.27	2.02	2.79
95	−0.55	0.14	0.85	1.58	2.33
90	−0.92	−0.25	0.44	1.15	1.87
85	−1.29	−0.64	0.03	0.71	1.41
80	−1.66	−1.03	−0.38	0.28	0.95
75	−2.02	−1.42	−0.79	−0.16	0.49
70	−2.39	−1.80	−1.21	−0.59	0.03
65	−2.76	−2.19	−1.62	−1.03	−0.43

Table 8: $T = 10$, $\hat{b} = 60\%$

θ / $b_\circ$	5.5	4.5	3.5	2.5	1.5
140	−0.95	−0.14	0.70	1.59	2.52
135	−1.09	−0.30	0.53	1.40	2.31
125	−1.36	−0.60	0.19	1.02	1.88
120	−1.49	−0.75	0.02	0.83	1.67
100	−2.04	−1.36	−0.66	0.06	0.82
95	−2.18	−1.52	−0.83	−0.13	0.61
90	−2.31	−1.67	−1.00	−0.32	0.39
85	−2.45	−1.82	−1.17	−0.51	0.18
80	−2.58	−1.97	−1.35	−0.70	−0.03
75	−2.72	−2.13	−1.52	−0.89	−0.25
70	−2.86	−2.28	−1.69	−1.08	−0.46
65	−2.99	−2.43	−1.86	−1.27	−0.67

Table 9: $T = 20$, $\hat{b} = 60\%$

b_0 \ θ	5.5	4.5	3.5	2.5	1.5
140	−1.65	−0.87	−0.04	0.82	1.74
135	−1.74	−0.97	−0.17	0.68	1.57
125	−1.92	−1.19	−0.42	0.39	1.24
120	−2.02	−1.29	−0.54	0.25	1.08
100	−2.39	−1.72	−1.04	−0.32	0.42
95	−2.48	−1.83	−1.16	−0.46	0.26
90	−2.57	−1.94	−1.28	−0.61	0.10
85	−2.66	−2.05	−1.41	−0.75	−0.07
80	−2.76	−2.15	−1.53	−0.89	−0.23
75	−2.85	−2.26	−1.66	−1.03	−0.39
70	−2.94	−2.37	−1.78	−1.18	−0.56
65	−3.03	−2.48	−1.91	−1.32	−0.72

Table 10: $T = 25, \quad \hat{b} = 60\%$

Let us compare tables 1 to 5 (determining the constant primary surplus) with the tables 6 - 10 (determining the overall budget deficit). As one example how these tables can be used, we focus on table 3 ($T = 10$ years). Select the row $b_0 = 100$. The numbers in this rows increase. This is caused by the increase in the value of $(r - \theta)$ requiring a higher primary surplus to achieve the debt target. Since r is kept constant in the tables, the increase in $(r - \theta)$ indicates a slowdown in the rate of growth. Selecting a specific value of $(r - \theta)$, i.e. a given column in table 3, we see that the numbers in this column decrease as we move downward. If we start with $b_0 = 135\%$, a higher primary surplus is necessary than if we start with an initial debt-GDP ratio of, say, 70%.

Look at the corresponding table 8 ($T = 10$). The positive numbers denote an overall surplus, the negative numbers an overall deficit. Start again with the row $b_0 = 100$. The elements in this row increase as we move from left to the right. This reflects the fact that the rate of growth of the economy θ is declining as we move from the left to the right. Selecting a specific column (for a given nominal rate of growth θ) shows that the numbers decrease as we move downwards. The reason is obvious. A smaller initial value of b_0 needs a smaller overall surplus or is even compatible with a (small) overall budget deficit. The noticeable change in the sign of the entries in the matrices of table 6 - 10 is explained by two conflicting forces determining a fiscal retrenchment policy: the primary surplus reducing the overall deficit and the interest payments on the existing debt working into the opposite direction.

A final comment on the relationship between the fiscal retrenchment policies 1 and 2 is necessary. Substituting $d_t^p = -\, s_t^p$ into (8) we obtain

$$d_t + s_t^p = \left(\frac{r}{1+\theta}\right) b_{t-1} \tag{26}$$

Taking into account that under both fiscal retrenchment policies b_{t-1} declines gradually it is obvious that

a) under the fiscal retrenchment policy 1 (where s_t^p is kept constant over time) the overall deficit to GDP ratio d_t declines gradually (i.e. $d_1 > d_2 > \ldots > d_T$), while

b) under the fiscal retrenchment policy 2 (where d_t is kept constant), the primary surplus to GDP ratio s_t^p decreases over time (i.e. $s_1^p > s_2^p > \ldots > s_T^p$).

3.3 Fiscal Retrenchment Policies — Type 3 and 4 (The Gros Rules)

The tables of the previous section show clearly that it is impossible for highly indebted countries to reduce their debt-GDP ratio to the Maastricht reference value of 60% in any foreseeable future. Daniel Gros (1996) emphasizes in a recent study that the second paragraph of Article 104c of the Treaty only requires that for the foreseeable future the debt-GDP ratio must be "approaching the reference value at a satisfactory pace." The crucial question is: What is a "satisfactory pace"? One can get two answers from Gros' analysis. First, he suggests that the following practical rule could be adopted informally by ECOFIN:

> "The debt to GDP ratio is considered "approaching the reference value at a satisfactory pace" if, over the last three years, it has been declining continuously, and the total reduction has been equal to three-twentieths of the difference between the debt ratio at the beginning of the three-year period and the reference value of 60%." [Gros (1996), p. 36]

Gros stresses "that this rule should also apply *after* a country has joined in order to ensure continued movement towards the 60% target." [Gros (1996), p. 38].

What is the mathematical representation of this rule? If the debt to GDP ratio declines every year by one-twentieth of the difference between the debt ratio at the *beginning* of the fiscal retrenchment phase and the reference value of 60%, then the evolution of b_t is given by:

$$b_t = b_{t-1} - \frac{1}{20}\,(b_0 - 0.6)\,, \qquad t = 1, \ldots, 20 \tag{27}$$

For reasons which will become obvious later, this rule will be called the *strong* Gros rule. In this paper we study a more general form of this rule. Just as in the sections on fiscal retrenchment type 1 and type 2 we assume that the government aims at reducing the debt to GDP ratio from the initial value b_0 to the target value $\hat{b}$ within T periods. Contrary to the preceding sections we now assume that this goal should be achieved by a policy which ensures that b_t declines every year by one-T'th of $\left(b_0 - \hat{b}\right)$, so that

$$b_t = b_{t-1} - \frac{1}{T}\left(b_o - \hat{b}\right), \qquad t = 1, \ldots, T \tag{28}$$

holds. From (28) follows that

$$b_t = b_0 - \frac{t}{T}\left(b_o - \hat{b}\right), \qquad t = 1, \ldots, T \tag{29}$$

In order to ensure that the time path of b_t implied by the generalized version of the strong Gros rule is realized, the government has to choose the primary surplus to GDP ratio s_t^p according to the following rule:

$$s_t^p = \left(\frac{r-\theta}{1+\theta}\right) b_0 - \left(\frac{-(1+r)+(r-\theta)\,t}{(1+\theta)\,T}\right)\left(b_o - \hat{b}\right) \tag{30}$$

(30) implies that the time path of the overall deficit to GDP ratio d_t is given by

$$d_t = \left(\frac{\theta}{1+\theta}\right) b_0 - \left(\frac{1+\theta t}{(1+\theta)\,T}\right)\left(b_o - \hat{b}\right) \tag{31}$$

From (30) and (31) follows that neither s_t^p nor d_t is constant over time under this generalized version of the strong Gros rule. In case that $b_0 > \hat{b}$ holds, the following results are obvious:

i) s_t^p declines over time if and only if $r > \theta$ holds,

ii) d_t decreases unambiguously as t increases.

Before illustrating the above-derived results by numerical calculations, we will study the second rule that can be inferred from Gros' analysis. The main ideas presented in Box III.1 (p. 37) and Box III.2 (p. 41) can be summarized as follows:[2] From (6) follows that

$$b_t = b_{t-1} - \left(\frac{\theta}{1+\theta}\right)\left[b_{t-1} - \left(\frac{1+\theta}{\theta}\right) d_t\right] \tag{32}$$

Substituting $d_t = 0.03$ for all t and $\theta = (0.05/0.95) \approx 0.053$ into (32) we obtain

$$b_t = b_{t-1} - 0.05\,(b_{t-1} - 0.6) \tag{33}$$

From (33) follows: If the government chooses a *constant* overall deficit to GDP ratio of 3% and the growth rate of GDP happens to be 5.3%, then the debt to GDP ratio converges to the value of 60%. Since (33) can be rewritten as

$$b_t - 0.6 = (1 - 0.05)\,(b_{t-1} - 0.6) \tag{34}$$

it is obvious that the time path of b_t is characterized by the fact

[2] Our respresentation differs slightly from that chosen by Gros in Box III.1. In our paper we use the exact discrete-time representations, while in Box III.1. equations (1) and (2) represent approximations which result if in the exact continuous-time representations

$$b'(t) = d(t) - b(t)\theta(t) \qquad \text{and} \qquad b'(t) = -0.05\,(b(t) - 0.6)$$

the derivative of b with respect to time, $b'(t)$, is replaced by $b_t - b_{t-1}$.

"that one-twentieth (0.05) of the discrepancy between the actual debt ratio and the Maastricht target would be automatically eliminated each year" [Gros (1996), Box III.1, page 37].

This result derived for the special case in which $d_t = 0.03$ for all t and $\theta = 0.05/0.95$ holds, forms the basis of the second Gros rule. In case that the growth rate θ is constant (as assumed in our paper), this rule has the following form: The debt to GDP ratio b_t is considered approaching the reference value at a satisfactory pace if b_t is governed by (34). Since (34) and (33) are equivalent, this rule can also be expressed as follows:

> The pace of fiscal retrenchment is satisfactory if the debt to GDP ratio declines every year by one-twentieth of the difference between the *one-period lagged* debt ratio and the reference value of 60%.

Illustrating the implications of (33) by graphing the path of b_t for the case in which b_0 is 120%, Gros stresses that even "after 30 years the ratio would still be above 70%." [Gros (1996), Box III.2, page 41]. Obviously, the speed of adjustment under the second version of the Gros rule given by (34) and (33), respectively, is lower than under the first version given by (27), where b_t reaches the 60% value exactly after 20 years. This is the reason why i) (27) was called the *strong* Gros rule, and ii) (33) will be labeled as *weak* Gros rule in the following.

Taking into account that a fall in θ lowers $\theta\,(1+\theta)^{-1}$ and consequently raises $(1+\theta)\,\theta^{-1}$, it is obvious from (32) and (33) that serious problems arise if the growth rate of GDP drops below the critical value of 0.053: If the government still chooses a constant overall deficit to GDP ratio of 3% (i.e. $d = 3\%$), the debt to GDP ratio will converge to a long-run value which is above the Maastricht reference level. This problem could be avoided by choosing a lower overall deficit ratio. The value of d which yields long-run convergence of b_t to the 60% level is given by

$$d|_{b_t \to 0.6} = 0.6\left(\frac{\theta}{1+\theta}\right) \tag{35}$$

Assume, for instance, that $\theta = 1/24 \approx 4.17\%$. In this case we have

$$\left(\frac{\theta}{1+\theta}\right) = 0.04, \qquad d|_{b_t \to 0.6} = 2.4\% \tag{36}$$

If the growth rate of GDP is 1/24 ($\approx$ 4.17%), then the debt ratio converges automatically to the Maastricht value of 60%, if the government sets $d = 2.4\%$. But (36) also implies that there is a second problem that cannot be remedied by appropriately choosing d: the speed of adjustment is too low. Since $\theta\,(1+\theta)^{-1} = 1/25$, only 1/25 instead of 1/20 of the discrepancy between the actual debt ratio and the Maastricht target would be automatically eliminated each year.

Now we show that also the second problem can be avoided if one does not restrict attention to fiscal policies where the overall deficit is constant over time. More specifically, we will derive a policy which ensures that the generalized weak Gros rule

$$b_t - b_{t-1} = -\alpha\left(b_{t-1} - \hat{b}\right) \tag{37}$$

holds, so that the discrepancy between the actual debt ratio b_t and the reference value $\hat{b}$ declines by $100\alpha\%$ p.a. regardless of whether θ happens to be 5.3% or not. (37) implies that

$$b_t = \hat{b} + \left(b_0 - \hat{b}\right)(1-\alpha)^t \tag{38}$$

In order to ensure that this time path of b_t implied by the generalized weak Gros rule is realized, the government has to choose the primary surplus to GDP ratio s_t^p according to the following rule:

$$s_t^p = \left(\frac{r-\theta}{1+\theta}\right)\hat{b} + \left[\left(\frac{1+r}{(1-\alpha)(1+\theta)}\right) - 1\right]\left(b_0 - \hat{b}\right)(1-\alpha)^t \tag{39}$$

(39) implies that the time path of the overall deficit to GDP ratio d_t is given by

$$d_t = \left(\frac{\theta}{1+\theta}\right)\hat{b} - \left[\left(\frac{1}{(1-\alpha)(1+\theta)}\right) - 1\right]\left(b_0 - \hat{b}\right)(1-\alpha)^t \tag{40}$$

(39) and (40) imply that in general s_t^p and d_t are not constant over time under the generalized weak Gros rule. In the case of fiscal retrenchment (i.e. $b_0 > \hat{b}$) we have the following results:

i) $r > \theta$ is a sufficient (but not necessary) condition for s_t^p to decline over time.

ii) With respect to the time path of d_t we have to distinguish three cases: As t increases, d_t

A) remains constant if

$$(1-\alpha)(1+\theta) = 1 \tag{41}$$

B) increases if

$$(1-\alpha)(1+\theta) < 1 \tag{42}$$

C) decreases if

$$(1-\alpha)(1+\theta) > 1 \tag{43}$$

The above-derived results are now illustrated by tables 11 and 12. In both tables it is assumed that the initial value of the debt to GDP ratio, b_0, is 135% ("Belgian" case). We further assume that $\alpha = 1/T = 1/20$. With respect to the growth rate of GDP, θ, we consider three cases. The primary surplus to GDP ratio is the same for both Gros rules in the *first* period of adjustment. s_t^p declines over time for both regimes, but it falls less under the strong Gros rule. Consequently, the debt to GDP ratio, b_t, declines faster under the strong rule. While under the strong rule b_t reaches the reference value of 60%

after 20 years, the weak rule yields $b_{20} \approx 87\%$. With respect to the overall deficit to GDP ratio, d_t, we have: Under the strong Gros rule, d_t declines over time for all values of θ. Under the weak Gros rule the behaviour of d_t is ambiguous. d_t remains constant over time if $\theta \approx 5.3\%$ [see case A), condition (41)], increases for $\theta = 4.5\%$ [see case B), condition (42)], and decreases for $\theta = 6.0\%$ [see case C), condition (43)]. Note that if $\theta = 6.0\%$ holds, a problem arises that was not yet mentioned in our theoretical considerations: Under the weak Gros rule, the overall deficit to GDP ratio is permanently above the Maastricht reference value of 3%.

			$\theta = \frac{0.05}{0.95} \approx 5.3\%,\ r = 6.5\%$			
t	b_{SGR}	b_{WGR}	d_{SGR}	d_{WGR}	s^p_{SGR}	s^p_{WGR}
1	131.25	131.25	3.00	3.00	5.34	5.34
2	127.50	127.69	2.81	3.00	5.29	5.10
3	123.75	124.30	2.63	3.00	5.25	4.88
4	120.00	121.09	2.44	3.00	5.20	4.68
5	116.25	118.03	2.25	3.00	5.16	4.48
6	112.50	115.13	2.06	3.00	5.12	4.29
7	108.75	112.38	1.88	3.00	5.07	4.11
8	105.00	109.76	1.69	3.00	5.03	3.94
9	101.25	107.27	1.50	3.00	4.98	3.78
10	97.50	104.91	1.31	3.00	4.94	3.62
11	93.75	102.66	1.13	3.00	4.90	3.48
12	90.00	100.53	0.94	3.00	4.85	3.34
13	86.25	98.50	0.75	3.00	4.81	3.21
14	82.50	96.58	0.56	3.00	4.76	3.08
15	78.75	94.75	0.38	3.00	4.72	2.96
16	75.00	93.01	0.19	3.00	4.68	2.85
17	71.25	91.36	0.00	3.00	4.63	2.74
18	67.50	89.79	−0.19	3.00	4.59	2.64
19	63.75	88.30	−0.38	3.00	4.54	2.54
20	60.00	86.89	−0.56	3.00	4.50	2.45
$b_0 = 135\%,\ \hat{b} = 60\%,\ T = 20,\ \alpha = 1/T = 0.05$						

Table 11: Strong and Weak Gros Rule — Example 1

	$\theta = 4.5\%$				$\theta = 6.0\%$			
t	d_{SGR}	d_{WGR}	s^p_{SGR}	s^p_{WGR}	d_{SGR}	d_{WGR}	s^p_{SGR}	s^p_{WGR}
1	2.06	2.06	6.33	6.33	3.89	3.89	4.39	4.39
2	1.90	2.09	6.26	6.07	3.68	3.87	4.37	4.18
3	1.74	2.11	6.19	5.83	3.47	3.84	4.35	3.99
4	1.58	2.14	6.12	5.59	3.25	3.82	4.33	3.80
5	1.42	2.16	6.05	5.37	3.04	3.80	4.32	3.63
6	1.26	2.18	5.97	5.16	2.83	3.78	4.30	3.46
7	1.09	2.20	5.90	4.96	2.62	3.76	4.28	3.30
8	0.93	2.22	5.83	4.77	2.41	3.74	4.26	3.15
9	0.77	2.24	5.76	4.59	2.19	3.72	4.25	3.01
10	0.61	2.26	5.69	4.42	1.98	3.71	4.23	2.87
11	0.45	2.27	5.62	4.25	1.77	3.69	4.21	2.74
12	0.29	2.29	5.54	4.10	1.56	3.68	4.19	2.62
13	0.13	2.30	5.47	3.95	1.34	3.66	4.17	2.50
14	−0.04	2.32	5.40	3.81	1.13	3.65	4.16	2.39
15	−0.20	2.33	5.33	3.68	0.92	3.64	4.14	2.28
16	−0.36	2.34	5.26	3.55	0.71	3.63	4.12	2.18
17	−0.52	2.35	5.19	3.43	0.50	3.61	4.10	2.09
18	−0.68	2.37	5.11	3.32	0.28	3.60	4.09	2.00
19	−0.84	2.38	5.04	3.21	0.07	3.59	4.07	1.91
20	−1.00	2.39	4.97	3.11	−0.14	3.58	4.05	1.83
$r = 6.5\%,\ b_0 = 135\%,\ \hat{b} = 60\%,\ T = 20,\ \alpha = 1/T = 0.05$								

Table 12: Strong and Weak Gros Rule — Examples 2 and 3

4 Conclusions

In this paper we analyze four types of fiscal retrenchment policies. Type 1 chooses a constant primary surplus to GDP ratio to achieve the target value of the debt to GDP ratio, while type 2 uses a constant overall deficit to GDP ratio to achieve this goal. Two sets of tables show the implications of a fiscal retrenchment program for various economic scenarios. The figures in the tables show that the achievement of the 60% target is impossible for highly indebted countries, in the short and medium run, and implies a massive fiscal retrenchment for countries in a more favorable position.

In the second part of the paper we analyze a fiscal retrenchment characterized by a "satisfactory pace" in the sense suggested by Gros (1996). Generalized versions of the strong Gros rule and weak Gros rule are studied and applied to the situation of Belgium. In contrast to the policies of type 1 and type 2, these rules imply that, in general, neither the primary surplus to GDP ratio nor the overall deficit to GDP ratio remain constant during fiscal adjustment. The pace of adjustment under the strong Gros rule exceeds that arising under the weak Gros rule and it implies — in terms of the required primary surplus — a "very tight" fiscal policy.

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