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INSURANCE MARKETS WITH
NOISY LOSS DISTRIBUTIONS

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Abstract

This paper examines insurance-purchasing behavior and insurance-market equilibrium when loss distributions are uncertain, or "noisy." We determine sufficient conditions for noise to lead to increased insurance coverage. These conditions place reasonably natural convexity constraints on marginal utility. We next examine how our results affect market equilibrium in the presence of adverse selection. In both a Rothschild and Stiglitz type of Nash equilibrium and in a Miyazaki type of subsidized equilibrium, the addition of noise is shown to lessen the disparity in coverage usually found between the good risks and the bad risks, but at cost of reduced overall welfare. We also show how so-called participating insurance contracts dominate standard fixed-premium contracts in cases where noise is correlated among individuals.

Keywords: Insurance, Adverse Selection, Prudence, Contract Design

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I. INTRODUCTION

In this paper, we consider the effects of uncertain, or "noisy" loss distributions in several models of decision making under· uncertainty.   In particular, we examine several models of insurance-purchasing behavior and insurance-market equilibrium; and we show how the addition of noise can alter several well-known results.

We model noise as part of a two-stage uncertainly.   For example, a location parameter for a particular loss distribution may be unknown.   This represents one stage in the uncertainty, which we label "noise".   However, even with full information about the loss distribution, we still have a random draw from that distribution to determine the actual loss; which is our second stage of uncertainty.   If the noise is actuarially neutral, it effects a "riskier" loss distribution, where "riskier" is defined in the sense of Rothschild and Stiglitz (1970).   Noise can be localized, such as facing a loss distribution that is known for small losses, but uncertain for very large losses (the so-called "tail" of the loss distribution).   Noise can also be uniform throughout the loss distribution, as is the case if white noise affects the location parameter.

The paper begins by examining the rational purchase of insurance in the presence of noise.   We determine sufficient conditions for noise to lead to an increase in insurance coverage in two particular cases.   If noise affects only the loss state of a loss distribution with a two-point support (i.e. loss vs. no-loss), more insurance is purchased in the presence of noise whenever marginal utility is convex in wealth.   In the second case considered, white noise affecting the location parameter of the loss distribution, a sufficient condition for increasing coverage is that marginal utility is not only convex,

but is relatively "more convex" at lower wealth levels. Both of these conditions are formalized using the degree of absolute prudence, $-U'''/U''$, developed by Kimball (1990), to measure convexity of marginal utility.

The effects of noisy loss distributions on two well-known models of insurance market equilibrium in the presence of adverse selection are considered next: the Rothschild and Stiglitz (1976) type of Nash equilibrium, and the subsidized equilibrium of Miyazaki (1977). In general, the addition of noise is seen to lessen the disparity in coverage between good risks and bad risks that is usually observed in equilibrium, although it does so at a cost of reduced welfare due to residual noise. The noise also reduces the likelihood that equilibrium exists in the Rothschild/Stiglitz model.

Finally, we examine a contract design that allows the individual to choose a mixture of a fixed-premium contract and a "pooling" arrangement in which a particular risk pool (often called a "cohort") is either paid a dividend or assessed an additional premium based on the actual loss experience within the group. We show how the optimal contract design always includes full insurance along with some degree of pooling.

The paper is organized as follows. Section II models the insurance-purchasing decision in the presence of noise. In sections III and IV, we consider the effects of noise in insurance markets with adverse selection. Section V examines contract design and section VI contains some concluding remarks.

II. RATIONAL INSURANCE PURCHASING

Consider a risk-averse individual with preferences represented by the twice continuously differentiable von Neumann-Morgenstern utility function of final wealth, $U(Y)$, where $U'>0$, $U''<0$. The individual has an initial nonrandom wealth of W, which is subject to a loss. Let L denote the random loss. To

protect against loss, the individual may purchase insurance against a loss by specifying the proportion, α, of the loss that is to be indemnified by the insurance company. We assume that the indemnity itself is noiseless--the promised indemnity is paid by the insurer following a loss (see Doherty and Schlesinger (1990)). The premium for this coverage is given by $\pi \equiv \alpha P$, where $P \equiv (1+\lambda)E(L)$ is the premium for full coverage, $\alpha = 1$. If $\lambda = 0$, the premium is called "actuarially fair." Noise is added via the introduction of the random variable A, $E(A|L) = 0$ $\forall L$, where E denotes the expectation operator. Total damages for a loss are assumed to be $L+A$, which by construction is riskier than L in the sense of Rothschild and Stiglitz (1970). We also assume that the insurance contract bases the indemnity on the realized loss, i.e. the indemnity is $\alpha(L+A)$. For example, most property insurance is written on a replacement cost or depreciated replacement cost basis. In this setting, A might represent the uncertainty associated with an item's replacement cost. Note that the purchase of insurance also mitigates the effect of noise, and there is no residual noise if $\alpha = 1$. We consider two particular specifications of noise.

First, suppose the loss distribution has a two-point support such that $L=0$ with probability $(1-p)$ and $L=D$ with probability p, $D>0$, $0<p<1$. We further assume that A is identically zero if $L=0$, but that A is nondegenerate with $E(A|L)=0$ when $L=D$. Expected utility is given by

(1) $EU \equiv (1-p)U(W-\alpha P) + pE\left\{ U\left[W-\alpha P - (1-\alpha)(D+A) \right] \right\}$.

The first-order condition for maximizing (1) is

(2) $dEU/d\alpha = -(1-p)PU'(W-\alpha P) + p(D-P)E\left\{ U'\left[W-\alpha-(1-\alpha)(D+A) \right] \right\}$
$+ p \, \mathrm{Cov}\left\{ A, U'\left[W-\alpha P - (1-\alpha)(D+A) \right] \right\} = 0$.

The second-order condition is easly verified. It is straightforward to verify that $\alpha* = 1$ [$\alpha*<1$] if $\lambda = 0$ [$\lambda>0$], which is a well-known result of Mossin [1968] results to include a noisy loss size.

If $\lambda>0$, the level of insurance coverage generally can be either higher or lower with noise than without noise. Note that the addition of noise adds the third term to right-hand side (RHS) of (2). This term is positive for $\alpha<1$, reflecting the added benefit of insurance's protection against the risk **A**. However, noise also affects marginal utility in the second-term. This effect will also be positive whenever marginal utility is convex. Such convexity is a sufficient condition for a precautionary savings motive, as shown by Kimball [1990], and follows for example under nonincreasing absolute risk aversion. Consequently, U' convex is sufficient to increase the demand for insurance in the presence of noise.[1]

As a second case, we consider a more general distribution of L (either discrete, continuous or mixed) and we assume that **A** is independent of L, with $E(A)=0$.

Expected utility in this case is

(3) $\quad EU = E\left\{U\left[W-\alpha P-(1-\alpha)(L+A)\right]\right\}.$

The first-order condition for maximizing (3) is

(4) $\quad dEU/d\alpha = -PE\left[U'\right] + E\left[(L+A)U'\right] = 0.$

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[1] If we add the constraint $\alpha\geq0$, no insurance will be purchased for high enough λ. For expositional ease, we assume that λ is not so high as to preclude the purchase of insurance.

The second-order condition is easily verified. If prices are fair, we have

$$(5) \quad dEU/d\alpha = Cov\left[(L+A),U'\right] = 0,$$

which verifies Mossin's result that full coverage is optimal. Similarly, it is easy to verify that if $\lambda > 0$, less than full coverage is optimal.[2]

Once again the optimal level of partial coverage when $\lambda > 0$ generally can be higher either with or without noise in the loss distribution. However, convex marginal utility is not sufficient to resolve the ambiguity in this setting. Instead, we need to guarantee that marginal utility is more convex at lower wealth levels, where increased coverage would be a benefit through a higher net indemnity; and that marginal utility is relatively less convex at higher wealth levels, where the net effect of increased coverage is a lower final wealth due to the higher premium. This leads to a relatively higher subjective valuation of the net benefit (in terms of marginal utility) and lower valuation of the net cost associated with an increase in coverage when noise is present. This condition on marginal utility is satisfied whenever preferences exhibit standard risk aversion, as defined by Kimball (1991).[3] Arguments as to why this property is a natural extension of decreasing absolute risk aversion are given by Kimball (1991).

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[2] To avoid dealing with negative values of L+A, we assume that the support of L has a lower bound exceeding zero and that the lower bound of the support of A does not exceed this level in absolute value.

[3] Standard risk aversion implies that "any risk that makes a small reduction in wealth more painful also makes any undesirable, statistically independent risk more painful." [Kimball 1991, pg. 2]. It is equivalent to decreasing absolute risk aversion and decreasing absolute prudence, where the latter measure is $-U'''/U''$. It also implies the weaker property of proper risk aversion, developed by Pratt and Zeckhauser (1987).

Our main results can be formalized as follows.

Proposition 1: *Given a positive premium loading, $\lambda>0$, a sufficient condition for more insurance to be purchased in the presence of noise in the loss distribution is*

(i) *marginal utility convex if noise affects only the loss state in a two-state model.*

(ii) *standard risk aversion if the noise is "white noise."*

Proof: *(see appendix)*

III. ADVERSE SELECTION: THE ROTHSCHILD AND STIGLITZ MODEL

In this section, we look at how a noisy loss distribution affects equilibrium in a competitive insurance market with asymmetric information. In particular, we consider the now classic model of Rothschild and Stiglitz (1976). Consider a simple two-state world in which a loss of size D occurs with probability p and no loss occurs with probability 1-p. As in the first part of section II, we introduce noise only in the loss state, so that the "noisy loss" is represented by D+A, where E(A)=0. Since adding A to D induces a mean-preserving spread of the loss distribution, expected utility in the presence of noise will be lower for every level of insurance, with the exception of full coverage. This has the effect of making indifference curves more concave in π-α space as illustrated in Figure 1. [See Wilson (1977).]

In Figure 1, the indifference curve labeled I_1 depicts all of the combinations of premia and coverage that yield expected utility identical to no coverage when no noise is present. Indifference curves are concave due to risk aversion. The fair price line in Figure 1 represents the premium schedule $\pi \equiv \alpha pD$. The figure illustrates the well-known result that the optimal

level of coverage is full coverage ($\alpha^*=1$) when the price is fair, and the level of expected utility attained is the level associated with the insurance contract E, on the indifference curve labeled I_2.[4]

Now suppose we introduce noise in the loss size, D. Following the purchase of insurance, the residual noise in the final wealth distribution in the loss state is $-(1-\alpha)A$. Thus, if full coverage insurance is purchased ($\alpha=1$), there is no residual noise and hence no change in expected utility. However, for $\alpha \neq 1$ residual noise leads to a lower expected utility for each insurance contract. Hence, the indifference curve through E in the presence of noise lies everywhere below E except at E itself. This is depicted as I_2' in Figure 1. Note that contracts along the locus I_2' in the presence of noise yield same expected utility as contracts along the locus I_2 in the absence of noise.

We now suppose that there exist two types of individuals, who differ in their loss probabilities. The good risks have loss probability p_g and the bad risks probability p_b, where $p_g < p_b$. Insurers know both probabilities, but cannot observe the probabilty-type of any particular individual. Both types of individuals face the same noise. One possible scenario is that people are actually heterogeneous with regards to their loss sizes. Consumers know their own loss probability, but know only the average loss size, D, and the distribution of loss sizes, D+A. If uncorrelated between individuals, this type of noise is diversifiable by the insurance company, which by the law of large numbers can treat all losses as worth D (assuming, of course, that the insurer also believes $E(A)=0$). However, consumer behavior would be affected by the noisy loss size. This behavior is taken into account by the insurer.

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[4] This is easily derived by fixing expected utility and calculating the marginal rate of substitution α for π. For details using this graphical representation, the reader is referred to Wilson (1977).

We consider first the Nash separating equilibrium of Rothschild and Stiglitz (1976) for a competitive insurance market. This is illustrated by the policy pair (B, C) in Figure 2, for the case without noise. In this equilibrium, the bad risks fully insure at a bad-risk fair price, at contract B. Insurance at a good-risk fair price is offered only in the limited quantity, α_c, and thus good risks self select contract C. Bad risks are on indifference curve I_B and good risks on I_G. We note that, as drawn in Figure 2, the Rothschild/Stiglitz separating equilibrium does exist since the pooled price line lies everywhere above the good-risk indifference curve, I_G.[5]

Now consider the introduction of a noisy loss size. The bad risk indifference curve through B would shift to I_B'. Given our discussion of Figure 1, it is easy to see that a new Rothschild/Stiglitz separating equilibrium would entail the pair of contracts (B, C') in Figure 2, if the equilibrium exists. At such an equilibrium, there is less of a penalty imposed on the good risks, in that they are allowed to purchase a higher level of coverage, $\alpha_{c'}$, at a good-risk fair price. However, the noise does more harm than good as we show next.

Consider the change in welfare when noise is introduced. The bad risks retain contract B (which has no residual noise) and so their expected utility does not change. Consequently, contract C without noise and contract C' with noise, which are both indifferent to contract B, would yield the same expected utility to bad-risk individuals. Denoting the premium and coverage changes between contracts C and C' as $\Delta\pi$ and $\Delta\alpha$ respectively, this implies

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[5] This condition requires there be a sufficient proportion of bad risks. If the pooled-risk price line intersected I_G, then it would be possible to offer a single contract that would be preferred by both types of consumers and would earn an expected profit. In such a case, no equilibrium would exist. See Rothschild and Stiglitz (1976). We also note that insurance purchases are assumed to be perfectly observable, so that good-risk individuals cannot make multiple purchases to achieve more coverage. See Hellwig (1988) for a discussion on relaxing this assumption.

(6) $(1-p_b)[U(W-\pi)-U(W-\pi-\Delta\pi)] = p_b(EU[W-\pi-\Delta\pi-(1-\alpha-\Delta\alpha)(D+A)]-U[W-\pi-(1-\alpha)D])$.

Since individuals are identical except for their loss probabilities, the left-hand side of (6) is easily seen to be strictly greater than the right-hand side if p_b is replaced by p_g. This implies that expected utility for the good-risk individuals is higher with less coverage at C and no noise, than at C' with more coverage at a fair price, but with noise added. Hence, the addition of noise to the loss size leaves the bad-risk individuals with no change in utility while lowering expected utility of the good risks.

The addition of noise also affects the existence of a separating equilibrium. This is easily illustrated using Figure 2 once again. When no noise is present, then for good-risk individuals, the full coverage contract labeled F is indifferent to the partial coverage contract C, as drawn in Figure 2. We also know that contract F will yield the same level of expected utility to good-risk individuals with or without noise. However, as we have just shown, contract C' with noise is less preferred than contract C without noise. Thus, contract C' is less preferred than contract F when noise is present. Consequently, contract C' must lie to the left of the indifference curve through contract F in the presence of noise. In particular, the indifference curve through contract F in the presence of noise, labeled I_G' in Figure 2, defines some contract C" to the right of C' on the good-risk fair price line. This means that the good-risk indifference curve through C' in the presence of noise (not drawn in Figure 2) must lie everywhere above I_G'. In particular, it may lie partly above the pooled-risk price line, in which case a Rothschild/Stiglitz separating equilibrium would not exist.

The results of this section are summarized in the following proposition:

Proposition 2:   In a two-state Rothschild-and-Stiglitz adverse-selection model, the addition of noise to the loss size will

(i) decrease the signalling cost to the good-risk individuals in the sense of allowing them to obtain a higher level of insurance coverage.

(ii) decrease Societal welfare since good-risk individuals are strictly worse off, while bad-risk individuals are no better off.

(iii) decrease the likelihood of an equilibrium. In particular, noise will raise the critical (i.e. the minimum) proportion of bad-risk individuals needed to ensure the existence of an equilibrium.

IV. ADVERSE SELECTION: THE MIYAZAKI MODEL

The Nash equilibrium concept used by Rothschild and Stiglitz (1976) has been criticized by Wilson (1977), Miyazaki (1977) and others as being "too myopic" in the sense that each insurance firm assumes that the set of contracts offered by its rivals is independent of its own actions. Wilson (1977) assumes that insurance firms will take the reaction of rival firms into account and will not offer a contract if they cannot make a profit following the elimination of unprofitable, rival-firm contracts from the market place. In Wilson's model, an equilibrium always exists. Miyazaki (1977) extends Wilson's equilibrium concept by allowing for subsidies between the good-risk and bad-risk individuals.[6]

Miyazaki shows that his equilibrium always exists and is a unique, separating equilibrium. An example of Miyazaki's equilibrium is illustrated in Figure 3. The locus of contracts on the curve passing through C, N, E and D defines a set of good-risk contracts which could be offered by the insurer,

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[6]Miyazaki does not consider an insurance market per se, but his model is easily adaptable. See, for example, Spence (1978).

together with full insurance to the bad-risk individuals at a subsidized price, which together earn a zero expected profit and which leave the bad-risk individuals indifferent between full insurance and the good-risk contract. For example, the potential Rothschild/Stiglitz equilibrium pair (B, C) satisfies these properties. The policy pair (M, N) represents another such a set of contracts. For this pair, the bad-risk individuals purchase full-coverage contract M, which loses money, but the good risks pay a premium loading, (i.e. a subsidy) to obtain contract N, which earns enough money to net the insurer a zero overall expected profit. Note also how this pair of contracts induces consumers to self select the appropriate contract. The bad risks are indifferent to M and N, while the good risks strictly prefer contract N. Another such contract "pair" is the pooling contract pair (D, D), which lies on the pooled-risk price line and fully insures everyone. Of course, as drawn, none of the contract pairs illustrated above would support an equilibrium. The Miyazaki equilibrium occurs at the contract pair (F, E). With contract E, the good-risk individuals achieve their highest expected utility (on indifference curve I_G) among their zero-profit alternatives. At contract F, the bad-risk individuals can reach indifference curve I_{B3}. Note that both good-risk and bad-risk individuals are better off in a Miyazaki equilibrium than in a Rothschild/Stiglitz equilibrium.[7]

If we now introduce noise in the loss size, indifference curves will become more concave, as discussed earlier. The zero-profit set of contracts for the good-risk individuals will now be those on the locus containing C', N' and D, as illustrated in Figure 4. Thus, for example, (M, N') will replace (M, N) as a zero-profit pair. Of course the pair (M, N) still earns zero

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[7] We should mention that the locus of zero-profit separating contracts for the good risks, CNED, need not be convex. If the most preferred good-risk contract is not unique, equilibrium will occur at the most preferred good-risk contract with the lowest subsidy. See Miyazaki (1977, p. 411).

expected profit, but it does not induce self-selection. Note that N and N' are the same distance above the good-risk fair price line, indicating the same subsidy. When the bad-risk individuals receive contract M and noise is present, the good-risk individuals are offered a zero-profit choice along I_{B2}' rather than I_{B2}. Thus, the good risks can purchase a higher level of coverage, which in turn implies that a lower subsidy per unit of coverage is needed to cover the bad-risk losses. As drawn, the Miyazaki equilibrium in the presence of a noisy loss size entails a good-risk contract along the locus of contracts drawn containing C', N' and D. The following results are obtained (proofs in the appendix):

Proposition 3: *In a Miyazaki equilibrium in the presence of noise:*

    (i) *the good risks are strictly worse off than without noise,*

    (ii) *the bad risks are no better off than without noise.*

Proposition 4: *The subsidy paid by the good risks per unit of coverage in a Miyazaki equilibrium is lower in the presence of loss-size noise.*

Thus, we see subsidy results and welfare results similar to those occurring in the Rothschild/Stiglitz model, except that bad risks might be adversely affected by noise in the Miyazaki model.

V. CONTRACT DESIGN

It has been shown that mutual insurance companies in which policyowners can share in the aggregate loss experience of the risk pool, are preferred in insurance markets where individual loss exposures are correlated.[8] In this section, we decompose an individual's risk of loss into two components: one

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[8] See Marshall (1974), Doherty and Dionne (1989) and Smith and Stutzer (1990). We also note that there are several types of real-world participating contracts, some of which differ in their timing of premiums, dividends and assessments. In our static model, the timing of premiums, assessments and dividends is irrelevant. The important feature is that the final premium is based upon the pooled loss experience, which is not known at the time the insurance-purchasing decision is made.

independent of other individuals' loss distributions and the other identical for all individuals. This enables us to use our results on noise (from Proposition 1) in addressing the issue of contract design.

The type of contract we have in mind is one in which the insurer charges a premium based on loss expectations. However, if the aggregate loss within a risk pool is lower [higher] than predicted, policyholders are paid a dividend [charged an extra premium assessment]. Such contracts, which are called "participating contracts," are relatively routine for mutual insurance companies and also exist for some stock companies. Admittedly, many contracts will have limits on dividends and/or assessments. For example, a contract paying only dividends but not charging assessments is essentially a participating policy issued jointly with a call option whose payoff equals the assessment. Also, an imperfect proxy for a participating policy can be assembled by purchasing insurance from a stock insurance company that pays shareholder dividends and simultaneously buying shares of the company's stock; "imperfect" due to the influence of factors other than underwriting profit on the insurer's stock dividends, and due to the fact that even the underwriting profit itself depends on multiple books of business. Although limitations on dividends and/or assessments as well as proxy contracts are interesting, they are beyond our scope here and we will assume no limitations on policy participation.

Consider a risk class with n seemingly homogeneous individuals.[9] Each individual faces the same loss distribution with the same noise. The full-coverage premium, which is random ex ante, is given by

(7) $P \equiv (1-\beta)(1+\lambda)E(L) + \beta S$

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[9] Noise might cause a heterogeneous group to seem homogeneous.

where

β = weight on participation component and

$S = \Sigma(L_i + A_i)/n$ = average pooled loss.

The individual choosing coverage level α pays $\alpha(1+\lambda)E(L)$ up front and has an assessment of $\alpha\beta[S-E(L)]$ after S is realized (where a negative assessment denotes a dividend). As before, λ is the premium loading and $E(A_i)=0$ for each individual, i. We assume that the L_i are independent and identically distributed.[10] The weight is restricted, $0\leq\beta\leq1$. When $\beta=0$, we have a standard fixed-premium contract and when $\beta=1$ we have full participation. We assume that the risk pool is very large; in the limit $n=\infty$.

To illustrate the importance of the noise variable A, consider liability insurance. From their loss records, insurers may be able to estimate the loss distributions for a given class of policyholders and make this information public. However, settled claims will have been made against liability standards that prevailed in the past. There remains the ever present prospect that the liability rules against which new claims are resolved may change through new precedents or new legislation. There is likely to be much uncertainty in estimating the effects of the changing liability rules. Moreover, the effect of any rule changes is common over groups of policyholders. For example, a legal precedent which extends the common law liability will apply to all subsequent suits in the same jurisdiction, unless overruled by a higher court. In effect, the noise factors, A's, are positively correlated across policyholders. This feature has been seen as

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[10] For simplicity, we assume that the premium loading is zero for the participating component, S. We only need to assume that the fixed-premium loading exceeds the loading on S to obtain similar results. If the noise risk has a price in a competitive market, we would expect the loading to be lower for the contractual form in which the consumer bears some of the noise risk.

central to the recent liability insurance crisis (see Priest (1987), Doherty, Kleindorfer and Kunreuther (1990) and Winter (1988)).

We consider the case where there is a perfect positive correlation between the individual noise terms. In particular, individuals are truly homogeneous with random loss $L_i + a_i$, where $a_i = a_j$ for all individuals i and j, but where a_i is not known with certainty. Rather, a_i is known to be generated by the random variable Λ. In a large risk pool, the average risk associated with the L_i will vanish but the risk associated with the noise Λ will remain. The average pooled loss is $S \equiv E(L) + \Lambda$.

In this setting, the full insurance premium as given in (7) is $P = [1 + (1-\beta)\lambda]E(L) + \beta\Lambda$. Note that if we restrict $\beta = 0$, we revert to the standard model with noise, from section II, with $\alpha^* = 1$ if $\lambda = 0$ and $\alpha^* < 1$ if $\lambda > 0$. More generally, final wealth can be written as

$$(8) \quad Y \equiv W - \alpha[1 + (1-\beta)\lambda]E(L) - (1-\alpha)L - [1 - (1-\beta)\alpha]\Lambda.$$

First-order conditions for expected-utility-maximizing choices of α and β are

$$(9) \quad \partial EU/\partial\alpha = -[1 + (1-\beta)\lambda]E(L)E(U') + E[(L + (1-\beta)\Lambda)U']$$

$$= Cov(L, U') - (1-\beta)[\lambda E(L)E(U') - Cov(\Lambda, U')] = 0.$$

and

$$(10) \quad \partial EU/\partial\beta = \alpha\lambda E(L)E(U') - \alpha E(\Lambda U') = \alpha[\lambda E(L)E(U') - Cov(\Lambda, U')] = 0.$$

If the fixed-premium component of the contract is fair, $\lambda = 0$, then it is straightforward to show that $(\alpha^* = 1, \beta^* = 0)$ is the optimal (α, β) pair. Since the fixed-premium contract has a zero loading, there no advantage to combining the fixed premium with the participating component. Indeed, by

setting $\beta*=0$, the individual eliminates the average noise risk completely and is left with the certain final wealth, $Y = W-E(L)$.

However, in an efficient market, it is reasonable to assume that the market will not reward an insurer for holding a diversifiable risk, the L_i; but it will assess a risk loading for bearing the risk of the correlated noise terms. Hence, the insurer will charge a risk loading, $\lambda>0$, on the portion of the insurance contract for which it bears the noise risk. If $\lambda>0$, then $\beta*>0$. To see this, set $\beta=0$ and solve for the optimal level of coverage in this setting, $\alpha*$. Given the first-order condition for the level of coverage, equation (9), and the results of section II (i.e. $\alpha*<1$), we must have $Cov(L,U')>0$ and hence $\lambda E(L)E(U')-Cov(\Lambda,U')>0$. But this implies $\partial EU/\partial\beta>0$ and β should increase. However, if we now allow β to increase to its optimal level, then from (10) either $\partial EU/\partial\beta=0$; or $\partial EU/\partial\beta>0$ and $\beta*=1$ as a constrained optimum. In either case, substituting this result back into equation (9) obtains $\partial EU/\partial\alpha=Cov(L,U')=0$; which implies that $\alpha*=1$. We have, therefore, established the following result:

Proposition 5:   *For a large homogeneous risk class ($n=\infty$) with perfect positive correlation of the individual noise, the optimal joint coverage level and premium weight satisfy:*

(i) *$\alpha*=1$ (full coverage)*

(ii) *$\beta*=0$ if $\lambda=0$ and $\beta*>0$ if $\lambda>0$.*

Given the results of Proposition 5, final wealth in equation (9) is $Y=W-E(L)-[(1-\beta)\lambda E(L)+\beta\Lambda]$. We can interpret the results of Proposition 5 by considering the effect on the two risky variables the individual faces, L_i and Λ. If the individual were to buy full coverage and set $\beta=1$, the risk L_i would be eliminated at a fair price (replaced with $E(L)$), but the risk of Λ

would remain. However, by adjusting β, the individual has a control for the risk in λ.

The premium weight in the above setting affects final wealth through an additive term, $\beta[\lambda E(L)-\lambda]$. It thus becomes clear that $\beta*=0$ when $\lambda=0$ and $\beta*>0$ when $\lambda>0$. If the riskiness of λ, in the sense of Rothschild and Stiglitz (1970), were to decrease, $\beta*$ would rise if $\lambda>0$. Indeed, if left unconstrained, $\beta* \rightarrow \infty$ as the risk vanishes. On the other hand, if the level of noise increases (i.e. a riskier λ), then $\beta*$ will fall. Of course, when beta is constrained not to exceed one, it is possible that $\beta*$ remains at one for changes in the noise level. More formally,

Corollary: If $0<\beta*<1$ for a given noise distribution λ, then an increase [decrease] in the level of noise will cause $\beta*$ to fall [rise].

VI. CONCLUDING REMARKS

This paper has examined several effects of noisy loss distributions in an insurance market. Noise in the size of a loss, in a simple two-state model, was shown to increase the demand for insurance if consumers were "prudent." For more general loss distributions, the condition of standard risk aversion was seen to be sufficient to conclude that noise in the loss distribution increases the level of insurance coverage.

For competitive insurance markets with adverse selection, noisy loss distributions were seen to be have an impact on market equilibrium. In the Rothschild and Stiglitz (1976) equilibrium model, noise was shown to enable the good risks to purchase more coverage, to decrease overall welfare, and to lessen the likelihood of an equilibrium's existence. For a Miyazaki (1977) type of equilibrium, in which the good risks subsidize the bad risks, the rate of the subsidy was shown to be lower; but with an accompanying decrease in

overall welfare. Of course, several other equilibrium concepts are possible as are other types of asymmetry of information. Noisy loss distributions will clearly affect equilibria in these markets, although such effects are beyond our scope in this paper.

Finally, we examined contract design in the presence of noise. The contract examined allowed for a mixture of a fixed premium and a dividend/assesment component. Even if a large risk pool can reduce the average risk of loss, noise risk remains a significant concern if the individual noise levels are correlated with one another. For the case examined, full coverage was always optimal and the individual adjusted the level of dividend participation to control the overall noise level.

Naturally, the consequences of noisy loss distributions extend well beyond those examined here. Given the pervasive effects of noise in the models considered in this paper, we hope that including the consequences of noisy loss distributions in other settings will lead to more robust models of insurance-purchasing behavior.

<div align="center">APPENDIX</div>

Proof of Proposition 1:

Since (i) was shown in the text, we provide here a proof of (ii).

Define the derived utility function, V, as follows

(A1) $V(X;\alpha) \equiv EU[X-(1-\alpha)A]$,

where X is some deterministic level of wealth. As in Kimball (1990), define the precautionary premium for the random variable $(1-\alpha)A$ when initial wealth is X as $\Psi(X,(1-\alpha)A)$, where Ψ satisfies

(A2) $V'(X;\alpha) \equiv dV/dX = U'(X-\Psi)$.

Now,

(A3) $\qquad V''(X;\alpha) = (1-d\Psi/dX)U''(X-\Psi).$

Since prudence implies $\Psi>0$ and decreasing prudence implies $d\Psi/dX<0$, we obtain the following inequalities:

(A4) $\qquad \dfrac{-V''(X)}{V'(X)} \quad > \quad \dfrac{-U''(X-\Psi)}{U'(X-\Psi)} \quad > \quad \dfrac{-U''(X)}{U'(X)}$,

where the second inequality follows from decreasing absolute risk aversion. Consequently, V is a more risk-averse function than U. This implies the existence of an increasing, concave function g ($g'>0$, $g''<0$) such that $V(X) = g[U(X)]$ $\forall X$.

Now, the first-order condition (6) can be written as,

(A5) $dEU/d\alpha = -PE(U')+E[LU']+E[AU'] = \int(-P+L)V'(X)dF(L)+Cov(A,U') = 0$

where $X \equiv W-\alpha P-(1-\alpha)L$. If there is no noise, $A\equiv 0$, then the first-order condition reduces to $\int(-P+L)U'(X)dF(L) = 0$, which is assumed satisfied at $\alpha*$.

Now, since utility can be altered via an affine transformation, we can assume without losing generality that $E[V'(X)] \equiv \int g'[U(X)]U'(X)dF(L) = E[U'(X)]$. Consequently,

$\int(-P+L)V'(X)dF(L) = \int(-P+L)g'[U(X)]U'(X)dF(L) = E(-P+L)E(g'U')+Cov(L,g'U').$

The first term in this last expression equals $-Cov(L,U')$ when $\alpha=\alpha*$ by the first-order condition in the noiseless case and our assumption that $E(g'U')=E(U')$. Now $Cov(L,U')$, $Cov(L,g'U')$ and $Cov(A,U')$ are zero when $\alpha*=1$

and positive when $\alpha^* < 1$. Since g is concave, we also have in the case where $\alpha^* < 1$ that $Cov(L, g'U') > Cov(L, U')$. Since $\alpha^* < 1$ when $\lambda > 0$, it follows from (A5) that $dEU/d\alpha > 0$ when evaluated at α^* with noise. Thus the optimal level, α^{**}, should be higher.

<div align="right">Q.E.D.</div>

Before proving Proposition 3, we first establish the following lemmata.

Lemma 1: Let X represent an insurance contract without noise and Y a contract with noise such that both X and Y have the same premium. Then $X \tilde{}_G Y$ if and only if $X \tilde{}_B Y$,where "$\tilde{}_i$" represents indifference for type-i individuals.

Proof: Since X and Y have the same premium, both types of individuals are indifferent to these contracts in the no-loss state. Therefore, $X \tilde{}_i Y$ for i=B or G implies we must also have indifference in the loss state. Since both types have identical preferences, the lemma follows.

<div align="right">Q.E.D.</div>

Lemma 2: Let X be a contract without noise on the good-risk set of subsidizing contracts for the Miyazaki model; i.e. X is a contract along the CD locus in Figure 4. Let Y be a contract with noise along the set of subsidizing contracts with noise; locus C'D in Figure 4. Then $X \tilde{}_B Y$ if and only if X and Y generate identical expected profit for good-risk individuals.

Proof: Trivial since X and Y generate an identical bad-risk subsidy.

<div align="right">Q.E.D</div>

Note that Lemma 2 implies that bad risks are indifferent to contracts along CD and C'D which lie on lines parallel to the good-risk fair price line (in Figure 4).

Proof of Proposition 3:

(i) Suppose not. Let E' denote the equilibrium good-risk contract with noise. Then $E' \geq_G E$, where "$\geq$" denotes "is weakly preferred." Define contract J without noise such that J has the same premium as E' and $E' \sim_G J$. Also, define contract H without noise such that H is on the Miyazaki good-risk contract locus, CD in Figure 4, and $E' \sim_B H$. This is illustrated in Figure 5 which replicates part of Figure 4. To avoid clutter, axes and price lines are not drawn in Figure 5. Note that H lies "southwest" of E' by Lemma 2.

Now $E' \sim_B J$ by Lemma 1. Therefore, for the two contracts without noise, H and J, we have $H \sim_B J$. However, it is easy to show that the bad-risk indifference curve through J without noise is steeper than the good-risk curve I_G. Hence J is strictly preferred to H by the bad risks--a contradiction! Therefore, the good risks are strictly worse off.

(ii) Let E represent the good-risk Miyazaki-equilibrium contract without noise and let E' denote the contract with noise on C'D such that $E' \sim_B E$. Now choose contracts K without noise on CD and K' with noise on C'D such that K entails a premium increase over E and $K' \sim_B K$. Thus, segments KK' and EE' are parallel to the good-risk fair-price line by Lemma 2. An illustration is provided in Figure 6. We claim $K' <_G E'$. If this holds the good-risk equilibrium contract with noise must lie southwest of E' on the C'D locus and, consequently, the bad risks cannot be better off in equilibrium.

To prove our claim, let Δ_1 denote the change in utility in the no-loss state for a switch from contract E to contract K. Clearly $\Delta_1 < 0$ due to the higher premium. Let Δ_2 denote the corresponding utility change in the loss

state, $\Delta_2 > 0$. Similarly, let Δ_1' and Δ_2' denote utility changes for a switch from E' to K' with noise, where Δ_2' takes expectations over λ. Applying Lemma 2, we obtain

(18) $(1-p_b) (\Delta_1 - \Delta_1') + p_b (\Delta_2 - \Delta_2') = 0.$

Since the premium, π, is higher at E' than at E, it follows from the concavity of utility that $(\Delta_1 - \Delta_1') > 0$. Consequently,

(19) $(1-p_g) (\Delta_1 - \Delta_1') + p_g (\Delta_2 - \Delta_2') > 0$

or equivalently

(20) $(1-p_g)\Delta_1 + p_g\Delta_2 > (1-p_g)\Delta_1' + p_g\Delta_2'.$

The inequality in (20) indicates that the change in expected utility for the good risks from E to K without noise exceeds the corresponding change from E' to K' with noise. But from the optimality of E for the good risks without noise, the left-hand side of (20) must be strictly negative. Consequently, (20) implies that $K' <_G E'$. Our claim and, hence, the Proposition follow.

Q.E.D.

Proof of Proposition 4:

Since the optimum good-risk contract lies southwest of E' in Figure 6, its slope when joined to the origin is less than that of E. The conclusion follows.

Q.E.D.

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Premium
π

fair price line

I₁

I₂

I'₂

E

α=1

Level of coverage, α

Figure 1

Premium
π

bad-risk
price line

pooled-risk
price line

I_B

I'_B

B

I_G

I'_G

good-risk
price line

F

C C' C"

α_c α_c'   α = 1

Level of coverage, α
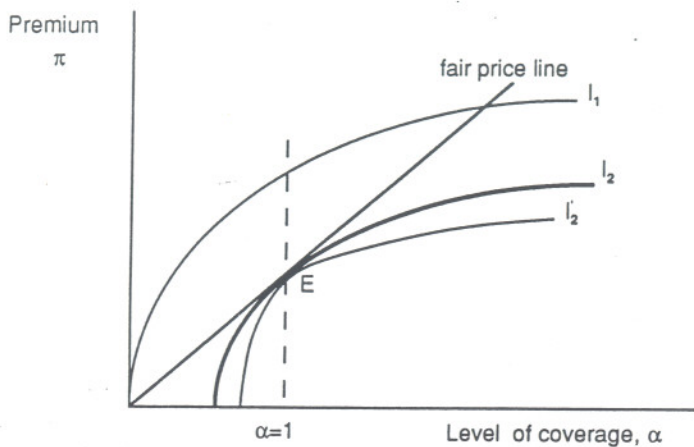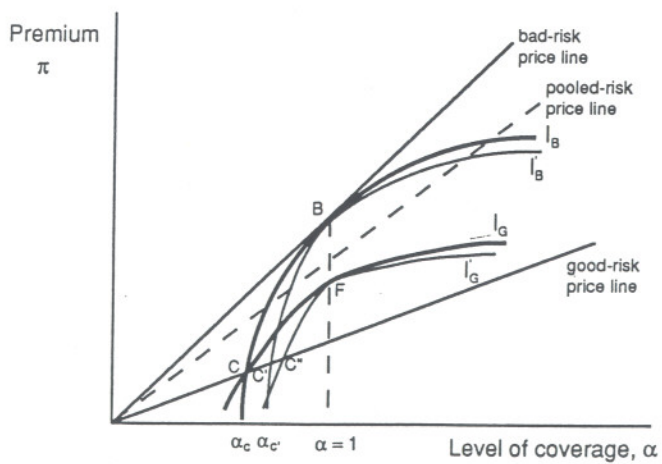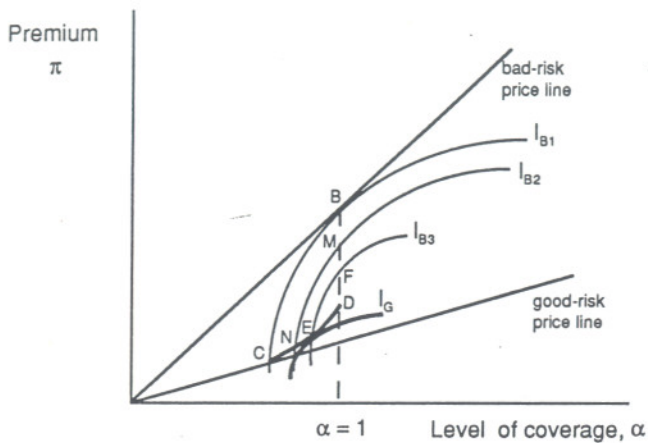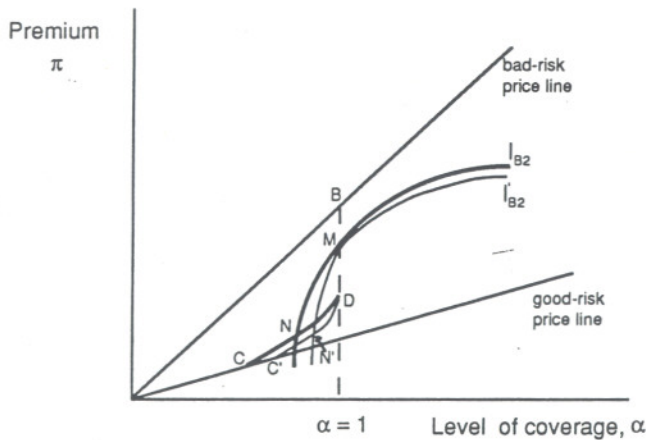
Figure 2

Figure 3

Figure 4
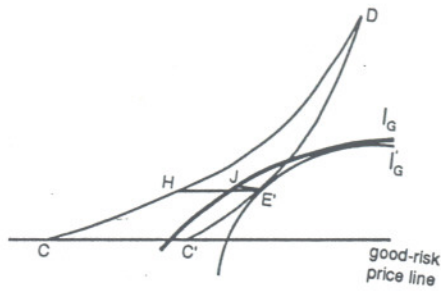
Figure 5
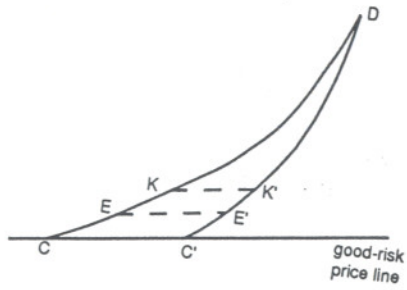
Figure 6

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