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THE GENERAL EQUILIBRIUM FOUNDATIONS OF MODERN FINANCE THEORY: AN EXPOSITION

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Abstract

This paper presents the mean-variance approach to modern finance from the perspective of general equilibrium theory. It provides a novel diagrammatic exposition to help students of finance appreciate the economic foundations of their discipline. The capital asset pricing formula (CAPM) is deduced in this framework in order to highlight the general equilibrium foundations of this widely applied asset pricing method.

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1. Introduction

There is a growing recognition among finance specialists and economists alike that finance is a sub-discipline of economics. Perhaps it was always understood implicitly that the two disciplines were closely related but, until recently, their research agendas and modes of enquiry were quite distinct. The converse is now true, however; indeed, the convergence of the research agendas in modern finance and the microeconomics of the firm, decision-making under uncertainty and the theory of industrial organisation is such that a separate field of "financial economics" has emerged with its own formal literature and professional journals. The "marriage" of the two disciplines was acknowledged formally in 1990 with the award of the Nobel Prize in economics to three finance theorists: Harry Markowitz, Merton Miller and Franco Modigliani.

The theory of finance is a particular application of the theory of general economic equilibrium. While this fact is appreciated in the upper reaches of the academic profession, it has not yet begun to shape the exposition of basic finance theory in introductory textbooks. There is a standard approach to the teaching of basic finance which owes a good deal to pioneering work by Harry Markowitz and James Tobin in the 1950s.[1] The approach relies almost exclusively on mean-variance analysis as the basis of the theory of portfolio choice and asset pricing.

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[1] The complementarity between the work of these two authors, one a finance specialist and the other an economist, shows how close the relationship was between the two disciplines from the outset.

Mean-variance analysis has served both disciplines well, both as an expository device and as a rich source of testable hypotheses, including the famous Capital Asset Pricing Model (CAPM). However, it tends to obscure rather than elucidate the theoretical foundations of portfolio choice theory. This occurs because mean-variance analysis rests on highly specialised assumptions about the preferences of agents over assets, a fact which is acknowledged in finance textbooks and then promptly forgotten. It may well be necessary to impose restrictive assumptions in order to generate testable conclusions from general theory. At least until recently, this has been the overriding objective of finance specialists, who have tended to be impatient with theoretical niceties, viewing such matters as more properly the concern of economists. But in order to gain a proper appreciation of the theoretical underpinnings of modern finance theory one must abandon the simplifying assumptions of mean-variance analysis, and seek out the roots of finance theory in the rich soil of general equilibrium economics.

This paper is an attempt to exposit the basic elements of modern finance theory from the perspective of general equilibrium economics. It develops the theory of portfolio choice and asset pricing in a model of general economic equilibrium. It then proceeds to show how these general results may be specialised to derive the familiar results of mean-variance analysis and the CAPM. A number of techniques are introduced to help students of finance appreciate the economic foundations of their discipline, and students of economics to understand how general equilibrium theory has been applied to such great effect in the field of finance. The hope is that teachers of both finance and economics will be encouraged to link the two disciplines explicitly, and to dispel the notion that economic principles are somehow left behind when one embarks upon the study of finance.

2. The General Equilibrium Theory of Portfolio Choice

Consider an economy with assets $k = 1, \ldots, K$. Assets are characterised by the non-negative payoffs r_{sk} they generate in different states:

$$r_k = \left(r_{1k}, \cdots, r_{sk}, \cdots, r_{Sk}\right)$$

Consumers can buy or sell assets in unlimited quantities at given positive prices q_k per unit of the asset. Denote by a_k the quantity of asset k which a consumer holds or wants to hold. For $a_k > 0$, the consumer holds an entitlement to receive payments in each state of the world according to the pattern indicated by the state-contingent payoff vector r_k. For $a_k < 0$, the consumer is obliged to make state-contingent payments according to the same pattern. A portfolio $a = \left(a_1, \cdots, a_k, \cdots, a_K\right)$ specifies quantities of the different assets held by a consumer. Each consumer is endowed with an initial portfolio of assets $\bar{a} = \left(\bar{a}_1, \cdots, \bar{a}_k, \cdots, \bar{a}_K\right)$.

As usual in an exchange economy, consumers trade assets freely at given asset prices $q = \left(q_1, \cdots, q_k, \cdots, q_K\right)$. The value of a consumer's initial endowment $W_0 \equiv \sum_{k=1}^{K} q_k \cdot \bar{a}_k$ is her initial wealth. Furthermore, associated with each portfolio a there is a state-dependent wealth vector:

$$W(a) = \left[W_1(a), \cdots, W_s(a), \cdots, W_S(a)\right]$$

where $W_s(a) \equiv \sum_{k=1}^{K} r_{sk} \cdot a_k$ is the wealth generated by portfolio a in state s.

Assuming that our consumer is an expected-wealth-maximiser, her net demand for the various assets can be derived as the solution to the following optimisation problem:

$$\underset{a}{\text{Max}} \sum_{s=1}^{S} p_s \cdot u\left[\mathcal{W}_s(a)\right]$$

$$\text{subject to } \sum_{k=1}^{\mathcal{K}} q_k \cdot a_k = \mathcal{W}_0$$

Note that there are no non-negativity constraints on the choice of assets since short sales are permissible.

A solution to this optimisation problem is an optimal portfolio a^* whose composition depends upon:

- the prices of all assets q_k
- the payoff vectors of all assets r_k
- the initial asset endowment $\overline{a}$ (or equivalently, the initial wealth $\mathcal{W}_0$).

Formally, for $k = 1, ..., \mathcal{K}$:

$$a_k^* = f_k\left(q_1, \cdots, q_{\mathcal{K}}; r_1, \cdots, r_{\mathcal{K}}; \overline{a}_1, \cdots, \overline{a}_{\mathcal{K}}\right).$$

Given a set of consumers, $i = 1, ..., I$:

- each endowed with an asset holding $\overline{a}^i = \left(\overline{a}_1^i, \cdots, \overline{a}_{\mathcal{K}}^i\right)$; and
- each with preferences represented by an expected utility index u^i

an equilibrium is an asset price vector $q^* = \left(q_1^*, \cdots, q_k^*, \cdots, q_{\mathcal{K}}^*\right)$ and an asset allocation $a^{i*} = \left(a_1^{i*}, \cdots, a_{\mathcal{K}}^{i*}\right)$ for each agent such that, for all asset markets $k = 1, ..., \mathcal{K}$:

$$\sum_{i=1}^{I} f_k^i\left(q_1, \cdots, q_{\mathcal{K}}; r_1, \cdots, r_{\mathcal{K}}; \overline{a}_1^i, \cdots, \overline{a}_{\mathcal{K}}^i\right) = \sum_{i=1}^{I} \overline{a}_k^i$$

In other words, asset markets experience a general equilibrium when asset prices are such that the quantities demanded by all consumers at the 'equilibrium prices' exactly match the quantities available in each asset market.

Some important properties of equilibrium or 'optimal' portfolios are most easily illustrated in the special case of two assets and two states which we now develop.

2.1 Optimal Portfolio Choice with Two Assets and Two States

For the two-asset/two-state world, the agent's choice problem is written formally as follows:

$$\max_{a_1, a_2} \sum_{s=1}^{S} u(r_{s1} \cdot a_1 + r_{s2} \cdot a_2) = p \cdot u(r_{11} \cdot a_1 + r_{12} \cdot a_2) + (1-p) \cdot u(r_{21} \cdot a_1 + r_{22} \cdot a_2)$$

subject to $$q_1 \cdot a_1 + q_2 \cdot a_2 = W_0 \qquad (1)$$

Note that the argument of the utility function $u(\cdot)$ is state-contingent wealth, i.e., $r_{s1} \cdot a_1 + r_{s2} \cdot a_2 = W_s$. Note also that there are no restrictions placed on the values of a_1 and a_2. In particular, a_1 and a_2 may be negative, i.e., either asset may be sold short.

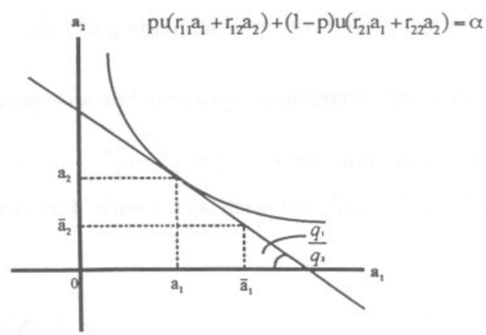

Figure 1

The agent's choice problem is represented in Figure 1. The shape of the indifference curves depends on the asset return vector since assets affect wealth through their state-contingent payoffs.

Figure 1 represents the portfolio choice problem as an optimisation problem in 'asset space'. There is an alternative representation in 'wealth space'. Recall that:

$$W_1 = r_{11} \cdot a_1 + r_{12} \cdot a_2 = \frac{r_{11}}{q_1} \cdot q_1 \cdot a_1 + \frac{r_{12}}{q_2} \cdot q_2 \cdot a_2$$

and

$$W_2 = r_{21} \cdot a_1 + r_{22} \cdot a_2 = \frac{r_{21}}{q_1} \cdot q_1 \cdot a_1 + \frac{r_{22}}{q_2} \cdot q_2 \cdot a_2$$

Solving these two equations for a_1 and a_2 and substituting into the budget constraint (1) yields :

$$W_2 = \frac{\left[\frac{r_{21}}{q_1} - \frac{r_{22}}{q_2}\right]}{\left[\frac{r_{11}}{q_1} - \frac{r_{12}}{q_2}\right]} \cdot W_1 + \frac{\left[\frac{r_{22} \cdot r_{11}}{q_2 \cdot q_1} - \frac{r_{12} \cdot r_{21}}{q_2 \cdot q_1}\right]}{\left[\frac{r_{11}}{q_1} - \frac{r_{12}}{q_2}\right]} \cdot W_0 \qquad \text{(1a)}$$

Now (1a) is a budget line in state-contingent wealth space (W_1, W_2), with a slope of $\frac{\left[\frac{r_{21}}{q_1} - \frac{r_{22}}{q_2}\right]}{\left[\frac{r_{11}}{q_1} - \frac{r_{12}}{q_2}\right]}$ and an intercept on the W_2 axis of $\frac{\left[\frac{r_{22} \cdot r_{11}}{q_2 \cdot q_1} - \frac{r_{12} \cdot r_{21}}{q_2 \cdot q_1}\right]}{\left[\frac{r_{11}}{q_1} - \frac{r_{12}}{q_2}\right]}$.

The portfolio choice problem can therefore be stated equivalently as:

$$\underset{W_1, W_2}{\text{Max}} \sum_{s=1}^{2} p_s \cdot u(W_s) = p \cdot u(W_1) + (1-p) \cdot u(W_2) \tag{2}$$

subject to (1a).

The representation of the budget constraint (1a) in state-contingent wealth space reveals an important necessary condition for the existence of a solution to the portfolio choice problem. Note that the slope of the budget line in (1a) depends on the payoffs r_1 and r_2 and the asset prices q_1 and q_2. The following diagram illustrates two possible configurations.

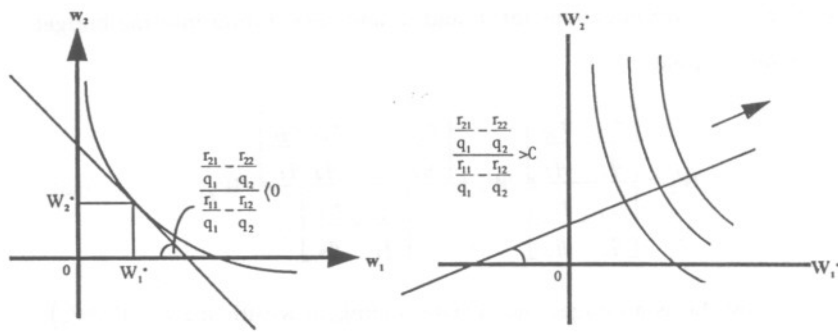

Figure 2

If the budget line is upward sloping (as in the right-hand panel of Figure 2), a consumer whose preferences increase in wealth can increase her utility infinitely by consuming further and further along the budget constraint, (i.e. , proceeding in a north-easterly direction). This follows from the fact that $\left(\frac{r_{21}}{q_1} - \frac{r_{22}}{q_2}\right) \Big/ \left(\frac{r_{11}}{q_1} - \frac{r_{12}}{q_2}\right) > 0$ implies:

$$\text{either } \frac{r_{21}}{q_1} > \frac{r_{22}}{q_2} \text{ and } \frac{r_{11}}{q_1} > \frac{r_{12}}{q_2};$$

$$\text{or } \frac{r_{21}}{q_1} < \frac{r_{22}}{q_2} \text{ and } \frac{r_{11}}{q_1} < \frac{r_{12}}{q_2}.$$

The first pair of inequalities implies that asset 1 provides a higher payoff than asset 2 in both states. Selling a_2 short and buying a_1 therefore allows the consumer to achieve arbitrarily high levels of wealth. Similarly, the second pair of inequalities shows the state-contingent payoffs of asset 2 dominating those of asset 1 and again riskless arbitrage is possible. In short, if the slope of the budget line were positive, it would be possible simultaneously to sell one asset and buy the other in unbounded quantities, so as to produce ever-increasing levels of state-contingent wealth and expected utility, without violating the initial wealth constraint.

A necessary condition for a well-defined solution to the portfolio choice problem, whether in asset space (a_1, a_2) or state-contingent wealth space $(\mathcal{W}_1, \mathcal{W}_2)$, is that $\left(\frac{r_{21}}{q_1} - \frac{r_{22}}{q_2}\right) \Big/ \left(\frac{r_{11}}{q_1} - \frac{r_{12}}{q_2}\right) < 0$ holds. This condition relating asset payoffs to their prices rules out the possibility of riskless arbitrage. It is consistent with a downward-sloping budget line in wealth space, as depicted in the left-hand panel of Figure 2.

If the budget line is downward sloping in $(\mathcal{W}_1, \mathcal{W}_2)$ space (i.e., if no arbitrage possibilities exist), the portfolio choice yielding $(\mathcal{W}_1^*, \mathcal{W}_2^*)$ is optimal so long as the marginal rate of substitution (the slope of an indifference curve of (2)) equals the slope of the budget line (1a), i.e.:

$$-\frac{p \cdot u'(\mathcal{W}_1)}{(1-p) \cdot u'(\mathcal{W}_2)} = \frac{\left[\frac{r_{21}}{q_1} - \frac{r_{22}}{q_2}\right]}{\left[\frac{r_{11}}{q_1} - \frac{r_{12}}{q_2}\right]} \tag{3}$$

Since $0 \leq p \leq 1$ and $u'(\cdot) > 0$, the MRS on the left-hand-side of (3) must be negative, and can be equal to the slope of the budget line on the right-hand-side only if it is negative. This confirms the necessity of the arbitrage condition.

While the basic aim is to solve for the optimal asset demands a_1^* and a_2^* the problem can be reduced to one in $(\mathcal{W}_1, \mathcal{W}_2)$ space, where we solve for the optimal levels of state-contingent wealth $\mathcal{W}_1^*$ and $\mathcal{W}_2^*$. There is always a one-to-one correspondence between the asset space (a_1, a_2) and the state-contingent wealth space $(\mathcal{W}_1, \mathcal{W}_2)$. In some cases it is more convenient for expository purposes to work in one space rather than the other.

2.2 Exchange equilibrium

The solution to the portfolio choice problem in (1) yields expected-utility-maximising values for a_1 and a_2. These will be functions of the asset payoffs, r_{sk} $(s, k = 1,2)$, the asset prices, q_1 and q_2, and the endowed quantities of each asset, $\bar{a}_1$ and $\bar{a}_2$ (recall that $\mathcal{W}_o = q_1 \cdot \bar{a}_1 + q_2 \cdot \bar{a}_2$). In a general exchange equilibrium where there is more than a single consumer, the prices of the assets are such that the quantities demanded by each consumer in general equilibrium equal the total endowed quantities of each asset. In other words, asset markets clear at the ruling equilibrium asset prices.

An example should help to clarify this point. Consider an economy with two consumers, each with identical von Neumann-Morgenstem preferences and expected utility indexes given by $u(\cdot) = \ln(\cdot)$. Assume that the payoff matrix for two assets is as follows :

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 2 \end{bmatrix}$$

and that the probabilities of the two states are $p_1 = p_2 = \frac{1}{2}$. The endowed quantities of each asset held by the two consumers are as follows :

$$\left(\bar{a}_1^1, \bar{a}_1^2\right) = (15, 5)$$
$$\left(\bar{a}_1^2, \bar{a}_2^2\right) = (-5, 15)$$

Since the consumers have identical preferences, they each have an expected utility function given by :

$$V(a) = \tfrac{1}{2}.ln\left(a_1 + \tfrac{1}{2}a_2\right) + \tfrac{1}{2}.ln\left(a_1 + 2a_2\right)$$

where $W_1 = a_1 + \frac{1}{2}a_2$ and $W_2 = a_1 + 2a_2$.

The general equilibrium asset allocation between these two consumers can be depicted in a "box" diagram of the familiar Edgeworth-Bowley type. Notice, however, that the equilibrium is in asset space and not in the final consumption or wealth space. Thus negative quantities of assets can be held in equilibrium (short sales are permitted) and the "box" need not lie wholly in the positive orthant of Euclidean space. In fact, the box will not be rectangular, as in the usual Edgeworth-Bowley case, but rather trapezoidal as depicted in Figure 3.

This follows because the boundaries of the box in asset space are determined by the inequalities $W_1 \geq 0$ and $W_2 \geq 0$ in wealth space. In asset space, these inequalities become :

$$a_1 + \tfrac{1}{2}a_2 \geq 0 \text{ and } a_1 + 2a_2 \geq 0$$

When these weak inequalities hold as equalities, they bound the space of feasible asset choices for each consumer.

The dimensions of the box are determined by the initial endowment (as in the usual Edgeworth-Bowley box) together with the inequalities derived from the payoff matrix. The total endowment of the two assets is found by summing the individual endowments :

$$\overline{\mathcal{A}}_1 = \bar{a}_1^2 + \bar{a}_1^2 = 10$$
$$\overline{\mathcal{A}}_2 = \bar{a}_2^1 + \bar{a}_2^2 = 20$$

The point (10, 20) becomes the origin for consumer 2, as depicted in Figure 3.

The indifference curves for each consumer are hyperbolas which asymptote to the boundaries of the respective sides of the box. A general equilibrium occurs where the indifference curves for the two consumers share a common tangency with a price line through the endowment point marked as E in Figure 3. At the point of common tangency, the marginal rates of substitution for each consumer are equal to the ratio of asset prices. The asset prices which support this equilibrium are $\left(q_1^*, q_2^*\right) = (7, 6.5)$; hence the common marginal rate of substitution is $\left(-\frac{14}{13}\right)$.

In the general equilibrium, the asset demands of each consumer are as follows :

$$\left(a_1^{*1}, a_2^{*1}\right) = (6.875, 13.75)$$
$$\left(a_1^{*2}, a_2^{*2}\right) = (3.125, 6.25)$$

Note that these demands exhaust the available supply of each asset, i.e.:

$$\sum_{i=1}^{2} a_1^{*i} = \overline{\mathcal{A}}_1 = 10;$$
$$\sum_{i=1}^{2} a_2^{*i} = \overline{\mathcal{A}}_2 = 20$$

These asset demands correspond to state-contingent wealth levels of:

$$\left(\mathcal{W}_1^{*1}, \mathcal{W}_2^{*1}\right) = (13.75, 34.375)$$
$$\left(\mathcal{W}_1^{*2}, \mathcal{W}_2^{*2}\right) = (6.25, 15.625)$$

Again, these exhaust the endowed state-contingent wealth available to the two consumers jointly of :

$$\left(\overline{W}_1, \overline{W}_2\right) = (20, 50)$$

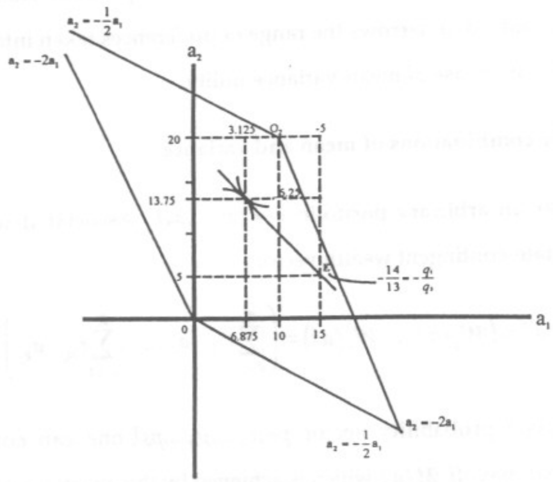

Figure 3

3. Mean-Variance Analysis

Much modern finance theory is grounded on the assumption that agents choose amongst assets on the basis of the mean and variance of their return distributions. In particular, mean-variance analysis provides the foundation for the renowned Capital Asset Pricing Model. The assumption that agents care only about the first two statistical moments of an asset's return distribution embodies a particularly narrow conception of the utility function in asset space. No such assumption is necessary, as we have seen, to derive a complete theory of portfolio choice. Nevertheless, it is *de rigueur* in finance theory to proceed in this fashion.

We follow this tradition to show how the theory of portfolio choice emerging from financial economics may be specialised to encompass the usual case studied in basic finance. Our point is that, while suggestive, mean-variance analysis adds nothing to the structure of the portfolio choice problem studied above; indeed, it narrows the range of preferences taken into account to the highly particular case of mean-variance utility.

3.1 Feasible combinations of mean and variance

Consider an arbitrary portfolio $a = (a_1, \dots, a_K)$. Associated with such a portfolio is a state-contingent wealth vector:

$$\mathcal{W}(a) = \left(\mathcal{W}_1(a), \dots, \mathcal{W}_s(a)\right) = \left(\sum_{k=1}^{K} r_{1k} \cdot a_k, \dots, \sum_{k=1}^{K} r_{sk} \cdot a_k \right)$$

For a given probability vector $p=(p_1, \dots, p_s, \dots, p_S)$, one can compute the expected or mean payoff $\mathcal{M}(a)$ which is achieved by this portfolio a as:

$$\begin{aligned} \mathcal{M}(a) &= \sum_{s=1}^{S} p_s \cdot \mathcal{W}_s(a) = \sum_{s=1}^{S} p_s \cdot \left(\sum_{k=1}^{K} r_{sk} \cdot a_k\right) \\ &= \sum_{k=1}^{K}\left(\sum_{s=1}^{S} p_s \cdot r_{sk}\right) \cdot a_k = \sum_{k=1}^{K} \mu_k \cdot a_k \end{aligned}$$

where $\mu_k = \sum_{s=1}^{S} p_s \cdot r_{sk}$ denotes the expected or mean payoff of asset k.

Similarly, one can compute the variance of the payoffs from portfolio a, $S^2(a)$. Let:

$$\sigma_{jk} = \sum_{s=1}^{S} p_s \cdot (r_{sj} - \mu_j) \cdot (r_{sk} - \mu_k)$$

be the covariance of the payoffs from assets j and k. For $j=k$, $\sigma_{jj} = \sum_{s=1}^{S} p_s \cdot (r_{sj} - \mu_j)^2$ denotes the variance of the payoff from asset j. The variance of the payoff from portfolio a, $S^2(a)$, can now be written as:

$$
\begin{aligned}
\mathcal{S}^2(a) &= \sum_{s=1}^{\mathcal{S}} p_s \cdot (\mathcal{W}_s(a) - \mathcal{M}(a))^2 = \sum_{s=1}^{\mathcal{S}} p_s \cdot \left(\sum_{k=1}^{\mathcal{K}} (r_{sk} - \mu_k) \cdot a_k \right)^2 \\
&= \sum_{s=1}^{\mathcal{S}} p_s \cdot \left(\sum_{j=1}^{\mathcal{K}} (r_{sj} - \mu_j) \cdot a_j \right) \cdot \left(\sum_{k=1}^{\mathcal{K}} (r_{sk} - \mu_k) \cdot a_k \right) \\
&= \sum_{k=1}^{\mathcal{K}} a_k \cdot \sum_{j=1}^{\mathcal{K}} a_j \cdot \left[\sum_{s=1}^{\mathcal{S}} p_s \cdot (r_{sj} - \mu_j) \cdot (r_{sk} - \mu_k) \right] \\
&= \sum_{k=1}^{\mathcal{K}} a_k \cdot \left(\sum_{j=1}^{\mathcal{K}} a_j \cdot \sigma_{jk} \right)
\end{aligned}
$$

The transformations of the mean $\mathcal{M}(a)$ and the variance $\mathcal{S}^2(a)$ show that the mean of a portfolio is the weighted sum of the mean payoffs of the individual assets, where the respective asset quantities act as weights. The variance of a portfolio is the quadratic form obtained from the vector of individual asset quantities applied to the matrix of covariances of asset payoffs, i.e.:

$$
\mathcal{S}^2(a) = a \cdot \Omega \cdot a' \text{ where } \Omega = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{k1} \\ \vdots & \ddots & \vdots \\ \sigma_{1k} & \cdots & \sigma_{kk} \end{bmatrix}
$$

Since $\mathcal{S}^2(a)$ is a quadratic form which is non-negative for any portfolio a, it follows that the covariance matrix must be positive semi-definite. This latter property implies that the determinant of Ω must be a non-negative number.

Every portfolio a has associated with it a mean $\mathcal{M}(a)$ and a variance $\mathcal{S}^2(a)$. There is however usually more than one portfolio for any given mean-variance combination. In particular, it is possible to determine the set of $(\mathcal{M}(a), \mathcal{S}^2(a))$ combinations which are feasible for the consumer since they correspond to portfolios satisfying the budget constraint:

$$
\sum_{k=1}^{\mathcal{K}} q_k \cdot a_k = \mathcal{W}_o
$$

The feasible set of mean-variance combinations is written formally as follows:

$$\left\{ \left(\mathcal{M}(a), S^2(a) \right) \middle| \sum_{k=1}^{K} q_k \cdot a_k = \mathcal{W}_o \right\}$$

Note the dependence of this set on the asset prices q_k and the initial endowment of wealth $\mathcal{W}_o$.

It is possible to represent the set of feasible mean-variance combinations in a mean-variance diagram or, as is more common in the finance literature, in a mean-standard deviation diagram. The set of feasible mean-standard deviation combinations has the general form displayed in Figure 4.

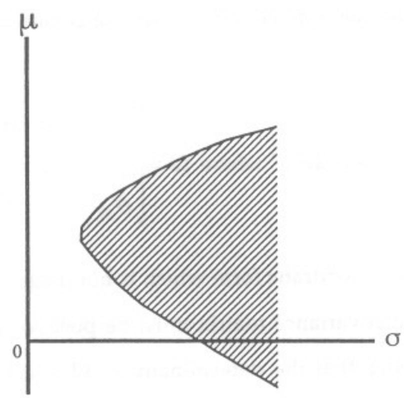

Figure 4

Note that, in general, all mean-standard deviation combinations in the shaded arc of Figure 4 are feasible. If there are only two assets, however, feasible mean-standard deviation combinations lie on the border of this set only and not in the interior. The restriction to two assets facilitates a neat diagrammatic derivation of the set of feasible mean-standard deviation combinations, and allows us to compare the mean-variance approach which dominates the finance literature with the general equilibrium economic approach developed in this paper.

3.2 Feasible combinations of mean and variance in a two-asset model

We begin by recognising that it is possible to construct iso-μ and iso-σ contours in (a_1, a_2) space. Iso-μ contours represent portfolios (a_1, a_2) with the same mean, say $\overline{\mu}$. Recalling the formulation of the mean of a portfolio given above:

$$\mu(a_1, a_2) = \mu_1 \cdot a_1 + \mu_2 \cdot a_2 = \overline{\mu}$$

Iso-μ contours are linear with slope and location parameters μ_1 and μ_2. Since $\mu_k = \sum_{s=1}^{S} p_s \cdot r_{sk}$, it is clear that iso-μ contours are drawn for a given probability distribution over states. Changing the probability distribution changes the position and slope of the iso-μ contours; however, they remain parallel linear functions in (a_1, a_2) space. Iso-μ contours are depicted in Figure 5.

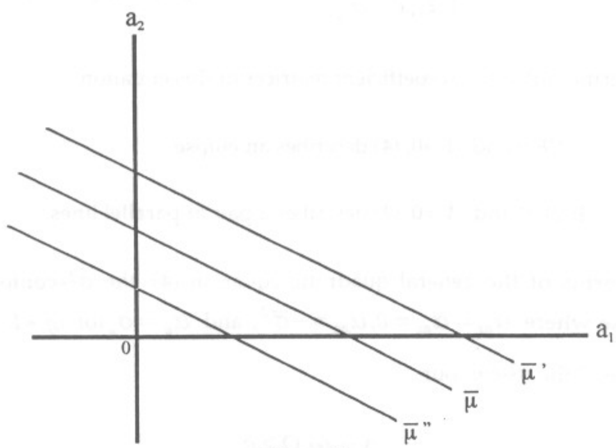

Figure 5

Iso-σ contours are obtained by fixing a level of variance $\overline{\sigma}^2$ (or equivalently of standard deviation $\overline{\sigma} = \sqrt{\overline{\sigma}^2}$):

$$\sigma^2(a_1,a_2)=\sigma_{11}\cdot a_1^2+2\cdot\sigma_{12}\cdot a_1\cdot a_2+\sigma_{22}\cdot a_2^2=\overline{\sigma}^2$$

Since squaring a function is a monotonic transformation, the contour curves of $\sigma^2(a_1, a_2)$ are identical to the contour curves of $\sigma(a_1, a_2)$. The equation of a σ^2 -contour, $\sigma_{11}\cdot a_1^2+2\cdot\sigma_{12}\cdot a_1\cdot a_2+\sigma_{22}\cdot a_2^2-\overline{\sigma}^2=0$, is a special case of the *general quadratic equation*. For such equations, the following result may be proven:

Lemma 1:

Consider the general quadratic equation:

$$\alpha_{11}\cdot x_1^2+2\cdot\alpha_{12}\cdot x_1\cdot x_2+\alpha_2\cdot x_2^2+2\cdot\alpha_{01}\cdot x_1+2\cdot\alpha_{02}\cdot x_2+\alpha_{00}=0 \qquad (4)$$

Let:

$$\mathcal{D}=det\begin{bmatrix}\alpha_{11}\,\alpha_{12}\,\alpha_{01}\\ \alpha_{12}\,\alpha_{22}\,\alpha_{02}\\ \alpha_{13}\,\alpha_{02}\,\alpha_{00}\end{bmatrix} \text{ and } \Delta=det\begin{bmatrix}\alpha_{11}\,\alpha_{12}\\ \alpha_{12}\,\alpha_{22}\end{bmatrix}$$

be the determinants of two coefficient matrices of this equation.

(i) If $\mathcal{D}<0$ and $\Delta>0$, (4) describes an ellipse.

(ii) If D=0 and $\Delta=0$, (4) describes a pair of parallel lines. ■

In terms of the general quadratic equation (4), the σ^2–contour is the special case where $\alpha_{01}=\alpha_{02}=0, \alpha_{00}=-\overline{\sigma}^2$, and $\alpha_{ij}=\sigma_{ij}$ for $i,j=1, 2$. Thus, for the case of the σ^2-contour:

$$\Delta=det\,\Omega\geq 0$$

since Ω is positive semi-definite. Substituting the respective parameters of the σ^2-contour into $\mathcal{D}$, we discover that:

$$\mathcal{D}=-\overline{\sigma}^2\cdot\Delta=-\overline{\sigma}^2\cdot det\,\Omega\leq 0$$

It follows from Lemma 1 that there can be only two cases:

(i) if $det\ \Omega > 0$, the contour of $\sigma^2(a_1,a_2)$ must be an ellipse; and

(ii) if $det\ \Omega = 0$, the contour of $\sigma^2(a_1,a_2)$ must be a pair of parallel lines.

Case (i) is illustrated in Figure 6. As expected, the contours are ellipses in (a_1,a_2) space, centred on the origin and symmetric about a ray through the origin. Successive ellipses radiating from the origin are loci of (a_1,a_2) pairs with successively greater σ-values (i.e., standard deviations of contingent wealth levels). Once again, the position of the family of ellipses depends on the probability distribution over states.

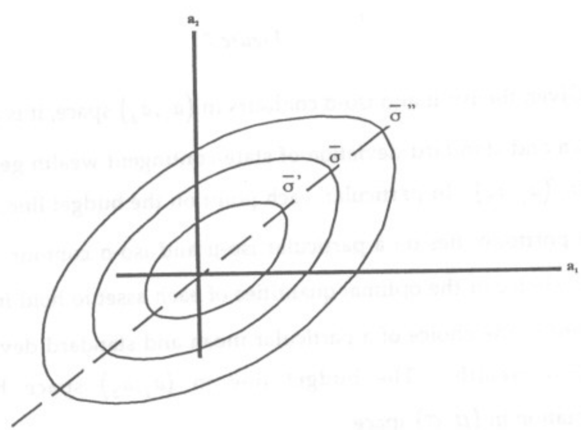

Figure 6

The budget line in (a_1,a_2) space is given by the equation $a_2 = \frac{W_0}{q_2} - \frac{q_1}{q_2} \cdot a_1$, which is derived from the budget constraint of the portfolio choice problem. With positive prices for both assets, the slope of the budget line is negative.

Combining the budget line with the iso-μ and iso-σ contours in a single diagram yields the left-hand panel of Figure 7 below.

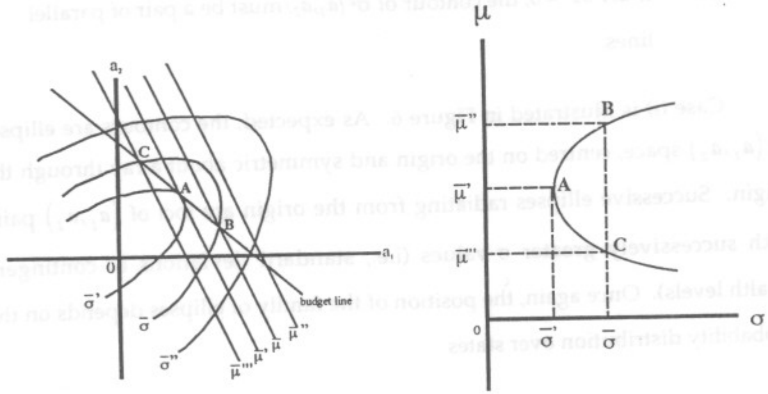

Figure 7

Given the iso-μ and iso-σ contours in (a_1,a_2) space, it is easy to read off the mean and standard deviation of state-contingent wealth generated by any portfolio, (a_1,a_2). In particular, each point on the budget line, representing a feasible portfolio, lies on a particular iso-μ and iso-σ contour. Thus one can view the choice of the optimal quantities of each asset to hold in a portfolio as equivalent to the choice of a particular mean and standard deviation of state-contingent wealth. The budget line in (a_1,a_2) space has a unique representation in (μ,σ) space.

Consider the budget line depicted in the left-hand panel of Figure 7. Beginning at the intercept of the budget line on the a_2-axis, the positions of the iso-μ and iso-σ contours reveal that successive points downwards and to the right along the budget line have successively higher means, and at first successively lower and then successively higher standard deviations. Moving down the budget line in (a_1,a_2) space traces out a locus in (μ,σ) space of the type depicted in the right-hand panel of Figure 7. Note that the point of minimum standard deviation corresponds to the point at which the budget line is tangent to the lowest iso-σ contour (shown as A in both panels).

3.3 Some special cases

The portfolio $(\tilde{a}_1, \tilde{a}_2)$ which achieves minimum standard deviation is known as the *minimum variance portfolio* (MVP). In general, the standard deviation or variance of the MVP will not be zero. This is the case depicted in Figure 7. The variance of the MVP will be zero, however, if one or other of the assets is riskless (in which case the MVP is trivially the portfolio consisting exclusively of the riskless asset), or if it is possible to create a riskless portfolio by combining the two risky assets in appropriate proportions.

This latter possibility will only arise if there are at least as many different assets as there are states of the world, a condition known as *complete markets.* Clearly, in the example we have used so far in this section, markets are *incomplete*, i.e., there are many states of the world but only two assets. It is therefore not possible to synthesise a riskless portfolio, and given that neither of the two assets available is riskless, the MVP will have positive variance.

If there is a riskless asset, or it is possible to create a riskless portfolio by combining risky assets, the iso-σ contours and the (μ, σ) frontier take on a special shape. To see this, we revert to the earlier example of two assets and two states of the world.

Case 1 : *riskless asset*

An asset is riskless if it pays the same amount regardless of the state of the world, i.e., if $r_{sk} = r_k$ for all $s = 1, \ldots, S$. Clearly, for a riskless asset $\mu_k = r_k$ and $\sigma_{jk} = 0$ for all j, k. Hence, $det\ \Omega = 0$ when there is a riskless asset, and the iso-σ contours must be pairs of parallel lines.

In our two-asset example, let asset 2 be the riskless asset. The determinant of the covariance matrix is :

$$det\,\Omega = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & 0 \end{bmatrix} = 0$$

All portfolios with $a_1 = 0$ will have a variance of zero in this case. The zero iso-σ contour coincides therefore with the a_2-axis. For any positive level of variance $\overline{\sigma} > 0$, the iso-σ contours consist of two lines parallel to the a_2-axis.

In general, when one of the two assets is riskless (i.e., offers the same payoff in each state), the iso-σ contours become straight lines parallel either to the a_1-axis or the a_2-axis, depending upon which of the two assets is riskless. The (σ=0) contour in this case is the a_1-axis or the a_2-axis itself, again depending upon which of the two assets is riskless.

Case 2: complete markets

Two asset markets are complete if there are exactly two states of the world and the asset payoffs are linearly independent. In this case, the iso-μ contours are given by the equation:

$$\begin{aligned} \mu(a_1, a_2) &\equiv p \cdot W_1 + (1-p) \cdot W_2 \\ &= [p \cdot r_{11} + (1-p) \cdot r_{21}] \cdot a_1 + [p \cdot r_{12} + (1-p) \cdot r_{22}] \cdot a_2 \end{aligned}$$

where, instead of p_1 and p_2, we write the probabilities of the two states as p and $(1-p)$. These are straight lines in (a_1, a_2) space.

The iso-σ contours are given by the equation:

$$\begin{aligned} \sigma(a_1, a_2) &\equiv \sqrt{p \cdot [W_1 - \mu]^2 + (1-p) \cdot [W_2 - \mu]^2} \\ &= \sqrt{p \cdot (1-p) \cdot [(r_{11} - r_{21}) \cdot a_1 + (r_{12} - r_{22}) \cdot a_2]^2} \\ &= \pm\sqrt{p \cdot (1-p)} \cdot [(r_{11} - r_{21}) \cdot a_1 + (r_{12} - r_{22}) \cdot a_2] \end{aligned}$$

which is also linear in (a_1,a_2) space. The iso-σ contours are a set of parallel straight lines in (a_1,a_2) space. The (σ=0) contour is a ray through the origin with slope equal to $-\frac{(r_{11}-r_{21})}{(r_{12}-r_{22})}$. Contours representing successively higher values of σ are straight lines parallel to the (σ=0) contour at equal vertical distances above and below it.

Recall that the iso-σ contours in the case of incomplete markets were ellipses centred on the origin and symmetric about a ray through the origin. In the case of complete markets, the ellipses are "stretched out" infinitely in the direction of the longer of their two axes, and thus become a set of parallel straight lines. The (σ=0) contour, instead of being a single point located at the origin, becomes a ray through the origin. The higher contours, instead of being ellipses radiating from the origin, become parallel straight lines extending either side of a ray through the origin.

The following diagram illustrates the case of complete markets in two assets.

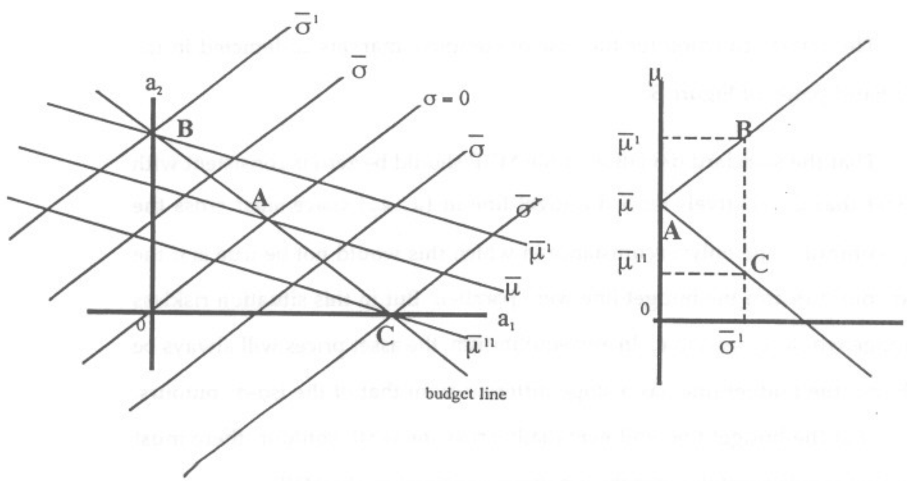

Figure 8

To deduce the shape of the (μ,σ) frontier, we note that the two linear equations for the iso-μ and iso-σ contours can be solved simultaneously to yield another linear equation relating μ and σ. This equation can in turn be solved simultaneously with the linear budget equation to obtain the equation of the (μ,σ) frontier. It is important to note that the position and slope of the (μ,σ) frontier will depend on q_1, q_2 and $\mathcal{W}_0$, since these are parameters of the budget line.

In the presence either of complete markets or a riskless asset, the (μ,σ) frontier is piecewise-linear. It has the same basic shape as the (μ,σ) frontier in the case of incomplete markets except that:

(i) the positive- and negative-sloped sections of the frontier are both linear; and

(ii) the standard deviation of the minimum variance portfolio is zero (i.e., the (μ,σ) frontier meets the μ-axis at a point equal to the expected payoff of the MVP (or the certain payoff from the riskless asset, if there is one).

The (μ,σ) function for the case of complete markets is depicted in the right-hand panel of Figure 8.

That the standard deviation of the MVP should be zero is consistent with the fact that a negatively-sloped budget line in (a_1,a_2) space *must* cross the (σ=0) contour. The only circumstance in which this would not be true is if the iso-σ contours and the budget line were parallel. But in this situation riskless arbitrage would be possible. In any equilibrium, the asset prices will always be such that the budget line has a slope different from that of the iso-σ contours. Given that the budget line will eventually cross the (σ=0) contour, there must be some feasible portfolio for which σ=0, and this will be the MVP.

3.4 Portfolio choice in mean-variance space

Having established the correspondence between the budget line in (a_1, a_2) space and the (μ, σ) frontier, we can proceed to discuss preferences and optimal portfolio choice. As noted in Section 2, selection of the optimal portfolio in (a_1, a_2) space is a matter of maximising expected utility subject to the budget constraint. The optimal portfolio will lie on the particular (μ, σ) frontier which is consistent with the assumed values of q_1, q_2 and $\mathcal{W}_0$.

To represent preferences in the mean-standard deviation space directly, a decision-maker's preferences over risky prospects must not depend on any characteristic of the prospect other than its mean and its variance, i.e., there must be a representation of the form $\mathcal{V}(\mu, \sigma)$. This amounts to the assumption that only mean and variance are relevant to the portfolio choice decision. Since such an assumption is generally incompatible with expected utility theory, and the only way to reconcile the two is to:

- restrict decision-making to probability distributions which are completely characterised by their means and variances (this is essentially the class of normal distributions); or
- assume a quadratic von Neumann-Morgenstem utility index (which has the inconvenient property that it is not monotonically increasing in wealth).

Whether or not the expected utility hypothesis is adopted, assuming that preferences over assets may be represented by a utility function with arguments μ and σ, where the marginal utility of μ is positive and the marginal utility of σ is negative, is sufficient to derive indifference curves in (μ, σ) space. They will be upward-sloping, reflecting the fact that an agent must be offered additional expected wealth in order to be indifferent to the prospect of bearing additional risk (where risk is measured by the standard deviation of state-contingent wealth). Figure 9 depicts portfolio choice in the mean-variance framework with a utility function $\mathcal{V}(\mu,\sigma)$.

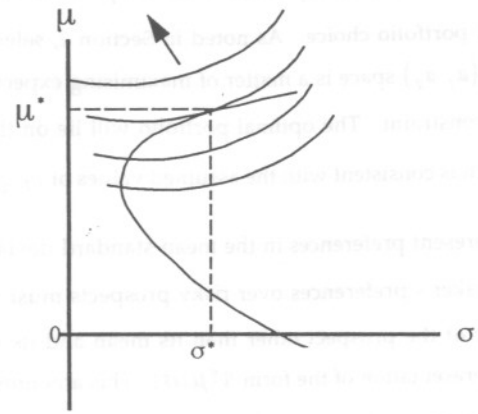

Figure 9

The assumption that $\mathcal{V}(\mu,\sigma)$ is increasing in μ and decreasing in σ implies positively-sloped indifference curves, but not convexity as shown in Figure 9. Utility increases in a north-westerly direction as indicated by the arrow. It follows from the slope of the indifference curves and the direction of increasing utility that the optimal portfolio must correspond to a μ–σ combination on the upper branch of the feasible μ–σ set.

In other words, since the consumer unambiguously prefers a portfolio with a higher expected value of wealth to one with a lower expected value *and the same standard deviation of wealth,* those points on the lower branch of the (μ,σ) frontier will never be chosen. Such points are said to be *mean-variance inefficient*. The set of *efficient* portfolios in (μ,σ) space consists of all those points on the (μ,σ) frontier which have *both a mean and a standard deviation at least as great as that of the minimum variance portfolio.*

A mean-variance utility function $\mathcal{V}(\mu,\sigma)$ induces a ranking in the (a_1,a_2) space as well, since mean and variance depend on the portfolio chosen, i.e., $\mathcal{V}(\mu(a_1,a_2), \sigma(a_1,a_2))$. For every preference relation over lotteries which can be represented by a mean-variance utility function $\mathcal{V}(\mu,\sigma)$, there is an induced preference relation on the space of prospects. The converse, however, does not generally hold, i.e., a preference relation on lotteries cannot generally be represented by a preference relation over mean and variance alone.

Of course, dispensing with the need to represent preferences in (μ,σ) space overcomes the need to consider a restricted range of utility functions. The fundamental space in which portfolio choice takes place is (a_1,a_2) space. The tradition in finance theory of using the (μ,σ) space to tell the story of portfolio selection unnecessarily restricts the range of preferences which consumers may display. This fact is rarely, if ever, made explicit. A distinguishing feature of financial economics as opposed to finance theory is a preference for revealing the fundamental *economic* forces at work in financial decision-making.

Restricting preferences to the (μ,σ) space has the advantage, however of enabling one to derive an explicit relationship among asset prices in an asset market equilibrium. This is the essence of the Capital Asset Pricing Model to which we turn now.

4. The Capital Asset Pricing Model

Consider once again the portfolio choice problem faced by individuals in a general equilibrium involving a finite number of agents and a finite number of assets. A famous result in the finance literature characterises the relationship among the prices of assets in a general equilibrium in which there are I consumers, $(\mathcal{K}\text{-}1)$ risky assets and one riskless asset. Known as the *Capital Asset*

Pricing Model (CAPM), the analysis proves the surprising result that the relationship among the prices of assets in a general equilibrium (in which agents select assets so as to maximise mean-variance utility) is linear. Apart from being surprising, the result is especially convenient, since it lends itself immediately to the application of linear regression estimation techniques, as the vast literature on empirical testing of the CAPM testifies.

Our purpose here is to derive the *capital asset pricing equation* (the general equilibrium pricing relation which emerges from the CAPM) from the microeconomic foundations of portfolio choice developed in the earlier sections of this paper.

We begin by recalling some definitions. The vector $a = (a_1, \ldots, a_{\mathcal{k}})$ represents a portfolio, where the elements correspond to quantities of each of the $\mathcal{K}$ assets held in portfolio. The assets have payoffs in each of the $\mathcal{S}$ states denoted $r_{s\mathcal{k}}\,(s = 1, \ldots, \mathcal{S}; \mathcal{k} = 1, \ldots, \mathcal{K})$. The wealth derived in each state of the world depends upon the quantity of each asset held and the payoff from each asset in the particular state, i.e.:

$$\mathcal{W}_s(a) = \sum_{\mathcal{k}=1}^{\mathcal{K}} r_{s\mathcal{k}} \cdot a_{\mathcal{k}}$$

for $s = 1, \ldots, \mathcal{S}$. The expected wealth derived from a portfolio a equals the sum of the expected payoffs from the individual assets weighted by the quantities of the assets held in portfolio:

$$\mathcal{M}(a) = \sum_{\mathcal{k}=1}^{\mathcal{K}} \mu_{\mathcal{k}} \cdot a_{\mathcal{k}}$$

where μ_k is the expected payoff from asset $\mathcal{k}\,(\mathcal{k} = 1, \ldots, \mathcal{K})$. The variance of state-contingent wealth derived from holding a portfolio a is expressed as:

$$\mathcal{S}^2(a) = \sum_{\mathcal{k}=1}^{\mathcal{K}} a_{\mathcal{k}} \cdot \left[\sum_{j=1}^{\mathcal{K}} a_j \cdot \sigma_{j\mathcal{k}} \right] = \sum_{\mathcal{k}=1}^{\mathcal{K}} \sum_{j=1}^{\mathcal{K}} a_{\mathcal{k}} \cdot a_j \cdot \sigma_{j\mathcal{k}}$$

If we differentiate $\mathcal{M}(a)$ and $\mathcal{S}^2(a)$ with respect to a_j, where a_j is the quantity of one of the $\mathcal{K}$ assets chosen arbitrarily, we obtain the following:

$$\mathcal{M}_j(a) = \mu_j$$

where $\mathcal{M}_j(a) = \partial \mathcal{M}(a) / \partial a_j$ denotes the partial derivative of $\mathcal{M}(a)$ with respect to a_j; and, denoting the partial derivative of $\mathcal{S}^2(a)$ with respect to a_j by $\mathcal{S}_j^2(a) = \partial \mathcal{S}^2(a) / \partial a_j$:

$$\begin{aligned} \mathcal{S}_j^2(a) &= 2 \cdot \left[\sum_{k=1}^{\mathcal{K}} a_k \cdot \sigma_{jk} \right] \\ &= 2 \cdot \mathcal{S}(a, j) \end{aligned}$$

Note that $\left[\sum_{k=1}^{\mathcal{K}} a_k \cdot \sigma_{jk} \right]$ is the covariance between the payoff of the entire portfolio a and the payoff of a single asset j:

$$\begin{aligned} \mathcal{S}(a, j) &= \sum_{s=1}^{\mathcal{S}} p_s \cdot (r_{sj} - \mu_j) \cdot (\mathcal{W}_s(a) - \mathcal{M}(a)) \\ &= \sum_{s=1}^{\mathcal{S}} p_s \cdot (r_{sj} - \mu_j) \cdot \left(\sum_{k=1}^{\mathcal{K}} (r_{sk} - \mu_k) \cdot a_k \right) \\ &= \sum_{k=1}^{\mathcal{K}} a_k \cdot \left[\sum_{s=1}^{\mathcal{S}} p_s \cdot (r_{sj} - \mu_j) \cdot (r_{sk} - \mu_k) \right] \\ &= \sum_{k=1}^{\mathcal{K}} a_k \cdot \sigma_{jk} \end{aligned}$$

Now consider the optimisation problem for the typical consumer $i \in \{1, 2, \cdots, I\}$:

$$\max_a \ \mathcal{V}^i\left[\mathcal{M}(a), \mathcal{S}^2(a)\right]$$

$$\text{subject to } \sum_{k=1}^{\mathcal{K}} q_k \cdot a_k = \sum_{k=1}^{\mathcal{K}} q_k \cdot \bar{a}_k$$

The first order conditions for this problem are:

$$\mathcal{V}_1^i\left[\mathcal{M}(a),\mathcal{S}^2(a)\right]\cdot\mathcal{M}_k(a)+\mathcal{V}_2^i\left[\mathcal{M}(a),\mathcal{S}^2(a)\right]\cdot\mathcal{S}_k^2(a)-\lambda\cdot q_k=0 \quad \text{for } k=1,\dots,\mathcal{K}$$

and

$$\sum_{k=1}^{\mathcal{K}} q_k\cdot a_k=\sum_{k=1}^{\mathcal{K}} q_k\cdot\bar{a}_k$$

where λ is the Lagrange multiplier, $\mathcal{V}_1^i(\cdot)\equiv\partial\mathcal{V}^i(\cdot)/\partial\mathcal{M}$ and $\mathcal{V}_2^i(\cdot)\equiv\partial\mathcal{V}^i(\cdot)/\partial\mathcal{S}^2$.

The first order conditions implicitly define asset demand functions of the following form $a_k^i=f_k^i\left(q_1,\dots,q_{\mathcal{K}};\bar{a}_1^i,\dots,\bar{a}_{\mathcal{K}}^i\right)$ for all $k=1,\dots,\mathcal{K}$. A general equilibrium in this exchange economy is a vector of asset prices $q^*=\left(q_1^*,\dots,q_{\mathcal{K}}^*\right)$ together with a vector of asset demands $a^{i*}=\left(a_1^{i*},\dots,a_{\mathcal{K}}^{i*}\right)$ for each consumer $i=1,2,\dots,I$ such that:

$$\sum_{i=1}^{I} a_k^{i*}=\sum_{i=1}^{I} f_k^i\left(q_1^*,\dots,q_{\mathcal{K}}^*;\bar{a}_1^i,\dots,\bar{a}_{\mathcal{K}}^i\right)=\sum_{i=1}^{I}\bar{a}_k^i=\mathcal{A}_k,$$

where $\mathcal{A}_k$ denotes the aggregate quantity of the asset available in the economy.

In words, the quantity of each asset demanded in equilibrium by all consumers precisely exhausts the available supply.

The capital asset pricing equation is derived from the first order conditions given above, evaluated at equilibrium, and assuming that one of the assets is riskless.

If asset $\mathcal{K}$ is the riskless asset, then $r_{s\mathcal{K}}=r$ for all $s=1,\dots,\mathcal{S}$. Therefore the partial derivatives of the expected payoff and variance functions with respect to changes in the quantity of asset $\mathcal{K}$ held in portfolio are, respectively, $\mathcal{M}_{\mathcal{K}}(a)=r$ and $\mathcal{S}_{\mathcal{K}}^2(a)=0$. Substituting these values into the first order conditions and choosing the riskless asset as numeraire, i.e., setting $q_{\mathcal{K}}=1$, the k-th first order condition may be solved for the Lagrange multiplier:

$$\lambda = \mathcal{V}_1^i\left[\mathcal{M}\left(a^{i*}\right), \mathcal{S}^2\left(a^{i*}\right)\right] \cdot r$$

Substituting expressions for λ and $\mathcal{S}_k^2(a)$, the first $\mathcal{K}$-1 first order conditions become:

$$\mathcal{V}_1^i\left[\mathcal{M}\left(a^{i*}\right), \mathcal{S}^2\left(a^{i*}\right)\right] \cdot \left(\mu_k - q_k^* \cdot r\right) + 2 \cdot \mathcal{V}_2^i\left[\mathcal{M}\left(a^{i*}\right), \mathcal{S}^2\left(a^{i*}\right)\right] \cdot \left(\sum_{j=1}^{\mathcal{K}-1} a_j^{i*} \cdot \sigma_{jk}\right) = 0$$

This equation may be re-written as:

$$\theta^i\left(a^{i*}\right) \cdot \left(\mu_k - q_k^* \cdot r\right) = \sum_{j=1}^{\mathcal{K}-1} a_j^{i*} \cdot \sigma_{jk} \tag{5}$$

where $\theta^i\left(a^{i*}\right) = -\mathcal{V}_1^i\left[\mathcal{M}\left(a^{i*}\right), \mathcal{S}^2\left(a^{i*}\right)\right] / 2 \cdot \mathcal{V}_2^i\left[\mathcal{M}\left(a^{i*}\right), \mathcal{S}^2\left(a^{i*}\right)\right]$ is the marginal rate of substitution along an individual agent's indifference curve in mean-standard deviation space.

Summing (5) over all consumers, and noting that $\sum_{i=1}^{I} a_k^{i*} = \mathcal{A}_k$ in equilibrium (market clearing), we obtain:

$$\Theta\left(a^*\right) \cdot \left(\mu_k - q_k^* \cdot r\right) = \sum_{j=1}^{\mathcal{K}-1} \mathcal{A}_j \cdot \sigma_{jk} \tag{6}$$

where $\Theta\left(a^*\right) = \sum_{i=1}^{I} \theta^i\left(a^{i*}\right)$ is the sum of the agents' marginal rates of substitution. Note the dependence of these marginal rates of substitution, and their summation, on the equilibrium asset allocation a^*. This reminds us that the capital asset pricing equation is strictly valid *only in a general equilibrium of the asset economy,* a point rarely acknowledged in the headlong rush to apply the CAPM to any and every traded security in the real world.

We now multiply both sides of (6) by $\mathcal{A}_k$ and sum over the risky assets (i.e., $k = 1, \ldots, \mathcal{K}-1$):

$$\sum_{\mathcal{k}=1}^{\mathcal{K}-1} \Theta(a^*) \cdot (\mu_{\mathcal{k}} - q_{\mathcal{k}}^* \cdot r) \cdot \mathcal{A}_{\mathcal{k}} = \sum_{\mathcal{k}=1}^{\mathcal{K}-1} \sum_{j=1}^{\mathcal{K}-1} \mathcal{A}_{\mathcal{k}} \cdot \mathcal{A}_j \cdot \sigma_{j,\mathcal{k}} \tag{7}$$

Equation (7) may be re-expressed as:

$$\Theta(a^*) \cdot [\mathcal{M}(\mathcal{A}) - \mathcal{Q}^*(\mathcal{A}) \cdot r] = \mathcal{S}^2(\mathcal{A}) \tag{8}$$

where:

- $\mathcal{M}(\mathcal{A}) = \sum_{\mathcal{k}=1}^{\mathcal{K}-1} \mu_{\mathcal{k}} \cdot \mathcal{A}_{\mathcal{k}} = \sum_{\mathcal{k}=1}^{\mathcal{K}-1} \sum_{s=1}^{S} p_s \cdot r_{s\mathcal{k}} \cdot \mathcal{A}_{\mathcal{k}} = \sum_{s=1}^{S} p_s \cdot \mathcal{W}_s(\mathcal{A})$ is the aggregate expected payoff in equilibrium of the market portfolio of risky assets;
- $\mathcal{Q}^*(\mathcal{A}) = \sum_{\mathcal{k}=1}^{\mathcal{K}-1} q_{\mathcal{k}}^* \cdot \mathcal{A}_{\mathcal{k}}$ is the aggregate expenditure by all consumers in equilibrium on the market portfolio of risky assets; and
- $\mathcal{S}^2(\mathcal{A}) = \sum_{s=1}^{S} p_s \cdot [\mathcal{W}_s(\mathcal{A}) - \mathcal{M}(\mathcal{A})]^2$ is the variance of the aggregate payoff in equilibrium of the market portfolio of risky assets.

Substituting the expression for $\Theta(a^*)$ in (6) into (8), and re-arranging terms yields:

$$(\mu_{\mathcal{k}} - q_{\mathcal{k}}^* \cdot r) = \frac{\mathcal{S}(\mathcal{A}, \mathcal{k})}{\mathcal{S}^2(\mathcal{A})} \cdot [\mathcal{M}(\mathcal{A}) - \mathcal{Q}^*(\mathcal{A}) \cdot r] \tag{9}$$

where $\mathcal{S}(\mathcal{A}, \mathcal{k}) = \sum_{j=1}^{\mathcal{K}-1} \mathcal{A}_j \cdot \sigma_{j\mathcal{k}}$ is the covariance of the payoff of asset $\mathcal{k}$ with the payoff of the market portfolio of risky assets.

Up to this point, we have maintained the economist's preferred mode of operation, in which asset payoffs are measured in absolute units of account and quantities of assets held in portfolio are measured in absolute units. To complete the derivation of the CAPM, we switch to the finance theorist's preferred mode of operation, and measure asset returns as payoffs *per unit invested* and asset quantities in *units of expenditure*. Thus, instead of an expected payoff, μ_k, an asset has an expected return in equilibrium of $\tilde{\mu}_k = \frac{\mu_k}{q_k^*}$. Similarly, instead of an optimal quantity of asset k in equilibrium, a_k^*, we speak of the optimal investment in (or expenditure on) asset k of $\tilde{a}_k^* = a_k^* \cdot q_k^*$. These equilibrium returns and expenditures clearly depend on the set of equilibrium asset prices, q_k^*.

Using these new concepts, we transform equation (9) by dividing both sides by q_k^* to obtain:

$$(\tilde{\mu}_k - r) = \beta_k \cdot \left[\tilde{\mathcal{M}}(\mathcal{A}) - r\right] \tag{10}$$

where $\beta_k = \frac{Q^*(\mathcal{A})}{q_k^*} \cdot \frac{S(\mathcal{A},k)}{S^2(\mathcal{A})} = \frac{\tilde{S}(\mathcal{A},k)}{\tilde{S}^2(\mathcal{A})}$ and $\tilde{\mathcal{M}}(\mathcal{A}) = \frac{\mathcal{M}(\mathcal{A})}{Q^*(\mathcal{A})}$.

Equation (10) is the *capital asset pricing equation*. It states that, in equilibrium, the difference between the expected return on each risky asset and the return on the riskless asset is proportional to the difference between the expected return on the market portfolio of risky assets and the return on the riskless asset. The factor of proportionality, β_k, varies directly with the covariance of the return on the market portfolio with that on the risky asset k, i.e., $\tilde{S}(\mathcal{A},k)$, and is scaled by the variance of the return on the market portfolio itself, i.e., $\tilde{S}^2(\mathcal{A})$. If the covariance of the kth risky asset with the market portfolio is greater than the covariance of the market portfolio with itself (i.e., the variance of the market portfolio), β_k will be greater than one, and the risk premium required by the market in equilibrium will exceed that required on the entire portfolio of risky assets.

5. Conclusion

A student of economics exposed to the theory of finance for the first time is invariably disconcerted by the lack of familiarity of much of the analysis. One would have thought that a common theoretical framework would have been employed in two disciplines ostensibly so closely related. And yet instead of payoffs we find returns, instead of quantities we find expenditures, and more confusing still, preferences are not general but highly particular, depending upon special functions of the asset return distributions.

It is in response to this experience, which the authors have had first-hand both as students and subsequently as teachers, that this paper was written. It is an attempt to link the disciplines of finance and economics by grounding elementary finance theory firmly on the foundation of general equilibrium economics. The basic problem studied in finance is a direct application of general equilibrium theory. While this fact may be appreciated by finance theorists, it is not reflected in the standard introductory textbooks. The authors' aim is to offer an accessible, diagrammatically amenable treatment of basic portfolio choice theory which allows one to exposit the economic foundations of mean-variance analysis and the CAPM in a theoretically satisfying manner.

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