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THE CHAOTIC DUPOLISTS

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Abstract

The adjustment process of a pair of Cournot duopolists is studied. It is demonstrated that with an iso-elastic demand function and constant marginal costs, the system can result in periodic or even in chaotic behaviour.

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The Chaotic Duopolists

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INTRODUCTION

Duopoly, though contextually the first step from monopoly towards perfect competition, is analytically not a case of intermediate complexity, but more complicated than any of the extremes. This is so as the duopolists have to take in consideration, not only the behaviour of all the consumers in terms of the entire market demand curve for the commodity. The duopolist must also take account of the behaviour of the competitor, inclusive possibe retaliation on his own actions.

The formal theory of duopoly goes as far back as back to 1838, when Augustin Cournot [1] treated the case where there is no retaliation at all, so that in every step each duopolist assumes that the latest step taken by the competitor is his last. The process was assumed to lead to a steady state, nowadays called Cournot equilibrium, though it is by no means certain that it is stable. As a matter of fact, the following discussion will show how cycles and chaos may arise from very simple Cournot adjustment procedures.

Exactly a Century later, in 1938, Heinrich von Stackelberg made some ingenious extensions of the Cournot model, by assuming that any of competitors might try to become a "leader", by taking the reactions of the competitor according to Cournot in explicit consideration when devising his own actions. The tenability of such a situation would, of course, depend on whether the competitor was content with adhering to his Cournot-like behaviour, i.e., being a "follower". He might take up the challenge by trying to become a leader himself. The outcome of such warfare would depend on long term production conditions for the competitors, and on their current financial strengths. In the end one of the duopolist might ultimately force the other out of the market, thus becoming a monopolist. They may agree on collusive behaviour, provided law admits this. They may also return to the Cournot equilibrium, or to a Stackelberg equilibrium, provided the duopolists tacitly agree to form a leader-follower pair.

At the event of game theory, duopoly was one of the obvious fields of application, and the theory was recast in terms of probabilistic strategies. This, however, falls outside the scope of the present discussion, because the points we want to make don't need more than the classical deterministic models.

It has been realised that the Cournot model may lead to cyclic behaviour, and David Rand [3] conjectured that under suitable conditions the outcome would be chaotic. His purely mathematical treatment does, however, not include any substantial economic assumptions under which this becomes true.

THE COURNOT MODEL

Assume an isoelastic demand function, such that price, denoted p, is reciprocal to the total demand. Provided demand equals supply it is made up of the supplies of the two competitors, denoted x and y. Thus

$$p = \frac{1}{x+y} \tag{1}$$

This demand function is not umproblematic, because it does not yield a resonable solution to the collusive case, the reason being that when the possibility of making total supply zero is considered price can go to infinity and total revenue remain constant. On the other hand, total costs would vanish, and the duopolists could get the entire revenue without incurring any costs. Such a solution is purely formal and does not carry any economic substance. This absurdity does not occur with any of the other solutions, as the presence of a positive supply by the competitor always keeps the price finite. We could also easily remedy this technical problem by adding any positive constant in the denominator of (1). As this would not change any substantial conclusions, but has the price of making all formulas much more messy, we abstain from this, and just point out this little complication.

Suppose next that the duopolists produce with constant marginal costs, denoted a and b respectively. The profits of the two firms become accordingly:

$$U(x,y) = \frac{x}{x+y} - ax \tag{2}$$

$$V(x,y) = \frac{y}{x+y} - by \tag{3}$$

The first firm would maximize $U(x,y)$ with respect to x, the second $V(x,y)$ with respect to y. Equating the partial derivatives to zero, we can solve for the reaction functions:

$$x = \sqrt{\frac{y}{a}} - y \tag{4}$$

$$y = \sqrt{\frac{x}{b}} - x \tag{5}$$

A check of the second order conditions, in fact, testifies that we always deal with local profit maxima, provided quantities are positive as indeed they should be. The reaction functions are displayed in Figure 1. Their general outline is that they start at the origin, have unique maxima, and drop to zero again. The intersection is thus unique.

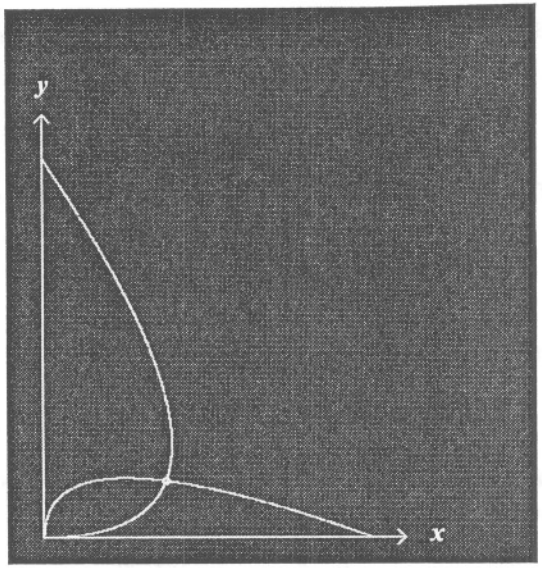

Fig. 1. Reaction curves and the Cournot point.

We can easily solve for the output quantities at the Cournot point, taking (4)-(5) as a simultaneous system of equations:

$$x = \frac{b}{(a+b)^2} \tag{6}$$

$$y = \frac{a}{(a+b)^2} \tag{7}$$

This point, of course, is the intersection of the reaction curves as shown in Figure 1. The profits of the duopolists at the Cournot point can be calculated by substituting back from (6)-(7) in (2)-(3):

$$U=\frac{b^2}{(a+b)^2} \tag{8}$$

$$V=\frac{a^2}{(a+b)^2} \tag{9}$$

STACKELBERG EQUILIBRIA

We can locate the Stackelberg equilibria as well. Supposing that the first firm is the leader, we can substitute for y from (5) into (2), which then becomes:

$$U=\sqrt{bx}-ax \tag{10}$$

This is a function of x alone, so equating the total derivative of (10) to zero we can solve for the optimal output of the leader:

$$x=\frac{b}{4a^2} \tag{11}$$

The output of the follower can be obtained by substituting from (11) into (5):

$$y=\frac{2a-b}{4a^2} \tag{12}$$

We can finally calculate the profits of the duopolists in the first Stackelberg equilbrium by substituting from (11)-(12) into (2)-(3). Thus:

$$U=\frac{b}{4a} \qquad V=\frac{(2a-b)^2}{4a^2} \tag{13}$$

As U according to (13) can be shown to be larger than according to (8), we conclude that it is always preferable for any firm to be a Stackelberg leader rather than to be at a Cournot equilibrium. Both may thus attempt to become leaders, irrespective of the marginal costs, which provide the only difference between the firms in the present model. Marginal costs, of course, would play a role for the chance of surviving economic warfare when both persist in their attempts to become leaders.

Let us just write down the equations for the other Stackelberg equilbrium for the sake of completeness. The leadership profit function for the second firm is:

$$V=\sqrt{ay}-by \tag{14}$$

and maximizing it yields the optimal solution:

$$y = \frac{a}{4b^2} \tag{15}$$

The corresponding passive reaction of the follower then is:

$$x = \frac{2b - a}{4b^2} \tag{16}$$

It remains to calculate the profits in the second Stackelberg equilibrium:

$$U = \frac{(2b-a)^2}{4b^2} \qquad V = \frac{a}{4b} \tag{17}$$

Both Stackeberg equilibria are feasible. What is not feasible is if both the duopolists attempt to be leaders simultaneously.

STABILITY OF THE COURNOT POINT

Let us now return to the Cournot case. Actually, once we are interested in the process of adjustment, we have to lag the variables. So, (4)-(5) must be written:

$$x_{t+1} = \sqrt{\frac{y_t}{a}} - y_t \tag{18}$$

$$y_{t+1} = \sqrt{\frac{x_t}{b}} - x_t \tag{19}$$

The fixed point of this iteration, of course, is the Cournot equilibrium. To find out something about the stability of it we calculate the derivatives of the functions (4)-(5):

$$\frac{dx}{dy} = \frac{1}{2}\sqrt{\frac{1}{ay}} - 1 \tag{20}$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1}{bx}} - 1 \tag{21}$$

Loss of stability occurs when

$$\left| \frac{dx}{dy}\frac{dy}{dx} \right| = 1 \tag{22}$$

i.e., substituting from (6)-(7), whenever:

$$(a-b)^2 = 4ab \tag{23}$$

We observe that this condition can be solved for the ratio:

$$\frac{a}{b} \quad \text{or} \quad \frac{b}{a} \quad = \quad 3 \pm 2\sqrt{2} \tag{24}$$

Thus, whenever one of the ratios of the marginal costs of the duopolists falls outside the interval bounded by the two values specified in (24), the Cournot point will not be stable. The two roots happen to be reciprocal, so there is nothing odd in stating this condition for *any* of the ratios. In fact all critical conditions depend on the ratio of marginal costs only, so there is just one single control parameter.

PERIODIC POINTS AND CHAOS

As this control parameter passes either of the critical values specified in equation (24), the fixed point is replaced by a two period cycle. This should be understood so that x and y each oscillate between two values. The two different variables themselves, of course, take on different values, even in the fixed point.

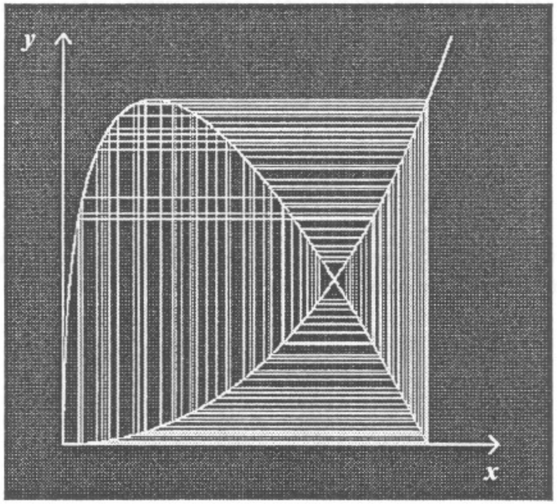

Fig. 2. Chaotic Cournot duopoly adjustmens.

Actually, we are dealing with a pair of independent iterations, which can be seen if (19) is substituted in (18) or vice versa. We then get iterations of each of the variables alone, without interference of the other one, though the delay is two periods in terms of the originally introduced lag. For the simplest cycle it then takes four of these periods to return to the initial value.

Implicit in this is also that, contrary to what may be assumed, it has no importance at all whether the duopolists adjust simultaneously or take turns in their adjustments. The only difference is how these two, essentially autonomous, time series are paired together.

After the first cycle appears there is a period doubling cascade to chaos. In Figure 2 we display the cob-web like chaotic process of adjustment in the same type of diagram as displayed in Figure 1.

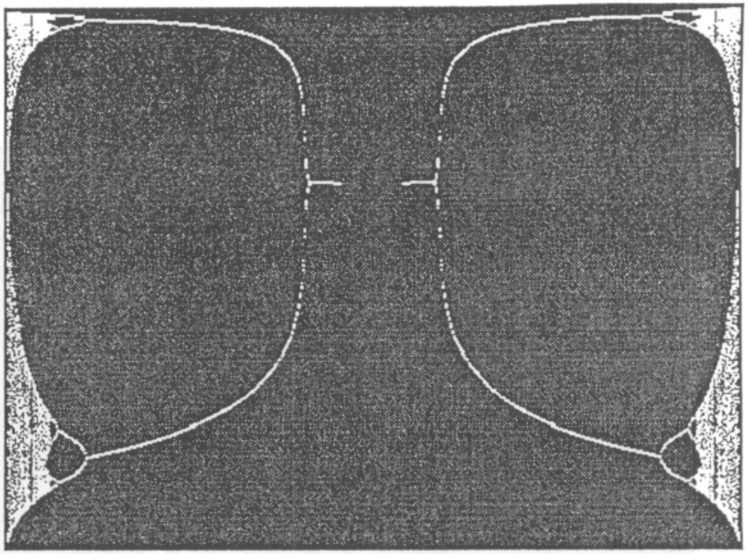

Fig. 3. Bifurcation diagram. Sum of amplitudes plotted versus arctangent of marginal cost ratio.

In Figure 3 we illustrate the bifurcation diagram. For the sake of symmetry we plot the sum of the phase variables, $x+y$, against the arctangent of the marginal cost ratio a/b. This sum of the phase variables is total supply or, according to (1), the reciprocal of commodity price. As mentioned, the arctangent angle of the cost ratio, rather than the ratio itself is chosen in order to make both ends of the diagram mirror images. It should also be mentioned, that a substantial section in the middle, corresponding to the stable fixed point, has been removed.

The general appearance of the bifurcation diagram is very like that of the extensively studied logistic map. We might say that this is to be expected from the general look of the reaction functions in Figure 1, but in reality substitution of (19) in (18) or vice versa results in a two-humped iteration in each variable. Thus, the result is not so obvious.

ADAPTIVE EXPECTATIONS

As mentioned, the iterations considered up to now are actually two independent ones. To make the map really two-dimensional, and bring in a little variation, we next assume that the duopolists do not immediatley jump to their new optimal positions, but adjust their previous decisions in the direction of the new optimum. Assume:

$$x_{t+1} = x_t + \lambda\left(\sqrt{\frac{y_t}{a}} - y_t - x_t\right) \tag{25}$$

$$y_{t+1} = y_t + \mu\left(\sqrt{\frac{x_t}{b}} - x_t - y_t\right) \tag{26}$$

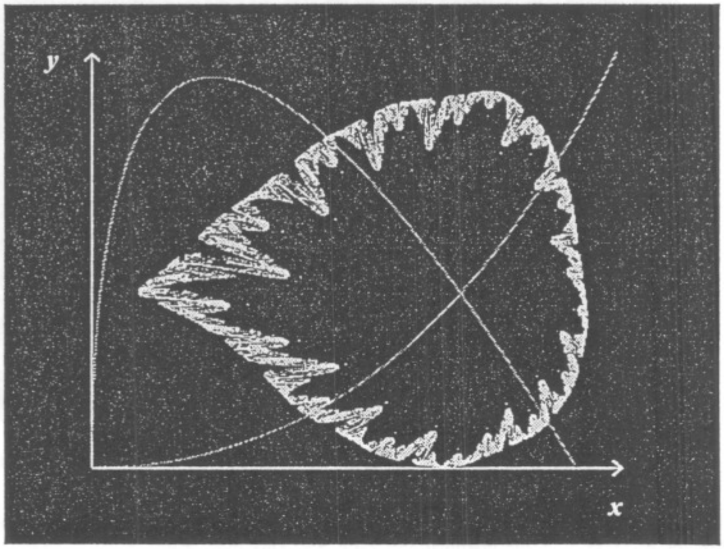

Fig. 5. Fractal attractor of the adaptive model

With the adjustment speeds λ, μ unitary, we are back at the case already treated, with those speeds zero, the duopolists will never revise any decision taken. Intermediate values bring a lot of new possibilities.

The stability of various outcomes, like the Cournot fixed point, and the cycles, now depend, not on the marginal cost ratio alone, but on the adjustment speeds as well. We can study the stability by linearising the system (25)-(26) around some point in phase space. The matrix of the linearized system will then be:

$$\begin{Bmatrix} 1-\lambda & \lambda\left(\dfrac{1}{2\sqrt{ay}}-1\right) \\ \mu\left(\dfrac{1}{2\sqrt{bx}}-1\right) & 1-\mu \end{Bmatrix} \tag{27}$$

where we for simplicity deleted the period indices. Considering the Cournot point, we susbstitute for the phase variables from (6)-(7). The matrix then becomes:

$$\begin{Bmatrix} 1-\lambda & \dfrac{\lambda(b-a)}{2a} \\ \dfrac{\mu(a-b)}{2b} & 1-\mu \end{Bmatrix} \tag{28}$$

At the threshold of loss of stability for the Cournot point the determinant of the matrix (28) becomes unitary. Simplifying, we get the condition:

$$(a-b)^2 = 4ab\left(\frac{1}{\lambda}+\frac{1}{\mu}-1\right) \tag{29}$$

We note that, with the adjustment speeds unitary, the condition (29) becomes identical with (23), as it indeed should.

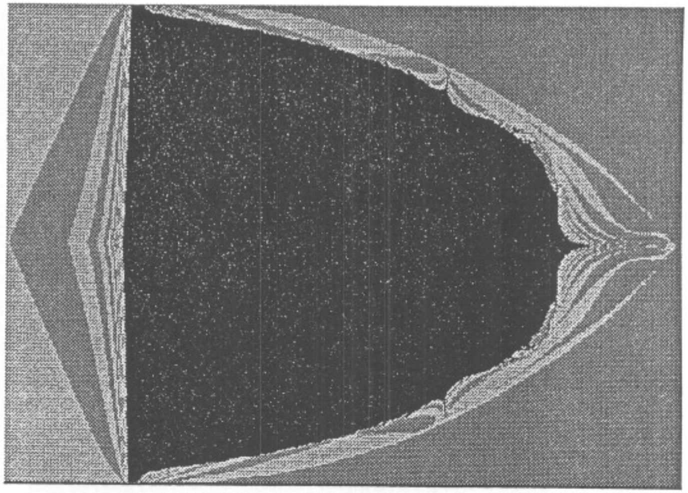

Fig. 5. The "Mandelbrot" set for the adaptive model.

The advantage of the present model, due to its higher dimension, is that we are now able to design fractal attractors in two dimensional embedding space. Such an attractor in the shape of a "leaf" is shown in Figure 4. We can now also display the stability of the model in terms of a "Mandelbrot" set in parameter space. We therefore need two parameters. For the moment we have three, the marginal cost ratio, and the two adjustment speeds. But in the preparation of Figure 5, the adjustment speeds were set equal. The horizontal coordinate is the adjustment speed for expectations, the vertical one is the cost ratio. The dark central body represents the stability of the Cournot point, whereas the shaded gray bands indicate the speed at which the model explodes for illicit parameter values. The period doubling cascades and chaos occur in the fractal border fringe of the central body.

The reader should be warned that we preferred to display the entire symmetric set. The feasible region in terms of economic substance is a subset of it, but includes sufficiently much of the border fringe, including a lot of intricacy at higher resolutions.

ADJUSTMENT INCLUDING STACKELBERG POINTS

Let us now take in consideration that the duopolists might consider trying to become Stackelberg leaders. We noted that Stackelberg leadership, according to (11) or (15), is always better than the Cournot point, described by (6)-(7). If thus the process is hunting for a, stable or unstable, Cournot point, according to the reaction functions (18)-(19), any of the duopolists might try Stackelberg leadership at any moment. If one duopolist has chosen to try this strategy, the other might accept this, and this is then the end of the story, but he might as well take up the challenge.

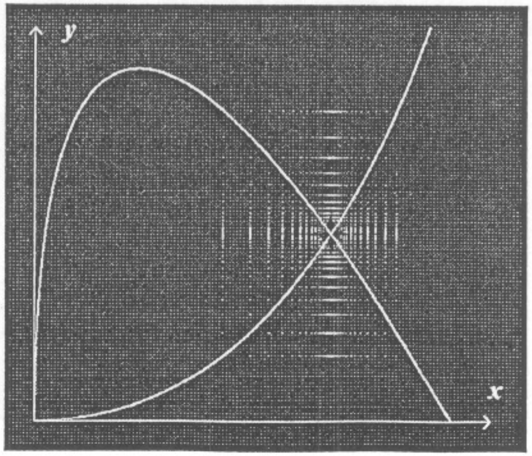

Fig 6. Adjustments including Stackelberg leadership

Formalizing, we would then have a generalized dynamical system (a nonlinear IFS), where:

(30

$$x_{t+1} = \begin{cases} \sqrt{\dfrac{y_t}{a}} - y_t \\ \dfrac{b}{4a^2} \end{cases}$$

(31

$$y_{t+1} = \begin{cases} \sqrt{\dfrac{x_t}{b}} - x_t \\ \dfrac{a}{4b^2} \end{cases}$$

and where the choices would be devised on a probabilistic basis. The attractor for such a process does generally not depend on the probabilities, provided they are positive.

We note that the attractor in Figure 6 is the product of two one-dimensional sets. This is so because we are again back at the situation where the mappings (30)-(31) are essentially independent, as substitution would show. Such a substitution, of course, results in four alternatives.

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