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OPTIMAL GAMBLING STRATEGIES

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Abstract

Optimal Gambling Strategies are defined as those that maximize the expected utility of the gambler's fortune after a given number N of trials, at each of which he can bet at most his current wealth. They are discussed in terms of betting on a biassed coin with probability $p > ½$ for heads. We also develop a Bayesian approach to the case of an unknown bias. Dynamic Programming is the principal tool of this analysis.

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1. Gambling brings to mind Bernard Shaw's advice to those about to be married: Don't. True enough for unfair games or fair games when the utility function is concave. But what about the case of favorable odds that crops up occasionally, e.g. in business situations with inside information. Concretely, assure a logarithmic utility function and consider betting on a biassed coin. Without restriction let heads (H) have the higher probability p and tails probability q

$$p \geq \frac{1}{2} \geq q = 1 - p .$$

What is it worth having n chances of playing this game and how much should one bet? This calls for a Dynamic Program (Bellman).

2. Let y be our wealth, the state variable and x, $0 \leq x \leq y$ be the size of the bet, our decision variable. Let value function $v_n(y)$ denote the (undiscounted) expected utility of our wealth expected from n optimal plays of this game. Let a budget constraint

$$0 \leq x \leq y$$

limit our bets and let $u(y)$ be the utility of wealth. Following a long tradition (Bernoulli) assume a logarithmic utility function

$$u(y) = \ln y .$$

The Bellman equation or "principle of optimality" for this sequential decision problem is then

$$v_0(y) = u(y) = \ln y \tag{1}$$

$$v_1(y) = \underset{0 \leq x \leq y}{Max} \;\; p v_0(y+x) + q v_0(y-x) = \underset{0 \leq x \leq y}{Max}\, p \ln(y+x) + q(y-x)$$

and generally

$$v_n(y) = \underset{0 \leq x \leq y}{Max} \;\; p v_{n-1}(y+x) + q v_{n-1}(y-x) \qquad n = 1,2,\ldots,N,\ldots \tag{1a}$$

Equations (1), (1a) are both necessary and sufficient in determining an optimal gambling strategy $x_n = x_n(y)$.
With a suitable change of (1) an optimal gambling strategy may be found for every given utility function $u(y)$.

3. Consider the last decision $x_1(y)$ and assume first that $0 < x_1 < y$. Since $u = \ln y$ is concave, a necessary and sufficient condition for x_1 to be maximizing is

$$0 = \frac{p}{y + x_1} - \frac{q}{y - x_1}$$

or $\qquad q(y + x_1) = p(y - x_1)$

from which

$$x_1 = (p-q)y \tag{2}$$

or

$$x_1 = (2p-1)y \tag{2a}$$

so that $0 < x_1 < y$ if and only if $\frac{1}{2} < p < 1$.

A corner solution

$$x_1 = 0$$

occurs whenever

$$\frac{d}{dx}\{p\ln(y+x) + q\ln(y-x)\} \le 0$$

or

$$\frac{p-q}{y} \le 0$$

hence

$$p \le q = 1-p$$

or

$$p \le \frac{1}{2}. \tag{2b}$$

Combining (2a) and (2b)

$$x_1(y) = [2p-1]_+ \cdot y \tag{3}$$

where $[a]_+ = Max(0,a)$

Notice that (2), (3) include also the case $p = 1$.

One's entire fortune y should be bet only on a "sure thing" $p = 1$, nothing should be bet in fair - or worse - unfair games $p \le \frac{1}{2}$, and with increasing odds a (linearly) increasing proportion of one's wealth should be ventured.

That bets should increase with p may be shown for all concave utility functions.

4. What is the economic value, if any, of one chance of playing this game $v_1(y)$? Substituting (2) in (1a) for $n = 1$ yields

$$\begin{aligned} v_1(y) &= p\ln([1+(2p-1)]y) + q\ln([1-(2p-1)]y) \\ &= \ln y + p\ln 2p + q\ln 2q \\ &= \ln y + c_1 \end{aligned} \tag{4}$$

with

$$c_1 = \ln 2 + p\ln p + q\ln q \ . \tag{4a}$$

Call $-p\ln p - q\ln q = E(p)$ the entropy of p.
It is well-known that

$$\begin{aligned}\ln 2 &= -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} \\ &= \max_{0\le p\le 1} E(p)\end{aligned}$$

and that this maximum is unique since $E(p)$ is strictly concave.
Hence

$$c_1 \ge 0$$

$$\text{"="} \qquad \text{if and only if} \qquad p = \frac{1}{2}.$$

A single chance of playing this game has thus a positive value $c_1 > 0$ if and only if $p > \frac{1}{2}$.
It is now readily seen that for logarithmic utility functions $u(y) = \ln y$

$$v_n(y) = \ln y + c_n$$

where

$$c_n = n[2 + p\ln p + q\ln q] = nc_1 \tag{4b}$$

each gamble contributes the same amount in expected utility, regardless of the intitial capital y. The shape (4a) of the value function shows that our betting strategy should be

$$x_n(y) = (p-q)y \tag{2c}$$

independent of n, the number of gambles allowed to us and that the ratio $\frac{x_n}{y}$ of bet to wealth should be independent of wealth. This decision rule has thus a particularly simple form.

5. That we should know the coin's bias and the other party presumably not, is almost too good to be true. A more challenging situation is that where a bias is suspected but not known and must be inferred from observation. To apply Dynamic Programming now requires a Bayesian approach. For simplicity let the prior distribution of p be flat

$$w(p)\,\mathrm{d}p = \mathrm{d}p \qquad 0 \le p \le 1$$

Suppose that k heads have occurred in m trials. Then the posterior probability of p is, using Bayes' formula

$$w(p|k,m)\,\mathrm{d}p = \frac{\binom{m}{k} p^k (1-p)^{m-k}\,\mathrm{d}p}{\int_0^1 \binom{m}{k} p^k (1-p)^{m-k}\,\mathrm{d}p}$$

and the posterior expected value of p equals

$$\bar{p}(k,m)=\int p w(p|k,m)\,\mathrm{d}\,p=\frac{\int_0^1 p^{k+1}(1-p)^{m-k}\,\mathrm{d}\,p}{\int_0^1 p^k(1-p)^{m-k}\,\mathrm{d}\,p}=\frac{B(k+2,m-k+1)}{B(k+1,m-k+1)}=\frac{k+1}{m+2} \tag{5}$$

by a well-known formula for the Beta function (Courant II).
Since $k=0$, $m=0$ implies

$$\bar{p}(0,0)=\frac{1}{2}$$

(2) means that one should not bet anything on the first toss of a coin.
From now on let $k \geq \frac{m}{2}$ denote the number of occurrences of the more frequent event heads or tails.
After one toss $k=1$, $m=1$ one has

$$\bar{p}(1,1)=\frac{2}{3}$$

so that we should bet $\frac{1}{3}$ of our wealth on the next toss coming out the same as the first. The economic value of this is, according to (4),

$$\begin{aligned} v_1(1,1,y) &= \ln y + 2 + \frac{2}{3}\ln\frac{2}{3}+\frac{1}{3}\ln\frac{1}{3} \\ &= \ln y + 0.74978 \qquad > 0. \end{aligned}$$

In view of $v_2(0,0,y)=v_1(1,1,y)$ it is seen that one should opt for at least two chances of playing this game, betting nothing at first and $\frac{1}{3}$ of one's wealth on a repeat of the first outcome. It may be shown that once more $v_n(k,m,y)=\ln y + c_n(k,m)$ where c_n increases with n and k and decreases with m. But it is no longer true that $c_n = nc_1$. Generally after m trials and k successes we should bet the fraction

$$\frac{x_n}{y}=\frac{2k-m}{m+2} \tag{6}$$

of our wealth regardless of the number of gambles yet to be made and regardless of our (current) wealth y.

In the "best case" of continued success

$$k=m$$

one bets the fraction $\frac{m}{m+2}$ which approaches unity.
This bet, if successful, multiplies our current wealth by

$$\frac{2m+2}{m+2} = 2\frac{m+1}{m+2}.$$

After M successful tosses we are worth

$$y^{(M)} = y\prod_{m=1}^{M} 2\frac{m+1}{m+2} = y\frac{2^{M+1}}{M+1}$$

This is $\frac{2}{M+1}$ as much as we would have under a reckless strategy of betting everything M times in a row.

The "worse case" is that of a strict alternation

HTHTHT... or
THTHTH...

For all odd m one has

$$m = 2k-1$$

implying

$$\bar{p}(k,m) = \frac{k+1}{2k+1} > \frac{1}{2}$$

and a betting strategy

$$\begin{aligned} x_n(y) &= \left(\frac{2k+2}{2k+1} - 1\right)y \\ &= \frac{1}{2k+1}y = \frac{1}{m+2}y \end{aligned}$$

and each bet is lost.
For even $m = 2k$

$$\bar{p}(k,2k) = \frac{1}{2}$$

and no bet is placed.
Though continued losses the initial capital y is reduced to $y\prod_{i=1}^{k}\frac{2i}{2i+1}$ after k bets.

Observe that

$$\prod_{i=1}^{k}\frac{2i}{2i+1} = \frac{\left(2^k k!\right)^2}{(2k+1)!}$$

$$= \frac{\sqrt{2}\pi}{e} \frac{1}{\sqrt{2k+1}} \left(\frac{k}{2k+2} \right)^{2k+1}$$
$$< \left(\frac{1}{2} \right)^{2k+1}$$

using Stirling's formula (Courant I).
The lesson is to be suspicious of any regular sequence, i. e. any sequence that is not random.

6. This analysis is readily extended to the case of all concave utility functions with constant risk aversion

$$u(y) = y^{\alpha} \qquad 0 \le \alpha \le 1.$$

The results are

$$\frac{x_n}{y} = \frac{\left[p^{\frac{1}{1-\alpha}} - q^{\frac{1}{1-\alpha}} \right]}{p^{\frac{1}{1-\alpha}} + q^{\frac{1}{1-\alpha}}} \tag{2d}$$

for all n and all y. Now

$$v_n(y) = b_n y^{\alpha}$$

where

$$b_n = \left[2^{\alpha} \left(p^{\frac{1}{1-\alpha}} + q^{\frac{1}{1-\alpha}} \right)^{1-\alpha} \right]^n = b_1^n \quad . \tag{4c}$$

Moreover

$$b_1 \ge 1$$
$$\text{"="} \qquad \text{if and only if} \qquad p = q = \frac{1}{2}$$

so that $b_{n+1} > b_n > 1$ whenever $p > \frac{1}{2}$, in which case each chance of gambling has a positive economic value. Observe that both value function and bet increase with α, i. e. decrease with risk aversion.
When p is unknown the Bayesian estimate remains

$$\overline{p} = E(p|k,m) = \frac{k+1}{m+2} \tag{5}$$

and

$$v_n(y,k,m) = b_n(k,m) y^{\alpha}$$

with

$$b_n(0,0) > 1$$

and b_n an increasing function of n, k, and α, and a decreasing function of m.

It is therefore still optimal to bet on the second throw. The logarithmic utility function is represented in theses formulae by $\alpha = 0$.
In conclusion we can only wish the reader interested in applications "lots of luck".

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