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Peter A. Zadrozny

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Linear Identification of Linear Rational-Expectations Models by Exogenous Variables Reconciles Lucas and Sims

Abstract

Linear rational-expectations models (LREMs) are usually “forwardly” estimated. Structural coefficients are restricted in terms of deep parameters. For given deep parameters, structural equations are solved for rational-expectations solution (RES) eqs. that determine endogenous variables. For given VAR eqs. that determine exogenous variables (XVAR), RES eqs. reduce to reduced-form VAR eqs. with exogenous variables (ERF). Combined XVAR and ERF eqs. comprise reduced-form (RF) overall VAR (OVAR) eqs. of all variables. The specified, solved, and combined eqs. define a mapping from deep parameters to OVAR coefficients used to forwardly est. a LREM in terms of deep parameters. Forwardly-est. deep parameters determine forwardly-est. RES eqs. that Lucas (1976) advocated for making policy predictions. Sims (1980) called identifying restrictions on deep parameters of forwardly-est. LREMs “incredible”, because he considered in-sample fits of forwardly-est. OVAR eqs. inadequate and out-of-sample policy predictions of forwardly-est. RES eqs. inaccurate. Sims (1980, 1986) instead advocated directly estimating OVAR eqs. restricted by statistical restrictions and directly using directly-est. OVAR eqs. To make policy predictions. However, if assumed or predicted out-of-sample exogenous policy variables differ significantly from predictions of their in-sample est. XVAR eqs., then, out-of-sample policy predictions of endogenous variables made with OVAR eqs. won’t satisfy Lucas’s critique. If directly-est. OVAR eqs. are RF eqs. of underlying RES eqs., then, identification 2 derived in the paper linearly “inversely” est. the underlying RES eqs. from the directly-est. OVAR eqs. and the inversely- est. RES eqs. can be used to make policy predictions that satisfy Lucas's critique. If Sims considered directly-est. OVAR eqs. to fit in-sample data adequately (credibly) and their inversely-est. RES eqs. To make accurate (credible) out-of-sample policy predictions, then, he should consider the inversely-est. RES eqs. and further underlying LREM- structural eqs. to be credible. Thus, inversely-est. RES eqs. by identification 2 would reconcile Lucas’s advocacy for making policy predictions with RES eqs. and Sims’s advocacy for directly estimating OVAR eqs.

EL-Codes: C320, C430, C530, C630.

Keywords: cross-equation restrictions of rational expectations, factorization of matrix polynomials, reconciliation of Lucas’s advocacy of rational-expectations modelling and policy predictions and Sims’s advocacy of VAR modelling.

Peter A. Zadrozny
Division of Price Index Number Research
Bureau of Labor Statistics
Washington, DC / USA
zadrozny.peter@bls.gov
peterzadrozny@gmail.com

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1. INTRODUCTION.

Linear rational-expectations models (LREMs; Lucas and Sargent, 1981) are now standard multivariate macroeconomic structural models used in macroeconomic analysis by academics, central banks, and others. LREMs were the first macroeconomic models in which average realizations of endogenous variables generated by a model equal expected-future values of endogenous variables according to the model, which is a statistical-consistency condition of a random process akin to stationarity. Initial LREMs were criticized for unrealistically omitting frictions and assuming that economic agents in the LREMs immediately and always have rational expectations. However, it was shown that LREMs can include frictions (Taylor, 1979, 1980) and staggered learning about rational expectations (Sargent, 1993; Zadrozny, 1997). LREMs are now mostly called dynamic stochastic general equilibrium models (DSGEMs; Smets and Wouters, 2003, 2007), because the models can be nonlinear in variables and disturbances and generalize original LREMs, although most REMs in macroeconomic applications are still either mostly or entirely linear in variables and disturbances. The models are here called LREMs because only models linear in variables and disturbances are considered.

Endogenous variables in LREMs are determined by rational-expectations solution (RES) equations of structural equations. (The paper often refers to vector or matrix equations in plural terms as scalar "equations", but this shouldn't cause any confusion.) Evaluating expected-future exogenous variables in RES equations according to vector autoregressive equations that determine the exogenous variables (XVAR) reduces the RES equations to reduced-form VAR equations for endogenous variables (ERF) with exogenous variables. The combined ERF and XVAR equations comprise the reduced-form overall VAR (OVAR) equations of all variables in a LREM.

LREM-structural equations such as equation (3.1) are usually specified by their coefficients being mapped nonlinearly from deep parameters. Part of specifying LREM-structural and XVAR equations involves specifying moments that determine probability distributions of unobserved disturbances, hence, of observed variables (data). LREMs have

5 levels of quantities: deep parameters in θ_1 at the lowest level, structural coefficients in θ_2 such as $\{\{A_i\}_{i=0}^{2}, B_0, C_0\}$ of LREM-structural equation (3.1) at the next higher level, RES coefficients in θ_3 such as $\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^{\infty}\}$ of RES equation (3.3), reduced-form ERF and XVAR coefficients and disturbance covariances in θ_4 such as $\{\Phi_1, \Theta_0, \{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}, \Sigma_\xi\}$ of reduced-form ERF and XVAR (OVAR) equations (5.2) and (3.2), and data moments in θ_5 at the highest level. Φ_1 and Θ_0 are both RES and reduced-form ERF coefficient matrices that are listed in both θ_3 and θ_4 and $\{D_i\}_{i=1}^{k}$ and Σ_ξ are both structural and reduced-form XVAR coefficient and disturbance-covariance matrices but are listed only in θ_4.

Specification of structural coefficients and determination of RES and ERF coefficients and data moments proceeds in a sequence of forward mappings, from lower-level to higher-level quantities, $\theta_{i+1} = f_i(\theta_i)$, for i = 1,…,4, that combine as the complete composite-forward mapping $\theta_5 = f_4(f_3(f_2(f_1(\theta_1))))$. If each forward mapping has an inverse mapping, $\theta_i = f_i^{-1}(\theta_{i+1})$, for i = 4,…,1, then, the individual inverse mappings combine as the complete composite-inverse mapping $\theta_1 = f_1^{-1}(f_2^{-1}(f_3^{-1}(f_4^{-1}(\theta_5))))$.

An identification is a unique determination of a lower-level $\theta_i = f_i^{-1}(\cdots f_{j-1}^{-1}(\theta_j)\cdots)$ from a higher-level θ_j, for i < j, according to an individual-inverse or composite-inverse mapping, so that an identification may map down from highest-level data moments or from lower-level quantities. Following Slutzky's theorem (Schmidt, 1976, p. 250), an identification is a consistent estimation when it maps down from consistently-estimated higher-level quantities with a continuous inverse mapping. Because all inverse mappings in identifications 1 and 2 derived in the paper are differentiable, hence, continuous, identifications 1 and 2 deliver consistently estimated structural and RES coefficients down from consistently estimated data moments.

Under assumptions A.1.i-iii, A.3, and A.4.i-ii, the paper derives identification 1 that linearly identifies structural coefficients from RES coefficients and, under assumptions A.1.i-iii to A.4.i-ii, the paper derives identification 2 that linearly identifies RES coefficients from ERF and XVAR coefficients. Assumptions A.5 to A.7 are added so that identifications 1 and 2 together are complete as identifications and consistent estimations of structural and RES coefficients down from assumed or consistently estimated data moments.

Recent macroeconomic literature discussed in section 2 emphasizes that identification of deep parameters of a LREM is a nonlinear problem. The present paper doesn't discuss identification of deep parameters, because it aims for general results and nonlinear identification of deep parameters depends on a particular mapping from deep parameters to structural coefficients of a particular model. Otherwise, nonlinearities in identification of LREM coefficients can occur only from the way data are generated and observed. For example, if data are subsampled or have mixed frequencies (Tank et al., 2019), then, identification can be nonlinear due to aliasing (Hansen and Sargent, 1983; Anderson et al., 2012, p. 188-189; Zadrozny, 2016, p. 439). Although combined identifications 2 and 1, in that order, strictly comprise a nonlinear composite-inverse mapping, we consider the composite identification to be linear because its individual-inverse mappings, hence, all of their computations are linear.

An identification holds if and only if its identifying equations have a unique solution. If there are too few, just enough, or more than enough scalar-level identifying equations, then, an identification is, respectively, an under- (un-), just-, or over-identification. Sections 4 and 5, respectively, derive identifications 1 and 2 and discuss their under-, just-, and over-identification. Section 5 discusses how identifying equations of identification 2 and their rank and order conditions differ according to three cases of how exogenous variables enter a structural equation in past, current, and expected-future values. Section 5 also discusses tradeoffs in identification 2 among the number of endogenous variables (n), the number of exogenous variables (m), and the number of lags of exogenous variables (k).

The paper continues as follows. Section 2 reviews some recent macroeconomic literature on identification of LREMs. Section 3 states the LREM-structural equation in the usual first-order form, a 3rd-order

XVAR(3) equation, the resulting RES equation, and assumptions A.1.i to A.7 on coefficients. Sections 4 and 5, respectively, derive identifications 1 and 2 and discuss their under-, just-, and over-identifications. Identification 1 doesn't depend on how exogenous variables are generated. Section 5 first derives identification 2 for a 3rd-order XVAR(3) equation and, then, extends the identification for any kth-order XVAR(k) equation, for any finite $k \geq 4$. Concluding section 6 explains the contributions of identifications 1 and 2. Identification 1 contributes mainly by showing that OVAR equations can be reduced-form equations of underlying RES and LREM-structural equations. Identification 2 makes the main contribution of the paper by showing that inversely-estimated RES equations can reconcile Lucas's advocacy for making policy predictions with RES equations and Sims's advocacy for directly estimating OVAR equations.

2. REVIEW OF SOME RECENT MACROECONOMIC LITERATURE.

If a quantity in a set of observationally-equivalent quantities is isolated, then, it's locally identified (LI); if a quantity in a set of observationally-equivalent quantities is a singleton (unique), then, it's globally identified (GI). If observationally-equivalent quantities are linearly related, then, LI and GI are equivalent. In the present discussion of identification down to structural coefficients, non-GI can occur only from the way data are generated and observed, such as aliasing due to subsampling or mixed frequencies.

Identification of LREMs has received insufficient attention in the literature possibly because LREMs have been estimated mostly by Bayesian methods (Smets & Wouters, 2004, 2007; An & Schorfheide, 2007). LI is crucial in sampling-theoretic estimation, because the estimation optimizes a criterion function over a parameter region. If the criterion function is twice differentiable and 2nd-order conditions of the optimization hold at parameter estimates, then, the estimates are locally unique and can be LI, GI, and consistently estimated (Rothenberg, 1971). In practice, Bayesian estimation doesn’t need LI or

GI, because it doesn't optimize a criterion function, but instead pseudo-randomly generates a numerical histogram over a parameter region and computes the histogram's statistics of central tendency and dispersion.

Identification of LREMs began being addressed at a general level more than a decade ago (Iskrev, 2010; Komunjer & Ng, 2011). Forward and inverse mappings between deep parameters and structural coefficients, specified according to economic reasoning, have usually been at least partly nonlinear. Forward mappings from structural to RES coefficients have also usually been nonlinear, because they involve solving a matrix polynomial equation. Exceptionally, the usual most-stable forward solution of the matrix polynomial equation is obtained linearly. For example, in structural equation (3.1), if $A_0 = 0_{nxn}$, then, the most-stable forward solution of the endogenous-feedback matrix is $\boldsymbol{\Phi}_1 = 0_{nxn}$ (Taylor, 1977).

Apparently for these reasons, Komunjer & Ng (2011), Qu & Tkachenko (2012, 2017, 2018), Kociecki & Kolasa (2018, 2020), and others considered identification of a LREM to be a nonlinear problem. However, sections 4 and 5 in the paper show that identification 1 of structural coefficients from RES coefficients and identification 2 of RES coefficients from ERF and XVAR coefficients are always linear problems.

Qu & Tkachenko (2012, 2017, 2018) and Kociecki & Kolasa (2018, 2020) contributed to numerically checking nonlinear GI of deep parameters. Komunjer & Ng (2011) stated propositions about necessary and sufficient conditions for LI of deep parameters of a LREM (or linear approximation of an nonlinear REM), but didn't and couldn't obtain the the present identifications 1 and 2, because, unlike in the present paper, they didn't exploit the cross-equation restrictions of rational expectations (CERRE) that are the central quantitative implication of rational expectations.

Let S denote a nonsingular similarity-transformation matrix of the state-transition matrix in a state-space representation of ERF and XVAR (OVAR) equations of a LREM. Kociecki & Kolasa (2018, 2020) used one of Komunjer & Ng's results, that if θ_1 and $\tilde{\theta}_1$ are observationally-equivalent values of deep parameters with nonsingular similarity-transformation matrices $S(\theta_1)$ and $\tilde{S}(\tilde{\theta}_1)$, then, $\theta_1 = \tilde{\theta}_1$ and GI holds if and only if $S(\theta_1) = \tilde{S}(\tilde{\theta}_1)$. Kociecki & Kolasa developed and illustrated numerical methods for checking $S(\theta_1) = \tilde{S}(\tilde{\theta}_1)$, hence, $\theta_1 = \tilde{\theta}_1$.

Komunjer & Ng, Qu & Tkachenko, and Kociecki & Kolasa included observation errors in a state-space representation of ERF and XVAR equations of a LREM, which here means inserting another mapping between θ_4 and θ_5. Observation errors are rarely a useful addition in macroeconomic modelling, because separately identifying observation-error covariances and disturbance covariances requires either having observation-error covariances given by a data producer or estimating them using data with multiple observations per variable per period, both rare in macroeconomics. Exceptionally, Zadrozny (1990a,b) used observation-error variances of GNP published by the Bureau of Economic Analysis. Also, observation errors are a data problem that has nothing to do with CERRE equations (3.4) to (3.7) that are central to deriving identifications 1 and 2. For these reasons, the present paper doesn't consider observation errors in its discussion.

Part of the recent literature on identification of a LREM arises from Sims's (2001) paper: in the present notation, identifying structural, RES, and ERF disturbance-coefficient matrices B_0 and Θ_0 in structural, RES, and ERF equations like (3.1), (3.3), and (5.2) from RES and ERF disturbance-covariance matrix $\Sigma_\varepsilon = \Theta_0\Theta_o^T$ when more disturbances than endogenous variables are classified as "shocks" versus "sunspots". See, e.g., Lubik & Schorfheide (2003), Farmer et al. (2015), Funovits (2017), and references therein. The present paper doesn't consider this identification problem because it can be resolved only by imposing exact theoretical (non-data-based) restrictions in a particular application and the paper aims to derive general results that don't depend on particular restrictions in a particular application.

3. STRUCTURAL, RES, CERRE EQUATIONS AND ASSUMPTIONS.

3.1. STRUCTURAL, RES, AND CERRE EQUATIONS.

LREMs can include expected-future variables in multiple future periods and realized variables and disturbances in multiple past periods, with expectations conditioned on both current and past information (Taylor, 1979; Zadrozny, 1998). Various ways have been proposed for expressing such higher-order LREMs in the most commonly used first-order form with one future period, one past period, no past disturbances, and expectations conditioned only on current information (e.g., Binder & Pesaran, 1995). The paper's results on first-order LREM-structural equations can be extended using the same reasoning to higher-order LREM-structural equations, with additional terms in expected-future and past variables, but only with more and more complicated algebraic details.

Consider an $n\times 1$ LREM-structural equation in first-order form,

$$(3.1) \qquad A_2 E_t y_{t+1} + A_1 y_t + A_0 y_{t-1} = B_0 \varepsilon_t + C_0 z_t ,$$

where $\{\{A_i\}_{i=0}^{2}, B_0, C_0\}$ denote constant (unchanging over time periods t), real-valued, $n\times n$ and $n\times m$, structural coefficient matrices; y_t and z_t denote time-varying, real-valued, $n\times 1$ and $m\times 1$ vectors of endogenous and exogenous variables generated in and observed for period t and remembered; E_t denotes expectations conditioned on information in period t; ε_t denotes a time-varying, real-valued, $n\times 1$ vector of disturbances generated in period t and never observed.

Disturbance vector ε_t is assumed to be generated stochastically, identically, and independently over periods t, with zero mean vector and identity covariance matrix, $\varepsilon_t \sim IID(0_{nx1}, I_n)$, where 0_{ixj} denotes the $i\times j$ zero vector (if i = 1 or j = 1) or matrix (if i > 1 and j > 1) and I_j denotes the $j\times j$ identity matrix.

To simplify algebraic details in the discussion, exogenous vector z_t is first stated as generated by XVAR(3) equation

(3.2) $$z_t = D_1 z_{t-1} + D_2 z_{t-2} + D_3 z_{t-3} + \xi_t ,$$

where $\{D_i\}_{i=1}^{3}$ denote constant, real-valued, $m \times m$ coefficient matrices, ξ_t denotes a time-varying, real-valued, $m \times 1$ vector of disturbances generated in period t, never observed, and distributed $IID(0_{mx1}, \Sigma_\xi)$, with real-valued, $m \times m$, symmetric positive-definite, covariance matrix Σ_ξ. In section 5, XVAR(3) equation (3.2) is extended to an XVAR(k) equation with coefficient matrices $\{D_i\}_{i=0}^{k}$, for any finite $k \geq 4$.

Vector z_t is exogenous in structural equation (3.1) if and only if disturbance vectors in structural and RES equations (3.1) and (3.2) are uncorrelated in and across all periods, $E\varepsilon_s \xi_t^T = 0_{nxm}$, for all s and t, where superscript T denotes vector or matrix transposition.

Under assumptions A.1.i-iii and A.2, structural equation (3.1) is solved uniquely for RES equation

(3.3) $$y_t = \Phi_1 y_{t-1} + \Theta_0 \varepsilon_t + \sum_{i=0}^{\infty} \Xi_i E_t z_{t+i} ,$$

where Φ_1, Θ_0, and $\{\Xi_i\}_{i=0}^{\infty}$ denote real-valued, $n \times n$ and $n \times m$, RES coefficient matrices and $E_t z_t = z_t$. The only difference between ε_t and z_t used in deriving RES equation (3.3) from structural equation (3.1) is that $\{E_t \varepsilon_{t+i}\}_{i=1}^{\infty}$ are zero but $\{E_t z_{t+i}\}_{i=1}^{\infty}$ are generally nonzero. It follows along the lines of the proof in section 5, that row rank(Υ_0) = full = n, that if z_t is limited nonstationary (defined below equation (4.3)), then, $\sum_{i=0}^{\infty} \Xi_i E_t z_{t+i}$ exists (converges).

If y_t and $E_t y_{t+1}$ are replaced in structural equation (3.1) using RES equation (3.3) and the resulting equation holds for all values of y_{t-1}, ε_t, and $\{E_t z_{t+i}\}_{i=0}^{\infty}$, then, the resulting equation implies the cross-equation restrictions of rational expectations (CERRE) between structural and RES coefficients,

$$(3.4)\qquad A_2\Phi_1^2 + A_1\Phi_1 + A_0 = 0_{n\times n},$$

$$(3.5)\qquad (A_2\Phi_1 + A_1)\Theta_0 = B_0,$$

$$(3.6)\qquad (A_2\Phi_1 + A_1)\Xi_0 = C_0,$$

$$(3.7)\qquad (A_2\Phi_1 + A_1)\Xi_i + A_2\Xi_{i-1} = 0_{n\times m},$$

for i = 1,2,3,…, where equations (3.4) and (3.5) are $n\times n$ and equations (3.6) and (3.7) are $n\times m$. The term "CERRE" comes from Hansen and Sargent (1980).

RES coefficients in $\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^{\infty}\}$ are rational-expectations solution coefficients if and only if they solve CERRE equations (3.4) to (3.7) forwardly, exactly, but not necessarily uniquely, for given values of structural coefficients in $\{\{A_i\}_{i=0}^{2}, B_0, C_0\}$. Conversely, structural coefficients are identified from RES coefficients by CERRE equations (3.4) to (3.7) if and only if they solve the CERRE equations inversely, exactly, and uniquely for given values of RES coefficients. On various methods for computing RES coefficients for given values of structural coefficients, see Blanchard and Kahn (1980), Hansen and Sargent (1980), Zadrozny (1998), Klein (2000), and Sims (2001).

3.2. ASSUMPTIONS ON COEFFICIENTS, DISTURBANCE COVARIANCES, AND DATA MOMENTS.

This subsection states and discusses assumptions A.1.i to A.7 that restrict coefficients, disturbance covariances, and data moments. The restrictions are classified as normalizing, generic, exact, combinatorial, specific, and nonspecific in order to clarify their possible stringencies.

"Normalizing" restrictions restrict coefficients to exact numerical values, but only to eliminate redundant (unidentifiable) coefficients, don't restrict the ability of ERF and XVAR (OVAR) equations to fit in-sample data or the ability of RES equations to predict out-of-sample data, and, therefore, aren't stringent. "Generic" restrictions restrict coefficients to be elements of supersets of open and dense subsets (Anderson et al., 2012, p. 185), usually hold in practice and usually aren't considered stringent. "Exact" restrictions restrict coefficients to exact numerical values, restrict the ability of ERF and XVAR equations to fit data, and restrict the ability of RES equations to predict data, but, depending on the application, may or may not be considered stringent. "Specific" restrictions are specified in the present paper and would also be in an application. "Nonspecific" restrictions aren't specified in the paper but would be in an application. All restrictions also implicitly restrict deep parameters, but, because the paper considers identifications only down to structural coefficients, the restrictions aren't also stated explicitly as restricting deep parameters.

Assumptions A.1.i-iii imply that RES coefficients solve CERRE equations (3.4) to (3.7) forwardly and uniquely for given values of structural coefficients. Assumptions A.1.i-iii, A.3, and A.4.i-ii imply that structural coefficients solve the CERRE equations inversely and uniquely for given values of RES coefficients, so that identification 1 holds. Assumptions A.1.i-iii to A.4.i imply, in cases I and II and possibly also in case III of the way exogenous variables enter structural equations, that RES coefficients solve identifying equations (5.8) and (5.16) and possibly equation (5.25) inversely and uniquely for given values of ERF and XVAR coefficients, so that identification 2 holds. Assumption A.1.ii is the only combinatorial assumption and chooses RES coefficients from a finite set of choices, such that endogenous-feedback matrix $\boldsymbol{\Phi}_1$ in RES and ERF equations (3.3) and (5.2) has the smallest possible absolute eigenvalues, usually less than one. Assumptions A.5 to A.7 are added so that identifications 1 and 2 are complete as identifications or as consistent estimations of structural and RES coefficients down from assumed or consistently estimated data moments.

Let $A(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$, $K(\lambda) = K_1\lambda + K_0$, and $\Phi(\lambda) = I_n\lambda - \Phi_1$ be $n\times n$ lambda matrices, where K_0, K_1, and Φ_1 are real-valued $n\times n$ matrices and λ is a complex-valued scalar. A latent root of $A(\lambda)$ is a value of λ that satisfies $|A(\lambda)| = 0$, where $|\cdot|$ denotes the determinant of a square matrix, and similarly for $K(\lambda)$ and $\Phi(\lambda)$. The latent roots of $\Phi(\lambda)$ are exactly the eigenvalues of Φ_1. See Dennis et al. (1976) for a review of lambda matrices.

ASSUMPTION A.1.i: The $n\times n$ structural coefficient matrices $\{A_i\}_{i=0}^{2}$ of endogenous variables in LREM-structural equation (3.1) are sufficiently restricted so that $A(\lambda)$ has a unique factorization $A(\lambda) = K(\lambda)\Phi(\lambda)$ in a forward rational-expectations solution.

Assumption A.1.i imposes <u>nonspecific</u> <u>generic</u> restrictions on structural coefficients in $\{A_i\}_{i=0}^{2}$.

ASSUMPTION A.1.ii: In the factorization $A(\lambda) = K(\lambda)\Phi(\lambda)$, the n latent roots of $A(\lambda)$ with the smallest absolute values and their associated latent vectors are assigned "right" to $\Phi(\lambda)$ and the remaining latent roots of $A(\lambda)$ and their associated latent vectors are assigned "left" to $K(\lambda)$.

Assumption A.1.ii imposes a <u>specific</u> <u>combinatorial</u> restriction on RES coefficients in $\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^{\infty}\}$.

ASSUMPTION A.1.iii: At most n latent roots of $A(\lambda)$ are zero.

Assumption A.1.iii imposes a <u>specific</u> <u>generic</u> restriction on structural coefficients in $\{A_i\}_{i=0}^{2}$.

In macroeconomic applications, the n latent roots of $A(\lambda)$ with the smallest absolute values are always assigned to $\Phi(\lambda)$ and usually have absolute values less than one. Neighboring left and right latent roots could have the same value (be repeated), in which case some additional rule would have to assign any distinct associated latent vectors left and right. Although it's not clear what economic or other reasoning could be used to make such an assignment, this case has apparently not occurred in a macroeconomic application.

No general, direct, generic or exact restrictions on $\{A_i\}_{i=0}^{2}$ are known that imply or are implied by unique factorization $A(\lambda) = K(\lambda)\Phi(\lambda)$ for any particular left-right latent root assignment, only indirect restrictions in terms of latent roots and latent vectors of $A(\lambda)$ obtained while computing the factorization (Dennis et al., 1976; Zadrozny, 1998). For example, Dennis et al. (1976, pp. 837-838) proved that if latent roots of $A(\lambda)$ are distinct, then, $A(\lambda)$ has a complete set of solvents and a factorization $A(\lambda) = K(\lambda)\Phi(\lambda)$, but their proof could contradict a particular desired left-right latent root assignment. Onatski (2006) determined that for the usual left-right latent root assignment of assumption A.1.ii a unique factorization $A(\lambda) = K(\lambda)\Phi(\lambda)$ is equivalent to a unique Wiener-Hopf factorization that satisfies a certain winding number criterion or, equivalently, certain values of partial indices. Al-Sadoon (2018) extended Onatski's partial-index results to include latent roots of $\Phi(\lambda)$ on the unit circle.

ASSUMPTION A.2: The $m\times 1$ exogenous vector z_t is generated by an XVAR(k) equation, for some finite $k \geq 1$, that is controllable and halfway limited nonstationary.

Equation (3.2) is the XVAR(k) equation for $k = 3$, which in section 5 is extended for any finite $k \geq 4$. Assumption A.2 imposes <u>specific generic</u> restrictions on XVAR(k) coefficients in $\{D_i\}_{i=1}^{k}$.

A XVAR(k) equation is controllable if and only if its highest-lag coefficient matrix D_k is nonsingular or $|D_k| \neq 0$. Controllability of an XVAR(k) equation means that for given past exogenous variables, $\{z_s\}_{s=t-k}^{t-1}$, and any given target, z^*, there's a finite sequence of current and future disturbances, $\{\xi_s^*\}_{s=t}^{t+\ell}$, such that $z_{t+\ell} = z^*$. Controllability is a basic concept in dynamic system theory. See Hannan & Deistler (1988) and Zadrozny (2016, pp. 440-441).

Halfway limited nonstationarity of an XVAR equation is defined below equation (4.3).

ASSUMPTION A.3: The $n\times m$ structural coefficient matrix C_0 of exogenous vector z_t in LREM-structural equation (3.1) has full row rank of n.

Assumption A.3 imposes a specific generic restriction on structural coefficients in C_0.

Assumption A.3 implies that $m \geq n$ and that there's an order of columns of C_0 (and corresponding order of elements of z_t) and partitions $C_0 = [C_{01}, C_{02}] = [n\times p, n\times(m-p)]$ and $C_{01} = [C_{011}, C_{012}] = [n\times n, n\times(p-n)]$, for $n \leq p \leq m$, such that row rank(C_{01}) = full = n and C_{011} is nonsingular. If $p = m$, then, C_{02} is null and $C_0 = C_{01}$; and, if $p = n$, then, C_{012} is null and $C_{01} = C_{011}$.

ASSUMPTION A.4.i: There's an order of columns of C_0 and a maximal $p = \bar{p}$, for $n \leq \bar{p} \leq m$, such that $C_{01} = n\times\bar{p}$ is known and $C_{011} = n\times n$ is nonsingular.

The part of assumption A.4.i that C_{01} is known imposes a nonspecific normalizing restriction and the part that C_{011} is nonsingular imposes a specific generic restriction.

ASSUMPTION A.4.ii: The $n\times n$ structural coefficient matrix B_0 of disturbance vector ε_t in structural equation (3.1) is known and nonsingular.

The part of assumption A.4.ii that B_0 is known imposes a nonspecific normalizing restriction and the part that B_0 is nonsingular imposes a specific generic restriction.

Identification 1 is derived in section 4 for two cases: (a) assumptions A.1.i-iii, A.3, and A.4.i hold <u>or</u> (b) assumptions A.1.i-iii, A.3, and A.4.ii hold (this inclusive "or" could be replaced by "and"). For identification 1 to hold in case (a), C_{01} must have full row rank and be known; for identification 1 to hold in case (b), C_{01} <u>and</u> B_0 must have full row rank <u>but</u> <u>only</u> B_0 needs to be known.

Depending on the application, the parts of assumptions A.4.i-ii that C_{01} and B_0 are known may or may not be considered stringent. Generally, assuming that C_{01} is known may be considered more stringent than assuming that B_0 is known, because C_{01} is the coefficient matrix of observed variables but B_0 is the coefficient matrix of unobserved disturbances.

ASSUMPTION A.5: The $n \times n$ disturbance-coefficient matrix Θ_0 in RES and ERF equations (3.3) and (5.2) is just-identified from the $n \times n$ symmetric positive-definite disturbance-covariance matrix $\Theta_0 \Theta_0^T$ in the RES and ERF equations by the $n(n-1)/2$ restrictions implied by the symmetry of $\Theta_0 \Theta_0^T$ and $n(n+1)/2$ additional restrictions on Θ_0, such that the n^2 restrictions are mutually independent (nonredundant).

Assumption A.5 imposes $n(n-1)/2$ <u>specific</u> <u>normalizing</u> symmetry restrictions on Θ_0 and $n(n+1)/2$ <u>nonspecific</u> <u>normalizing</u> additional restrictions on Θ_0. Cholesky factorization (Golub & Van Loan, 1983, p. 89) of $\Theta_0 \Theta_0^T$ in terms of lower-triangular Θ_0 is one source of the n^2 symmetry and additional restrictions on Θ_0.

ASSUMPTION A.6: ERF and XVAR coefficients and disturbance covariances in $\{\Phi_1, \{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}, \Theta_0 \Theta_0^T, \Sigma_\xi\}$ of ERF and XVAR(k) equations are identified and consistently estimated from assumed or consistently estimated data moments, for some finite $k \geq 1$.

Assumption A.6 imposes nonspecific generic restrictions on data moments. For example, ERF and XVAR coefficients and disturbance covariances could be identified and consistently estimated by applying linear least-squares estimation to ERF and XVAR equations (5.2) and (3.2) under standard linear least-squares assumptions on data moments (Theil, 1971; Schmidt, 1976).

ASSUMPTION A.7: Higher-level coefficients in θ_j from which lower-level coefficients in θ_i, for i < j, are identified, derive from or have underlying lower-level coefficients.

Assumption A.7 is nonspecific and in any particular application would be a combination of normalizing, generic, and exact restrictions on coefficients, disturbance covariances, and data moments in $\{\theta_i\}_{i=2}^{5}$. Whereas none of the assumptions explicitly impose any exact restrictions, assumption A.7 implicitly allows exact restrictions to be imposed.

Assumptions A.5 to A.7 are added so that identifications 1 and 2 are complete as identifications and consistent estimations of structural and RES coefficients down from assumed and consistently estimated data moments. Even if identifications 1 and 2 can be computed in practice as estimations, strictly, they aren't consistent estimations without this underpinning. Assumptions A.1.i to A.7 overlap to varying degrees, because they're considered only as sufficient conditions for identifications 1 and 2 and not also as necessary conditions for the identifications.

4. IDENTIFICATION 1 OF STRUCTURAL COEFFICIENTS FROM RES COEFFICIENTS.

This section derives linear identification 1 of structural coefficients in $\{\{A_i\}_{i=0}^{2}, B_0, C_0\}$ from RES coefficients in $\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^{\infty}\}$ by

CERRE equations (3.4) to (3.7). To minimize notation, we reuse $XM = N$ as the key identifying equation in each case (a) and (b) in identification 1 in this section 4 and in each case I to III in identification 2 in next section 5. This shouldn't cause confusion, because X, M, and N are defined separately for each use.

In identification 1, because A_0 appears only in equation (3.4), the equation is reserved for identifying A_0. $X = [A_2, A_1]$ is identified from M and N in cases (a) and (b) by solving $XM = N$. In case (a), $XM = N$ is formed by combining equations (3.6) and (3.7), for i = 1; and, in case (b), $XM = N$ is formed by combining equations (3.5) and (3.7), for i = 1. $X = [A_2, A_1]$ can't be identified using only equations (3.7), because for any number of combined equations (3.7) the resulting M has less than full row rank. In both cases (a) and (b), for identified $X = [A_2, A_1]$, remaining structural coefficients in $\{A_0, B_0, C_0\}$ are identified by equations (3.4) to (3.6).

The derivations of both identifications 1 and 2 require $A_2\Phi_1 + A_1$ to be nonsingular, which is now proved. Assumption A.1.i implies that $A(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ factors as $K(\lambda)\Phi(\lambda)$, where $K(\lambda) = K_1\lambda + K_0$, $\Phi(\lambda) = I_n\lambda - \Phi_1$, K_0, K_1, and Φ_1 are $n\times n$ real-valued matrices, and λ is a complex-valued scalar. Assumptions A.1.ii-iii imply that the latent roots of $K(\lambda)$ are nonzero, so that K_0 is nonsingular. Multiplying out $K(\lambda)\Phi(\lambda)$ and equating coefficients of powers of λ with those in $A(\lambda)$, implies that $A_2\Phi_1 + A_1 = K_0$, so that $A_2\Phi_1 + A_1$ is nonsingular.

Although the state representation of an XVAR equation is needed only in identification 2 in section 5, it's convenient now to introduce the state representation of XVAR(3) equation (3.2). The state representation has observation equation

(4.1) $$z_t = Hx_t,$$

observation matrix $H = [I_m, 0_{mx2m}] = mx3m$, state vector $x_t = (z_t^T, z_{t-1}^T, z_{t-2}^T)^T = 3mx1$, state equation

$$(4.2)\qquad x_t = Fx_{t-1} + G\xi_t,$$

state-transition matrix $F = \begin{bmatrix} D_1 & D_2 & D_3 \\ I_m & 0_{mxm} & 0_{mxm} \\ 0_{mxm} & I_m & 0_{mxm} \end{bmatrix} = 3m\times 3m$, disturbance-coefficient matrix $G = [I_m, 0_{mx2m}]^T = 3mxm$, and the same $mx1$ disturbance vector ξ_t as in XVAR(3) equation (3.2). State representation (4.1) and (4.2) extends in the obvious way for an XVAR(k) equation, for any finite $k \geq 4$. Section 5 first uses state representation (4.1) and (4.2) to derive identification 2 and, then, extends the identification for an XVAR(k) equation, for any finite $k \geq 4$.

Nonsingular $A_2\Phi_1 + A_1$ implies that equations (3.7) can be written equivalently as

$$(4.3)\qquad \Xi_i = \Pi\Xi_{i-1},$$

for i = 1,2,3,…, where $\Pi = -\left(A_2\Phi_1 + A_1\right)^{-1}A_2$.

An XVAR(k) equation, for any finite $k \geq 1$, like XVAR(3) equation (3.2), is stationary, nonstationary, limited nonstationary (LN), and halfway limited nonstationary (HLN), respectively, if and only if $\bar{\lambda}(F) < 1$, $\bar{\lambda}(F) \geq 1$, $\bar{\lambda}(F)\bar{\lambda}(\Pi) < 1$, and $\bar{\lambda}(F)\bar{\lambda}(\Pi) < 1/2$, where $\bar{\lambda}(\cdot)$ denotes the largest absolute eigenvalue of a square matrix.

In practice, sufficiently differenced data are stationary, so that $\bar{\lambda}(F) < 1$, and usual economic restrictions on a LREM imply that $\bar{\lambda}(\Pi) < 1$, which together imply LN. If $\bar{\lambda}(F) = \bar{\lambda}(\Pi)$, then, HLN implies that $\bar{\lambda}(F)$ and $\bar{\lambda}(\Pi) \leq \sqrt{1/2} \cong .71$, which suggests that HLN can hold for many applications with differenced data.

In case (a), for an order of columns and partitions of C_0 for which assumptions A.3 and A.4.i imply that C_{01} has full row rank and is known, let columns of $\{\Xi_i\}_{i=0}^{1}$ be ordered conformably and partitioned as Ξ_i =

$[\Xi_{i1}, \Xi_{i2}] = [n\times\bar{p}, n\times(m-\bar{p})]$. Then, because only the $\bar{p}$ leftmost columns of C_0 in C_{01} are known and can be used in equation (3.6) to identify $\{A_i\}_{i=1}^{2}$, we temporarily ignore any $m-\bar{p}$ rightmost columns of equations (3.6) and (3.7), for i = 1, and combine the equations as $n\times 2\bar{p}$ linear equation

(4.4) $\quad XM = N$,

where $X = [A_2, A_1] = n\times 2n$, $M = \begin{bmatrix} \Phi_1\Xi_{01} & \Phi_1\Xi_{11} + \Xi_{01} \\ \Xi_{01} & \Xi_{11} \end{bmatrix} = 2n\times 2\bar{p}$, and $N = \begin{bmatrix} C_{01}, & 0_{nx\bar{p}} \end{bmatrix} = n\times 2\bar{p}$.

Four conditions must hold to identify X by equation $XM = N$: (i) M and N must exist; (ii) M and N must be known; (iii) X must solve the equation; and, (iv) the solution X must be unique. Strictly, a solution X is either exact or inexact, i.e., exists or doesn't exist, but, in practice, in an identification as estimation, an inexact solution X may be an acceptable approximation if it's not too inaccurate.

M and N in equation (4.4) exist and are known because they're given inputs to the identification. X solves equation (4.4) exactly if and only if rows(N) $\subset$ row span(M) and uniquely as $X = NM^T(MM^T)^{-1}$ if and only if row rank(M) = full = $2n$. Assumption A.7 implies that rows(N) $\subset$ row span(M) and, except in some applications, there's nothing more to say about when this condition holds. Even if $X = NM^T(MM^T)^{-1}$ doesn't solve equation (4.4) exactly, it solves it approximately with error equal to least-squares residual $N - NM^T(MM^T)^{-1}M$ (Golub & Van Loan, 1983, ch. 6, pp. 162-169). If $n = m = \bar{p}$, M is square, and row rank(M) = full = $2n$, then, M is nonsingular and $X = NM^{-1}$ solves equation (4.4) exactly and uniquely.

We now abstract from existence of a solution X of equation (4.4) and look more closely at conditions under which a solution X of equation (4.4) is unique. Postmultiplying equation (4.4) by $\text{diag}[\Xi_{01}^T(\Xi_{01}\Xi_{01}^T)^{-1}, \Xi_{01}^T(\Xi_{01}\Xi_{01}^T)^{-1}] = 2\bar{p}\times 2n$ implies $nx2n$ equation

$$(4.5) \qquad XM = N,$$

where $X = [A_2, A_1] = n\times 2n$, $M = \begin{bmatrix} I_n & \Phi_1 \\ 0_{nxn} & I_n \end{bmatrix}\begin{bmatrix} 0_{nxn} & I_n \\ I_n & \Pi \end{bmatrix} = 2n\times 2n$, $\Pi = \Xi_{11}\Xi_{01}^T(\Xi_{01}\Xi_{01}^T)^{-1} = n\times n$, and $N = \begin{bmatrix} C_{01}\Xi_{01}^T(\Xi_{01}\Xi_{01}^T)^{-1}, & 0_{nxn} \end{bmatrix} = n\times 2n$.

Π, M, and N in equation (4.5) exist because $(\Xi_{01}\Xi_{01}^T)^{-1}$ exists, because row rank(Ξ_{01}) = full = n, because in equation (3.6) $A_2\Phi_1 + A_1$ is nonsingular and row rank(C_0) = full = n by assumptions A.3 and A.4.i. M and N are known because they're given inputs to the identification. M is nonsingular because $\begin{bmatrix} I_n & \Phi_1 \\ 0_{nxn} & I_n \end{bmatrix}$ and $\begin{bmatrix} 0_{nxn} & I_n \\ I_n & \Pi \end{bmatrix}$ are always nonsingular. Therefore, $X = NM^{-1}$ is the exact, unique, and computable solution of equation (4.5) that identifies $\{A_i\}_{i=1}^2$ from $\{C_{01}, \Phi_1, \{\Xi_{i1}\}_{i=0}^1\}$ by equation (4.4), exactly or approximately depending on how closely M and N in equation (4.4) satisfy rows(N) $\subset$ row span(M).

For exactly or approximately identified $\{A_i\}_{i=1}^2$ by equation (4.4), A_0 is identified exactly from $\{\{A_i\}_{i=1}^2, \Phi_1\}$ by equation (3.4), B_0 is identified exactly from $\{\{A_i\}_{i=1}^2, \Phi_1, \Theta_0\}$ by equation (3.5), and, if it's non-null, C_{02} is identified exactly from $\{\{A_i\}_{i=1}^2, \Phi_1, \Xi_{02}\}$ by the $m-\bar{p}$ rightmost columns of equation (3.6). Thus, in case (a), all structural coefficients in $\{\{A_i\}_{i=0}^2, B_0, C_{02}\}$ are identified exactly or approximately from normalized structural coefficients in C_{01} and RES coefficients in $\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^\infty\}$ by CERRE equations (3.4) to (3.7).

In case (b), in which assumption A.4.i and equation (3.6) of case (a) are replaced by assumption A.4.ii and equation (3.5), the equation that corresponds to equation (4.4) is $n\times(n+\bar{p})$ equation

$$(4.6) \qquad XM = N,$$

where $X = [A_2, A_1] = n\times 2n$, $M = \begin{bmatrix} \Phi_1\Theta_0 & \Phi_1\Xi_{11} + \Xi_{01} \\ \Theta_0 & \Xi_{11} \end{bmatrix} = 2n\times(n+\bar{p})$, and $N = \begin{bmatrix} B_0, 0_{nx\bar{p}} \end{bmatrix} = n\times(n+\bar{p})$. Postmultiplying equation (4.6) by $\mathrm{diag}[I_n, \Xi_{01}^T(\Xi_{01}\Xi_{01}^T)^{-1}] = (n+\bar{p})x2n$ implies $nx2n$ equation

(4.7) $XM = N$,

where $X = [A_2, A_1] = n\times 2n$, $M = \begin{bmatrix} I_n & \Phi_1 \\ 0_{nxn} & I_n \end{bmatrix}\begin{bmatrix} 0_{nxn} & I_n \\ I_n & \Pi \end{bmatrix}\begin{bmatrix} \Theta_0 & 0_{nxn} \\ 0_{nxn} & I_n \end{bmatrix} = 2n\times 2n$, $\Pi = \Xi_{11}\Xi_{01}^T(\Xi_{01}\Xi_{01}^T)^{-1} = n\times n$, and $N = \begin{bmatrix} B_0, & 0_{nxn} \end{bmatrix} = n\times 2n$.

Π, M, and N in equation (4.7) exist for the same reason that $(\Xi_{01}\Xi_{01}^T)^{-1}$ exists in equation (4.5). M and N are known because they're given inputs to the identification. M is nonsingular because $\begin{bmatrix} I_n & \Phi_1 \\ 0_{nxn} & I_n \end{bmatrix}$ and $\begin{bmatrix} 0_{nxn} & I_n \\ I_n & \Pi \end{bmatrix}$ are always nonsingular and because Θ_0 is nonsingular, because in equation (3.5) $A_2\Phi_1 + A_1$ is nonsingular and B_0 is nonsingular by assumption A.4.ii. Therefore, $X = NM^{-1}$ is the exact, unique, and computable solution of equation (4.7) that identifies $\{A_i\}_{i=1}^2$ from $\{B_0, \Phi_1, \Theta_0, \{\Xi_{i1}\}_{i=0}^1\}$ by equation (4.6), exactly or approximately depending on how closely M and N in equation (4.6) satisfy rows(N) $\subset$ row span(M).

For exactly or approximately identified $\{A_i\}_{i=1}^2$ by equation (4.6), A_0 is identified exactly from $\{\{A_i\}_{i=1}^2, \Phi_1\}$ by equation (3.4) and all of C_0 is identified exactly from $\{\{A_i\}_{i=1}^2, \Phi_1, \Xi_0\}$ by all of equation (3.6). Thus, in case (b), all structural coefficients in $\{\{A_i\}_{i=0}^2, C_0\}$ are identified exactly or approximately from normalized structural coefficients in B_0 and RES coefficients in $\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^\infty\}$ by CERRE equations (3.4) to (3.7).

In both cases (a) and (b), there are up to $\sum_{q=n}^{\bar{p}} \binom{\bar{p}}{q}$ possible values of M and N that can be used to identify X = $[A_2, A_1]$ by equations (4.4) to (4.7), where $\binom{\bar{p}}{q}$ denotes the binomial coefficient. If $\bar{p} > n$, then, $\{A_i\}_{i=1}^{2}$ are over-identified by equations (4.4) to (4.7), because Π is over-identified by $\Xi_{11}\Xi_{01}^{T}(\Xi_{01}\Xi_{01}^{T})^{-1}$. Schmidt's (1976, pp. 145-149) discussion suggests that over-identified estimated $\{A_i\}_{i=1}^{2}$ are inconsistently estimated, but assumption A.7 overrules this conclusion, so that both just- and over-identified estimated $\{A_i\}_{i=1}^{2}$ can be considered consistently estimated. Of course, in practice, in identification 2 as estimation, different combinations of M and N will result in different identified values of X.

In both cases (a) and (b), if $\bar{p} > n$, then, $\bar{p} - n$ columns of Ξ_0 needn't be used in the identifications. However, if the identifications are estimations, then, it's better to use all $\bar{p}$ columns of Ξ_0 that correspond to known columns of C_0, because then Π = $\Xi_{11}\Xi_{01}^{T}(\Xi_{01}\Xi_{01}^{T})^{-1}$ averages the maximum number of outer products of columns of estimated $\{\Xi_{i1}\}_{i=0}^{1}$, so that, according to a law of large numbers, Π and $\{A_i\}_{i=1}^{2}$ should have lower sampling covariances.

In both cases (a) and (b), if M has less than full row rank, then, X = $[A_2, A_1]$ could still be identified by equations (4.4) to (4.7) as X = $NM^{T}(MM^{T})^{-1}$ if enough exact, independent, and linear restrictions are added as columns of M and N that raise the row rank of M to full rank $2n$ (Al-Sadoon & Zwiernik, 2019). Al-Sadoon & Zwiernik emphasized affine (linear nonhomogeneous) restrictions, but, because $N \neq 0_{nxn}$, equations (4.4) to (4.7) are affine and remain affine for any added linear homogeneous or nonhomogeneous restrictions.

In both cases (a) and (b), because row rank(C_0) = full = n is the necessary rank condition for solving identifying equations (4.4) to (4.7) uniquely, the necessary order condition for identification 1 is

(4.8) $m \geq n$,

for m exogenous variables and n endogenous variables.

In both cases (a) and (b), if, for given RES coefficients, structural-coefficient matrices that satisfy CERRE equations (3.4) to (3.7) are premultiplied by any and the same nonsingular $n \times n$ matrix, then, the resulting premultiplied structural-coefficient matrices also satisfy the CERRE equations and are observationally equivalent to the unpremultiplied structural-coefficient matrices. Therefore, because assumptions A.3 and A.4.i restrict C_{011} to be nonsingular for a suitable order of columns of C_0 and assumption A.4.ii restricts B_0 to be nonsingular, either C_{011} or B_0, but not both, can be normalized to any nonsingular nxn matrix, including the nxn identity matrix.

Identification 1 can be set up alternatively and similarly to cases (a) and (b) by instead normalizing any sufficient combination of structural coefficients in $\{\{A_i\}_{i=0}^{2}, B_0, C_0\}$.

5. IDENTIFICATION 2 OF RES COEFFICIENTS FROM ERF AND XVAR COEFFICIENTS.

This section derives linear identification 2 of RES coefficients in $\{\Xi_i\}_{i=0}^{\infty}$ from ERF and XVAR coefficients in $\{\{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}\}$ for three cases in the way exogenous variables enter the structural equation: case I in current (period t) values in LREM-structural equation (3.1); case II in current and previous-period values in modified equation (3.1); and, case III in current and expected-next-period values in modified equation (3.1). See Martinez-Garcia (2020) for case III.

Because $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Theta}_0$ are both RES and ERF coefficients, it suffices to identify $\{\Xi_i\}_{i=0}^{\infty}$ from $\{\{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}\}$. In each case I to III, identification 2 is first derived for an XVAR(3) equation and is, then, extended to an XVAR(k) equation, for any finite $k \geq 4$.

5.1. CASE I: CURRENT EXOGENOUS VARIABLES.

In case I of identification 2, exogenous variables are in structural equation (3.1) in current values, z_t, are first considered to be generated by XVAR(3) equation (3.2), and are predicted as

$$(5.1) \qquad E_t z_{t+i} = HF^i \left(z_t^T, z_{t-1}^T, z_{t-2}^T\right)^T,$$

for i = 1,2,3,…, according to state representation (4.1) and (4.2).

Inserting predictions (5.1) into RES equation (3.3) implies ERF equation

$$(5.2) \qquad y_t = \Phi_1 y_{t-1} + \Theta_0 \varepsilon_t + \Upsilon_0 z_t + \Upsilon_1 z_{t-1} + \Upsilon_2 z_{t-2},$$

where $\Upsilon = [\Upsilon_0, \Upsilon_1, \Upsilon_2] = [n\times m, n\times m, n\times m] = nx3m$ is defined by

$$(5.3) \qquad \Upsilon = \sum_{i=0}^{\infty} \Pi^i [\Xi_0, 0_{nx2m}] F^i .$$

If the XVAR equation is limited nonstationary (LN), then, Υ exists and is the unique solution of the asymmetric Stein equation (Lancaster and Rodman, 1995, p. 100)

$$(5.4) \qquad \Upsilon = \Pi \Upsilon F + [\Xi_0, 0_{nx2m}] .$$

In identification 1, Π satisfies equation (4.3) for given $\{\Xi_{i1}\}_{i=0}^{1}$. Now, in all cases I to III of identification 2, Π is identified jointly with Ξ_0 from ERF and XVAR coefficients in $\{\{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}\}$.

Using the details of F to multiply out $\Pi\Upsilon F$, Stein equation (5.4) implies equations

$$(5.5) \qquad \Upsilon_0 = \Pi(\Upsilon_0 D_1 + \Upsilon_1) + \Xi_0 ,$$

(5.6) $\Upsilon_1 = \Pi(\Upsilon_0 D_2 + \Upsilon_2)$,

(5.7) $\Upsilon_2 = \Pi\Upsilon_0 D_3$.

Equations (5.5) to (5.7) combine as $n\times 3m$ equation

(5.8) $XM = N$,

where $X = [\Pi, \Xi_0] = nx(n+m)$, $M = \begin{bmatrix} \Upsilon_0 D_1 + \Upsilon_1 & \Upsilon_0 D_2 + \Upsilon_2 & \Upsilon_0 D_3 \\ I_m & 0_{mxm} & 0_{mxm} \end{bmatrix} = (n+m)x3m$, and $N = [\Upsilon_0, \Upsilon_1, \Upsilon_2] = nx3m$.

In all cases I to III of identification 2 by equations (5.8), (5.16), (5.25), and their extensions to any finite $k \geq 4$, M and N exist and are known because $\{\{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}\}$ are given inputs to the identifications. For M and N in equation (5.8), $X = [\Pi, \Xi_0]$ is identified exactly from $\{\{\Upsilon_i\}_{i=0}^{2}, \{D_i\}_{i=1}^{3}\}$ if and only if $X = NM^T(MM^T)^{-1}$ solves equation (5.8) exactly and uniquely, which occurs, respectively, if and only if rows(N) $\subset$ row span(M) and row rank(M) = full = $n+m$.

If M is square and row rank(M) = $n+m$, then, rows(N) $\subset$ row span(M), M is nonsingular, and $X = NM^{-1}$ solves equation (5.8) exactly and uniquely. If M isn't square, rows(N) $\subset$ row span(M) and row rank(M) = $n+m$, then, $X = NM^T(MM^T)^{-1}$ solves equation (5.8) exactly and uniquely. If M isn't square, rows(N) $\not\subset$ row span(M), and row rank(M) = $n+m$, then, $X = NM^T(MM^T)^{-1}$ solves equation (5.8) uniquely, but only approximately in a least-squares sense (Golub & Van Loan, 1983, ch. 6, pp. 162-169). Like in identification 1, assumption A.7 implies that rows(N) $\subset$ row span(M) and, except in some applications, there's nothing more to say about when rows(N) $\subset$ row span(M).

$X = NM^T(MM^T)^{-1}$ solves equation (5.8) uniquely if and only if row rank(M) = full = $n + m$, which occurs if and only if row rank(M_{12}) =

n, where M_{12} = $[\Upsilon_0 D_2+\Upsilon_2, \Upsilon_0 D_3]$ = $nx2m$, which holds if but not only if row rank($\Upsilon_0 D_2+\Upsilon_2$) = n or row rank($\Upsilon_0 D_3$) = n, which holds if but not only if row rank(Υ_0) = n and $|D_3| \neq 0$. Controllability of XVAR(3) equation (3.2) implies that $|D_3| \neq 0$.

Unlike in identification 1, where $\{A_i\}_{i=1}^{2}$ can be identified using fewer than $\bar{p}$ columns of $\{\Xi_i\}_{i=0}^{1}$, in all cases I to III of identification 2, all m columns of $\{\Upsilon_i\}_{i=0}^{2}$ must be used because otherwise the $XM = N$ identifying equations will be distorted.

We now prove that halfway limited nonstationarity (HLN) of an XVAR(k) equation, for any finite $k \geq 1$, implies that row rank(Υ_0) = n. The proof is important because it shows that there are values of LREM-structural coefficients that imply that identifying equations (5.8), (5.16), possibly (5.25), and their extensions for $k \geq 4$ can have unique solutions, so that identification 2 can hold in cases I and II and possibly in case III.

Row rank(Υ_0) = n if and only if $x \neq 0_{nx1}$ implies that $x^T\Upsilon_0 \neq 0_{1xm}$. Infinite sum (5.3) implies that $x^T\Upsilon_0 = \sum_{i=0}^{\infty} v_i^T$, where v_i^T = $x^T\Pi^i\left[\Xi_0, 0_{nx2m}\right]F^i\left[I_m, 0_{mx2m}\right]^T$ = $1xm$. Because HLN of an XVAR(k) equation implies that $x^T\Pi^i\left[\Xi_0, 0_{nx2m}\right]F^i\left[I_m, 0_{mx2m}\right]^T$ declines to zero at a geometric rate less than 1/2 as i increases, $\|v_{i+1}\| = \rho_i\|v_i\|$, for i = 0,1,2,…, where $0 < \rho_i \leq \bar{\rho} = \bar{\lambda}(\Pi)\bar{\lambda}(F) < 1/2$ and $\|\cdot\|$ denotes the Frobenius norm of a vector (Golub & Van Loan, 1983, p. 14). Because equation (3.6), row rank(C_0) = full = n by assumptions A.3 and A.4.i, and nonsingularity of $A_2\Phi_1 + A_1$ imply that row rank(Ξ_0) = full = n, $x \neq 0_{nx1}$ implies that v_0^T = $x^T\Xi_0 \neq 0_{1xm}$. Without loss of generality, we assume that v_0 is normalized as $\|v_0\| = 1$.

Because $1-\rho_0 > 0$, the triangle inequality implies that $\|v_0 + v_1\| \geq \left|\|v_0\| - \|v_1\|\right| = 1-\rho_0 > 0$. Because $1-\rho_0-\rho_0\rho_1 \geq 1-\bar{\rho}-\bar{\rho}^2$ and HLN implies that $1-\bar{\rho}-\bar{\rho}^2 > 0$, the triangle inequality and $\|v_2\| = \rho_0\rho_1$ imply that

$\|v_0 + v_1 + v_2\| \geq \big|\|v_0 + v_1\| - \|v_2\|\big| \geq 1 - \rho_0 - \rho_0\rho_1 \geq 1 - \bar{\rho} - \bar{\rho}^2 > 0$. Continuing like this, $\|x^T\Upsilon_0\| = \|\sum_{i=0}^{\infty} v_i\| \geq 1 - \sum_{i=1}^{\infty}\bar{\rho}^i = (1 - 2\bar{\rho})/(1 - \bar{\rho}) > 0$, so that HLN implies that $x^T\Upsilon_0 \neq 0_{1xm}$, hence, that row rank($\Upsilon_0$) = n.

The above result appears to hold under some weakening of HLN, so that HLN apparently isn't necessary for row rank(Υ_0) = n, but exactly by how much weakening hasn't been determined or found. Of course, row rank(Υ_0) = n could just be assumed, but we prefer to restrict only structural coefficients.

Extending M and N in equation (5.8) for z_t ~ XVAR(k), for any finite $k \geq 4$,

(5.9) $$M = \begin{bmatrix} \Upsilon_0 D_1 + \Upsilon_1 & \Upsilon_0 D_2 + \Upsilon_2 & \cdots & \Upsilon_0 D_{k-1} + \Upsilon_{k-1} & \Upsilon_0 D_k \\ I_m & 0_{mxm} & \cdots & 0_{mxm} & 0_{mxm} \end{bmatrix} = (n+m)xkm$$

and $N = [\Upsilon_0, \cdots, \Upsilon_{k-1}] = nxkm$.

For extended M and N given by and below equation (5.9), X = $[\Pi, \Xi_0]$ is identified from $\{\{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}\}$ if and only if $X = NM^T(MM^T)^{-1}$ solves extended equation (5.8) exactly and uniquely. $X = NM^T(MM^T)^{-1}$ solves extended equation (5.8) uniquely, but not necessarily exactly, if and only if row rank(M) = full = $n + m$, which holds if and only if row rank(M_{12}) = n, where $M_{12} = [\Upsilon_0 D_2 + \Upsilon_2, \ldots, \Upsilon_0 D_{k-1} + \Upsilon_{k-1}, \Upsilon_0 D_k] = nx(k-1)m$, which holds if but not only if row rank($\Upsilon_0 D_i + \Upsilon_i$) = n, for one or more i = $2, \ldots, k-1$, or row rank($\Upsilon_0 D_k$) = n, which holds if but not only if row rank(Υ_0) = n and $|D_k| \neq 0$. Controllability and HLN of an XVAR(k) equation imply that $|D_k| \neq 0$ and row rank(Υ_0) = n, for any finite $k \geq 1$.

For identified $\{\Pi, \Xi_0\}$, remaining RES coefficients in $\{\Xi_i\}_{i=1}^{\infty}$ are identified by iterating on equation (4.3).

In case I of identification 2, for z_t ~ XVAR(k) and any finite $k \geq 2$, because row rank(M) = full = $n + m$ is the necessary rank condition for solving identifying equation (5.8) or its extension uniquely, the necessary order condition for identification is

(5.10) $m \geq \lceil n/(k-1) \rceil$,

for finite and positive $\lceil n/(k-1) \rceil$, where $\lceil n/(k-1) \rceil$ denotes the smallest integer $\geq n/(k-1)$. Consider the following three cases.

1. If $k = 1$, then, row rank(M) < $n+m$ for any m and n, $X = [\Pi, \Xi_0]$ is unidentified by equation (5.8), and order condition (5.10) doesn't hold because $\lceil n/(k-1) \rceil$ isn't finite and positive.

2. If $k = 2$, then, row rank(M) = $n+m$ if <u>and</u> only if row rank($\Upsilon_0 D_2$) = n; and, if so, then, $X = [\Pi, \Xi_0]$ is, respectively, just- or over-identified by solving equation (5.8) uniquely as $X = NM^{-1}$ or as $X = NM^T(MM^T)^{-1}$ and order condition (5.10) holds as $m = n$ or as $m > n$.

3. If $k \geq 3$, then, row rank(M) = $n+m$ if <u>but</u> <u>not</u> only if row rank($\Upsilon_0 D_k$) = n; and, if so, then, $X = [\Pi, \Xi_0]$ is, respectively, just- or over-identified by solving equation (5.8) or its extension uniquely as $X = NM^{-1}$ or as $X = NM^T(MM^T)^{-1}$ and order condition (5.10) holds as $m = n/(k-1)$ or as $m > n/(k-1)$.

Order condition (5.10) suggests that identification 2 could hold for any m and n if k is large enough, in fact, for any n even if $m = 1$. However, in practice, the highest-lag XVAR(k) coefficients at lag k are unlikely to be significantly nonzero for k greater than about 16, even for undifferenced and seasonally-varying monthly data, so that order condition (5.10) limits n to about $15m$.

However, k could be much larger if the XVAR(k) equation is considered an approximation of the true z_t-generating equation. For example, if z_t is generated by an invertible vector autoregressive moving-average (VARMA) equation with a VAR(∞) representation whose coefficients decline geometrically by 10% per increased lag, then, an approximate XVAR(k) equation that's within 1% accuracy of the invertible-VARMA equation needs k greater than about 44, so that order condition (5.10) limits n to about $43m$.

5.2. CASE II: CURRENT AND LAGGED EXOGENOUS VARIABLES.

In case II of identification 2, exogenous variables are in the structural equation in current and one-period-lagged values, z_t and z_{t-1}, so that the exogenous term in structural equation (3.1) becomes

$$(5.11) \qquad C_0 \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix} = C_{01}z_t + C_{02}z_{t-1},$$

where $C_0 = [C_{01}, C_{02}] = [nxm, nxm] = nx2m$, and RES equation (3.3) becomes

$$(5.12) \qquad y_t = \Phi_1 y_{t-1} + \Theta_0\varepsilon_t + \textstyle\sum_{i=0}^{\infty} [\Xi_{i1}E_t z_{t+i} + \Xi_{i2}E_t z_{t+i-1}],$$

where $\Xi_i = [\Xi_{i1}, \Xi_{i2}] = [nxm, nxm] = nx2m$.

Exogenous z_t is again first assumed to be generated by XVAR(3) equation (3.2), again has state equation (4.2), but now has observation equation $\begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix} = Hx_t$, where $H = \begin{bmatrix} I_m & 0_{mxm} & 0_{mxm} \\ 0_{mxm} & I_m & 0_{mxm} \end{bmatrix} = 2m\times 3m$, so that z_t is predicted as

$$(5.13) \qquad \begin{pmatrix} E_t z_{t+i} \\ E_t z_{t+i-1} \end{pmatrix} = HF^i x_t,$$

for i = 1,2,3,….

Inserting predictions (5.13) into RES equation (5.12) continues to imply ERF equation (5.2), Υ_2 continues to satisfy equation (5.7), but now Υ_0 and Υ_1 satisfy

(5.14) $\Upsilon_0 = \Pi(\Upsilon_0 D_1 + \Upsilon_1) + \Xi_{01}$,

(5.15) $\Upsilon_1 = \Pi(\Upsilon_0 D_2 + \Upsilon_2) + \Xi_{02}$.

Equations (5.7), (5.14), and (5.15) combine as $n\times 3m$ equation

(5.16) $XM = N$,

where $X = [\Pi, \Xi_{01}, \Xi_{02}] = nx(n+2m)$, $M = \begin{bmatrix} \Upsilon_0 D_1 + \Upsilon_1 & \Upsilon_0 D_2 + \Upsilon_2 & \Upsilon_0 D_3 \\ I_m & 0_{mxm} & 0_{mxm} \\ 0_{mxm} & I_m & 0_{mxm} \end{bmatrix} =$ $(n+2m)x3m$, and $N = [\Upsilon_0, \Upsilon_1, \Upsilon_2] = nx3m$.

For M and N in equation (5.16), $X = [\Pi, \Xi_{01}, \Xi_{02}]$ is identified from $\{\{\Upsilon_i\}_{i=0}^{2}, \{D_i\}_{i=1}^{3}\}$ if and only if $X = NM^T(MM^T)^{-1}$ solves equation (5.16) exactly and uniquely. $X = NM^T(MM^T)^{-1}$ solves equation (5.16) uniquely, but not necessarily exactly, if and only if row rank(M) = full = $n + 2m$, which holds if and only if row rank($\Upsilon_0 D_3$) = n, which holds if but not only if row rank(Υ_0) = n and $|D_3| \neq 0$.

Extending M and N in equation (5.16) for z_t ~ XVAR(k), for any finite $k \geq 4$,

(5.17) $$M = \begin{bmatrix} \Upsilon_0 D_1 + \Upsilon_1 & \Upsilon_0 D_2 + \Upsilon_2 & \Upsilon_0 D_3 + \Upsilon_3 & \cdots & \Upsilon_0 D_{k-1} + \Upsilon_{k-1} & \Upsilon_0 D_k \\ I_m & 0_{mxm} & 0_{mxm} & \cdots & 0_{mxm} & 0_{mxm} \\ 0_{mxm} & I_m & 0_{mxm} & \cdots & 0_{mxm} & 0_{mxm} \end{bmatrix} = (n+2m)xkm$$

and $N = [\Upsilon_0, \cdots, \Upsilon_{k-1}] = nxkm$.

For extended M and N given by and below equation (5.17), $X = [\Pi, \Xi_{01}, \Xi_{02}]$ is identified from $\{\{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}\}$ if and only $X = NM^T(MM^T)^{-1}$ solves extended equation (5.16) exactly and uniquely. $X = NM^T(MM^T)^{-1}$ solves extended equation (5.16) uniquely, but not necessarily exactly, if and only if row rank(M) = full = $n + 2m$, which holds if and only if row rank(M_{12}) = n, where M_{12} =

$[\Upsilon_0 D_3+\Upsilon_3, \ldots, \Upsilon_0 D_{k-1}+\Upsilon_{k-1}, \Upsilon_0 D_k] = nx(k-2)m$, which holds if but not only if row rank$(\Upsilon_0 D_i+\Upsilon_i) = n$, for one or more i = 3,..., $k-1$, or row rank$(\Upsilon_0 D_k)$ $= n$, which holds if but not only if row rank$(\Upsilon_0) = n$ and $|D_k| \neq 0$. Controllability and HLN of an XVAR(k) equation imply that $|D_k| \neq 0$ and row rank$(\Upsilon_0) = n$, for any finite $k \geq 1$.

As in case I of identification 2, for identified $\{\Pi, \Xi_0\}$, remaining RES coefficients in $\{\Xi_i\}_{i=1}^{\infty}$ are identified by iterating on equation (4.3).

In case II of identification 2, for z_t ~ XVAR(k) and any finite k $\geq$ 3, because row rank(M) = full = $n + 2m$ is the necessary rank condition for solving identifying equation (5.16) or its extension uniquely, the necessary order condition for identification is

(5.18) $$m \geq \lceil n/(k-2) \rceil,$$

for finite and positive $\lceil n/(k-2) \rceil$, which has implications analogous to those discussed in case I below order condition (5.10). Consider the following three cases.

1. If k = 1 or 2, then, row rank(M) < $n + 2m$ for any m and n, $X = [\Pi, \Xi_0]$ is unidentified by equation (5.16), and order condition (5.18) doesn't hold because $\lceil n/(k-2) \rceil$ isn't finite and positive.

2. If k = 3, then, row rank(M) = $n + 2m$ if <u>and</u> only if row rank$(\Upsilon_0 D_3) = n$; and, if so, then, $X = [\Pi, \Xi_{01}, \Xi_{02}]$ is, respectively, just- or over-identified by solving equation (5.16) uniquely as $X = NM^{-1}$ or as $X = NM^T(MM^T)^{-1}$ and order condition (5.18) holds as $m = n$ or as $m > n$.

3. If $k \geq 4$, then, row rank(M) = $n + 2m$ if <u>but</u> <u>not</u> only if row rank$(\Upsilon_0 D_k) = n$; and, if so, then, $X = [\Pi, \Xi_{01}, \Xi_{02}]$ is, respectively, just- or over-identified by solving equation (5.16) or its extension

uniquely as $X = NM^{-1}$ or as $X = NM^T(MM^T)^{-1}$ and order condition (5.18) holds as $m = n/(k-2)$ or as $m > n/(k-2)$.

5.3. CASE III: EXPECTED-FUTURE AND CURRENT EXOGENOUS VARIABLES.

In case III, exogenous variables are in the structural equation in expected-next-period and current values, $E_t z_{t+1}$ and z_t, so that the exogenous term in structural equation (3.1) becomes

$$(5.19) \qquad C_0 \begin{pmatrix} E_t z_{t+1} \\ z_t \end{pmatrix} = C_{01} E_t z_{t+1} + C_{02} z_t ,$$

where $C_0 = [C_{01}, C_{02}] = [nxm, nxm] = nx2m$, and RES equation (3.3) becomes

$$(5.20) \qquad y_t = \Phi_1 y_{t-1} + \Theta_0 \varepsilon_t + \textstyle\sum_{i=0}^{\infty} [\Xi_{i1} E_t z_{t+1+i} + \Xi_{i2} E_t z_{t+i}] ,$$

where $\Xi_i = [\Xi_{i1}, \Xi_{i2}] = [nxm, nxm] = nx2m$.

Exogenous z_t is again first assumed to be generated by XVAR(3) equation (3.2), again has state equation (4.2), but now has observation equation $\begin{pmatrix} z_{t+1} \\ z_t \end{pmatrix} = Hx_t$, where $H = \begin{bmatrix} D_1 & D_2 & D_3 \\ I_m & 0_{mxm} & 0_{mxm} \end{bmatrix} = 2m \times 3m$, so that z_t is predicted as

$$(5.21) \qquad \begin{pmatrix} E_t z_{t+i+1} \\ E_t z_{t+i} \end{pmatrix} = HF^i x_t ,$$

for i = 1,2,3,....

Inserting predictions (5.21) into RES equation (5.20) continues to imply ERF equation (5.2), but now Υ_0, Υ_1, and Υ_2 satisfy

$$(5.22) \qquad \Upsilon_0 = \Pi(\Upsilon_0 D_1 + \Upsilon_1) + \Xi_{01} D_1 + \Xi_{02} ,$$

,

(5.23) $\Upsilon_1 = \Pi(\Upsilon_0 D_2 + \Upsilon_2) + \Xi_{01}D_2$,

(5.24) $\Upsilon_2 = \Pi\Upsilon_0 D_3 + \Xi_{01}D_3$.

Equations (5.22) to (5.24) combine as $n \times 3m$ equation

(5.25) $XM = N$,

where $X = [\Pi, \Xi_{01}, \Xi_{02}] = nx(n+2m)$, $M = \begin{bmatrix} \Upsilon_0 D_1 + \Upsilon_1 & \Upsilon_0 D_2 + \Upsilon_2 & \Upsilon_0 D_3 \\ D_1 & D_2 & D_3 \\ I_m & 0_{mxm} & 0_{mxm} \end{bmatrix} =$ $(n+2m)x3m$, and $N = [\Upsilon_0, \Upsilon_1, \Upsilon_2] = nx3m$.

For M and N in equation (5.25), $X = [\Pi, \Xi_{01}, \Xi_{02}]$ is identified from $\{\{\Upsilon_i\}_{i=0}^{2}, \{D_i\}_{i=1}^{3}\}$ if and only if $X = NM^T(MM^T)^{-1}$ solves equation (5.25) exactly and uniquely. $X = NM^T(MM^T)^{-1}$ solves equation (5.25) uniquely, but not necessarily exactly, if and only if rank(M) = full = $n + 2m$, which holds if and only if row rank(M_{12}) = $n+m$, where M_{12} = $\begin{bmatrix} \Upsilon_0 D_2 + \Upsilon_2 & \Upsilon_0 D_3 \\ D_2 & D_3 \end{bmatrix} = (n+m)x2m$.

Extending M and N in equation (5.25) for z_t ~ VAR(k), for any finite $k \geq 4$,

(5.26) $$M = \begin{bmatrix} \Upsilon_0 D_1 + \Upsilon_1 & \Upsilon_0 D_2 + \Upsilon_2 & \cdots & \Upsilon_0 D_{k-1} + \Upsilon_{k-1} & \Upsilon_0 D_k \\ D_1 & D_2 & \cdots & D_{k-1} & D_k \\ I_m & 0_{mxm} & \cdots & 0_{mxm} & 0_{mxm} \end{bmatrix} = (n+2m)xkm$$

and $N = [\Upsilon_0, \cdots, \Upsilon_{k-1}] = nxkm$.

For extended M and N given by and below equation (5.26), X = $[\Pi, \Xi_{01}, \Xi_{02}]$ is identified from $\{\{\Upsilon_i\}_{i=0}^{k-1}, \{D_i\}_{i=1}^{k}\}$ if and only if X = $NM^T(MM^T)^{-1}$ solves extended equation (5.25) exactly and uniquely. X =

$NM^T(MM^T)^{-1}$ solves extended equation (5.25) uniquely, but not necessarily exactly, if and only if row rank(M) = full = $n + 2m$, which holds if and only if row rank(M_{12}) = $n + m$, where M_{12} = $\begin{bmatrix} \Upsilon_0 D_2 + \Upsilon_2 & \cdots & \Upsilon_0 D_{k-1} + \Upsilon_{k-1} & \Upsilon_0 D_k \\ D_2 & \cdots & D_{k-1} & D_k \end{bmatrix}$ = $(n+m)x(k-1)m$.

Controllability and HLN of an XVAR(k) equation imply that $|D_k| \neq 0$ and row rank(Υ_0) = n, for any finite $k \geq 1$, but now in case III controllability and HLN are insufficient to imply that row rank(M_{12}) = $n + m$. $|D_k| \neq 0$ is now a necessary condition for row rank(M_{12}) = $n + m$ if $m = n$, $k = 3$, and $M = 2nx2n$ is square. Without further analysis, it's unclear what restrictions on structural, ERF, and XVAR coefficients imply or are implied by row rank(M_{12}) = $n + m$.

As in cases I and II of identification 2, for identified $\{\Pi, \Xi_0\}$, remaining RES coefficients in $\{\Xi_i\}_{i=1}^{\infty}$ are identified by iterating on equation (4.3).

In case III of identification 2, for z_t ~ VAR(k) and any finite $k \geq 3$, row rank(M) = full = $n + 2m$ implies that inequality (5.18) continues to be the necessary order condition for identification, with implications analogous to those discussed in case I below order condition (5.10).

Order condition (4.7) of identification 1 always implies order conditions (5.10) or (5.18) of identification 2 but the reverse isn't the case. Does this mean that if identification 2 holds but identification 1 doesn't hold, then, there's a contradiction? Not necessarily, because, first, even if LREM-structural coefficients solve their identifying equations exactly, they may not solve them uniquely, so that the LREM-structural coefficients aren't identified and order condition (4.7) doesn't hold; and, second, if only RES coefficients are needed to make policy predictions and LREM-structural coefficients aren't needed for anything else, then, LREM-structural equations can be disregarded and RES equations and their coefficients can be considered structural. Therefore, if identification 2 holds but identification 1 doesn't hold, then, there's not necessarily a contradiction.

6. CONCLUSION: RECONCILIATION OF LUCAS AND SIMS.

Linear rational-expectations models (LREMs) are conventionally "forwardly" estimated as follows. Structural coefficients are restricted by economic restrictions in terms of deep parameters. For given deep parameters, LREM-structural equations are solved for rational-expectations solution (RES) equations that determine endogenous variables. For given exogenous vector autoregressive (XVAR) equations that determine exogenous variables, RES equations are reduced to reduced-form VAR equations for endogenous variables (ERF) with exogenous variables. The combined ERF and XVAR equations are the reduced-form overall VAR (OVAR) equations of all variables in a LREM. The sequence of specified, solved, and combined equations defines a mapping from deep parameters to OVAR coefficients that is used to forwardly estimate a LREM in terms of deep parameters. Forwardly-estimated deep parameters determine forwardly-estimated RES equations that Lucas (1976) advocated for making policy predictions in his critique of policy predictions made with reduced-form equations.

Sims (1980) called economic identifying restrictions on deep parameters of forwardly-estimated LREMs "incredible", because he considered in-sample fits of forwardly-estimated OVAR equations inadequate and out-of-sample policy predictions of forwardly-estimated RES equations inaccurate. Sims (1980, 1986) instead advocated directly estimating OVAR equations restricted by statistical shrinkage restrictions and directly using the directly-estimated OVAR equations to make policy predictions. However, if assumed or predicted out-of-sample policy variables in the directly made policy predictions differ significantly from in-sample values, then, the out-of-sample policy predictions won't satisfy Lucas's (1976) critique.

If directly-estimated OVAR equations are reduced-form equations of underlying RES equations and further underlying LREM-structural equations, then, identification 2 derived in the paper linearly "inversely" estimates underlying RES equations from the directly-estimated OVAR equations and the inversely-estimated RES equations can be used to make policy predictions that satisfy Lucas's critique. Inversely-estimated RES equations satisfy Lucas's critique if they have underlying LREM-structural equations, because then they have policy-invariant distributed-lead coefficients inherited from policy-invariant underlying LREM-structural coefficients.

If Sims considered directly-estimated OVAR equations to fit in-sample data adequately (credibly) and their inversely-estimated RES equations to make accurate (credible) out-of-sample policy predictions, then, he should consider the RES equations and their underlying LREM-structural equations to be credible. Thus, inversely-estimated RES equations by identification 2 can reconcile Lucas's (1976) advocacy for making policy predictions with RES equations of underlying LREM-structural equations and Sims's (1980, 1986) advocacy for directly estimating OVAR equations. However, identification 2 doesn't reconcile Lucas's advocacy for making policy predictions with RES equations and Sims's advocacy for making policy predictions directly with directly-estimated OVAR equations.

Sims (1980, 1986) proposed two methods for making policy predictions directly with directly-estimated OVAR equations: (a) for assumed or predicted out-of-sample current and expected-future policy variables, setting out-of-sample disturbances of nonpolicy variables to zero and out-of-sample disturbances of policy variables and nonpolicy variables according to the estimated OVAR equations; (b) setting all out-of-sample variables to impulse responses of uncorrelated disturbances, considered structural disturbances, in the estimated OVAR equations.

Policy predictions (a) and (b) and policy predictions of RES equations have the following relative advantages:

1. Policy predictions (a) and policy predictions of RES equations can be strictly correct only if policy variables are exogenous in reduced-form OVAR equations. Policy variables needn't be exogenous in policy predictions (b), but policy predictions (b) are much more limited

than possible policy predictions (a) or policy predictions of RES equations.

2. If out-of-sample current and expected-future (exogenous) policy variables differ significantly from in-sample values, then, Lucas's (1976) critique says that out-of-sample policy predictions should be made only with RES or RES-like equations and not with policy predictions (a) and (b) of reduced-form equations.

Lucas (1976) discussed examples in which LREM-structural equations are or include first-order conditions of dynamic-optimization problems, so that their implied RES equations are or include forward-looking dynamic-optimal decision rules for endogenous variables. Even the most ordinary daily decisions such as whether to go and buy food and how much to buy are forward-looking dynamic-optimal decisions, because they are subject to frictions such as adjustment costs and delays. Both forwardly- and inversely-estimated RES equations have these same forward-looking properties that reflect through distributed-lead coefficients how economic agents use information on current and expected-future exogenous variables to make current forward-looking dynamic-optimal decisions on endogenous variables. By contrast, although directly-estimated backward-looking OVAR equations may correctly account for in-sample forward-looking decisions on endogenous variables, by not having any explicit connections to forward-looking decisions, they and their policy predictions generally won't satisfy -- and can't be modified to satisfy -- Lucas's critique.

Forwardly estimating RES equations requires knowing and using the composite forward mapping from deep parameters to overall OVAR coefficients, but inversely estimating RES equations doesn't require this knowledge. Inversely estimating RES equations by identification 2 requires only knowing or assuming the number of endogenous variables (n), the number of exogenous variables (m), and the number of lags of exogenous variables (k). More generally, inverse estimation requires knowing or assuming the number of leads of expected-future endogenous variables in LREM-structural equations and the numbers of lags of endogenous variables and disturbances in LREM-structural, RES, and ERF equations, respectively, r, p, and q in Zadrozny (1998). However, strictly, in order for policy predictions of inversely-estimated RES equations to satisfy Lucas's critique, the inversely-estimated RES equations should either have underlying policy-invariant LREM-structural equations or themselves be considered the structural starting point of analysis.

So far, no general statistically-consistent alternative to rational expectations has been proposed for economic modelling. The empirical scientific principle says that a model and its assumptions should together be accepted or rejected according to the accuracy of the model's out-of-sample predictions (Friedman, 1953, chapter 1). Out-of-sample prediction accuracies of forwardly- or inversely-estimated RES equations and of predictions (a) and (b) can and should be compared accordingly. Although impulse responses (b) are often made only for purposes of analysis, being de facto predictions, they can be compared as such with any other predictions of the same variables.

Identification 1 of LREM-structural coefficients from RES coefficients has no direct role in making policy predictions with RES equations, contributes mainly by showing that directly-estimated OVAR equations can be reduced-form equations of underlying LREM-structural equations, and could also possibly be used to guide specification of restrictions on LREM-structural coefficients in terms of deep parameters.

The paper has derived linear identifications 1 and 2 that in the order 2 and 1 can be used to consistently, easily, and quickly estimate RES and structural equations of a LREM. If the goal is to estimate a LREM only to make policy predictions, then, only RES equations need to be inversely estimated by identification 2, LREM-structural equations can be disregarded, and conventional, generally nonlinear, more arduous, and much slower forward estimation of a LREM in terms of deep parameters can be avoided.

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