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Abstract

We introduce endogenous fire sales into a simple network model. For any given initial distribution of shocks across the network, we develop a clearing algorithm to solve for the financial equilibrium. We then utilise the results to perform ex ante risk assessment and derive risk premia for every balance sheet item where liabilities are differentiated according to priority rights. We find that risk premia reflect both idiosyncratic risk and risk of contagion (network risk). Moreover, we show that network risk magnifies the gap between the risk premia of equity and debt. We also perform comparative statics, showing that changes to the distribution of shocks and network structure can have substantial effects on the level of systemic losses.

JEL-Codes: G330, G320, D850.

Keywords: networks, fire sales, systemic risk premia, risk assessment.

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We thank seminar participants at University of Nottingham. All remaining errors are our own.

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Spiros Bougheas[†] Adam Hal Spencer[‡]

November 16, 2022

1 Introduction

A firm with insufficient liquidity to meet its obligations to outside creditors might maintain solvency through selling its illiquid assets. Clearly, its survival depends on these assets' market value. Most likely, their market value will drop below their book value, on account of the urgency to sell. The extent of this drop can be significantly influenced by general market conditions. Times of crisis may involve many firms attempting to offload their assets, with fewer potential buyers, as firms focus their efforts on consolidating their current positions rather than expanding their operations. These firms in financial distress will then face 'fire sale' values of their assets; should they be unable to raise the required funds, they will become insolvent. The creditors of these insolvent firms, at least those without priority claims, might then be unable to meet their own obligations. As this process continues, insolvencies cascade through a network of connections that link firms and their creditors, thereby magnifying the impact of the shocks that catalysed the initial failures.

This description of bankruptcy cascades highlights the two types of risks a firm faces. The first type of risk is idiosyncratic. Firms in production networks can fail for technological reasons, for a lack of demand for their products or for the inability of final goods consumers who made their purchases on credit to repay their debts. Similarly, banks in financial networks can fail when households and firms, to whom they extent credit, are unable to meet the associated obligations. The second type of risk arises from being part of a network — we will refer to this type as *network risk*. In production networks, firms provide inputs to other firms and the links represent trade credit offered to buyers. In financial networks, as for example in the interbank market, the links are loans offered from one financial institution to another. We make an important distinction between counterparty risk and network risk. When firms belong to a network, they can fail due to the default of their counterparties. These counterparties may have failed either because of a direct idiosyncratic shock, or because they were caught in the middle of a cascade of defaults. The distinction between these two scenarios highlights potential risk assessment complexities that arise due to network effects.

Making accurate risk assessments is crucial for a firm's creditors, (irrespective of the firm's sector), as well as policymakers, in particular for regulators. For example, keeping systemic risk low in financial markets requires regulators to assess risks arising from potential failures of counterparties — such risks are hidden from balance sheets and other financial statements. Nevertheless, many popular measures of systemic risk rely entirely on information collected from such documents.[1] Knowing the structure of the entire network is a necessity for the regulator's risk assessments; this structure is not static, but changes over time in an uncertain way. Two other issues that further complicate the task of regulators are (a) some creditors might hold priority claims, and (b) potential liquidation values will depend on uncertain future market conditions.

In this paper, we ask the question of how are balance sheet risk assessments affected by the presence of network risk? Our answer comes in the form of a relatively simple model, which allows us to derive explicit expressions for the risk premia of each entry in the balance sheet. Yet the model has enough complexity such that we can study the impact on these premia of (a) the distribution of shocks across the network and the associated fire sales and (b) the structure of the network. There are two important features of the model. Firstly, on the liability side of the balance sheet, there are creditors that are differentiated according to the priority status of their claims.[2] Clearly, risk premia on those claims higher on the priority ladder command lower risk premia. But a main insight of our results is that network effects increase the wedge between the risk premium on equity and those on other liabilities, providing a novel explanation for the Equity Premium Puzzle (EPP). Secondly, liquidation values in our model are endogenous and depend on the total amount of assets offered for sale. It is well known in the context of financial network clearing

[1]Examples of such measures used predominantly in financial markets include Systemic Risk Measure (Acharya *et al.*, 2012), Marginal Expected Shortfall (Acharya *et al.,* 2017), CoVaR (Adrian and Brunnermeier,2016), SRISK (Brownless and Engle, 2016), the Granger-causality method (Billio *et al.*, 2012) and the Aggregate Vulnerability Index (Duarte and Eisenbach, 2021). For a review of this literature see Benoit *et al.* (2017).

[2]Two important papers in the corporate finance literature on the role of seniority are Bolton and Scharfstein (1990) and Hart & Moore (1995).

models without fire sales, that network effects do not amplify initial shocks, but rather only affect the number of firms that are liquidated. In our model with fire sales, we instead observe amplification and show that systemic losses and risk premia depend on both the distribution of shocks across the network, as well as its structure.

In the next section we present the model network economy. The model setup and analysis can be applied to both financial and production networks. That is — the results pertain equally to a network of firms interconnected through trade credit links, or to a network of banks participating in the interbank market. To keep the exposition clear, we will follow the financial interpretation, however later in the paper, we provide some empirical evidence from a production network that is consistent with one of the main predictions of our model.

Our model builds-off the banking network framework of Caballero and Simsek (2013), where all banks are symmetric and are located on a circle (each obtains a loan from exactly one bank and offers a loan to exactly one bank). We introduce multiple circles that can vary according to the number of banks they contain. This structure has many advantages. It is significantly richer than the one circle model and allows us to perform meaningful comparative static exercises with respect to the distribution of shocks and the network structure. Furthermore, it facilitates derivations of closed-form solutions for risk premia, which is a main contribution of our study. But of course, this simplified structure abstracts from several complexities of real networks. It is however, exactly because of real networks' complexity, that studies which replicate them do not admit analytical solutions, instead requiring numerical characterisations. There are a limited number of analytical results available for network models, without either/both endogenous fire sales and/or priority claims. Moreover, none of the existing models offer any explicit solutions for risk premia.

In Section 3, we devise and utilise a clearing algorithm to find the financial equilibrium of the model. Endogenous fire sales add considerable complication to this clearing algorithm. To clarify this added complexity, we first clear the system under the supposition that fire sales are exogenous, before building-up to the scenario where they can vary endogenously. In models with fire sales, there can be a multiplicity of clearing equilibria. In this scenario, these equilibria can always be ranked. We focus on the 'best' equilibrium that is the one with the lowest number of bankruptcies.[3]

Our clearing algorithm, under exogenous fire sales, begins by clearing the balance sheets of those banks that were hit by the initial idiosyncratic shock. The next step clears the balance sheets of the banks, which become insolvent because they had offered a loan to one of those hit by an initial shock. The algorithm repeats this process and terminates when all remaining banks are solvent. In contrast, when fire sales are endogenous, the value of liquidated assets depends on the volume of such assets offered for sale — we introduce a liquidation function, which captures an inverse relationship between these two

[3]Barucca *et al.* (2020) also focus on the algorithm that solves for the best equilibrium, arguing that it is the most natural given that at it takes as its starting point for equity values the corresponding book values. In contrast, Jackson and Pernoud, (2022) focus on optimal bailout policies that are designed so that all other equilibria can be avoided. In their model there are no priority claims and as long as other banks make full repayments on their loans and the value of its assets is sufficiently high to cover its liabilities. In our model, the existence of other claims with priority over bank debt implies that even in the best equilibrium there is the possibility of cascades.

objects. The algorithm then involves an additional iterative process, on the total volume of liquidated assets, to find a fixed point. As a first stage, we conjecture this equilibrium volume to be only the assets of the banks hit by the initial idiosyncratic shocks — this then implies a corresponding liquidation value. At any subsequent stage, if there are any additional insolvencies given this conjecture, the value of liquidated assets is adjusted to reflect the higher supply and then the algorithm goes back to the first stage.

Our model is simple enough to show analytically how the number of bank insolvencies, hence systemic losses, depend on the distribution of shocks across the network and also its underlying structure. We employ simple examples to illustrate that small changes in either the distribution of shocks for a given network structure, or in the structure of the network itself, can have a remarkably large effect on the system. That is — complexity of the network can drive significant residual uncertainty regarding final outcomes — even if the magnitudes of the initial idiosyncratic shocks are known. Moreover we show that, in the presence of claims with differing seniority status, changes in the structure of the network can also affect the distribution of losses among creditors.

In Section 4, we derive bank survival probabilities for any initial distribution of shocks. Next, we use these probabilities to calculate risk premia for each entry in a bank's balance sheet. Then we decompose risk premia across the two sources of risk — idiosyncratic and network risk. We further show that network risk widens the gap between the risk premium on equity and those on other liabilities, thus providing a new explanation for the EPP. In section 5, we offer some evidence from production networks in the data. In particular, we compare the equity risk premia for US firms and find that, on average, they are higher for firms that belong to more connected sectors. To keep the exposition of the analysis of our model clear, we focus on a benchmark case with several simplifying assumptions. In section 6, we relax these assumptions and in the process show that the benchmark model provides lower bounds for both the number of bankruptcies and the size of systemic losses in most cases. Finally, we provide concluding comments in Section 7.

Literature Review[4] Our paper contributes to that part of the literature that views fire sales as an important amplification mechanism of exogenous shocks. In their review article on fire sales, Shleifer and Vishny (2011) emphasize that 'because of fire sales, risk becomes systemic', (p.30). Their insight is that in financial network models without fire sales, or other 'wedges' such as bankruptcy costs, aggregate losses are simply equal to those losses incurred by the initial shocks. In that case, the only thing left for the clearing process to determine is the number of institutions that will be liquidated, where the prices of assets to be sold are equal to their book values by assumption. However, evidence is mounting that fire sales played a prominent role in amplifying shocks during the Great Depression (Mitchener and Richardson, 2019) and during the 2009 Global Financial Crisis (Adrian and Shin, 2010; Brunnermeier, 2009; Gorton and Metrick, 2012). There is also evidence on the impact of fire sales in the manufacturing (Benmelech and Bergman, 2009) and in the housing sector (Campbell *et al.*, 2011). Much theoretical work on fire sales has

[4]The literature on contagion in financial networks is vast and there are already a number of review articles that provide good cover; Allen and Babus (2009), Bougheas and Kirman (2015), Glasserman and Payton (2016) and Jackson and Pernoud (2022).

been based on models that do not explicitly account for the connections between firms (Acharya and Shin, 2010; Diamond and Rajan, 2011; Greenwood *et al.*, 2015; Guerrieri and Shimer, 2014; Kurlat, 2021; Shleifer and Vishny, 1992).

Our work is more closely related to those papers that investigate how fire sales amplify shocks in networks. A vast literature has followed the work by Eisenberg and Noe (2001), by developing algorithms for clearing financial networks hit by some initial shocks, under a variety of suppositions about the transmission of shocks between institutions and the relationships between balance sheet entries. Many papers introduce liquidation costs, but assume that these costs, (like those of bankruptcy), are independent of the number of liquidated institutions (see, for example, Bardoscia *et al.*, 2015; Barucca *et al.*, 2018; Furfine, 2003; Rogers and Veraart, 2013). Our work belongs to that group of papers where liquidation costs (fire sales) depend on the supply of assets offered for liquidation.[5] We generalise the parsimonious setup of Caballero and Simsek (2013), allowing for sufficiently more complex network structures that still allow for the assessment of *ex ante* balance sheet risk when there exist liabilities with different priority rights. Amini *et al.* (2016), Cifuentes *et al.* (2005), Feinstein (2017) and Jackson and Pernoud (2022) develop clearing algorithms for networks with arbitrary structures and endogenous fire sales. However, none of these papers develop methods for *ex ante* balance sheet risk assessment and with the exception of Amini *et al.* (2016) do not allow for claims that have priority rights.[6]

Our work is also related to a small group of papers that develop interbank clearing mechanisms, similar to that in Eisenberg and Noe (2001), then use them to provide *ex ante* risk assessments. The models developed differ in the asset valuation functions they employ. In Veraart (2020) and in Elsinger *et al.* (2006), there is a common valuation function for the whole network. In Barucca *et al.* (2016), each bank does its own evaluation, but those valuations and the clearing algorithm ensures that the evaluations are consistent across banks. Glasserman and Young (2015) follow a different approach, by comparing an interbank network with a financial system consisting of the same number of banks but without any interbank obligations. They derive upper bounds for the probability of contagion and also for expected losses due to network effects. These papers allow for bankruptcy costs, which are independent of the number of liquidated institutions. Our study goes beyond this approach, by developing a model with endogenous liquidation costs, which is tractable yet sufficiently rich to allow for interesting comparative statics on the shock distribution and network structure. In our framework, each round of liquidations in the clearing algorithm has to go through another round of iterations after an adjustment is made for the change in liquidation values. We also cover new ground by introducing priority rights we are also able to provide separate *ex ante* risk assessments for outside debt, interbank obligations and equity.

[5]There is another group of papers where liquidation costs are triggered by mark-to-market pricing of assets when institutions hold over-lapping portfolios (Caccioli *et al.*, 2014; Cont and Schaanning, 2017; Elliott *et al.*, 2014).

[6]Acemoglu *et al.* (2015) in an interbank network and Fisher (2014) in a cross-ownership model also allow for priority claims.

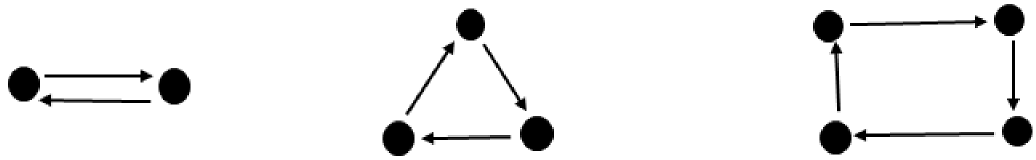

Figure 1: Interbank network example. $N = 9, n^* = 3, n(2) = 1, n(3) = 1, n(4) = 1$.

2 The Interbank Network

An interbank network G has N symmetric banks that correspond to its nodes and a set of directed links. A link ij denotes that bank i has offered a loan to bank j. We focus on networks that consist of a set of distinct circle subnetworks where for each node there is one incoming and one outgoing link. We will refer to each of these subnetworks as a *circle*. The number of banks on each circle in the network can vary, (i.e. not all circles must be the same) and can assume any integer value in the interval $[2, N]$. Let $n(m)$ denote the number of circles with m banks and n^* denote the total number of circles. Then,

$$N = \sum_{m=2}^{N} mn(m),$$

and

$$n^* = \sum_{m=2}^{N} n(m),$$

where for N even $n^* \in [1, \frac{N}{2}]$ and for N odd $n^* \in [1, \frac{N-2}{2}]$.[7] Despite the above restrictions the number of potential networks of size N is very large and can be calculated recursively.[8] In our comparative statics analysis we will consider variations of the network structure across the whole network space. Figure 1 shows a simple network with three circles.

Initial Shocks To simplify the derivation of aggregate outcomes and the comparisons of the effects of various network structures we initially make the following assumptions:

Assumption 1: There is no aggregate uncertainty about initial shocks.[9]

Therefore, we assume that exactly $\phi < n^*$ banks will fail and, thus, the *ex ante* probability that a bank will fail is equal to $\frac{\phi}{N}$.

[7]For N odd the maximum number of possible circles is equal to $\frac{N-1}{2}$ with $\frac{N-2}{2}$ circles of size 2 and 1 circle of size 3.

[8]See Appendix 1 for the derivation.

[9]It is straightforward (although computationally more demanding) to extend the analysis to the case when there is aggregate uncertainty. Under aggregate uncertainty, the number of shocks ϕ is a discrete random variable that can take any integer value in the interval $[0, m]$. By repeating the analysis below for all possible values, multiplying the results for each value with its corresponding relative frequency and summing up we can obtain the corresponding expected values.

Remark 1 *We will show below that even if there is no aggregate uncertainty with respect to initial shocks there is still aggregate uncertainty with respect to final outcomes. Although financial institutions have a number of instruments at their disposal to insure against risks (e.g. credit defaults swaps and other derivatives), for various reasons, including that the providers went bust, these proved inadequate during the 2009 Global Financial Crisis. To keep things simple, we assume that such insurance opportunities are not available. For example, this could be because it is costly to verify bank returns; this would also explain why the outside liabilities of banks are debt contracts and verification is only triggered when the bank becomes insolvent (see, for example, Townsend, 1979).*

Assumption 2: At most one bank in each circle can be hit by a shock. The probability that a circle is hit by a shock is independent of its size.

As long as n^* is much larger relatively to ϕ, this assumption has minimal impact. In fact, we will show below that our results would remain exactly the same if we assume that two banks that belong to the same circle can both be hit by a shock, as long as the distance between them is not too short.

Remark 2 *Our main interest is too understand the implications of network structure for systemic risk. We will show that the amplification of shocks is stronger when they hit larger circles. As we argue below, without Assumption 2, the amplification of shocks would be even stronger. We will also show that an extension of the analysis to the case when there is aggregate uncertainty is straightforward.*

Balance Sheets Given that all banks are symmetric we focus on the balance sheet of a representative bank after the realization of shocks but before the settlement of any obligations to depositors and other banks. The status of the balance sheet will depend on whether or not the bank was hit by a shock. For a bank, which was not hit by a shock, the balance sheet is shown in table 1 below:

Assets	Liabilities
Revenues (R)	Deposits (F)
Bank Loans (D)	Bank Deposits (D)
Non-liquid Assets (K)	Equity (E)

Table 1: Balance sheet.

The assets include the revenues from loans to households and firms, R, loans offered to its debtor bank, D, and other non-liquid assets, K. The liabilities include funding from depositors, F, the value of its obligations to its creditor bank, D, and the value of its equity, E. Given that all banks are symmetric, the value of interbank debt, D, is the same on the two sides of the balance sheet. We assume that in the case of insolvency, depositors and other have a priority claim on the bank's assets. A bank that is not hit by a shock, which also has a solvent creditor bank, is itself solvent: $E = K + R - F > 0$. Initially, we introduce the following assumptions:

Assumption 3: $R - F > 0$

Assumption 4: $D - F > 0$

Assumption 3 implies that a bank not hit by a shock has enough liquidity to cover its obligations to depositors. For a bank that is hit by a shock, without any loss of generality, we set revenues equal to 0. Such a bank will have to sell its tangible assets in the secondary market. Assumption 4 implies that as long the bank loan is fully repaid so will be the depositors. This last assumption is introduced to reduce the number of types of clearing equilibria in the main part of the paper. Below, we will show that none of the main conclusions of the paper depend on any of the above three assumptions. We only introduce them at this stage to reduce the number of cases that we need to consider.

3 Account Settlements, Liquidations and Cascades

In this section, we derive the financial equilibrium of the model; that is for any number of initial shocks we derive the total number of liquidated firms and the final balance sheets after the settlement of all accounts. We begin with the case where fire sales are exogenous, that is, they do not depend on the total number of assets offered for sale. Then we move to analyse the case when fire sales are endogenous. In the last part of the section we show how our results will be affected by (a) changes in the initial distribution of shocks, and (b) changes in the structure of the network.

3.1 Exogenous Fire Sales

Let $L < K$ denote the value of tangible assets in liquidation.

We begin by considering circles that are sufficiently large, where their minimum size will be defined below.

Proposition 1 *Let* $\delta = \frac{R-F+L}{F}$ *and let* v^* *denote the number of additional banks that are liquidated (that is other than the one initially hit by the shock). Let* $Int[x]$ *denote the integer part of a real number* x*. Then, for sufficiently large circles*

(a) if δ *is an integer* $v^* = \delta - 1$*, and*

(b) if δ *is not an integer* $v^* = Int[\delta]$*.*

Proof We will first show that v^* satisfies the *Solvency Condition* (SC)

$$v^*R - (v^*+1)\,F + v^*L < 0 \leqslant (v^*+1)R - (v^*+2)\,F + (v^*+1)\,L. \tag{1}$$

We number the m banks of the circle hit by a shock so that the bank hit by the shock is bank 1, any bank i ($i = 2, .., m$) is a creditor to bank $i-1$ and bank 1 is a creditor to bank m. Suppose that the total number of banks that have already been liquidated, (including the one hit by the shock), is equal to λ and consider the status of bank $\lambda + 1$. Then the partial repayment of bank λ to bank $\lambda + 1$ is given by $D + (\lambda - 1)R + \lambda L - \lambda F$. Given that depositors hold priority claims and given that bank λ is liquidated the repayment must be partial. The supposition that the circle is large implies that then bank m will fully repay its loan to bank 1. Given that all banks from bank 1 to bank λ are liquidated, the repayment to bank $\lambda+1$ must include all the assets of the liquidated banks (the repayment from bank m, all the revenues of the $\lambda-1$ banks not hit by the shock and the liquidation proceeds of all liquidated banks minus the compensation received by all depositors who had deposits at the liquidated funds). Then, the total liquid assets of bank $\lambda+1$ are equal to $D+\lambda R+\lambda L-\lambda F$ and its liabilities are equal to $D+F$. Then, if $\lambda R+\lambda L-(\lambda+1)F<0$, bank $\lambda+1$ will be liquidated. We need to consider two cases:

(a) if $\lambda R+(\lambda+1)L-(\lambda+1)F \geqslant 0$, bank $\lambda+1$ will fully repay bank $\lambda+2$, and therefore the number of banks liquidated is equal to $\lambda+1$. Given that $\lambda R+(\lambda+1)L-(\lambda+1)F < (\lambda+1)R+(\lambda+1)L-(\lambda+2)F$ and letting $v^*=\lambda$ we find that (1) holds.

(b) if $\lambda R+(\lambda+1)L-(\lambda+1)F < 0$, bank $\lambda+1$ will not fully repay bank $\lambda+2$. Repeating the steps above we find that as long as $(\lambda+1)R-(\lambda+2)\,F+(\lambda+1)\,L = (v^*+1)R-(v^*+2)\,F+(v^*+1)\,L \geqslant 0$, bank $\lambda+2$ will not be liquidated and thus (1) holds again.

Then the proof is completed by finding a real number δ such that $\delta R-(\delta+1)F+\delta L=0$. □

Condition (1) holds as long as the circle size is at least v^*+1. If the circle has either exactly v^*+1 banks and the last bank is unable to meet its obligations in full, or it has less than v^*+1 banks the settlements will depend on both v^* and the size of the circle. What changes now is that the bank hit by the shock will also have to write off some of the loan offered to the last bank.

Suppose that $m \leqslant v^*+1$ and $(m-1)R-mF+mL<0$. The second condition means that the last bank is liquidated and cannot meet its obligations. It also means that the value of total assets, excluding the inter-bank obligations, is not sufficiently high to cover the obligations to all depositors. The depositors of the banks that are not hit by a shock will be fully compensated. The depositors of the bank hit by the shock will be partially compensated by receiving $(m-1)R-(m-1)F+mL<F$.

The next proposition summarizes the above results:

Proposition 2 *Suppose that a bank belonging to a circle of size m is hit by a shock. Then the total number of banks in that circle that will be liquidated, $\hat{v}$, is given by*

$$\hat{v} = \begin{matrix} v^*+1 \\ m \end{matrix} \quad for \quad \begin{matrix} m > v^* \\ m \leqslant v^* \end{matrix} \tag{2}$$

Aggregate Outcomes The effect of the ϕ shocks, on the the total number of insolvencies depends on (a) the structure of the network, that is its partition into circles, and (b) the distribution of shocks across these circles. For given values of R, F and L we can use the results above to derive v^*. Let $\hat{n}(m)$ denote the number of circles of size m

that were hit by a shock and let $\hat{N}$ denote the total number of banks in the economy that will be liquidated. Notice that $\sum_{m=2}^{N} \hat{n}(m) = \phi$. Then, the total number of banks in the economy that will be liquidated is given by

$$\hat{N} = \sum_{m=2}^{v^*} m\hat{n}(m) + (v^* + 1) \sum_{m=v^*+1}^{N} \hat{n}(m) \equiv \Psi(v^*). \tag{3}$$

Notice that v^* is non-increasing in L. As liquidation values drop the number of banks being liquidated either stays the same or goes up. In the second case the right-hand side of (3) increases as all banks of even bigger circles get liquidated.

3.2 Endogenous Fire Sales

Up to now, we have treated the liquidation value of assets, L, as an exogenous variable that can take any value in the interval $[0, K]$. In reality this value will depend on the price that these assets will be exchanged in a secondary market. Other things equal, we would expect that the larger the number of banks that are liquidated, the lower will be the equilibrium price of these assets and hence the lower the value of L. To capture this relationship we introduce the liquidation function

$$L = l\left(\hat{N}\right), \tag{4}$$

where $l(0) = K$, $l' < 0$ and $l'' > 0$ and $l(N) = 0$. Of course, the number of banks being liquidated will itself depend on L. With endogenous fire sales, the RHS of (3) is also a function of $\hat{N}$, denoted as $\psi\left(\hat{N}\right)$, given that v^* depends on the liquidation value of assets. Thus, the number of banks that will be liquidated in equilibrium, $\hat{N}^*$, is given by the fixed points of

$$\hat{N} = \psi\left(\hat{N}\right) = \Psi\left(V\left(l\left(\hat{N}\right)\right)\right), \tag{5}$$

where the last expression recognizes that changes in $\hat{N}$ will directly affect L through (4), which in turn will affect v^* through $V(L)$, a step-function that can be derived using the solvency condition (1), which in turn will affect $\psi\left(\hat{N}\right)$ through $\Psi(v^*)$.

The Function $\Psi(v^*)$ We begin the analysis of (5) by taking a closer look at the function $\Psi(v^*)$ given by (3). The table below shows the values of this function for different values of v^*.

When $v^* = 0$, the only banks that get liquidated are those that were hit by a shock. Given that the minimum size of a circle is 2 and given that each circle can be hit by at most one shock, when $v^* = 1$ the number of banks that will be liquidated will be twice the number of initial shocks. For $v^* \geqslant 2$, as long as there are circles of size less than v^* hit by a shock, then the total number of liquidations on such circles is restricted by their size. For example, for $v^* = 3$ all the banks of circles of size less or equal to 3 that were hit by a shock will be liquidated, as shown by the first two terms in the table. In contrast,

v^*	$\Psi(v^*)$
0	ϕ
1	2ϕ
2	$2\hat{n}(2) + 3(\phi - \hat{n}(2))$
3	$2\hat{n}(2) + 3\hat{n}(3) + 4(\phi - \hat{n}(2) - \hat{n}(3))$
$\vdots$	$\vdots$
v^*	$2\hat{n}(2) + \cdots + v^*\hat{n}(v^*) + (v^*+1)(\phi - \hat{n}(2) - \cdots - \hat{n}(v^*))$

Table 2: The function $\Psi(v^*)$

for all circles of size greater or equal than 4 that were hit by a shock exactly 4 banks will be liquidated, as shown by the last term of the table.

By collecting terms we get[10]

$$\Psi(v^*) = (v^* + 1)\phi - \sum_{\gamma=1}^{v^*-1} (v^* - \gamma)\hat{n}(\gamma + 1). \tag{6}$$

Clearly, (6) implies that both the total number of banks that will get liquidated and the size of aggregate losses will depend on the structure of the network.

The Function $V(L)$ Next, we turn our attention to the function $V(L)$. Given that at least ϕ banks will be liquidated, the maximum value of liquidated assets is equal to $l(\phi) < K$. Then, from the discussion of account settlements, we find that if $R - 2F + l(\phi) \geqslant 0$ then $v^* = 0$. Let δ be a real number such that the LHS of the solvency condition (1) is equal to 0, that is $\delta R - (\delta + 1)F + \delta L = 0$. Then,

$$v^* = V(L) = Int(\delta) = Int\left[\frac{F}{R - F + L}\right]. \tag{7}$$

The Solution By substituting (4) in (7) and (7) in (6) we obtain a closed form expression for (5). Depending on the parameter values of the model, we can get two types of solutions shown in Figure 2 and described below.

Proposition 3 *Given a network G and ϕ shocks the number of banks that are liquidated in equilibrium are given by:*

(a) If $R - 2F + l(\phi) \geqslant 0$ then $\hat{N}^* = \phi$; (Type 1: Without-Cascades Equilibrium)

(b) If $R - 2F + l(\phi) < 0$ then there exists $\hat{N}^* > \phi$ such that (5) is satisfied; (Type 2: With-Cascades Equilibrium)

In part (a), the inequality implies that if the only banks liquidated are those hit by the initial shock, the liquidation value of the assets is sufficiently high for $v^* = 0$ (no cascades)

[10] We can write the bottom expression of the table as $(v^* + 1)\phi - (v^* - 1)\hat{n}(2) - (v^* - 2)\hat{n}(3) - \cdots - \hat{n}(v^*)$.

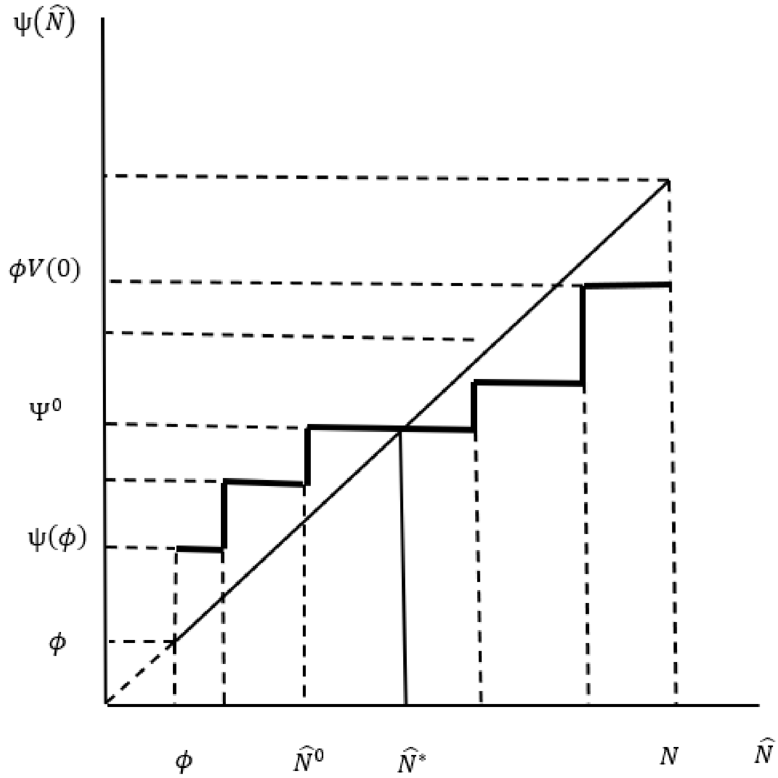

Figure 2: The function $\psi(\hat{N})$

which in turn means that $\hat{N}^* = \phi$. Figure 2 shows the determination of the equilibrium where the 45^0 line stands for the left side of (5) and the bold step function stands for the right side of (5). When $R - 2F + l(\phi) < 0$ then for $\hat{N} = \phi$ we have $v^* > 0$. As the number of liquidated banks increases, the liquidation value of assets drops. However, given that v^* can only take integer values, as long as the second inequality in the solvency condition (1) are satisfied, $V\left(l\left(\hat{N}\right)\right)$ remains stays the same (we move horizontally). Given that $\phi < n^*$ and therefore $\hat{N}^* < N$, the step function will eventually cross the 45^0 line. The crossing (there might be multiple) corresponds to the equilibrium of the model. From Figure 2 we find that when $\hat{N} = \hat{N}^0$, $\psi^0 = \psi\left(\hat{N}^0\right) > \hat{N}^0$. However, $V\left(l\left(\psi\left(\hat{N}^0\right)\right)\right) = V\left(l\left(\hat{N}^*\right)\right)$ which implies that $\psi\left(\hat{N}^0\right) = \hat{N}^*$.

Remark 3 *When the parameters are such that case (b) of Proposition 3 is relevant there might be multiple crossings of the step function with the 45^0 line. In that case, the equilibria can be ranked according to their corresponding liquidation values.*[11] *When there are multiple equilibria our solution above always identifies the 'best' one, that is the one with the smaller number of liquidations (higher liquidation values).*

[11] For a more thorough discussion of multiplicity see Jackson and Pernoud (2022).

3.3 Network Structure and Endogenous Uncertainty

In our model, a state of nature, s, corresponds to a distribution of ϕ shocks across the n^*circles. The total number of such states S is equal to $\binom{n^*}{\phi} = \frac{n^*!}{\phi!(n^*-\phi)!}$. Let $\hat{N}_s$ denote the total number of liquidated banks, L_s denote the liquidation value of assets and v_s^* the size of cascades for state s. We have assumed, for now, that there is no aggregate uncertainty, that is, the number of shocks and their size are known. However, aggregate outcomes are uncertain as they will depend on (a) the distribution of shocks across the network, and (b) on the structure of the network.

Let f^1 and f^2 denote two distributions of shocks that correspond to two states on nature s^1 and s^2. A distribution of shocks specifies the values $\frac{\hat{n}(m)}{\phi}$ for all possible circle sizes m (that is integer values in the interval $[2, N]$), where $\sum_m \frac{\hat{n}(m)}{\phi} = 1$. Consider two networks G^1 and G^2 with the same number of banks N and let g^1 and g^2 denote the two corresponding distributions of circle sizes. A distribution of circle sizes specifies the values $\frac{n(m)}{n^*}$ for all possible circle sizes m (that is integer values in the interval $[2, N]$),where $\sum_m \frac{n(m)}{n^*} = 1$. The following two results highlight the impact of initial shocks on aggregate outcomes.

Proposition 4 *Consider a network G and two distribution of shocks f^1 and f^2 such that f^1 first-order stochastic dominates f^2. Then, $\hat{N}_1 \geqslant \hat{N}_2$, $L_1 \leqslant L_2$ and $v_1^* \geqslant v_2^*$.*

Proof First-order stochastic dominance implies that the average size of those circles hit by shocks is larger under f^1 than f^2. We need to consider three cases:

(a) $\hat{N}_1 = \hat{N}_2$, $L_1 = L_2$ and $v_1^* = v_2^*$. These results obtain when the circles of size less than v_2^*+1 hit by a shock are the same under f^1 and f^2. Then, the number of liquidations are exactly the same under f^1 than f^2.

(b) $\hat{N}_1 > \hat{N}_2$, $L_1 < L_2$ and $v_1^* = v_2^*$. These results obtain when the circles of size less than v_2^*+1 hit by a shock are higher under f^1 than f^2 but the larger number of liquidations under f^1 have not induced a sufficiently high decline in liquidation values to push v_1^* above v_2^*.

(c) $\hat{N}_1 > \hat{N}_2$, $L_1 < L_2$ and $v_1^* > v_2^*$. These results obtain when the circles of size less than $v_2^* + 1$ hit by a shock are higher under f^1 than f^2 and the larger number of liquidations under f^1 have induced a sufficiently high decline in liquidation values to push v_1^* above v_2^*. □

Proposition 5 *Consider two networks G^1 and G^2 with the same number of banks N where g^1 first-order stochastic dominates g^2. Then, the average number of liquidations across states of network G^1 is higher than the corresponding average of network G^2.*

Proof First-order stochastic dominance implies that we can obtain g^1 from g^2 by taking away some banks from smaller circles and placing them in bigger circles. This also implies that the average circle size across states hit by a shock is higher under g^1than g^2. □

	Number of circles					
		FOSD			SOSD	
Circle Size	Ex. 1	Ex. 1A	Ex. 1B	Ex. 1C	Ex. 1D	Ex. 1E
2	6	0	0	0	10	11
3	8	12	0	0	8	8
4	7	7	16	0	1	1
5	3	3	3	3	3	1
6	1	1	1	1	1	1
7	1	1	1	1	1	1
8	1	1	1	9	3	4

Table 3: Network for Example 1 and variant examples. Note that Examples 1A–1C pertain to an exercise for first order stochastic dominance, while Examples 1D–1E pertain to an exercise for second order stochastic dominance. Column heading 'Ex.' is shorthand for example.

The following is a numerical example highlighting the above results.

Example 1 *Consider a network with* 100 *banks and structure described in Table 3. The balance sheet parameters are:* $R = 1$, $F = 0.95$, $K = 0.25$, $D = 0.8$, *and* $\phi = 3$. *The liquidation function is given by* $l\left(\hat{N}\right) = K\left(1 - \sqrt{\frac{\hat{N}}{N}}\right)$. *The results are shown in Figure 3 where we compare the number of liquidations for alternative network structures. Thus, each line shows for each network structure the distribution of liquidations across all states of nature. The total number of states for this example are equal to* $\binom{n^*}{\phi} = \binom{27}{3} = 2,925$. *For these parameter values* $v^* = 4$.

Using Example 1, we study the distribution of the number of defaults, where the frequency measure is in terms of the fraction of all possible states of the world for the network. For instance, for the 2,925 total states, there are 20 states with 6 defaults in aggregate, giving a corresponding fraction of 0.007. The solid black line in Figure 3 corresponds to the network of example 1. We then progressively change the network in a way that first order stochastically dominates. We create a network 1A, which removes the circles of 2 and redistributes the displaced banks to new circles of 3. Example 1B then removes the circles of 3 and redistributes the banks to new circles of 4. Example 1C removes the circles of 4 and redistributes the banks into circles of 8. Note that since $v^* = 4$ in this example, any other distribution of the banks from the circles of 4 from Example 1B would be have identical results to Example 1C. Each new configuration first order stochastically dominates the previous.

As we change the network structure, distributions that first-order dominate have a higher number of average liquidations. For the network of Example 1, we find that the lower number of liquidations, 6, is obtained when all shocks hit circles of size 2 and the higher number of liquidations, 15, is obtained when all shocks hit circles of sizes greater or equal to 5. When all shocks hit circles of size 2, the total number of liquidations cannot exceed 6. As we move to states of nature where shocks hit larger circles, liquidations increase.

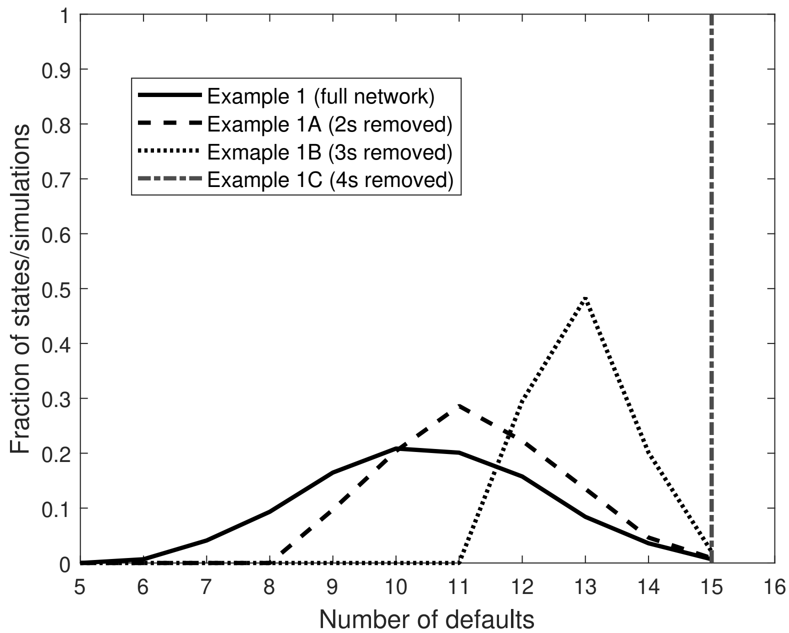

Figure 3: Distribution of aggregate defaults for network 1 and re-configurations that first order stochastically dominate. Example 1A takes the network of example 1 and re-distributes the firms in the circles of 2 to circles of 3. Example 1B does the same with the 3s from Example 1A. Example 1C does the same with the 4s from Example 1B.

Then as we move to Example 1A, we we find that the lower number of liquidations is equal to 9 while the higher number is still 15. The key observation is that as the distribution of circle sizes moves to the right so does the distribution of liquidations across the states of nature. The distribution continues to shift rightwards as we move from Example 1A to Example 1B. Finally for Example 1C, again since $v^* = 4$, it does not matter which circles are hit by shocks. Independent of circle size, there are 5 liquidations in each one of those circles. For the same reason, further shifts of the distribution of circle sizes will not affect the number of liquidations.

Having explored network changes that dominate stochastically in the first order, we now use Example 1 to study networks that dominate in the second order. We re-configure this network to create another with a larger spread and the same mean. Specifically, Example 1D takes banks from circles of size 4 and creates new circles of sizes 2 and 8. Example 1E then takes circles of size 5 from Example 1D and again creates new circles of size 2 and 8. Table 3 shows the exact configurations of these variants. Example 1 dominates Example 1D in the second order, while the latter dominates Example 1E. Figure 4 illustrates the distributions of the number of defaults across the possible states of the world for each configuration.

Figure 4 shows that, although the new networks of Examples 1D and 1E exhibit higher variance, their average number of defaults shifts to the left. This follows since more circles of 2s are created with the re-distributed banks than circles of 8s. Since only one shock

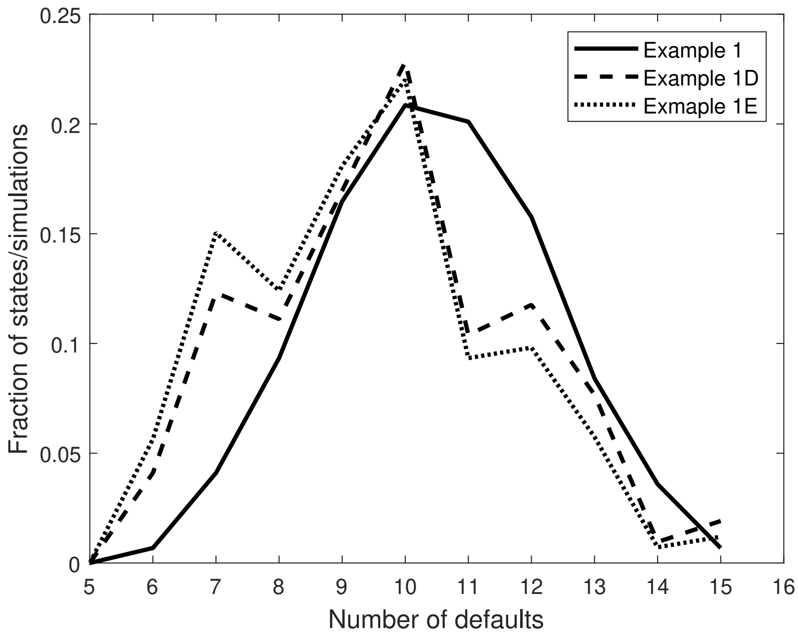

Figure 4: Distribution of aggregate defaults for network 1 and re-configurations that are second-order stochastically dominated.

can hit a given circle, this increases the likelihood of contained liquidation cascades (where circle size is less than or equal to $v^* + 1$).[12] However, more circles at the larger end of the spectrum gives more scope for the maximum number of defaults of 15. The frequency of 15 defaults is higher for both Examples 1D and 1E than Example 1.

Network Structure and the Distribution of Losses Lastly, network structure can affect the distribution of losses between those who hold priority claims (depositors) and those who hold subordinate claims (banks). To see this consider two networks; one network with only very small circles (all sizes less or equal than $v^* + 1$) and the other network with very large circles (all sizes higher than $v^* + 1$). The total number of liquidations in the former network will be less than $\phi(v^* + 1)$ since some of the circles hit will have size less than $v^* + 1$. In contrast, the number of liquidations for the network with the larger circles will be equal to $\phi(v^* + 1)$. In this setting, the total losses are higher in the network with the larger circles but depositors are fully protected (as long as $D + L > F$). Although the total losses are lower for the network with the smaller circles, the depositors of the bank hit by the shock will not be fully compensated. Thus, there is a trade-off between minimising aggregate losses and minimising losses to depositors. Clearly, recognizing the trade-off is only a first step. Designing optimal policies would require a model where the interbank network is endogenously formed so that any effects of policy on the incentives to form interbank links can be internalized.[13]

[12] Again, this assumption is relaxed in the Extensions Section.

[13] See Babus (2016) for a model where a financial network is endogenously formed.

4 Balance Sheet Risk

The uncertainty about revenues implies that all balance sheet items are risky, however, the level of risk varies significantly among them. In addition to the direct risk related to revenues, a bank faces two additional indirect risks on the assets side of its balance sheet. The first is related to the ability of the bank that borrowed from it to repay its loan — this depends on whether that bank is caught in the middle of a chain of bankruptcies. The second risk is related to the value of its non-liquid assets, which can drop below its book value due to fire sales. On the liability side, the commitments of the bank to its creditors are also risky. Depositors face lower risk given that they hold priority claims. Symmetry implies that the risk on the interbank loans on either side of the balance sheet bear the same risk. Equity holders, being residual claimholders, face the highest risk.

For reasons that will become clear below, we begin by analysisng the risk of interbank loans. As a first step, we calculate the probability that a bank survives. Then we decompose the implicit risk between direct idiosyncratic risk due to a shock on revenues and indirect risk due to network effects.

Remark 4 *For the moment, we conduct the analysis under the supposition that the network structure is known but the position of each bank in the network is not known (i.e. the size of the corresponding circle). Later in the paper, we will consider the case when the bank's position in the network is known and also the case when there is uncertainty about the structure of the network.*

4.1 Probability of Survival

As shown above, in sufficiently large circles that were hit by a shock, the last surviving bank will only be partially compensated. From an *ex ante* point of view, this contingency is very unlikely. As such, in this section we focus on an approximate solution, where in such contingencies the last surviving bank is assumed to be fully repaid yet its debtor bank is still liquidated. Given that D appears on both sides of the balance sheet, thus cancelling out, this assumption greatly simplifies the exposition of results. However, the cancellation is not valid anymore when we allow for partial repayments. In the 'Extensions' section, we modify the solution method to allow for partial repayments and show that they indeed have minimal impact on our results.

As indicated by figure 3, there are several states of the world, which are indistinguishable in terms of final outcomes. Here we leverage this point to simplify the construction of the algorithm. What matters is how many of the ϕ shocks hit circles of a given size and not which particular circles were hit. We let μ denote the number of distinct circle sizes.

Lower liquidation values materialise when shocks hit larger circles, leading to more bank failures and higher values of v_s^*. As such, given that circles have at least two banks, if $v_s^* = 0$ or $v_s^* = 1$ for some s, then $v_s^* = 0$ or $v_s^* = 1$, respectively in every s. This follows since the highest possible number of liquidations is either 1 (for $v_s^* = 0$) or 2 (for $v_s^* = 1$) for every circle hit by a shock, bringing the total number of liquidations to either ϕ or 2ϕ, respectively. That is, the total number of liquidations does not depend on the distribution of shocks. In contrast, at higher values, v_s^* can vary from state to state since the number of banks liquidated, and therefore the size of fire sales, will depend on the distribution of shocks.

We use versions of the network in the example below to illustrate our results.

Example 2 *Consider a network with* $N = 21$*,* $n^* = 7$ *and* $\mu = 4$*, where* $n(2) = 3$*,* $n(3) = 2$*,* $n(4) = 1$ *and* $n(5) = 1$*. Further,* $\phi = 3$*. For this example* $S = \binom{n^*}{\phi} = 35$*.*

The following algorithm calculates the probability of survival for $v_s^* \leqslant 3$, $\forall s$. For values of $v^* > 3$ the steps are the same, however, the combinatorics get considerably more complicated as the size of cascades increases. According to the solvency condition (1), this condition on $v^* \leq 3$ implies a restriction on the parameters and liquidation function. Specifically that the inequality $4R - 5F + 4l\,(4\phi) \geqslant 0$ holds; where $l\,(4\phi)$ equals the lowest possible liquidation value of assets (or maximum number of liquidations) that is obtained when the sizes of all the circles hit by a shock are greater than 4.

4.1.1 The Approximate Solution for $v_s^* \leqslant 3$, $\forall s$

Step 1: If $R - 2F + l\,(\phi) \geqslant 0$ then from (1) $v_s^* = 0$, $\forall s$ is the solution and $\hat{N}_s = \phi$ and $L_s = l\,(\phi)$. The probability of survival, $\pi_{v_s^*=0}$, is given by

$$\pi_{v_s^*=0} = \left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*}\left(\sum_{m=2}^{N}\left(\frac{mn(m)}{N}\frac{m-1}{m}\right)\right). \tag{8}$$

With probability $1 - \frac{\phi}{n^*}$ a bank's circle is not hit by a shock. With probability $\frac{\phi}{n^*}$ a bank's circle will be hit by a shock. The probability that a bank belongs to a circle of size m is given by $\frac{mn(m)}{N}$. The probability that a bank belonging to a circle of size m will survive is equal to $\frac{m-1}{m}$.

If $R - 2F + l\,(\phi) < 0$ then go to step 2.

Step 2: If $2R - 3F + 2l\,(2\phi) \geqslant 0$ then $v_s^* = 1$, $\forall s$ is the solution and $\hat{N}_s = 2\phi$ and $L_s = l\,(2\phi)$. The probability of survival is given by

$$\pi_{v_s^*=1} = \left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*}\left(\sum_{m=3}^{N}\left(\frac{mn(m)}{N}\frac{m-2}{m}\right)\right). \tag{9}$$

All banks in circles of size 2 that are hit by shocks are liquidated, as such the summation in the second term starts at $m = 3$. In larger circles hit by shocks, there are two banks that are liquidated.

If $2R - 3F + 2l\,(2\phi) < 0$ then go to step 3.

Step 3: Set $v_s^* = 2$ $\forall s$. Then from Table 2 we have $\hat{N}_s = 2\hat{n}(2) + 3(\phi - \hat{n}(2))$ and $L_s = l\,(2\hat{n}(2) + 3(\phi - \hat{n}(2)))$. States that have the same number of size 2 circles being hit by the initial idiosyncratic shocks, are indistinguishable in terms of final outcomes.[14] We

[14]The reason is that in all other circles hit by a shock there will be 3 liquidated firms, independently, of circle size.

$s^I = \hat{n}(2)$	S^I	$\hat{N}_s$
0	$1 \times \binom{n^*-n(2)}{\phi}$	3ϕ
1	$n(2) \times \binom{n^*-n(2)}{\phi-1}$	$2+3(\phi-1)$
2	$\binom{n(2)}{2} \times \binom{n^*-n(2)}{\phi-2}$	$4+3(\phi-2)$
$\vdots$	$\vdots$	$\vdots$
ϕ	$\binom{n(2)}{\phi} \times 1$	2ϕ

Table 4: States for $v_s^* = 2$.

use Table 4 that shows the number of indistinguishable states, S^I, and the total number of liquidated banks.[15]

For instance if $\hat{n}(2) = 2$, there are $\binom{n(2)}{2}$ ways to allocate 2 shocks among $n(2)$ circles of size 2 and $\binom{n^*-n(2)}{\phi-2}$ ways to allocate the remaining $\phi - 2$ shocks among the remaining $n^* - n(2)$ circles. There are 4 banks that will be liquidated because they belong to the two circles of size 2 that were hit by shocks and for each of the remaining circles hit by shocks there will be 3 banks liquidated.

Example 3 *For the network of Example 2 for* $\hat{n}(2) = (0, 1, 2, 3)$ *the number of corresponding states are equal to* $(4, 18, 12, 1)$ *which add up to* 35. *For* $\hat{n}(2) = 2$*, there are* 4 *banks that will be liquidated, since they belong to the two circles of size* 2 *hit by shocks. For each of the remaining circles hit by shocks there will be* 3 *banks liquidated.*

Notice that, even now if $v_s^* = 2$ for some s, there might be other states with higher values of v_s^*. This follows since, when shocks hit larger circles, more liquidations take place, which depresses liquidation values. This has the potential to increase v_s^* further. If $3R - 4F + 3l(3\phi) \geqslant 0$, then (given that $\hat{N}_s \leqslant 3\phi$), $v_s^* = 2$, $\forall s$ is the solution. The probability of survival is given by

$$\pi_{v_s^*=2} = \left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*}\left(\sum_{m=4}^{N}\left(\frac{mn(m)}{N}\frac{m-3}{m}\right)\right). \tag{9}$$

If $3R - 4F + 3l(3\phi) < 0$, there exist states such that (1) is not satisfied for $v_s^* = 2$.[16] Then go to step 4.

Step 4: Given that there will be some states with $v_s^* = 3$ the number of indistinguishable states will increase and will depend on both $\hat{n}(2)$ and $\hat{n}(3)$. Then, for those states where $v_s^* = 3$, according to Table 2, we have $\hat{N}_s = 2\hat{n}(2) + 3\hat{n}(3) - 4(\phi - \hat{n}(2) - \hat{n}(3))$ and $L_s = l(2\hat{n}(2) + 3\hat{n}(3) - 4(\phi - \hat{n}(2) - \hat{n}(3)))$. The new set of indistinguishable states along with their corresponding numbers and the corresponding number of liquidated banks are shown in Table 5.

[15] Clearly, if $n(2) < \phi$ then all entries for $\hat{n}(2) > n(2)$ will be equal to 0.

[16] This is the case if the number of circles of size greater than 2 is higher than ϕ. If that is not the case then we move to step 4 if there is some state such that $2R - 3W + l(2\hat{n}(2) + 3(\phi - \hat{n}(2))) < 0$.

$s^I = (\hat{n}(2), \hat{n}(3))$	S^I	$\hat{N}_s$
$0 = (0,0)$	$1 \times 1 \times \binom{n^* - n(2)}{\phi}$	4ϕ
$1 = (0,1)$	$1 \times n(3) \times \binom{n^* - n(2) - n(3)}{\phi - 1}$	$3 + 4(\phi - 1)$
$2 = (1,0)$	$n(2) \times 1 \times \binom{n^* - n(2) - n(3)}{\phi - 1}$	$2 + 4(\phi - 1)$
$3 = (0,2)$	$1 \times \binom{n(3)}{2} \times \binom{n^* - n(2) - n(3)}{\phi - 2}$	$6 + 4(\phi - 2)$
$4 = (1,1)$	$n(2) \times n(3) \times \binom{n^* - n(2) - n(3)}{\phi - 2}$	$5 + 4(\phi - 2)$
$5 = (2,0)$	$\binom{n(2)}{2} \times 1 \times \binom{n^* - n(2) - n(3)}{\phi - 2}$	$4 + 4(\phi - 2)$
$6 = (1,2)$	$\binom{n(2)}{1} \times 1 \times \binom{n^* - n(2) - n(3)}{\phi - 2}$	$8 + 4(\phi - 3)$
$\vdots$	$\vdots$	$\vdots$
$s' = (\hat{n}(2), \hat{n}(3))$	$\binom{n(2)}{\hat{n}(2)} \times \binom{n(3)}{\hat{n}(3)} \times \binom{n^* - n(2) - n(3)}{\phi - \hat{n}(2) - \hat{n}(3)}$	$2\hat{n}(2) + 3\hat{n}(3) + 4(\phi - \hat{n}(2) - \hat{n}(3))$
$\vdots$	$\vdots$	$\vdots$
$\bar{s} = (\phi, 0)$	$\binom{n(2)}{\phi} \times 1 \times 1$	2ϕ

Table 5: States for $v_s^* \leqslant 3$.

The number of total composite states $\bar{s} + 1$, is given by

$$\bar{s} + 1 = \sum_{z=0}^{\phi} (z + 1) = \frac{(\phi + 1)(\phi + 2)}{2}$$

There are $z + 1$ ways to allocate z shocks among the size 2 and size 3 circles, where $0 \leqslant z \leqslant \phi$.[17] The above table lists the states in descending order of the total number of liquidated banks. Nevertheless, there are still some states with the same number of liquidated banks (e.g. 5 and 6).

Example 4 *For the network of Example 2, the corresponding states are shown in Table 6.*

Given that $l\left(\hat{N}_s\right)$ is decreasing in $\hat{N}_s$, there exists a state say, γ, such that for all $s < \gamma$, $v_s^* = 3$ and for all $s \geqslant \gamma, v_s^* = 2$ (assuming descending order of liquidations by state as in Table 5). The probability of survival is given by

$$\pi_{v_s^* \in \{2,3\}} = \left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*} \left(\sum_{s=0}^{\gamma} p_s \sum_{m=5}^{N} \left(\frac{mn(m)}{N} \frac{m-4}{m} \right) + \sum_{s=\gamma+1}^{\bar{s}} p_s \sum_{m=4}^{N} \left(\frac{mn(m)}{N} \frac{m-3}{m} \right) \right) \tag{10}$$

where $p_s = \frac{S^I}{\binom{n^*}{\phi}}$, that is equal to the number of indistinguishable states divided by the total number of states. The first term in the first bracket is equal to the probability of belonging to a circle not hit by a shock. The first double summation in the second bracket

[17] Once more, this is correct as long as $n(2)$, $n(3)$ and the number of circles greater than 3 are all greater or equal to ϕ. If that is not the case some states will not exist.

$s = (\hat{n}(2), \hat{n}(3))$	S_s^I	$\hat{N}_s$
$0 = (0, 1)$	2	11
$1 = (1, 0)$	3	10
$2 = (0, 2)$	2	10
$3 = (1, 1)$	12	9
$4 = (2, 0)$	6	8
$5 = (1, 2)$	3	8
$6 = (2, 1)$	6	7
$7 = (3, 0)$	1	6

Table 6: States for $v_s^* \leqslant 3$ for Example 2

captures the likelihood of surviving given that the circle is hit by a shock and $v_s^* = 3$, while the second is for $v_s^* = 2$. When $v_s^* = 3$ all banks in circles of size less than or equal to 4 are liquidated. Thus, to survive a bank must belong to a circle of size greater than or equal to 5 and at a distance of at least 4 from the bank hit by the shock. When $v_s^* = 2$ all banks in circles of size less of equal to 3 are liquidated. So to survive, a bank must belong to a circle of size greater or equal to 4 and at a distance of at least 3 from the bank hit by the shock. Notice that the limit of the summation is N to allow for the possibility where the network has only one circle, in which case $\phi = 1$.

4.2 Survival, Idiosyncratic Risk and Network Risk

Let π_i, where $i \in \{v_s^* = 0, v_s^* = 1, v_s^* = 2, v_s^* \in \{2, 3\}\}$, denote the probability of survival. $\pi_{v_s^*=0}$ is equal to the probability of survival when there are no cascades, which is equal to the probability that the bank is not hit by a shock. Then, the implicit gross idiosyncratic risk premium on revenues ρ^R is given by

$$\rho^R = \frac{1}{\pi_{v_s^*=0}}. \tag{11}$$

The probability that the bank will be liquidated because of network effects, that is, it will be liquidated when is not hit by a shock is equal to $\pi_{v_s^*=0} - \pi_i$. Note that this difference is positive since longer cascades imply a lower chance of survival. The corresponding implicit gross risk premium, ρ^D, is given by

$$\rho^D = \frac{1}{1 - \pi_{v_s^*=0} + \pi_i}. \tag{12}$$

The last expression is a pricing formula for interbank loans that includes both the risk of the counterparty being hit by a shock and the risk of the counterparty being liquidated because another bank down the chain was liquidated.

Remark 5 *By referring to indirect risk as network risk, rather than as counterparty risk, we make a distinction between cases where borrowers might also default because their own borrowers defaulted in addition to defaults cause by direct shocks.*

4.3 Risk Premia for Liabilities

There are three types of creditors: depositors, banks and equityholders. Depositors hold the safest claim given that they have priority. Depositors always get fully compensated, as long as their bank belongs to a circle of size greater than $v^* + 1$. The only depositors that will not be fully compesated are those whose banks were hit by a shock and belong to circles of sizes less or equal to $v^* + 1$. We can derive an upper bound for the risk premium of deposits that corresponds to the case where the depositors of banks hit by shocks do not receive any compensation. Using (10) we can derive the probabilty that depositors will receive full compensation for the case when $v_s^* \in \{2, 3\}$, as

$$\pi_F = \left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*} \left(\begin{array}{c} \sum_{s=0}^{\gamma} p_s \left(\sum_{m=5}^{N} \frac{mn(m)}{N} + \sum_{m=2}^{4} \frac{mn(m)}{N} \frac{m-1}{m} \right) + \\ \sum_{s=\gamma+1}^{\bar{s}} p_s \left(\sum_{m=4}^{N} \frac{mn(m)}{N} + \sum_{m=2}^{3} \frac{mn(m)}{N} \frac{m-1}{m} \right) \end{array} \right) \tag{13}$$

This says that depositors receive full compensation as long as (a) their circle is not hit by a shock, or (b) their circle is hit by a shock but the circle is sufficiently large (greater than $v^* + 1$), or (c) their circle is hit by a shock, their circle is small but their bank is not the one hit by the shock. Then, an upper bound for the deposit gross risk premium, ρ^F is given by

$$\rho^F \leqslant \frac{1}{\pi_F}. \tag{14}$$

This is an upper bound since even depositors at banks belonging to small circles that were hit by a shock would still receive some compensation.

Given symmetry the implicit gross risk premium for the bank deposit (interbank loan) is equal to ρ^D.

Lastly, at the approximate solution, the bank is always fully compensated as long as it survives. Then, the equity risk premium, ρ^E, is given by

$$\rho^E = \frac{1}{\pi^i}, \tag{15}$$

where $i \in \{v_s^* = 0, v_s^* = 1, v_s^* = 2, v_s^* \in \{2, 3\}\}$. Notice, that $\rho^E > \rho^R$, so the overall risk premium on equity is higher than the risk premium due to idiosyncratic risk, the difference capturing the additional network risk.

4.4 Equity Risk Premium Puzzle

Around four decades ago, Mehra and Prescott (1985) observed that the risk premium on U.S. equities exceeds by an order of magnitude what should be expected the premium to be from neo-classical finance. The article planted the seeds for an extensive literature offering various solutions to the puzzle.[18] In our model, risk premia respond to two sources

[18] For a useful review the reader is referred to Mehra (2006).

of risk. One is common idiosyncratic risk, capturing any risk arising within an entity (in our case revenues from loans granted to housolds and firms). The second type of risk is due to network effects arising because of potential losses resulting from the liquidation of other entities belonging to the same network. While all liability holders are exposed to this second type of risk, those who hold priority claims are less exposed.

In our model, the gross equity risk premium in the absence of cascades is given by the premium due to idiosyncratic shocks, ρ^R. The corresponding risk premium on priority claims is equal to 1 (in the absence of cascades these claims are fully repaid). The corresponding overall risk premia that also include the risk due to cascades are given by ρ^E and ρ^F. The latter, especially if we allow for partial repayments, is very close to 1. In contrast, ρ^E is significantly higher than ρ^R. Thus, the difference between equity and priority claims, the risk premium, increases significantly when we account for a premium for systemic risk.

4.5 Network Structure and Risk Premia[19]

Above we derived the risk premia associated with all items on a bank's balance sheet for a given network structure. In this section, we examine how changes in the network structure affect these risk premia. We consider the same Example 1 and ensuing configurations that first order stochastically dominate as before (see table 3). From Proposition 5, we know that the expected number of liquidations will increase because now the shocks hit on average larger circles. Therefore, the risk premia will go up.

	Ex. 1	Ex. 1A	Ex. 1B	Ex. 1C
$\pi_{v^*=0}$	0.970	0.970	0.970	0.970
$\pi_{v^*=4}$	0.896	0.887	0.872	0.850
ρ_R	1.031	1.031	1.031	1.031
ρ_D	1.080	1.090	1.109	1.136
ρ_E	1.117	1.127	1.147	1.176

Table 7: Probabilities of survival (top rows) and balance sheet risk (bottom rows) for examples 1, 1A, 1B and 1C. Abbreviation 'Ex.' is shorthand for example. Recall that example 1 pertains to the full network, while example 1A removes circles of 2s, example 1B removes circles of 3s and example 1C removes circles of 4s.

Table 7 displays survival probabilities and the corresponding balance sheet risk premia. Recall that, in Example 1 and Examples 1A–1C, $v^* = 4$. As the smaller circles are removed, the likelihood of being involved in a cascade of defaults rises. Notice the non-linearities in these numbers as the removed circles increase in size. The probability of survival $\pi_{v^*=4}$ decreases at an increasing rate, falling by 0.9% as the 2s are removed, 1.5% as the 3s are

[19] In the above derivations there is an implicit assumption that all agents know the network structure. However, it is more likely that there is uncertainty about the overall structure of the network. In the Appendix we show how to derive the size of the network space. As long as agents know the distribution of possible networks then risk premia can be calculated by taking the expected value of all the risk premia that correspond to each network structure.

removed and 2.2% as the 4s are removed. This translates into an equity risk premium that increases at an increasing rate — rising by 1%, 2% and nearly 3% when removing the 2s, 3s and 4s respectively.

5 Application: Production Networks

As was mentioned in the Introduction, our model can be applied to both financial and production networks alike — nothing in the setup pertains exclusively to the former. In particular, by renaming the items on the balance sheet shown in Table 1, we can directly apply our results to production networks. In this context, R denotes accounts receivable from final consumers, D on the asset side becomes trade credit provided to other firms (input buyers), D on the liability side becomes trade credit received from other firms (input suppliers) and F becomes obligations to banks and bondholders that hold priority claims.

Our model predicts that a network with more interconnected firms has higher equity risk premia than a network with firms operating in isolation. An interpretation of this result, which allows for a mapping to the data, is that firms in sectors that are more central face higher such risks. In this section, we leverage firm-level equities data and sectoral input-output information to provide supporting evidence that firms in more connected sectors have higher equity risk premia. We take the stance that equity betas, pertaining to the CAPM model, give a reasonable approximation to the systematic risk faced by shareholders and hence can be used to estimate the corresponding risk premium.

We begin by identifying centrality using input-use tables from the Bureau of Economic Analysis (BEA), which classify production into 14 non-governmental sectors. We use these data for the year 2019 — the last before the instability induced by the COVID pandemic. For each sector i (buyer), we then calculate the share of its intermediates expenditure on goods from sector j (seller). Then for each selling sector j, we take the average expenditure share of all the industries i who buy its products. A firm is then classified as central if it has a high average expenditure share — its output constitutes a large share of its buyers' input use on average.

We then obtain estimates of equity betas from Damodaran (2022). These data estimate firm-level levered betas, using a sample period of January 2017–January 2022, which are averaged across firms within different industries. We then average these beta estimates across all industries that fall under a given NAICS code classification, consistent with the BEA tables. We then provide predicted risk premia for the equities of a particular sector by utilising this beta estimate, in conjunction with data on the market return and riskless rate. For the market return, we find the average annual return on the S&P500 index over the period since the end of the global financial crisis (January 01 2009 onwards). For the riskless rate, we use the 1 year t-bill rate over the same period. The former is found to be 11.1%, while the latter is 0.735%, giving a market risk premium estimate of 10.4%.

Table 8 provides the results of this exercise. Three predominant sectors stand-out with regard to centrality — professional services, the financial and manufacturing sectors — each with average buyer input shares above 20%. Several sectors are spread over the buyer share range of 2%–7%, while four sectors fall below 2%. The estimated betas vary strongly across sectors; wholesale trade assumes the maximum value of 1.4. Other services takes the lowest beta estimate of 0.9 — we use the ensuing estimate of the equity risk premium as a benchmark for the other sectors. Specifically, we present the predicted risk premia of all sectors as percentage point differences from that of other services for ease of interpretability.

Sector	**Centrality**	**Beta**	**Premium**
Professional & business services	23.76	1.2917	4.004
Finance, insurance, real estate, rental & leasing	21.12	1.0690	1.698
Manufacturing	20.17	1.1486	2.522
Transportation & warehousing	6.810	1.1100	2.123
Wholesale trade	5.910	1.4000	5.126
Information	4.590	1.1767	2.813
Utilities	3.350	1.0000	0.984
Agriculture, forestry fishing & hunting	3.170	1.0300	1.294
Mining	3.050	1.2700	3.780
Arts, entertainment, recreation, accommodation & food services	2.730	1.3433	4.534
Other services	1.830	0.9050	0.000
Retail trade	1.820	1.1438	2.472
Construction	1.190	1.0600	1.605
Educational services, health care & social assistance	0.500	1.1300	2.323

Table 8: Sectors by centrality measure, average equity beta and predicted equity premium. Centrality is the average expenditure share of all sectors on goods/services from this particular sector for the year 2009. The beta column is the equity beta averaged over those reported by Damodaran (2022) that pertain to a particular 2 digit NAICS code. The premium column gives the predicted premium, which utilises the equity beta and estimate of market risk premium over 2009-2022. The number presented is the premium for the sector in question less the premium for the other services sector (that with the smallest predicted premium).

The average beta for the top 7 most central sectors is estimated to be 1.17, while that for the bottom 7 is 1.13. Using the 10.4% value for the market premium, this translates into an expected equity risk premium for the top sectors, which is 0.47% higher than that of those at the bottom. We undertake a similar set of calculations for the top 3 sectors, who all have average expenditure shares above 20%, to the bottom 4 sectors, whose shares are below 2%. The average beta for the top sectors is 1.17, while that for the bottom sectors is 1.06, giving an expected equity premium that is 1.15% higher for the former group than the latter. Taken together, these results give some suggestive evidence consistent with the idea that the shareholders of firms in sectors that are more connected face higher risk than those less connected.

6 Extensions

6.1 $D < F$

Now we allow for the obligation to the supplier to be less than the obligation to the depositors. When the bank that is hit by a shock cannot meet fully its obligations, its creditor's compensation will be equal to $\max\{0, D + L - F\}$. We will first show that if $D + L - F > 0$ then the analysis above is still valid. Then, we will show the changes that we need to make when $D + L - F < 0$.

Suppose that $D + L - F > 0$, meaning that the creditor bank receives a payment. Clearly, if the creditor bank can fully meet its obligations to its depositors and its creditor, the analysis remains the same. If it cannot, then its creditor will be given $R+D-2F+2L > 0$, where the inequality follows from $D+L-F > 0$ and $R > F$. It follows that all depositors in subsequent rounds will also be fully compensated and thus the analysis is still the same.[20]

Next, suppose that $D + L - F < 0$. Now the creditor of the bank hit by the shock receives nothing. Then the creditor's assets assets are equal to $R+L$. As long as $R \geqslant D+F$, it will fully meet its obligations. If $R < D + F$ then the bank will be liquidated but it will still fully meet its obligations as long as $R+L \geqslant D+F$. If, in contrast, $R+L-D-F < 0$ its own creditor bank will receive $R+L-F$. The assets of this creditor bank are equal to $2R + 2L - F$. Then as long as $2R + L - F \geqslant D + F$, it will fully meet its obligations. If $2R + L - F < D + F$, then it will be liquidated but will still fully meet its obligations as long as $2R+2L-F \geqslant D+F$. If, in contrast, $2R+2L-D-2F < 0$, its own creditor bank will receive $2R+2L-2F$. By induction we conclude that the number of additional banks that are liquidated, v^*, (that is other than the one initially hit by the shock) satisfies the new *Solvency Condition* (SC)

$$v^*R - v^*F - D + v^*L < 0 \leqslant (v^* + 1)\,R - (v^* + 1)\,F - D + (v^* + 1)\,L. \qquad (1')$$

Once more, this condition holds as long as the size of the circle is at least $v^* + 1$.

6.2 Multiple Shocks

One of the aims of this paper is to demonstrate how, for a given network structure, the distribution of shocks across the network is an important determinant of the number of liquidations in equilibrium and thus the level of aggregate losses. In the benchmark case, we have assumed that each circle can be hit by at most one shock. It was briefly mentioned that, while this assumption simplified a lot of the account settlements analysis, it has also weakened our results. In this section, we allow for multiple shocks to hit a given circle and show that this can strengthen the impact of the distribution of shocks on aggregate outcomes.

The introduction of multiple shocks enlarges the state space. Now the total number of states is equal to $\binom{N}{\phi} = \frac{N!}{\phi!(N-\phi)!}$. We are going to focus on the case where $D < F$. The implication of this restriction is that the creditor of a bank hit by a shock, will not receive

[20] For circles with size smaller than $v^* + 1$ the analysis also remains the same given that all creditor banks do not receive any compensation.

any compensation, even if the latter bank's debtor survives. In this case, the assets of the bank hit by the shock are equal to D, which are not sufficient to cover the claims of depositors.[21]

Next consider what happens for a fixed value of v^* when a circle is hit by a second shock. For circles of size less than $v^* + 1$, all banks get liquidated when even only one of them is hit by a shock. A second shock will not affect the number of liquidated banks, but it will have distributional effects. In contrast, when a circle of size greater than $v^* + 1$ is hit by a shock then the number of banks that are liquidated will depend on both the exact size of the circle and the position of the two banks on the circle. The assumption $D < F$ implies that the number of banks liquidated after a shock hits, is independent of what happened to the shocked bank's creditor (i.e. whether the creditor was hit by a shock or was partially or fully compensated). Clearly, the number of banks that will be liquidated will be between $v^* + 2$ (this will be the case when the two banks hit by shocks are next to each other) and $2(v^* + 1)$ (when the shortest path between them is at least $v^* + 1$.

The above discussion implies that, when we introduce multiple shocks, the distribution of the number of liquidated banks across states is affected in two ways. Firstly its variance increases since there are more ways to have shocks concentrated on small size circles, as there are similarly for large circles. Secondly, this variance effect is asymmetric, as the upper bound of liquidations has now increased. Furthermore, so far we have assumed that the value of v^* has not been affected with this extension. However, as the losses due to fire sales increase with the number of liquidations, v^* can be even higher in those states where there are multiple shocks in large circles. In summary, allowing for multiple shocks exacerbates the impact of network effects on systemic risk and aggregate outcomes.

6.3 Risk Premium for D under Partial Compensation

In the benchmark case, we have assumed that all banks that are not liquidated have the interbank loans that they offered to other banks fully repaid. However, in circles where $m > z^* + 1$, the first bank to survive ($z^* + 1$ places down from the bank hit by the shock) will only be partially compensated. The partial compensation will be state dependent and we denote it as $D^s < D$. We will consider the more complicated case where $v_s^* \in \{2, 3\}$. Let $E(D^P)$ denote the expected value of compensation to be received from the debtor bank when we allow for partial compensation. Then, we have

$$E(D^P) = \left(\left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*} \left(\begin{array}{c} \sum_{s=0}^{\gamma} p_s \sum_{m=6}^{N} \left(\frac{mn(m)}{N} \frac{m-5}{m} \right) + \\ \sum_{s=\gamma+1}^{\bar{s}} p_s \sum_{m=5}^{N} \left(\frac{mn(m)}{N} \frac{m-4}{m} \right) \end{array} \right) \right) D$$

[21] For the case when $D > F$, the analysis is more complicated. However it will become clear that, in that case, our conclusions about the impact of the shock distribution on aggregate outcomes would be even stronger.

$$+ \quad \frac{\phi}{n^*}\left(\sum_{s=0}^{\gamma} p_s \sum_{m=5}^{N} \frac{mn(m)}{N}\frac{1}{m}D_s + \sum_{s=\gamma+1}^{\bar{s}} p_s \sum_{m=4}^{N} \frac{mn(m)}{N}\frac{1}{m}D_s\right)$$

The first term on the right-hand side is equal to the probability that the bank will survive conditional on receiving full compensation times the value of the claim. The term $1 - \frac{\phi}{n^*}$ is equal to the probability of survival conditional that the circle where the bank belongs is not hit by a shock. The next term in brackets is equal to the probability of survival conditional on (a) that the circle where the bank belongs is hit by a shock, and (b) the bank is fully compensated. In (10), the second condition was absent given that all banks were assumed to be fully compensated. The first double summation is over those states where $v^* = 3$. Then, banks whose circles where hit by a shock and were fully compensated by their buyers must belong to circles of size gretaed than 6.[22] With probability $\frac{mn(m)}{N}$, the bank belongs to a circle of size m and given that 4 banks are liquidated and one survives but is only partially compensated, the probability that the bank survives and is fully compensated is equal to $\frac{m-5}{m}$. The second double summation is over those states where $v^* = 2$ and is derived in a similar way.

The second term on the right hand side is equal to the expected value of partial compensation. The expressions are similarly derived but there are two differences. In circles of size equal to $v^* + 2$ that were hit by a shock there is one surviving fim that is partially compensated. For this reason the counter has decreased by 1. The second is that in circles of size greater than $v^* + 2$ there is only one bank that will receive partial compensation and this accounts for the term $\frac{1}{m}$.

Under partial compensation, the implicit risk premium on inter-bank liabilities, ϱ^{D^P}, is given by

$$\varrho^{D^P} = \frac{D}{E(D^P)}.$$

6.4 The Bank's Position in the Network is Known

In the main text, we have assumed that agents know the structure of the network but do not know the exact position (circle size) of their bank in the network. Let m^* denote the size of the circle that a given bank belongs to. If $m^* \leqslant v_s^* + 1$ then the bank will only survive if its circle is not hit by a shock, which happens with probability $1 - \frac{\phi}{n^*}$. If $m^* > v_s^* + 1$ the probability that the bank survives is now given by:[23]

$$\pi_{v_s^*} = \left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*}\frac{m^* - (v_s^* + 1)}{m^*}.$$

In the case when the bank's position was unknown we had to take the expectation of the last term over all possible circle sizes. Once more we can use the probabilities of survival to calculate the corresponding risk premia.

[22] In circles of size 5 only one firm survives but is only partially compensated.

[23] Strictly speaking this is correct if the value of v_s^* does not depend on the state of nature. If it does then we need to adjust the formula as we did in the derivation of (10) above.

7 Concluding Comments

Assessing systemic risk when the entities of the system are connected together by credit obligations is a very complex task for financial regulators, as evident from the 2009 Global Financial Crisis. There is a comprehensive body of work in the finance literature that develops methods for clearing a financial system following some initial shocks. There is much less work on developing methods to assess *ex ante* underlying risks. Those works that do exist in this area, have covered only the case where asset valuation is independent of the extent of the crisis (i.e. the number of institutions in trouble). Our work shows that modelling endogenous asset values, that depend on market conditions, is crucial as an amplification mechanism for the initial idiosyncratic shocks. Accounting for this feature can yield radically different inferences regarding aggregate outcomes from a framework where these losses are exogenous. Moreover, an accurate evaluation of risk for a set of liabilities requires a careful analysis of seniority, a further innovative feature of our framework.

We presented a stylized financial network to highlight the complexities that arise and need to be addressed in order for regulators to make proper risk assessments. This simple framework delivers insights regarding the impact of systemic risk on market participants and also corroborates results previously found using numerical methods. We find that small changes either in the distribution of shocks or in structure of the network can have large effects on aggregate losses. We also find, somewhat counter-intuitively, that network structures with lower expected aggregate losses, can actually have higher losses for those holding seniority claims. We then used a clearing algorithm to derive survival probabilities for each bank in the network for arbitrary distributions of initial shocks; we then used these probabilities to derive risk premia for each item of the balance sheet. Our framework allows for the decomposition of these premia into a part reflecting idiosyncratic risk and a part capturing network contagion. Finally, we showed that any gap between the equity risk premia and the risk premia on other liabilities is further magnified by network effects.

Given that our framework is stylised, it has some limitations. Firstly, we do not account for multiple creditors; a more connected system offers more paths for contagion to spread but can also offer opportunities for diversification by spreading the losses among multiple creditors.[24] Secondly, our model uses a financial equilibrium approach, where we have taken balance sheets and hence the network structure as fixed. While this methodology can be useful for a regulator who would like to know the underlying risks of a networked system, it might not be entirely appropriate for the design of policies. As Beale *et al.* (2011) stress, regulators also need to take into consideration how a new policy might alter the incentives of participants to form links.[25] In spite of these caveats, our study paves the way for understanding network risk in a variety of contexts. We leveraged firm and sector level data for an application to general production networks. A future potential application could be to understanding risk across multinational supply chains.[26] It is hoped that our

[24]See Elliott *et al.* (2014) for the trade-off between integration and diversification and Battiston *et al.* (2012) for the limits of diversification.

[25]For endogenous formation see Acemoglu and Azar (2020) for production networks and Babus (2016) for financial networks.

[26]For a recent example of a model exploring the impact of financial frictions on multinational firm

work will drive further awareness of the importance of accounting for endogenous fire sales, when directing both regulatory inferences and academic research more generally.

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Appendix

Network Space[27]

We begin by calculating the number of possible networks for a given number of circles, n^*. This problem is equivalent to finding the number of ways of allocating N unlabeled

[27]This is a modified version of the procedure described in https://janmr.com/blog/2008/12/twelve-ways-of-counting/

balls into n^* unlabeled urns. The allocation problem is equivalent to the number of ways of writing the integer N as the sum of n^* positive integers each integer greater or equal to 2. The problem reduces to finding the number of ways of allocating $N - n^*$ balls into n^* urns and then add 1 ball in each urn. The arrangements we get before we add the extra balls are called partitions of $N - n^*$ into n^* parts. Let

$$\left| \begin{matrix} N - n^* \\ n^* \end{matrix} \right|$$

denote the number of partitions. We have

$$\left| \begin{matrix} \alpha \\ \alpha \end{matrix} \right| = \left| \begin{matrix} \alpha \\ 1 \end{matrix} \right| = 1, \alpha \geqslant 1 \text{ and } \left| \begin{matrix} \beta \\ \alpha \end{matrix} \right| = 0, \alpha > \beta > 0,$$

and the boundary conditions

$$\left| \begin{matrix} 0 \\ 0 \end{matrix} \right| = 1 \text{ and } \left| \begin{matrix} \alpha \\ 0 \end{matrix} \right| = 0, \alpha > 1.$$

For the general case of partitioning α into β parts, we split the partitions into those that have at least one 1 among the parts and those where each part is greater than 1. The first group of partitions is obtained by including all partitions of $n - 1$ into $m - 1$ parts, and then add 1 in the empty arrangement, and the second group of partitions is obtained by including all partitions of $n - m$ into m parts, where 1 could be added to each part. We then have

$$\left| \begin{matrix} N - n^* \\ n^* \end{matrix} \right| = \left| \begin{matrix} N - n^* - 1 \\ n^* - 1 \end{matrix} \right| + \left| \begin{matrix} N - 2n^* \\ n^* \end{matrix} \right|.$$

By repeating this procedure we will end up with a sum of 0s and 1s.

The above calculation derives the number of possible networks when the number of circles are restricted to be equal to n^*. Then, the number of all possible networks is given by

$$\sum_{n^*=1}^{\frac{N}{2}} \left| \begin{matrix} N - n^* \\ n^* \end{matrix} \right| \text{ for } N \text{ even and } \sum_{n^*=1}^{\frac{N-2}{2}} \left| \begin{matrix} N - n^* \\ n^* \end{matrix} \right| \text{ for } N \text{ odd.}$$