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Abstract

We model competition on a credence market governed by an imperfect label, signaling high quality, as a rank-order tournament between firms. In this market interaction, asymmetric firms jointly and competitively control the underlying quality ranking's precision by releasing individual information. While the labels and the information they are based on can be seen as a public good guiding the consumers' purchasing decisions, individual firms have incentives to strategically amplify or counteract the competitors' information emission, thereby manipulating the label's (or ranking's) discriminatory power. Elements of the introduced theory are applicable to several (credence-good) industries which employ labels or rankings, including academic departments, books, music, and investment opportunities.

JEL-Codes: C700, D700, H400, M300.

Keywords: labelling, credence goods, contests, product differentiation.

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Contesting fake news

Daniel Rehsmann[§] Béatrice Roussillon[†] Paul Schweinzer[♮]

1 Introduction

Contests and tournaments are typically modeled such that a given "black box" technology ranks some dimension of the contestants' competitive efforts, expenditures, or qualities. The precise properties of this ranking technology, such as, for instance, the precision with which the underlying strategic variables are translated into ranks, are usually kept exogenous. This paper formalizes a novel type of interaction that allows contestants to strategically and jointly control the precision of the employed ranking technology through costly "marketing" outlays. Intuitively, the creation of an imprecise ranking is viewed as less costly than the assembly of the perfect ranking embodied in, for instance, auction models, which can translate even small differences in monetary bids into significant changes of assignment probabilities.

To put this idea to work, we consider a market of vertically differentiated goods or services with a credence or experience aspect.[1] In this market, different qualities of some credence good cannot be discerned by consumers absent further information. An ordinal ranking (i.e., labeling) of these qualities or goods—created, for instance, by experts or public servants on the basis of information provided by the producers of the goods—would, however, be perfectly useful to inform the public's consumption decisions. In our model, such an exogenous ranking benefits the firms and is assumed to spontaneously arise as firms release potentially competing information. Formally, this ranking translates market-provided information into probabilities of which product quality is ranked first, second, etc. Hence, in the modeled strategic interaction, a firm's decision is how much costly information to release in order to positively or negatively manipulate the industry's ranking "standard."

As an example, consider a ranking of academic departments to guide, for instance, ex-ante uninformed students on the relative merits of potential future almae matres.[2] Two departments are asked on the basis of which factors they wish to be ranked. If one department is highly successful in research, it may insist that the number of top-five publications enters the ranking prominently, while another department that excels in student satisfaction will want this information included in a nationwide ranking of departments. Obviously, if the ranking is supposed to capture research quality, the number of high-quality publications is more informative than student satisfaction. Hence, a ranking based exclusively on the number of top publications is "more precise" in picking out a high research-quality department. Conversely, if educational experiences are to be evaluated, the satisfaction data may be more useful, and including the number of top-five publications makes such a ranking "less precise."

The recent designation of nuclear energy and natural gas as sustainably "green" by the European Commission serves as another example. Costly political and industry lobbying activities on which "taxonomy" to use to assign energy forms a "green label" are well-documented, and several European Union member states sued the Commission over the implemented rules.[3,4] More generally, lobbying expenditures by firms and interest groups, as well as taxes paying for public certification agencies and their activities, may be seen as fitting the framework of our analysis. Such payments may indirectly translate into the quality of public ranking institutions, thereby shifting the status quo evaluation standard toward more or less adequate (or precise) certification.

Sports competitions provide further examples of situations in which ranking precision is a strategic choice. Wang (2010) reports on the rationale behind the change of the points scoring system by the International Table Tennis Federation from 21 to 11 as the "domination of China meant that

[1] "Credence qualities are those which, although worthwhile, *cannot be evaluated in normal use*. Instead, the assessment of their value requires additional costly information. An example would be the claimed advantages of the removal of an appendix, which will be correct or not according to whether the organ is diseased." (Darby and Karni, 1973). In contrast to experience goods, the utility of such goods is hard to ascertain, even after consumption.

[2] The United Kingdom's Research Excellence Framework ranks individual departments based on criteria established by consultation of Universities UK and the University & College Union (among others).

[3] The European Commission (EC) explicitly states the anticipated consumer reaction as one of the central aims of the taxonomy, which it sees as a "list of environmentally sustainable economic activities" whose overall purpose is to provide a "science-based classification system that allows financial and non-financial companies to share a common definition of sustainability when determining their investment choices." EC press release 7-Mar-2022.

[4] See, for instance, EC press release 1-Jan-2022. The rejected "amber category" would have, in addition to the implemented green and red labels, resulted in a more precise ranking (EC press release 2-Feb-2022).

there was little incentive for the other teams. Reducing the accuracy level increases the chance that a team other than China will win, thus inducing more effort from the other teams. This increased competition could, in turn, result in greater effort from the Chinese team." Deng et al. (2021) argue that similar reasoning has led to changing the best-of-three finals structure in the Israeli basketball league to a single game. "From 1970 to 2006, the Maccabi Tel Aviv team lost only one championship, while after the change, it lost six." Similarly, Yildirim (2015) reports on the slow adoption of obviously accuracy-improving video replay technology to support refereeing decisions in European soccer competitions as corresponding to non-aligned competitor interests.

We view the contribution of this paper as twofold. On the theoretical side, we introduce a family of contests that endogenize the degree of discrimination the underlying relative ranking is built on. This makes applications to (partial) credence or experience goods markets possible in which consumer-impenetrable product descriptions are translated into a simple ordinal ranking or label. In these stylized applications, we integrate both market sides into a standard equilibrium model of vertical product differentiation in which consumer demand reacts endogenously to the firms' choice of information dissemination. This allows us to analyze the competitive effects of the variation of ranking precision in credence markets.

Related literature

The idea that, in many circumstances, rank-order tournaments achieve socially beneficial outcomes in competitive situations is due to Lazear and Rosen (1981) and the contest literature they initiated. For detailed and recent surveys of this literature, see, for instance, Corchón and Serena (2018) and Fu and Wu (2019). To our knowledge, there is no prior contribution that allows contestants to endogenously control the precision of the underlying ranking technology in a strategic fashion.

A small set of papers, however, endogenize some aspects of a ranking's precision into a contest. Michaels (1988) is, as far as we know, the first to allow a "monopoly politician" the ability to set the discriminatory power of the Tullock contest success function to optimally extract rents from symmetric constituents. Into a similar setup, Dasgupta and Nti (1998) add a designer's own intrinsic valuation of the prize in addition to the valuation of the competitors. Wang (2010) allows for two asymmetric contestants, deriving the designer's optimal choice of accuracy, depending on the asymmetric contestants' ability spread. Yildirim (2015) models accuracy as the elasticity of contestants' efforts and derives comparative statics related to the heterogeneous players' payoffs. Ewerhart (2017) derives further revenue rankings for asymmetric Tullock contests, depending on an employed decisiveness (or discriminatory power) parameter. Bruckner and Sahm (2023) explore the optimal accuracy choice problem in multistage political competitions, finding that "a decisive primary might actually decrease the chances of winning the general election." Deng et al. (2021) allow for the use of contest precision as a competition instrument between contest organizers who compete to attract contestants among their potentially heterogeneous contests. None of these papers allows for the endogenous control of ranking precision as a competitive dimension between asymmetric players. Allowing for a significantly richer class of ranking technologies than explored by the existing literature, we embed this strategic competition into a vertically differentiated, labeled (credence) goods market.

The industrial organization literature on (competitive) labeling is well-developed and has been surveyed by Bergès-Sennou et al. (2004) and Sheldon (2017).[5] The papers closest to our idea of labeling are Roe and Sheldon (2007), Bonroy and Constantatos (2015), and Scott and Sesmero (2022).[6] They use a market-share approach which could be interpreted as a fixed-precision contest. The link to our contribution is that the degree of label-induced vertical product differentiation depends on the information disclosed by the label. Roe and Sheldon (2007) analyze how the practical implementation of a label (e.g., mandatory/voluntary, continuous/discrete, using a private certifier or public agency) can affect the size and distribution of surplus created in a vertically differentiated credence market.[7] In their models, governments can influence quality disclosure and the information a label communicates. Hence, a firm's strategic choice is to decide whether or not to hire a private certifier on top of some existing governmental labeling. This differs from (and complements) the approach of the present paper, which varies the level of information a label transfers to the consumers. Bonroy and Constantatos (2015) survey a variety of industrial organization models investigating how different label implementations affect welfare. They discuss questions of labeling policy with implications on firms' lobbying activities and incentives to develop labels of particular forms and stringency. Scott and Sesmero (2022) study the efficiency and distributional effects of consumers' misperception of product quality—similar to our imperfect rankings—in a vertically differentiated food market, both theoretically and empirically. They show that information-based policies aimed at curbing quality misperception (e.g., stricter labeling policies, nudging, changes in the labeling format) may have deleterious effects on efficiency and, perhaps most importantly, hurt the consumers they strive to protect. The competitive ranking precision aspects introduced in this paper are not included in any of these contributions and are, as far as we are aware, a novel approach to the labeling problem.

We define the formal structure of the market interaction in the following section 2 and characterize the firms' equilibrium behavior in section 3, which also contains all results, including elements of a welfare analysis. We provide illustrate of our results through several examples in section 3.5. All proofs can be found in the appendix.

2 Model of a labeled credence market

2.1 Supply side

There are two risk-neutral firms, $\mathcal{N} = \{1, 2\}$, each of which produces a good of quality θ_i, $i \in \mathcal{N}$. These qualities are assumed to be independently distributed according to $\theta_i \sim F_{[0,\bar{\theta}]}$, $\bar{\theta} \in \mathbb{R}_{++}$, with continuous and strictly positive density $f(\theta_i)$. We write $\boldsymbol{\theta} = (\theta_1, \theta_2)$ for the quality vector

[5] More distantly, our paper also relates to the literature on information disclosure and unraveling (Milgrom, 2008).

[6] Lehmann-Grube (1997) is, in some sense, diametrically opposite to our paper in investigating pure quality competition in a vertically differentiated market.

[7] Recent interest in credence goods has been spurred by applications to competition policy, health care, and the regulation of legal counseling. Comprehensive surveys include Dulleck and Kerschbamer (2006) and Balafoutas and Kerschbamer (2020). We are unaware of a previous application of contest-driven consumer demand to the analysis of competitive credence or experience markets.

and, without loss of generality, reindex firms such that $\theta_1 \geq \theta_2$. We assume that qualities are commonly known among the two firms but only the distribution of qualities F is known to the consumers. Denoting the mean and variance of the quality distribution F by (m, σ^2), we assume that the following holds:[8]

$$\frac{m + \sigma/\sqrt{3}}{m - \sigma/\sqrt{3}} \leq 2. \tag{A1}$$

For simplicity, production and distribution of the goods are assumed to be costless. Once the good is produced, there is nothing a firm can do to alter its quality. While qualities are fixed, firms can release (dis-)information $\rho_i \in \mathbb{R}$ allowing the two products to be ranked by means of a label. The absolute sum $r = |\rho_1 + \rho_2|$ of this emitted information is observed by the consumers and determines the precision used to rank the products.[9,10] This observed market information, r, affects the extent of product differentiation in the consumer market and, thus, the firms' expected profits represented by the winner's and loser's prizes $P_1(r)$, $P_2(r)$ in the contest for being ranked first. More precisely, we assume that firm $i \in \mathcal{N}$ maximizes

$$\max_{\rho_i} u_i(\boldsymbol{\theta}, r) = q_i(\boldsymbol{\theta}, r)P_1(r) + (1 - q_i(\boldsymbol{\theta}, r))P_2(r) - c(|\rho_i|) \tag{1}$$

in which $q_i(\boldsymbol{\theta}, r = |\rho_1 + \rho_2|)$ is player i's probability of being ranked first. The firms' information dissemination cost, $c(|\rho_i|)$, is assumed to be symmetric, strictly increasing, and strictly convex with $c(0) = 0$. For $i \in \mathcal{N}$ and $j = 3 - i$, we make the following assumptions on the noisy ranking of the firms' qualities $q_i(\boldsymbol{\theta}, r)$ which we collectively refer to as (Q):

(Q1) the ranking of firms' qualities $\boldsymbol{\theta}$ is observable and verifiable with $q_i(\boldsymbol{\theta}, r) + q_j(\boldsymbol{\theta}, r) = 1$;

(Q2) for $r > 0$, $q_i(\boldsymbol{\theta}, r)$ is strictly increasing in θ_i, strictly decreasing in θ_j;

(Q3) $q_i(\boldsymbol{\theta}, r)$ is continuous and strictly increasing in $r = |\rho_i + \rho_j|$ if $\theta_i > \theta_j$, strictly decreasing if $\theta_i < \theta_j$, and $q_i(\boldsymbol{\theta}, 0) = {}^1\!/_2$;

(Q4) $q_i(\boldsymbol{\theta}, r) = {}^1\!/_2$ for $\theta_i = \theta_j$; hence, $\partial q_i(\theta_1 = \theta_2, r)/\partial r = 0$;

(Q5) $q_i(\boldsymbol{\theta}, r)$ is sufficiently continuously differentiable in θ_i and has at most one inflection point at fixed $\theta_j > 0$, with $\partial^2 q_i/\partial\theta_i^2 \geq 0$ for $\theta_i \leq \theta_j$ and $\partial^2 q_i/\partial\theta_i^2 \leq 0$ for $\theta_i \geq \theta_j$;[11]

[8] The standard distributional restriction to the class of increasing failure rate distributions fails to provide sufficient conditions for our purposes of bounding ratios of expected order statistics. (A1) is satisfied for a broad class of distributions, as e.g., for the general uniform distribution, for arbitrary intervals of the standard triangular distribution, for standard values of the (truncated) normal with $m = \bar{\theta}/2$ and $\sigma > \bar{\theta}/5$, the log-normal distribution with an appropriate shifted mean-variance pair, and the beta distribution with parameters $1 \leq \alpha < 2$, $1 \leq \beta < 2$. It is violated by U-shaped distributions as, e.g., the beta distribution with $\alpha < 1$, $\beta < 1$.

[9] Dulleck and Kerschbamer (2006) discuss the incentives of strategic experts in credence markets. Since our focus is on the firms' information emission, we view the ranking as emerging spontaneously through the firms' activities. The recent White (2018) provides an overview of the literature on rating agencies, including several examples in which rankings arise from the (strategic) operations of market intermediaries.

[10] Allowing consumers to observe the sign of the total information introduces additional anti-symmetric equilibria on the negative information orthant (in which rational consumers invert the observed rankings) but otherwise adds little insight to the analysis.

[11] We adopt the convention that curvature changes at an inflection point. Therefore, the piecewise linear functions we define in form in subsection 3.5.3 have no inflection point in the ranges of interest.

(Q6) $q_i(\boldsymbol{\theta}, r)$ is sufficiently continuously differentiable in r and satisfies $\partial^2 q_i/\partial r^2 \leq 0$ for $\theta_i \geq \theta_j$.

Some of these assumptions are relaxed to accommodate our examples of a difference-based ranking of subsection 3.5.2 as well as the piecewise constant form of subsection 3.5.3.

2.2 Demand side

A unit mass of consumers, each with demand for a single good, is represented through a distribution of valuations $\mu \sim G_{[0,s]}$, $s \in \mathbb{R}_{++}$, with continuous and strictly positive density $g(\mu) > 0$.[12] Throughout the analysis of the labeling application, we restrict attention to the case in which $G_{[0,s]}$ follows a uniform distribution. We interpret the upper bound, s, as a measure of the consumers' preference heterogeneity. The utility of a type-μ consumer is assumed to be quasi-linear with

$$v(\mu, \theta) = \mu\tilde{\theta} - \tilde{p} \tag{2}$$

in which $\tilde{p}$ is the price paid for a product of (expected) quality $\tilde{\theta}$. Outside options are zero.

Apart from these individual preferences, the main element informing consumer demand is the commonly known outcome of a public ranking of the qualities θ_1 and θ_2, arising spontaneously following the firms' observed release of information $r = |\rho_1 + \rho_2|$.[13] In the absence of a ranking for the underlying credence good (or the case of observed $r = 0$), consumers cannot distinguish between products. Products of identical expected qualities are then assumed to be sold (under Bertrand competition) at the same price with the firms sharing expected profits equally.

Consumers do not know the realization of product qualities but form expectations of these, based on the commonly known distribution F. They observe the absolute market information $r = |\rho_1 + \rho_2|$ which they know to correspond to the ranking precision, i.e., the probability with which the first-ranked (or labeled) good actually has the higher quality. Consumers cannot observe the individually emitted components ρ_i.

2.3 Labeling contest

In our labeling contest, a firm's prize for coming first (second) is the expected consumer demand captured by the first-labeled (second-labeled) product, given the observed ranking based on $(q_1(\boldsymbol{\theta}, r), 1 - q_1(\boldsymbol{\theta}, r) = q_2(\boldsymbol{\theta}, r))$. To determine this demand, we use a standard vertical product differentiation model in which the expected quality is signaled through the product rank.[14] Given their mutually known qualities, firms decide on both their optimal, rank-dependent prices and the amount of information to release, ρ_i. Thus, first, firm i chooses and announces prices (p_i^1, p_i^2),

[12] In general, there are technical problems associated with the use of a continuum of independent random variables. These play no role in our analysis and could be resolved along the lines discussed by Lang (2019).

[13] In this paper, consumers know that the assigned ranks or labels are correct only with some probability $q(r)$. This contrasts with Scott and Sesmero (2022) who allow for consumers to misperceive labels, resulting in suboptimal purchasing decisions.

[14] The classic reference on vertical differentiation is Gabszewicz et al. (1981), succinctly summarized by Tirole (1988).

conditional on each possible rank.[15] This defines the labeling contest prizes $P_1(r)$ and $P_2(r)$ in which the firms subsequently choose individual information release, ρ_i.

It is useful to define the consumers' expectation of the first-ranked (second-ranked) product quality, given an observed ranking of precision r. This expectation is denoted by:

$$\begin{aligned}\Lambda_1(r) &= \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\left(q_1(\tilde{\boldsymbol{\theta}},r)\tilde{\theta}_1 + (1-q_1(\tilde{\boldsymbol{\theta}},r))\tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}})d\tilde{\theta}_2 d\tilde{\theta}_1,\\ \Lambda_2(r) &= \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\left(q_2(\tilde{\boldsymbol{\theta}},r)\tilde{\theta}_1 + (1-q_2(\tilde{\boldsymbol{\theta}},r))\tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}})d\tilde{\theta}_2 d\tilde{\theta}_1,\end{aligned} \tag{3}$$

in which $f_{(1,2:2)}(\boldsymbol{\theta})$ denotes the joint order probability density of randomly drawn quality θ_1 exceeding quality θ_2, and $q_i(\boldsymbol{\theta},r) = 1 - q_j(\boldsymbol{\theta},r)$, $j = 3 - i$.[16] In principle, these expectations are based on consumers' beliefs, conditional on the observed absolute total market information r.[17] In our independent qualities setting, the relevant joint order densities simplify to:

$$f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) = 2f(\tilde{\theta}_1)f(\tilde{\theta}_2). \tag{4}$$

Given an observed ranking $(q_1(\boldsymbol{\theta},r), 1-q_1(\boldsymbol{\theta},r))$ and announced prices $p = \left((p_i^1,p_i^2),(p_j^1,p_j^2)\right)$, a marginal consumer of valuation $\hat{\mu}_2^1$ is indifferent between buying the first- and second-ranked products if

$$\mu\Lambda_1(r) - p_1 = \mu\Lambda_2(r) - p_2, \tag{5}$$

resulting in the vector of cutoffs:

$$\hat{\mu} = \left(\hat{\mu}_1^0 = s,\ \hat{\mu}_2^1 = \frac{p_1 - p_2}{\Lambda_1(r) - \Lambda_2(r)},\ \hat{\mu}_3^2 = \frac{p_2}{\Lambda_2(r)}\right). \tag{6}$$

Because $\hat{\mu}_3^2 \geq 0$, the market is not generally fully served. Given these cutoffs, the first- and second-ranked firms maximize their profits by choosing p_1^* and p_2^*, respectively, such as to:

$$\begin{aligned}\max_{p_1} P_1(r) &= p_1\int_{\hat{\mu}_2^1}^{\hat{\mu}_1^0} g(\mu)d\mu = p_1\left(G(\hat{\mu}_1^0) - G(\hat{\mu}_2^1)\right),\\ \max_{p_2} P_2(r) &= p_2\int_{\hat{\mu}_3^2}^{\hat{\mu}_2^1} g(\mu)d\mu = p_2\left(G(\hat{\mu}_2^1) - G(\hat{\mu}_3^2)\right).\end{aligned} \tag{7}$$

A result due to Gabszewicz et al. (1981) establishes that such an equilibrium price vector exists and has the required properties.

Proposition 1. *For any market information $r > 0$ and any distribution of consumer tastes G with strictly positive and weakly concave density g, there exists an equilibrium vector of announced prices $p_1^* > p_2^* > 0$, provided that the failure rate*

$$\frac{g(\mu)}{1-G(\mu)} \text{ is strictly increasing.} \tag{8}$$

[15] Since consumers' valuation only depend on the ranking of the firms—everything else being uninformative—each firm faces the same optimization problem for choosing rank-dependent prices and thus selects the same equilibrium vector of conditional product prices. Therefore, it makes no difference whether firms decide on prices before or after the ranking realizes.

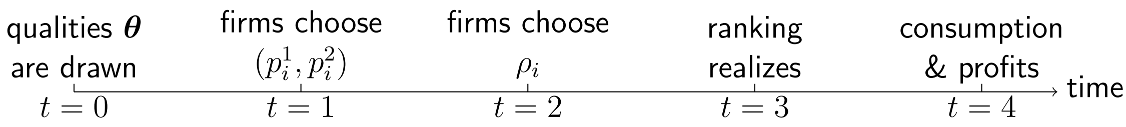
[16] For details on stochastic orders, see David and Nagaraja, 2003 or Shaked and Shanthikumar, 2007.

[17] For the timing and definition of consumer beliefs, see subsection 2.4.

Hence, $P_1(r)$ and $P_2(r)$ defined in (7) can be interpreted as endogenized prizes in a labeling contest in which firms manipulate the precision of the labels which partially inform consumers on the offered qualities.

2.4 Timing and information

Both prizes $P_i(r)$ and prices $p_i(r)$ are functions of the available information, i.e., the ranking precision $r = |\rho_1 + \rho_2|$. We are looking for (Bayesian Perfect) asymmetric, pure strategy Nash equilibria in which each firm chooses pairs $((p_i^k)_{k=1}^2, \rho_i)_{i=1}^2$. The complete timing of the interaction is:



Consumers cannot observe the firms' individually released information, ρ_i, but they can observe the absolute total amount of available information $r = |\rho_1 + \rho_2|$. Consumers realize that this total information determines the precision with which the ranking of product labels is correct, i.e., corresponds to the true order of qualities. Firms, by contrast, are assumed to fully understand the technology the industry is based on and therefore know each others' product qualities θ_i.

Off-equilibrium path beliefs: If consumers observe some joint information $\hat{r}$ which is incompatible with equilibrium play, e.g., a very high number $\hat{r}$ which necessarily results in negative utility for at least one player, we define $q_1(\boldsymbol{\theta}, \hat{r}) = q_2(\boldsymbol{\theta}, \hat{r}) = 1/2$.

3 Results

3.1 Preliminaries

We begin the analysis by collecting some simple results which will be used and referred to repeatedly in the subsequent reasoning. All proofs can be found in the Appendix.

We denote expected orders by:

$$\mathbb{E}[\Theta_{(1:2)}] = \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \tilde{\theta}_1 f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1, \ \ \mathbb{E}[\Theta_{(2:2)}] = \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \tilde{\theta}_2 f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 \tag{9}$$

and their sum of by:

$$\hat{\theta} = \mathbb{E}[\Theta_{(1:2)}] + \mathbb{E}[\Theta_{(2:2)}] = 2m. \tag{10}$$

Note that $\hat{\theta} = \bar{\theta}$ for all symmetric distributions.

Lemma 1. *For any distribution of product qualities $F(\cdot)$ with associated positive density $f(\cdot)$, the following "bookkeeping" results hold for all $r > 0$:*

$$\Lambda_1(r) + \Lambda_2(r) = \mathbb{E}[\Theta_{(1:2)} + \Theta_{(2:2)}] = \hat{\theta}, \tag{11}$$

$$\mathbb{E}[\Theta_{(1:2)}] \geq \Lambda_1(r) > \hat{\theta}/2 > \Lambda_2(r) \geq \mathbb{E}[\Theta_{(2:2)}] > 0, \tag{12}$$

$$\lim_{r\to\infty}\Lambda_1(r)=\mathbb{E}[\Theta_{(1:2)}],\quad \lim_{r\to\infty}\Lambda_2(r)=\mathbb{E}[\Theta_{(2:2)}], \tag{13}$$

$$\Lambda_1^{(n)}(r)=\int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}(\tilde{\theta}_1-\tilde{\theta}_2)\frac{\partial^n q_1(\tilde{\boldsymbol{\theta}},r)}{\partial r^n}f_{(1,2:2)}(\tilde{\boldsymbol{\theta}})d\tilde{\theta}_2 d\tilde{\theta}_1\ \ \forall n\in\mathbb{N}^+. \tag{14}$$

Moreover, in case of $r=0$, we have

$$\Lambda_1(0)=\Lambda_2(0)=\hat{\theta}/2. \tag{15}$$

The following inequality links properties of the underlying quality distribution with (spacings of) expected values of statistical orders and is repeatedly used in our analysis.

Lemma 2. *For any continuous quality distribution $F(\cdot)$ with mean m and variance σ^2 bounded on a positive interval, the ratio of expected order statistics satisfies:*

$$\frac{\mathbb{E}[\Theta_{(1:2)}]}{\mathbb{E}[\Theta_{(2:2)}]}\le\frac{m+\sigma/\sqrt{3}}{m-\sigma/\sqrt{3}}, \tag{16}$$

in which σ denotes the standard deviation.

The bound (16) is binding for the case of the standard uniform distribution.

3.2 The labeling application

To allow for the explicit calculation of market prices, cutoffs, and prizes, we restrict attention to uniformly distributed consumer preferences, i.e., $G(\mu)=\mu/s$, throughout this subsection. Quality distributions $F(\cdot)$ remain general, as defined in the model section.
For uniform preferences, (7) simplifies to:

$$\begin{aligned}\underset{p_1}{max}\ P_1 &= p_1\int_{\hat{\mu}_2^1}^{\hat{\mu}_1^0}g(\mu)d\mu=p_1\left(G(\hat{\mu}_1^0)-G(\hat{\mu}_2^1)\right)=p_1(\hat{\mu}_1^0-\hat{\mu}_2^1)\frac{1}{s},\\ \underset{p_2}{max}\ P_2 &= p_2\int_{\hat{\mu}_3^2}^{\hat{\mu}_2^1}g(\mu)d\mu=p_2\left(G(\hat{\mu}_2^1)-G(\hat{\mu}_3^2)\right)=p_2(\hat{\mu}_2^1-\hat{\mu}_3^2)\frac{1}{s}.\end{aligned} \tag{17}$$

Maximization with respect to p_i gives the optimal, rank-dependent prices as:

$$p_1^*(r)=2s\frac{\Lambda_1(r)\left(\Lambda_1(r)-\Lambda_2(r)\right)}{4\Lambda_1(r)-\Lambda_2(r)},\ \ p_2^*(r)=s\frac{\Lambda_2(r)\left(\Lambda_1(r)-\Lambda_2(r)\right)}{4\Lambda_1(r)-\Lambda_2(r)} \tag{18}$$

resulting, from (6), in the equilibrium cutoffs:

$$\hat{\mu}_1^0=s,\quad \hat{\mu}_2^1=s\frac{2\Lambda_1(r)-\Lambda_2(r)}{4\Lambda_1(r)-\Lambda_2(r)},\quad \hat{\mu}_3^2=s\frac{\Lambda_1(r)-\Lambda_2(r)}{4\Lambda_1(r)-\Lambda_2(r)} \tag{19}$$

giving, in turn, the rank dependent contest prizes as functions of the available information as:

$$\begin{aligned}P_1(r) &= (\hat{\mu}_1^0-\hat{\mu}_2^1)p_1^*(r)\frac{1}{s}=4s\frac{\Lambda_1(r)^2(\Lambda_1(r)-\Lambda_2(r))}{(\Lambda_2(r)-4\Lambda_1(r))^2},\\ P_2(r) &= (\hat{\mu}_2^1-\hat{\mu}_3^2)p_2^*(r)\frac{1}{s}=s\frac{\Lambda_1(r)\Lambda_2(r)(\Lambda_1(r)-\Lambda_2(r))}{(\Lambda_2(r)-4\Lambda_1(r))^2}.\end{aligned} \tag{20}$$

Since $\Lambda_1(r) + \Lambda_2(r) = \hat{\theta}$ from (10) and Lemma 1, (11), this results in contest prizes capturing the labeled market segments of:

$$P_1(r) = 4s\frac{\Lambda_1(r)^2\left(2\Lambda_1(r) - \hat{\theta}\right)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2}, \quad P_2(r) = s\frac{\left(\hat{\theta} - \Lambda_1(r)\right)\Lambda_1(r)\left(2\Lambda_1(r) - \hat{\theta}\right)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2}. \tag{21}$$

The endogenous emergence of contest prizes resulting from consumer demand is illustrated in Figure 1. Lemma 3 establishes some properties of the labeled market-segment prizes (21).

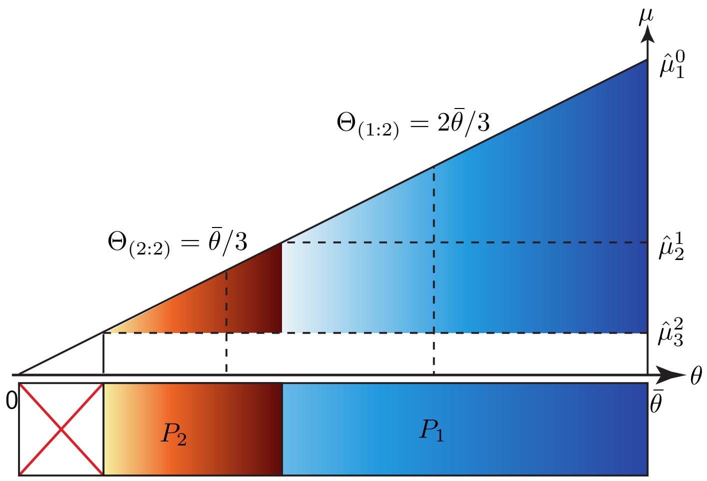


Figure 1: The labeled consumer market for uniform product qualities. The red-crossed consumer segment remains unserved.

Lemma 3. *For any distribution $F(\cdot)$ of product qualities and uniformly distributed consumer tastes μ, the following properties hold for $r > 0$:*

$$P_1(r) > P_2(r) > 0, \tag{22}$$

$$P_1'(r) > 0, P_1''(r) < 0 \text{ as well as } P_1'(r) > P_2'(r), \tag{23}$$

$$P_2'(r) > 0 \text{ and } P_2''(r) < 0, \tag{24}$$

in which only (24) *requires $F(\cdot)$ to satisfy* (A1)*. Moreover, in case of $r = 0$, $P_1(0) = P_2(0) = 0$.*

Remark 1 (Price competition)**.** *Notice that, from* (22)*, the prizes $P_i(r)$ accruing to competing firms are zero for the (unlabeled) no-information case of $r = 0$. The reason is that Bertrand competition for identically perceived products (albeit of positive expected qualities) drives prices $p_1 = p_2$ down to marginal cost which is zero in our environment.*

Notice that, for $r > 0$, neither player leaves the market since expected payoffs (1) are strictly positive in equilibrium. This follows from (21), continuity, and the fact that emission of no information is costless. We are now ready for the characterization of equilibrium information release.

Proposition 2. *Consider player $i \in \mathcal{N}$ with objective* (1). *A necessary condition for equilibrium is*

$$P_1{}'(\rho_1 + \rho_2) + P_2{}'(\rho_1 + \rho_2) = c'(|\rho_1|) + c'(|\rho_2|). \tag{25}$$

Since (25) is independent of qualities $\boldsymbol{\theta}$, an immediate corollary is that firms emit the same total amount of information r^* in any equilibrium, irrespective of individual qualities (and, in particular, in case of $\theta_1 = \theta_2$). Hence, in equilibrium, consumers cannot learn and update their beliefs about qualities (3), after observing the total market-provided ranking precision, r. This simplifies the analysis considerably and would not generally be the case if, for instance, firms could choose quality and information simultaneously. Moreover, the same invariance does not hold with respect to changes in preference heterogeneity, s, since prizes (21) react directly to this parameter.

The fact that the level of information in the market stays constant, irrespective of the product quality spread, is in line with numerous studies showing how difficult it is to reduce informational asymmetries in a credence market (Balafoutas and Kerschbamer, 2020; Dulleck and Kerschbamer, 2006). Evidence shows that companies may want to limit the level of information they provide, and consumers may be willfully ignorant. Our model underscores difficulties in information provision in such markets, and the firms' interest to strategically manipulate information disclosure.

Remark 2 (Positive precision)**.** *The total precision of the market-provided product ranking is positive in any equilibrium with*

$$\rho_1 \geq |\rho_2|. \tag{26}$$

To see this, consider firm 1's first-order condition

$$q_1(\boldsymbol{\theta}, r)P_1{}'(r) + q_2(\boldsymbol{\theta}, r)P_2{}'(r) + \frac{\partial q_1(\boldsymbol{\theta}, r)}{\partial \rho_1}\left(P_1(r) - P_2(r)\right) - c'(|\rho_1|) = 0. \tag{27}$$

For $r > 0$, we know that $q_1(\boldsymbol{\theta}, r) > q_2(\boldsymbol{\theta}, r)$ as well as $P_1(r) > P_2(r) > 0$ and $P_1{}'(r) > P_2{}'(r) > 0$ from Lemmata 1 & 3. Therefore, it must be the case that $c'(|\rho_1|) > c'(|\rho_2|)$. Moreover, since prizes are positive and $q_1(\boldsymbol{\theta}, r) > q_2(\boldsymbol{\theta}, r)$, it must be the case that equilibrium $\rho_1 > 0$. Hence, in equilibrium, "markets do not lie" and competition results in the provision of useful total information to consumers.

The following two results establish sufficient conditions for the existence of equilibrium information release policies in the labeling application.

Proposition 3. *Firm 1's utility function* (1) *is concave in ρ_1 for any fixed ρ_2^*.*

The proof shows that firm 1's benefit function—i.e., utility (1) without costs—is concave and increasing in ρ_1. Conceptually, the monotonicity of firm 1's benefits is attributed to a dual interplay of effects: 1) profits $P_i(r)$ increase monotonically in r, with $P_1(r) > P_2(r) > 0$, and 2) the ranking function $q(\boldsymbol{\theta}, r)$ can discriminate more correctly between qualities as precision r increases. The combined effect allows firm 1 to assert the higher profit P_1, as precision increases. Together with the assumed concavity of $-c(|\rho_1|)$ the result follows. Since firm 1's benefit function is concave and increasing for any ρ_2^*, standard methods imply that an optimal best response ρ_1^* exists to any of firm 2's choices of ρ_2 (Rosen, 1965).

Proposition 4. *A sufficient condition for asymmetric equilibrium behavior of firm 2 is that the ranking's hazard rate*

$$h(r) = \frac{\frac{\partial q_1(\boldsymbol{\theta}, r)}{\partial r}}{1 - q_1(\boldsymbol{\theta}, r)} \tag{28}$$

is strictly increasing in r for any $\theta_1 > \theta_2$.

The proof establishes quasi-concavity of firm 2's benefits in r. Intuitively, firm 2's benefits may be non-monotonic due to two opposing effects: 1) both Profits $P_1(r)$ and $P_2(r)$ increase with precision r, with $P_1(r) > P_2(r)$; 2) the increased precision of the ranking $q(\boldsymbol{\theta}, r)$ labels firm 2 with increasing precision as possessing the lower quality. Consequently, with increased r, firm 2 realizes the lower profit $P_2(r)$ with a higher probability. Hence, firm 2's benefits vary smoothly from the pooled prize $(P_1(0) + P_2(0))/2$ for zero precision to $P_2(r)$, for r large.

On a more technical level, the proof establishes strict-quasi-concavity of firm 2's benefit function ϕ_2 whenever the corresponding marginal benefits satisfies a strict single crossing condition, $-\phi_2'(\boldsymbol{\theta}, \rho_1^* + \rho_2') > 0 \implies -\phi_2'(\boldsymbol{\theta}, \rho_1^* + \rho_2'') > 0$ when $\rho_2'' > \rho_2'$. For this purpose, Quah and Strulovici (2012, Proposition 1)—characterizing when the single crossing property is stable under aggregation—is applied several times. To establish the single-crossing condition for the component function $q'(\boldsymbol{\theta}, \rho_1^* + \rho_2)P_1(\rho_1^* + \rho_2) - (1 - q(\boldsymbol{\theta}, \rho_1^* + \rho_2))P_1{}'(\rho_1^* + \rho_2)$ the ranking's hazard rate (28) must be strictly increasing in r. Based on quasi-concavity of firm 2's benefits we then show that firm 2's utility preserves quasi-concavity when subtracting strictly convex costs $c_2(|\rho_2|)$.

The ranking's hazard rate (28) is interpreted as the instantaneous increase in the chance of being ranked correctly at precision r, divided by the overall probability of being mis-ranked at r. The requirement for this rate of two decreasing functions to increase with r (for $\theta_1 > \theta_2$) implies that the instantaneous chance of being ranked correctly cannot decrease too quickly (see, e.g., Barlow and Proschan, 1965).

Remark 3 (Off-equilibrium path beliefs)**.** *The underlying Bayesian Perfect Nash equilibrium requires uninformed consumers to hold beliefs whenever they make their product choice. Along the equilibrium path, the observed signal $r > 0$ is uninformative of qualities, as Proposition 2 ensures that firms always emit the same total information, whatever the realization of $\boldsymbol{\theta}$. The same applies to rank-dependent prices, which are chosen before labels are assigned. If some observed total information $\hat{r}$ is incompatible with equilibrium play we assume that consumers' loose all confidence in the label's informational content. Accordingly, we define consumers' beliefs in such cases as $q_1(\boldsymbol{\theta}, \hat{r}) = q_2(\boldsymbol{\theta}, \hat{r}) = 1/2$ leading to Bertrand price competition and zero profits for both firms, dissuading such deviations.*

We now establish an interesting economic property of the defined credence-market interaction. The high-quality firm 1 always emits useful (i.e., positive) information which increases the ranking's precision while, for any fixed quality spread $\theta_1 > \theta_2$, there is a certain degree of heterogeneity, s, at which the low-quality firm 2 switches to obfuscating the ranking (i.e., releasing negative information), by releasing "fake news."

Proposition 5. *For sufficiently high preference heterogeneity $\tilde{s}$, there exists a set of parameters such that $\rho_1^*(\tilde{s}) > 0$ and $\rho_2^*(\tilde{s}) = 0$.*

Intuitively, the proof utilizes the fact that benefits are linear in s to establish the existence of a parameter pair $(\boldsymbol{\theta}, s)$ at which firm 2's marginal benefits are zero, resulting in $\rho_2^* = 0$. We then apply Cauchy's mean value theorem to firm 2's benefit function to show that there exists a pair of product qualities $\boldsymbol{\theta}$, at which a maximum of the function $y(\boldsymbol{\theta}) = (1-q(\boldsymbol{\theta}, \rho_1^*))P_1(\rho_1^*)/s$ in ρ_1^* exceeds the limit of $P_2(r)$ at infinity. The result then follows from the observation that this maximum of $y(\boldsymbol{\theta})$ increases with diminishing type spread $\boldsymbol{\theta}$, and corresponds to the limit of $P_1(r)/2$ at infinity, whenever $\theta_2 \to \theta_1$.

Our above results establish that similar-quality firms always jointly provide useful (i.e., precision-enhancing) information to the market. Proposition 5 shows that there exists a heterogeneity threshold $\tilde{s}$, such that for $s > \tilde{s}$, the lower-quality firm switches to strategic obfuscation (since firm 2's benefits are shown to be quasi-concave in Proposition 4). In principle, a similar result could also be shown for fixed consumer heterogeneity, s, and increasing quality spreads but, in that case, the potentially restrictive upper bound on available qualities implies that θ_1 cannot be raised indefinitely.

Proposition 5 is what partly motivates the "fake news" in the title of this paper. Firms cooperate to improve consumer information for relatively similar quality types on "homogeneous" markets while they choose to go at loggerheads for larger type differences on more heterogeneous markets. As a corollary, it can be shown that total market information, r, is improving with the heterogeneity of the consumer market.

The above Propositions 2–5 cannot generally accommodate ranking technologies $q_i(\boldsymbol{\theta}, r)$ with non-differentiabilities in precision or quality. Nevertheless, we show in our examples in section 3.5 that obfuscation-equilibria may still exist in those cases (see subsections 3.5.2 and 3.5.3).

Remark 4 (Maximum product differentiation)**.** *The maximum differentiation principle—i.e., concavity of $P_1(r)$ and $P_2(r)$ from* (23) *and* (24)*—is a well-known result saying that firms differentiate qualities in order to avoid "Bertrand" price-competition (Economides, 1986). Something similar is happening in the present model: for modest quality dispersion or low heterogeneity, both firms reinforce the market segmentation by improving market information. But since the quality dispersion also enters the probability with which the assigned labels are (perceived as) correct, there is a point at which the lower-quality firm starts to strategically obfuscate and partially offsets the better-quality firm's information release.*

Remark 5 (Asymmetric costs)**.** *Propositions 4 and 5 allow insights into the case of heterogeneous information dissemination costs. Consider asymmetric cost functions of the form $c(|\rho_1|)$ and $\gamma c(|\rho_2|)$, $\gamma \in [0, 1]$, for firms 1 and 2, respectively. For a suitably chosen quality pair $\boldsymbol{\theta}$, the boundary case of $\gamma = 0$ results in positive equilibrium information $r^* = \arg\max_r u_2(\boldsymbol{\theta}, r)$ because firm 2's benefits are single-peaked whenever* (28) *is satisfied. Reducing firm 1's costs to zero, by contrast, increases equilibrium information release without bound since firm 1's benefits are strictly increasing.*

3.3 Welfare properties

We start with the observation that, in any equilibrium, the precision of the market-provided product ranking is higher than what a cartel provides.

Proposition 6. *In any equilibrium of the labeling interaction, the precision of the market-provided product ranking* (25) *is higher than what a cartel (or multi-product monopoly) would choose.*

Intuitively, a single decision maker (or two cartelized firms) will optimally choose symmetric information emission costs and therefore $\rho_1 = \rho_2$. With strictly convex costs this leads to less precision than what the competitive duopoly provides.
Consumer welfare for the labeled market results from (2) and (3) as

$$\begin{aligned} W_H(r) &= \int_{\hat{\mu}_2^1}^{\hat{\mu}_1^0} (\mu \Lambda_1(r) - p_1^*)\, dG(\mu), \\ W_L(r) &= \int_{\hat{\mu}_3^2}^{\hat{\mu}_2^1} (\mu \Lambda_2(r) - p_2^*)\, dG(\mu) \end{aligned} \tag{29}$$

summing to served consumer welfare

$$W_C(r) = W_H(r) + W_L(r) \tag{30}$$

with the measure of unserved consumer dead-weight loss given by

$$W_0(r) = \int_0^{\hat{\mu}_3^2} \mu dG(\mu). \tag{31}$$

We show that consumer welfare is strictly decreasing in r, due to (i) firms recovering their costs of information by charging increased equilibrium prices p_i^* and (ii) the served market segments monotonically decreasing with increasing information r.

Proposition 7. *Consumer welfare is strictly decreasing in r.*

Together with firm utility (1), we define total welfare as

$$W(r) = u_1(r) + u_2(r) + W_C(r) \tag{32}$$

in which firm utilities are defined in (1). We compare this ranked welfare measure to the unlabeled market, characterized by Bertrand-competition and zero prices, resulting in

$$W_U = \int_0^s (\mathbb{E}[\Theta]\mu - 0)\, dG(\mu), \text{ with } \mathbb{E}[\Theta] = \int_0^{\bar{\theta}} \theta dF(\theta). \tag{33}$$

Notice that dead-weight loss in the competitive, unranked case is zero; all consumers are served. Moreover, the labeled market approaches the unlabeled case as information vanishes

$$\lim_{r \to 0} W(r) = W_U \tag{34}$$

also in terms of welfare. Our next result shows that adding information to an unlabeled market increases total welfare, with the gain in producer surplus exceeding the loss in consumer welfare.

Proposition 8. *Total welfare is strictly increasing in information release* ρ_i *at the unlabeled point* $\rho_1 = \rho_2 = 0 = r$.

Our minimal welfare analysis shows that, if possible, profit opportunities induce both firms to escape Bertrand competition through the introduction of a label. Therefore, it is in the firms' interests to implement our assumption of a "spontaneously" arising ranking. Moreover, since total welfare is increasing through the introduction of the label, (unserved) consumers could in principle be compensated while still allowing for vertical differentiation.

Remark 6 (Social planner benchmark)**.** *Consider a benevolent social planner who knows product qualities and is able to label product qualities with infinite precision. Consumers therefore know that the two labeled products are of expected qualities* (9)*, with no danger of mislabeling. Firms have no means of manipulating the infinitely precise ranking and compete costlessly on a vertically differentiated market with exogenous quality through the optimal choice of prices as in Gabszewicz et al. (1981). The resulting welfare is, for infinitely precise ranking in* (32)*,*

$$u_1 + u_2 + W_C = 4s\frac{\Lambda_1^2\left(2\Lambda_1 - \hat{\theta}\right)}{\left(\hat{\theta} - 5\Lambda_1\right)^2} + s\frac{\left(\hat{\theta} - \Lambda_1\right)\Lambda_1\left(2\Lambda_1 - \hat{\theta}\right)}{\left(\hat{\theta} - 5\Lambda_1\right)^2} + s\frac{\Lambda_1^2\left(5\hat{\theta} - \Lambda_1\right)}{2\left(\hat{\theta} - 5\Lambda_1\right)^2} \qquad (35)$$

simplifying to

$$W^* = s\frac{\mathbb{E}[\Theta_{(1:2)}]\left(11\,\mathbb{E}[\Theta_{(1:2)}]^2 + 3\hat{\theta}\,\mathbb{E}[\Theta_{(1:2)}] - 2\hat{\theta}^2\right)}{2\left(\hat{\theta} - 5\,\mathbb{E}[\Theta_{(1:2)}]\right)^2} \qquad (36)$$

which unambiguously improves over market welfare (32) *for any finite (equilibrium) precision* r.

For more stringent assumptions on the quality distribution further properties can be derived. For instance, it can be shown that for uniformly distributed qualities, there exist positive prices—below equilibrium prices (18)—such that consumers as well as producers strictly benefit from the introduction of labels. As this is generally not the case if firms choose prices competitively and strategically, there may well be scope for government regulatory action.

3.4 Other precision contests

Although the paper's main economic interest is the labeling application, we also want to point out some general properties of the precision contests we define. Proposition 2, (25), describes an equilibrium fixed point relationship which is responsible for many of the complications in the full labeling application. If we disentangle the underlying precision contest from the market reaction, however, analytic equilibrium strategies can be easily obtained.

Remark 7 (Exogenous Prizes)**.** *If we take demand to be exogenous of information, i.e.,* $P_1 > P_2 > 0$, *a direct consequence of* (25) *is*

$$c'(|\rho_1|) + c'(|\rho_2|) = 0 = r^* \qquad (37)$$

and the information emitted by the two firms in equilibrium exactly cancels out. Individual information dissemination is determined by the first-order conditions

$$c'(|\rho_1|) = \frac{\partial q_1(\boldsymbol{\theta}, r^* = 0)}{\partial \rho_i}(P_1 - P_2) = -c'(|\rho_2|). \tag{38}$$

Since costs are invertible, this directly determines the equilibrium information strategies as $\rho_i^* = \pm c'^{-1}(q'(\boldsymbol{\theta}, 0))$*, with the higher-quality firm emitting positive information while the lower-quality firm obfuscates.*

Remark 8 (Simple endogenous Prizes)**.** *We now allow prizes* $P_1(r), P_2(r)$ *to react to aggregate information, but in a less involved fashion than in subsection 3.2. In particular, we assume prizes to be monomials of the form*

$$P_1(r) = \alpha(r)^\beta, \quad P_2(r) = \frac{\alpha}{\gamma}(r)^\beta, \quad \alpha > 0,\ \beta > 1,\ \gamma > 1, \tag{39}$$

and costs to be quadratic $c(\rho) = (\rho^2)/2$*. These functional forms fix the left-hand side of* (25) *as*

$$P_1{}'(r) + P_2{}'(r) = \alpha\beta\frac{\gamma+1}{\gamma}r^{\beta-1} \tag{40}$$

implying that total available market information is

$$r^* = \left(\frac{\alpha\beta + \alpha\beta\gamma}{\gamma}\right)^{\frac{1}{2-\beta}}. \tag{41}$$

Given this total precision, individual maximization (27) *then leads to the pair of equilibrium information dissemination strategies*

$$\begin{aligned} \rho_1^* &= \alpha(r^*)^{\beta-1}\left(\beta + \beta(\gamma-1)q_1(\boldsymbol{\theta}, r^*) + (\gamma-1)r^*q'(\boldsymbol{\theta}, r^*)\right)/\gamma, \\ \rho_2^* &= \alpha(r^*)^{\beta-1}\left(\beta\gamma - \beta(\gamma-1)q_1(\boldsymbol{\theta}, r^*) - (\gamma-1)r^*q'(\boldsymbol{\theta}, r^*)\right)/\gamma. \end{aligned} \tag{42}$$

Figure 2 illustrates this result for a simple Tullock-ranking example with $\theta_2 = 1/4$, $\alpha = 1$, $\beta = 3/2$, *and* $\gamma = 2$*. Notice that information dissemination strategies are non-monotonic in* θ_1 *and, for any pair* $\boldsymbol{\theta}$*, total market-provided information* r^* *is constant.*

3.5 Examples of precision contests in the labeling application

We now illustrate the general results of the previous section by means of several examples for the transformation of firms' emitted information into (credence good) market ranking precision, i.e., the mapping from quality difference to label probability assignment. The Tullock-ratio, difference, piecewise-constant, and noise-based forms we use below to rank qualities $\theta = (\theta_1, \theta_2)$ are illustrating the class on which we base the general analysis of the previous subsections. The common example properties are uniformly drawn qualities on $[0, \bar{\theta}]$, uniformly distributed consumer tastes on $[0, s]$, and quadratic costs of information emission.

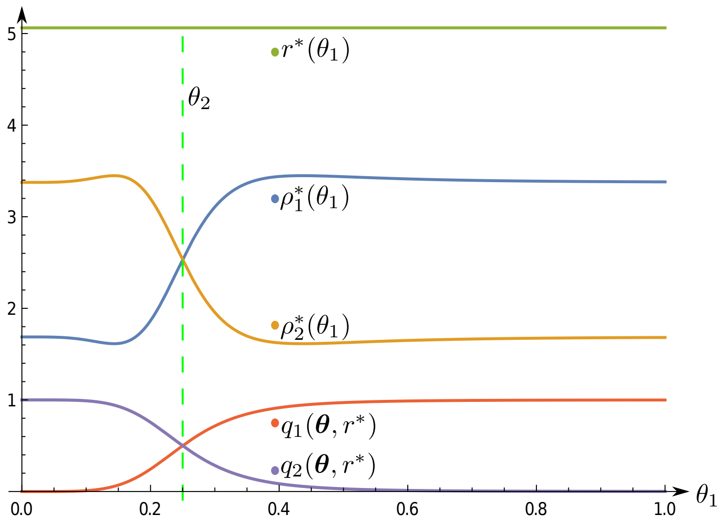


Figure 2: Information dissemination (42) for $\theta_2 = 1/4$ as a function of $\theta_1 \in [0, 1]$ is non-monotonic: ρ_1^* is plotted blue, ρ_2^* gold, total information $r^* = \rho_1^* + \rho_2^*$ green. For comparison, ranking probabilities $q_1(\boldsymbol{\theta}, r^*)$ and $q_2(\boldsymbol{\theta}, r^*)$ are drawn red and purple, respectively.

The following Table 1 summarizes the relationship of our examples to the set of assumptions (Q):

	Ratio-based 3.5.1	Difference-based 3.5.2	Piecewise-constant 3.5.3	Noise-based 3.5.4
(Q1)	✓	✓	✓	✓
(Q2)	✓	✓ locally	✗	✓
(Q3)	✓	✓ locally	✓	✓
(Q4)	✓	✓ locally	✓	✓
(Q5)	✓	✓ locally	✗	✓
(Q6)	✓	✓ locally	✓	✓

Table 1: Matching assumptions to example properties.

Despite the fact that some examples of ranking functions violate several of the assumed properties (Q), all examples verify the full set of properties derived in Propositions 1–5.

3.5.1 Ratio-based ranking

We consider the example of uniformly drawn qualities, $\theta_i \sim U_{[0,\bar{\theta}]}$, and uniform consumer preferences, $\mu \sim U_{[0,s]}$. Qualities θ_i are unobservable to the consumers but are known among competitors. Thus, to a consumer, the expected quality of a product can only be inferred from its observed label (or rank). We award a single "best-product" label to the firm ranked first, using the Tullock ranking technology, specifying the probabilities with which firms are ranked first as:

$$q_1(\boldsymbol{\theta}, r) = 1/\left(1 + x^{-r}\right), \quad q_2(\boldsymbol{\theta}, r) = 1/(1 + x^r)\,, \tag{43}$$

for $x = \theta_1/\theta_2$. Note that the Tullock ranking function—illustrated in Figure 3—satisfies Assumptions Q, as well as the sufficient condition for the firms' equilibrium existence (28), by the fact that

$$\frac{\partial}{\partial r}\left(\frac{\partial q_1(\boldsymbol{\theta}, r)}{\partial r}/(1 - q_1(\boldsymbol{\theta}, r))\right) = \frac{x^r \log(x)^2}{(1 + x^r)^2} \tag{44}$$

which is strictly positive for any $x > 1$ and $r > 0$.

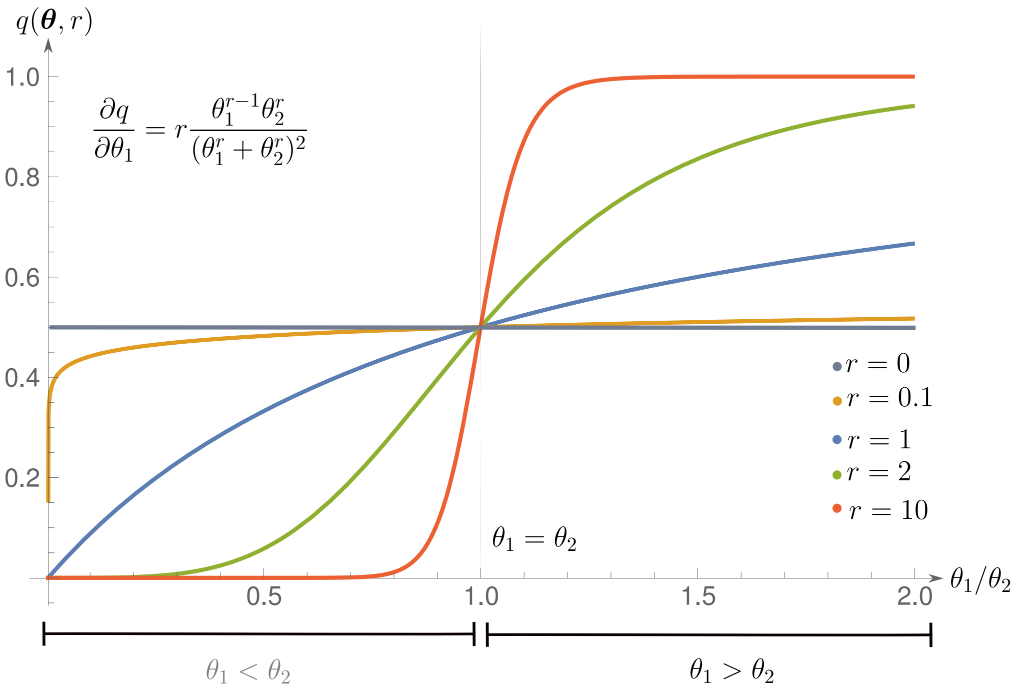


Figure 3: Reaction of Tullock ranking probabilities (43) to quality-ratio and precision.

A firm chooses i) the price of her product conditional on the product's rank and ii) the resources ρ_i it wishes to expend on influencing the overall ranking precision and consumer demand.
We start by modeling the demand side. Expected qualities are obtained using (3). For the uniform distribution on the $\bar{\theta}$-scaled unit interval, the joint density (4) is

$$f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) = 2f(\tilde{\theta}_1)f(\tilde{\theta}_2) = 2/\bar{\theta}^2. \tag{45}$$

The consumers assess the expected qualities of the first- and second-ranked products (3), given an observed ranking, for $\tilde{x} = \tilde{\theta}_1/\tilde{\theta}_2$ as:

$$\begin{aligned}
\Lambda_1(r) &= \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\left(\frac{1}{1+\tilde{x}^{-r}}\tilde{\theta}_1 + \frac{1}{1+\tilde{x}^r}\tilde{\theta}_2\right)\frac{2}{\bar{\theta}^2}d\tilde{\theta}_2 d\tilde{\theta}_1 \\
&= \frac{2}{\bar{\theta}^2}\left(\int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\frac{\tilde{\theta}_1}{1+\tilde{x}^{-r}}d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\frac{\tilde{\theta}_2}{1+\tilde{x}^r}d\tilde{\theta}_2 d\tilde{\theta}_1\right), \\
\Lambda_2(r) &= \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\left(\frac{1}{1+\tilde{x}^r}\tilde{\theta}_1 + \frac{1}{1+\tilde{x}^{-r}}\tilde{\theta}_2\right)\frac{2}{\bar{\theta}^2}d\tilde{\theta}_2 d\tilde{\theta}_1 \\
&= \frac{2}{\bar{\theta}^2}\left(\int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\frac{\tilde{\theta}_1}{1+\tilde{x}^r}d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\frac{\tilde{\theta}_2}{1+\tilde{x}^{-r}}d\tilde{\theta}_2 d\tilde{\theta}_1\right).
\end{aligned} \tag{46}$$

Following the general derivation in subsection 3.2 we obtain the contest prizes, using $\mathbb{E}[\Theta_{(1:2)}] + \mathbb{E}[\Theta_{(2:2)}] = \hat{\theta} = \bar{\theta}$, yielding contest prizes (21). We turn to the supply side and state firm $i \in \mathcal{N}$'s maximization problem (under mutually known $x_i = \theta_i/\theta_j$) on the basis of (7) as

$$\underset{\rho_i}{max}\ \frac{1}{1+x_i^{-r}}P_1(r) + \frac{1}{1+x_i^r}P_2(r) - \frac{\rho_i^2}{2}, \tag{47}$$

in which $r = |\rho_i + \rho_j|$. In order to get numerical values, we fix $\theta_1 = {}^3/_4$, $\theta_2 = {}^1/_4$ (i.e., $x = 3$), and the distributional parameters at $\bar{\theta} = 1$, $s = 10$. After taking derivatives with respect to ρ_i, we find the asymmetric equilibrium candidate[18]

$$\rho_1^* \approx 0.480, \;\; \rho_2^* \approx 0.131, \;\text{implying that } r^* \approx 0.612. \tag{48}$$

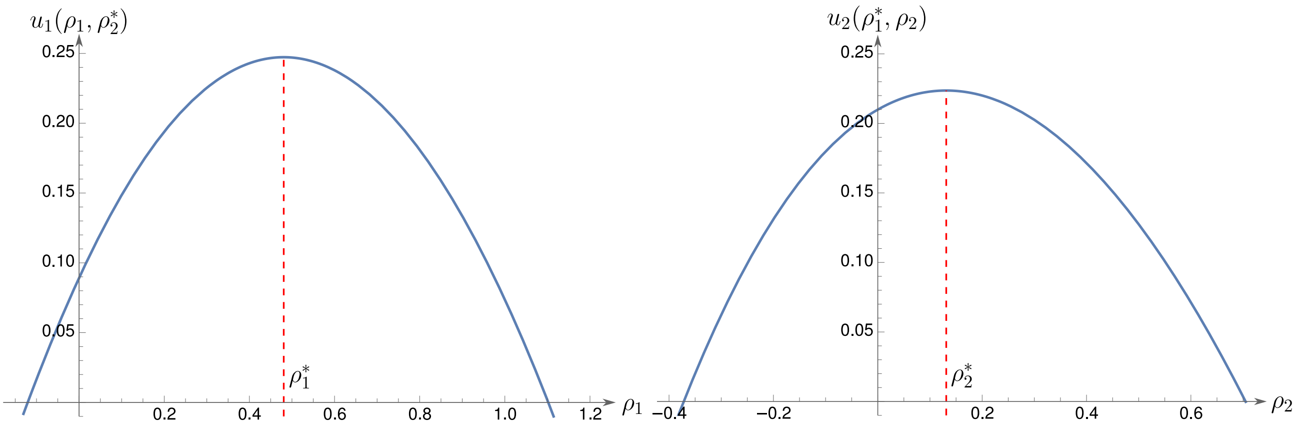


Figure 4: The two players' optimal choice of ρ_i in asymmetric equilibrium for ratio-based ranking.

Notice that both firms choose to *improve* the ranking precision by supplying positive amounts of information in equilibrium. Hence, the market-supplied ranking precision is an informational improvement over the status quo of the unranked market with information $r = 0$. For other parameterizations, however, it turns out that this behavior depends on both the ratio x and the consumer heterogeneity, s. The left-hand panel of Figure 5 fixes the heterogeneity at $s = 20$ and shows information emission as a function of x. For sufficiently wide type-difference, there exists a

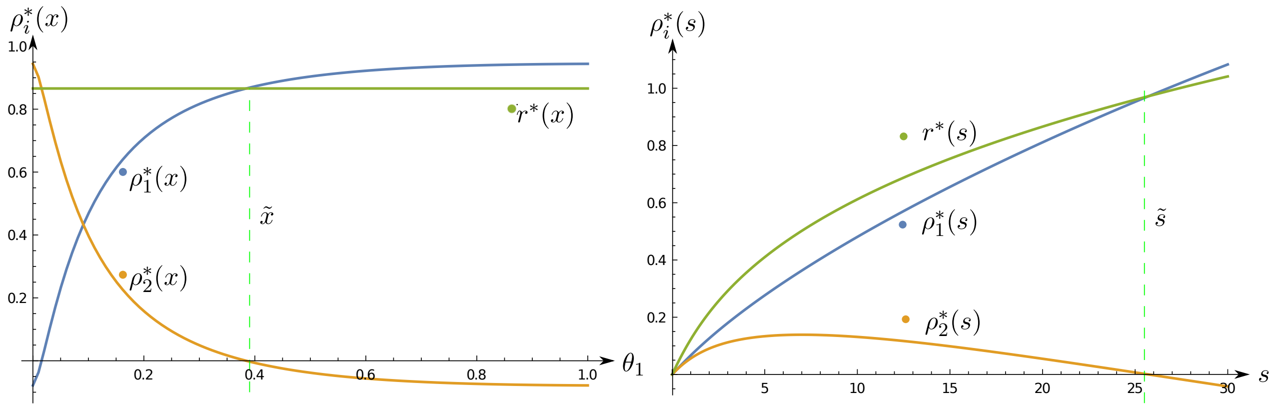


Figure 5: Left: information dissemination ρ_i^* for the ratio-form example, for heterogeneity $s = 20$, as functions of $\theta_1 \in [0, 1]$ for fixed $\theta_2 = 1/10$: $\rho_1^*(x)$ is plotted blue, $\rho_2^*(x)$ is shown gold, and total information $r^*(x) = \rho_1^*(x) + \rho_2^*(x)$ is green. The vertical dashed line indicates the ratio $\tilde{x}$ at which $\rho_2^*(x) = 0$. Right: information dissemination $\rho_i^*(s)$ as a function of the consumer heterogeneity s, for $\bar{\theta} = 1$. The dashed green line shows the critical heterogeneity $\tilde{s} \approx 25.5$ at which $\rho_2^* = 0$.

point $\tilde{x}$ at which the lower-quality firm turns to obfuscation. Moreover, the figure illustrates that the

[18] These are numerical results. If qualities are Beta-distributed with parameters $\beta_1 = 2, \beta_2 = 2$, the asymmetric equilibrium is $\rho_1^* \approx 0.344$, $\rho_2^* \approx 0.156$, implying that $r^* \approx 0.5$.

symmetric case of $\theta_1 = \theta_2$ (and therefore $\rho_1 = \rho_2 > 0$) is sufficient to determine market information r. The right-hand panel of Figure 5 shows that such an obfuscation-switching strategy also exists for fixed x when the consumer heterogeneity s is varied (Proposition 5).

3.5.2 Difference-based ranking

Consider the same uniform two-firms example as in the previous subsection, but governed now by a difference-form ranking technology in a piece-wise linear form, as discussed, e.g., by Che and Gale (2000).[19]

$$q_1(\boldsymbol{\theta}, r) = \max\left[\min\left[1/2 + r\left(\theta_1 - \theta_2\right), 1\right], 0\right]. \tag{49}$$

We tentatively simplify this ranking technology—illustrated in Figure 6—using $\theta_1 > \theta_2$ to define:

$$q_1(\boldsymbol{\theta}, r) = \min\left[1/2 + rx, 1\right], \quad q_2(\boldsymbol{\theta}, r) = \max\left[1/2 - rx, 0\right] \tag{50}$$

in which $x = \theta_1 - \theta_2$ and we tentatively guess that $xr^* < 1/2$.[20] This allows (50) to locally satisfy Assumptions (Q2)—(Q6), as well as the sufficient condition

$$\frac{\partial}{\partial r}\left(\frac{\partial q_1(\boldsymbol{\theta}, r)}{\partial r} / (1 - q_1(\boldsymbol{\theta}, r))\right) = \frac{4x^2}{(1 - 2rx)^2} \tag{51}$$

which is strictly positive for any $x > 0$ and $r > 0$ whenever $xr^* < 1/2$.

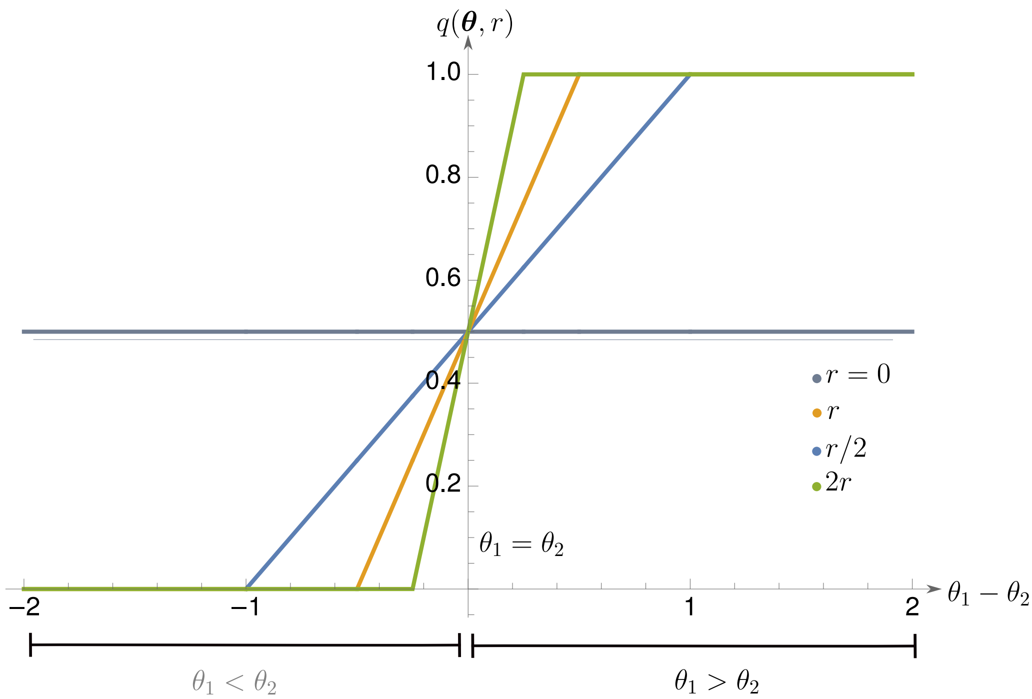


Figure 6: Reaction of ranking probabilities (49) to quality-difference and precision.

[19] A very similar analysis can be performed (with similar results) for a ranking based on 1) the "serial" contest success function introduced by Alcalde and Dahm (2007) that combines aspects of both the ratio form (43) and the difference form (49) and 2) a ranking based on the exponential type difference.

[20] Since the relevant equilibrium candidates fall into this region, this guess turns out to be correct in the present example.

We obtain expected qualities (3), given the observed ranking, for $\tilde{x} = \tilde{\theta}_1 - \tilde{\theta}_2$ as:

$$\begin{aligned}
\Lambda_1(r) &= \frac{2}{\bar{\theta}^2}\left(\int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}(1/2 + r\tilde{x})\,\tilde{\theta}_1 d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}(1/2 - r\tilde{x})\,\tilde{\theta}_2 d\tilde{\theta}_2 d\tilde{\theta}_1\right)\\
&= \frac{\bar{\theta}}{6}\left(3 + r\bar{\theta}\right),\\
\Lambda_2(r) &= \frac{2}{\bar{\theta}^2}\left(\int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}(1/2 - r\tilde{x})\,\tilde{\theta}_1 d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}(1/2 + r\tilde{x})\,\tilde{\theta}_2 d\tilde{\theta}_2 d\tilde{\theta}_1\right)\\
&= \frac{\bar{\theta}}{6}\left(3 - r\bar{\theta}\right).
\end{aligned} \tag{52}$$

Following the steps of the general derivation in subsection 3.2 results in the same prizes (21), parameterized by $\Lambda_i(r)$ and for $\hat{\theta} = \bar{\theta}$, as in the previous subsection. The resulting supply-side firms' maximization problem (under mutually known $x = \theta_1 - \theta_2$) is

$$\underset{\rho_i}{max}\; q_i(\boldsymbol{\theta}, r)P_1(r) + \left(1 - q_i(\boldsymbol{\theta}, r)\right)P_2(r) - \frac{\rho_i^2}{2}. \tag{53}$$

Under the simplified ranking and assumed $xr^* < 1/2$ in (50), the first-order conditions for this problem are

$$\begin{aligned}
\bar{\theta}^2 s\left(\frac{\bar{\theta}r\left(\bar{\theta}r\left(5\bar{\theta}r + 27\right) + 69\right) + 135}{2\left(5\bar{\theta}r + 9\right)^3} + \frac{2rx\left(\bar{\theta}r\left(5\bar{\theta}r + 21\right) + 27\right)}{3\left(5\bar{\theta}r + 9\right)^2}\right) &= \rho_1,\\
\bar{\theta}^2 s\left(\frac{\bar{\theta}r\left(\bar{\theta}r\left(5\bar{\theta}r + 27\right) + 69\right) + 135}{2\left(5\bar{\theta}r + 9\right)^3} - \frac{2rx\left(\bar{\theta}r\left(5\bar{\theta}r + 21\right) + 27\right)}{3\left(5\bar{\theta}r + 9\right)^2}\right) &= \rho_2.
\end{aligned} \tag{54}$$

Guessing affine solutions $\rho_i = \alpha \pm \beta x$ to (54) for $s = 10$ and $\bar{\theta} = 1$ results in the two functions

$$\rho_1^*(x) \approx 0.452 + 1.650x,\;\; \rho_2^*(x) \approx 0.452 - 1.650x,\; \text{implying that } r^* \approx 0.903, \tag{55}$$

which are indeed affine. Hence, the critical difference $\tilde{x} = \theta_1 - \theta_2$ at which $\rho_2^*(x) = 0$ equals

$$\tilde{x} = \alpha/\beta \approx 0.274. \tag{56}$$

For the same example values of $\theta_1 = 3/4$, $\theta_2 = 1/4$ as in the previous subsection, this yields the asymmetric equilibrium candidate[21]

$$\rho_1^* \approx 1.276, \quad \rho_2^* \approx -0.373. \tag{57}$$

This candidate satisfies the assumed $0.452 < 1/2 = xr^*$. Together with global cost convexity, Figure 7 verifies this candidate as pure strategy equilibrium. While both players choose to improve the ranking precision in the Tullock-example of the previous subsection, the lower-quality firm obfuscates the ranking in this second example. The (indirect) positive influence of total information r on the prizes outweighs the (direct) competitive effect in the difference-form example.

The left-hand panel of Figure 8 fixes the consumer heterogeneity at $s = 10$ and shows equilibrium information emission as a function of x. The left-hand panel of Figure 8 shows that for sufficiently

[21] If both qualities are Beta-distributed with parameters $\beta_1 = 2, \beta_2 = 2$, the asymmetric equilibrium is $\rho_1^* \approx 0.799$, $\rho_2^* \approx -0.068$, implying that $r^* \approx 0.731$.

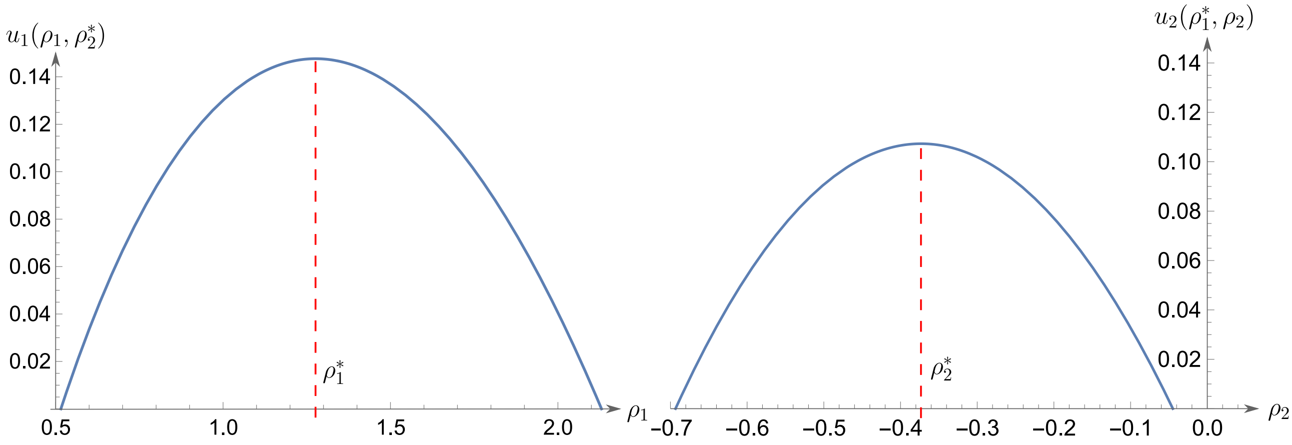


Figure 7: The two players' optimal choice of ρ_i in asymmetric equilibrium for difference-based ranking.

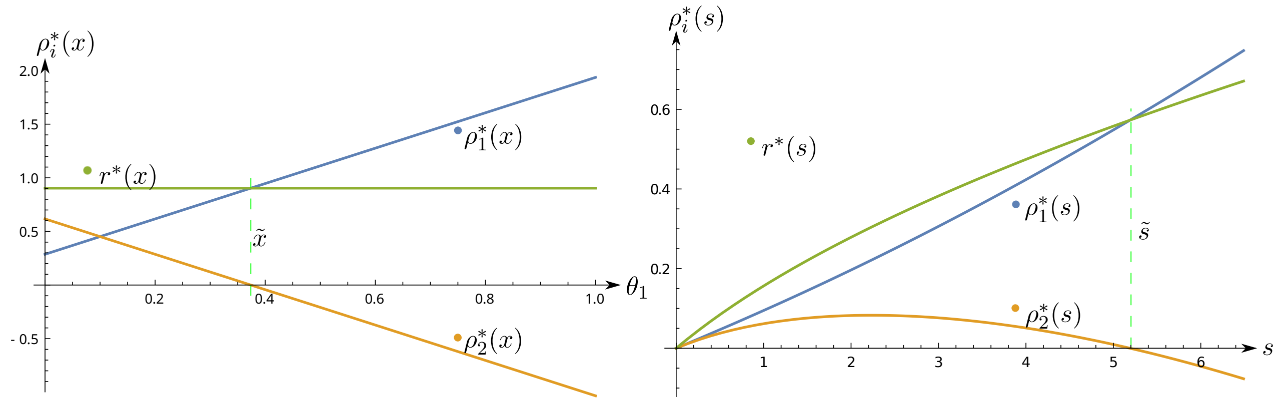


Figure 8: Left: information dissemination ρ_i^* for the difference-form example, for heterogeneity $s = 10$, as functions of $\theta_1 \in [0, 1]$ for fixed $\theta_2 = 1/10$: $\rho_1^*(x)$ is plotted blue, $\rho_2^*(x)$ is shown gold, and total information $r^*(x) = \rho_1^*(x) + \rho_2^*(x)$ is green. The vertical dashed line indicates the difference $\tilde{x}$ at which $\rho_2^*(x) = 0$. Right: information dissemination $\rho_i^*(s)$ as a function of the consumer heterogeneity s, for $\bar{\theta} = 1$. The dashed green line shows the critical heterogeneity $\tilde{s} \approx 5.192$ at which $\rho_2^* = 0$.

wide quality-difference there exists a point $\tilde{x}$ at which the lower-quality firm turns to obfuscation. A similar effect is illustrated in the right-hand panel of Figure 8 for a sufficiently large heterogeneity s. Notice that $\rho_1^*(s)$ is convex in this example.

3.5.3 Piecewise-constant ranking

In our third example we give up the stipulation that probabilities must strictly increase in the quality spread (Q2) and the differentiability requirement (Q5). We define a—to our knowledge novel—ranking technology $q_i(\boldsymbol{\theta}, r)$ that is only weakly increasing in θ_i and weakly decreasing in θ_j. Inspired by the all-pay auction, we use a piecewise constant function to define

$$q_i(\boldsymbol{\theta}, r) = \begin{cases} d(r), & x > 0 \\ 1/2, & x = 0 \\ 1 - d(r), & x < 0 \end{cases}, \tag{58}$$

for $x = \theta_i - \theta_j$, in which $d(r)$ is constant in x, satisfying $\lim_{r\to\infty} d(r) = 1$ and $\lim_{r\to 0} d(r) = 1/2$. The responsiveness of this ranking technology is illustrated in Figure 9. To fix ideas we set

$$d(r) = \frac{\delta r + 1}{\delta r + 2}, \quad \delta \in \mathbb{R}_+. \tag{59}$$

Observe that the sufficient condition for equilibrium existence (28) is violated, since for $x > 0$ we have

$$\frac{\partial}{\partial r}\left(\frac{\partial q_1(\boldsymbol{\theta}, r)}{\partial r}/(1 - q_1(\boldsymbol{\theta}, r))\right) = -\frac{\delta^2}{(2 + r\delta)^2} \tag{60}$$

which is strictly negative for any $r > 0$ and $\delta > 0$. Consumers assess expected qualities of the first-

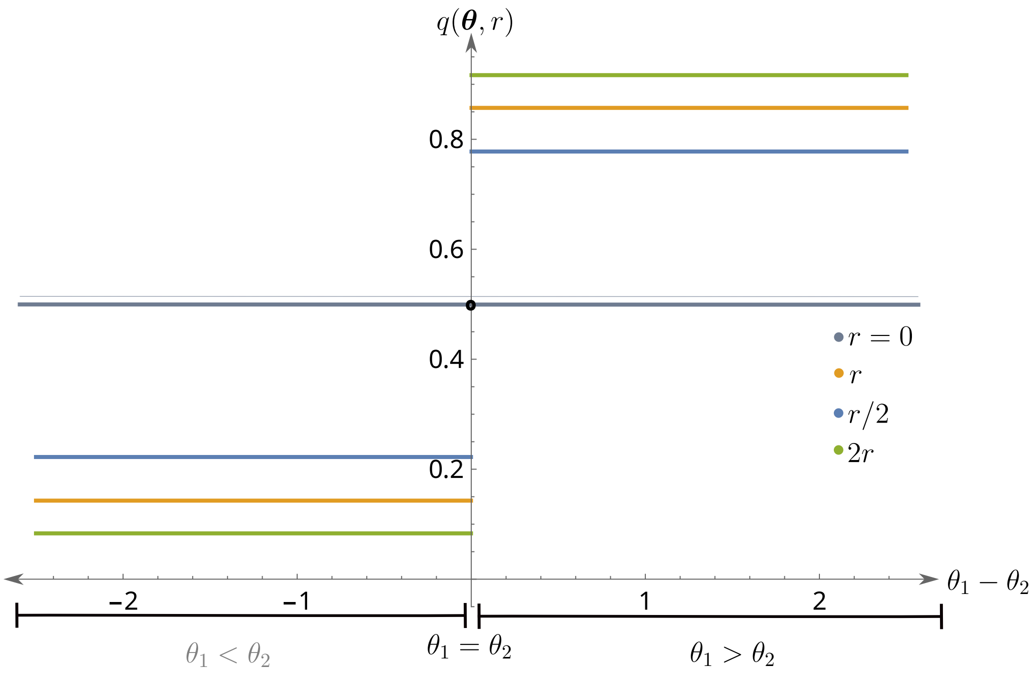


Figure 9: Reaction of ranking probabilities (58) to quality-difference and precision.

and second-ranked products (3), given the observed ranking, as:

$$\begin{aligned} \Lambda_1(r) &= \frac{2}{\bar{\theta}^2}\left(\int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1} q_1(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\left(1 - q_1(\tilde{\boldsymbol{\theta}}, r)\right)\tilde{\theta}_2 d\tilde{\theta}_2 d\tilde{\theta}_1\right) \\ &= \frac{\bar{\theta}\,(2\delta r + 3)}{3r + 6}, \\ \Lambda_2(r) &= \frac{2}{\bar{\theta}^2}\left(\int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1}\left(1 - q_1(\tilde{\boldsymbol{\theta}}, r)\right)\tilde{\theta}_1 d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\bar{\theta}}\int_0^{\tilde{\theta}_1} q_1(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_2 d\tilde{\theta}_2 d\tilde{\theta}_1\right) \\ &= \frac{\bar{\theta}\,(r + \delta 3)}{3r + 6}. \end{aligned} \tag{61}$$

The same steps as in the previous subsections, with standard parameters ($\bar{\theta} = 1, s = 10$) and $\delta = 10$, result in the two information release functions:

$$\rho_1^*(x) \approx \begin{cases} 0.531, & x > 0 \\ (0.531 - 0.068)/2, & x = 0; \\ -0.068, & x < 0 \end{cases} \quad \rho_2^*(x) \approx \begin{cases} -0.068, & x > 0 \\ (0.531 - 0.068)/2, & x = 0. \\ 0.531, & x < 0 \end{cases} \tag{62}$$

For the same parameter values of $\theta_1 = 3/4$, $\theta_2 = 1/4$ as in the previous subsection (and all other pairs for which $\theta_1 > \theta_2$), this yields the asymmetric equilibrium candidate

$$\rho_1^* \approx 0.531, \ \ \rho_2^* \approx -0.068, \text{ resulting in } r^* \approx 0.463, \tag{63}$$

which is verified in Figure 10. As in the previous subsection for the difference ranking, and in contrast

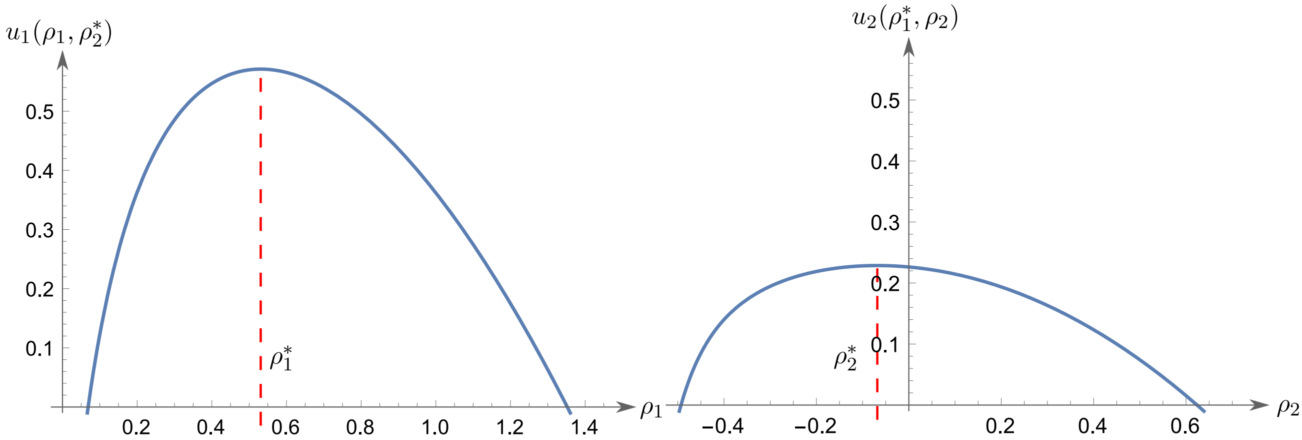


Figure 10: The two players' optimal choice of ρ_i in asymmetric equilibrium for piecewise-constant ranking.

to the ratio-form example, the lower-quality firm obfuscates the ranking in this example.

The left panel of Figure 11 fixes the heterogeneity at $s = 10$ and shows the firms' (almost everywhere constant) information emission as a function of x. Although information dissemination is piecewise constant in types, the right-hand panel of Figure 11 illustrates that information release is still a function of heterogeneity s. As with the quality dependency in the other examples (Figures 5, 8, and 14), there is a critical heterogeneity $\tilde{s} \approx 2.667$ beyond which ρ_2^* turns negative.

3.5.4 Noise-based ranking

In a final example of a ranking function satisfying our assumptions (Q), we now consider a form of noise-based assignment à la Lazear and Rosen (1981). In this case, the idea is that random "noise" in addition to qualities determines the ranking and total firm-emitted information controls the variance of this noise. More precisely, the two firms' probabilities of ranking first depend on the difference of qualities x (as in subsection 3.5.2) and independent normally distributed noise with expectation zero and variance[22]

$$r = \frac{1}{|\rho_1 + \rho_2|}. \tag{64}$$

[22] Notice that Proposition 2 ensures that $r > 0$ in equilibrium.

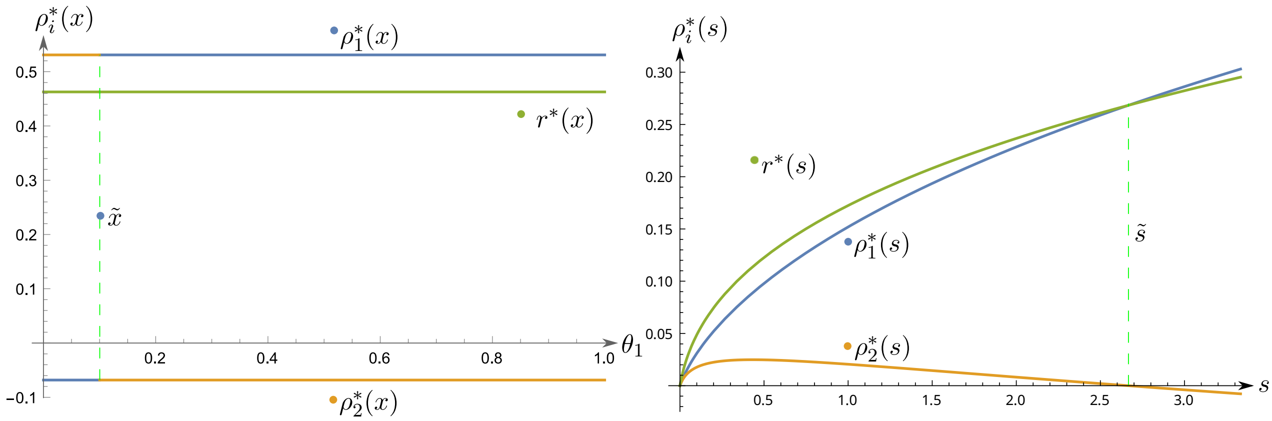


Figure 11: Left: information dissemination ρ_i^* for the piecewise-constant ranking, for parameters $d = 10$ and $s = 10$, as functions of $\theta_1 \in [0, 1]$ for fixed $\theta_2 = 1/10$: $\rho_1^*(x)$ is plotted blue, $\rho_2^*(x)$ is shown gold, and total information $r^*(x) = \rho_1^*(x) + \rho_2^*(x)$ is green. The vertical dashed line indicates the step at which qualities swap ranks. Right: information dissemination $\rho_i^*(s)$ as a function of the consumer heterogeneity s, $d = 10$, $\bar{\theta} = 1$. The dashed green line shows the critical heterogeneity $\tilde{s} \approx 2.667$ at which $\rho_2^* = 0$.

Under such a ranking, firm i is ranked first with probability

$$\Pr\left[\theta_i + \varepsilon_i \geq \theta_j + \varepsilon_j\right] = \Pr\left[\varepsilon_j - \varepsilon_i \leq \theta_i - \theta_j\right] \tag{65}$$

which can be expressed using the normally distributed random variable $\varepsilon = \varepsilon_j - \varepsilon_i$ with expectation zero and variance r for i.i.d. noise terms denoted by N_r (Lazear and Rosen, 1981). The responsiveness of this technology is illustrated in Figure 12.

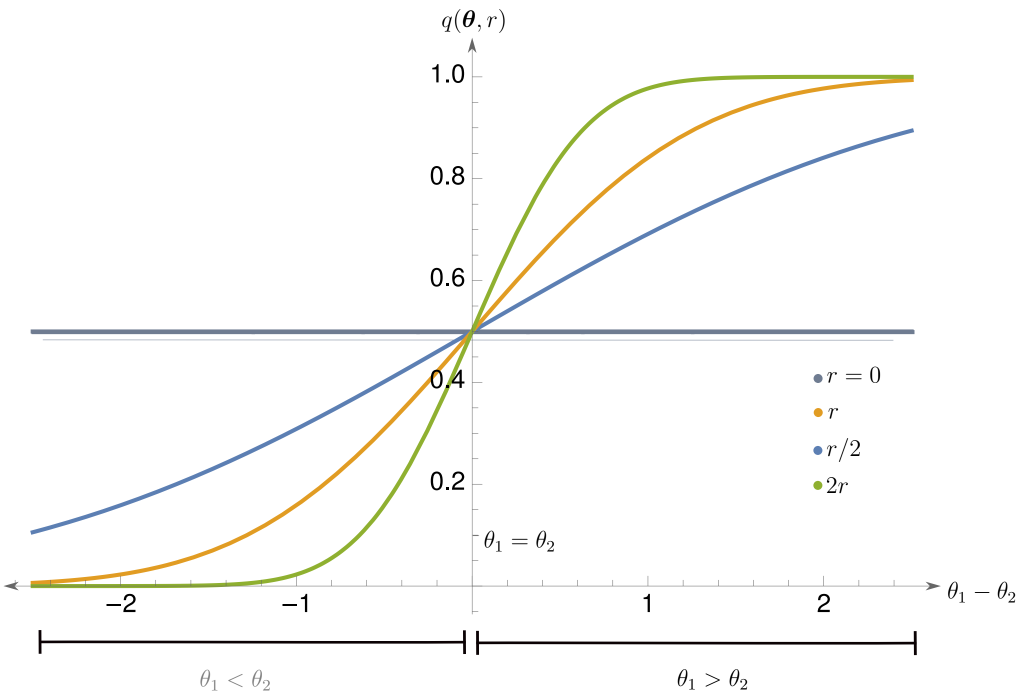


Figure 12: Reaction of noise-based ranking probabilities (65) to quality-difference and precision.

This yields firm 1's probability to be ranked first as

$$q_1(\boldsymbol{\theta}, r) = \Pr\left[\theta_i + \varepsilon_i \geq \theta_j + \varepsilon_j\right] = N_r(x), \tag{66}$$

for $x = \theta_1 - \theta_2$. Under the current assumptions this simplifies to

$$q_1(\boldsymbol{\theta}, r) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{-rx}{\sqrt{2}}\right)\right) \text{ for error function } \text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2} dt. \tag{67}$$

This function satisfies the increasing hazard rate property (28) whenever

$$\frac{\partial}{\partial r}\left(\frac{\partial q_1(\boldsymbol{\theta}, r)}{\partial r}/(1 - q_1(\boldsymbol{\theta}, r))\right) > 0 \;\Rightarrow\; e^{\frac{\theta^2 r^2}{2}}\sqrt{2\pi}xr\left(1 - \text{erf}\left(\frac{xr}{\sqrt{2}}\right)\right) < 2 \tag{68}$$

which can be numerically confirmed for $r > 0$ and $x > 0$. For the usual $q_1 = q$ and $q_2 = 1 - q$ in (3), $\tilde{x} = \tilde{\theta}_1 - \tilde{\theta}_2$, and joint order statistic $f_{(1,2:2)}$ from (4) simplifying to $2/\bar{\theta}^2$, consumer expectations under the noise-based ranking are:

$$\begin{aligned}
\Lambda_1(r) &= \frac{4\sqrt{\frac{2}{\pi}} + 3r^3\bar{\theta}^3 + e^{-\frac{1}{2}r^2\bar{\theta}^2}\sqrt{\frac{2}{\pi}}\left(r^2\bar{\theta}^2 - 4\right) + r\bar{\theta}\left(r^2\bar{\theta}^2 - 3\right)\text{erf}\left(\frac{r\bar{\theta}}{\sqrt{2}}\right)}{6r^3\bar{\theta}^2}, \\
\Lambda_2(r) &= \frac{-4\sqrt{\frac{2}{\pi}} + 3r^3\bar{\theta}^3 + e^{-\frac{1}{2}r^2\bar{\theta}^2}\sqrt{\frac{2}{\pi}}\left(4 - r^2\bar{\theta}^2\right) + r\bar{\theta}\left(3 - r^2\bar{\theta}^2\right)\text{erf}\left(\frac{r\bar{\theta}}{\sqrt{2}}\right)}{6r^3\bar{\theta}^2}.
\end{aligned} \tag{69}$$

Similar steps as for the other ranking functions lead to the same demand side behavior and the resulting supply-side firms' maximization problem (under mutually known $x = \theta_1 - \theta_2$) is

$$\max_{\rho_i} q_i(\boldsymbol{\theta}, r)P_1(r) + (1 - q_i(\boldsymbol{\theta}, r))\, P_2(r) - \frac{\rho_i^2}{2}. \tag{70}$$

Solving the resulting first-order conditions numerically in the standard example for $\bar{\theta} = 1$, $s = 10$, $\theta_1 = 3/4$, and $\theta_2 = 1/4$ gives

$$\rho_1^* \approx 0.363,\;\; \rho_2^* \approx 0.191,\;\; \text{implying that } r^* \approx 0.553. \tag{71}$$

Together with global cost convexity, Figure 13 verifies this candidate as pure strategy equilibrium.

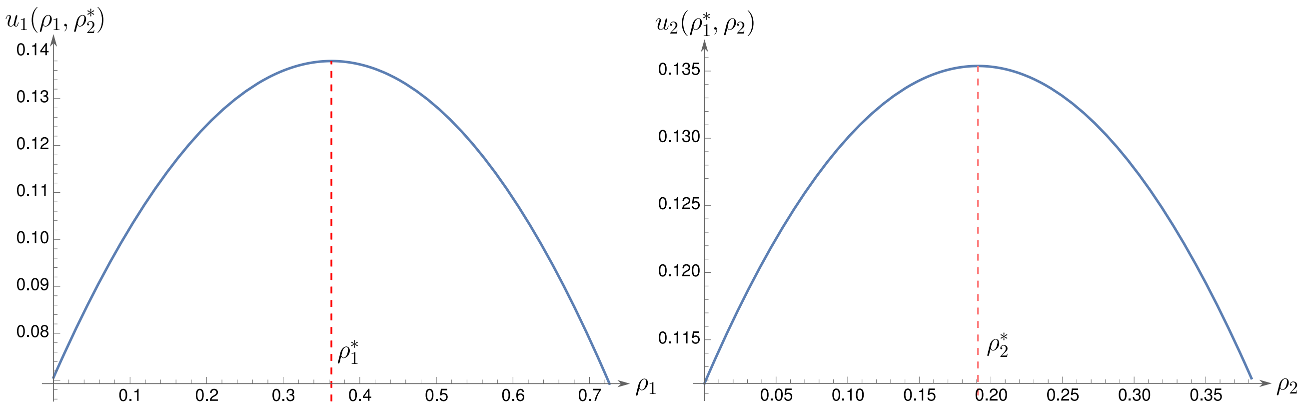


Figure 13: The two players' optimal choice of ρ_i in asymmetric equilibrium for noise-based ranking.

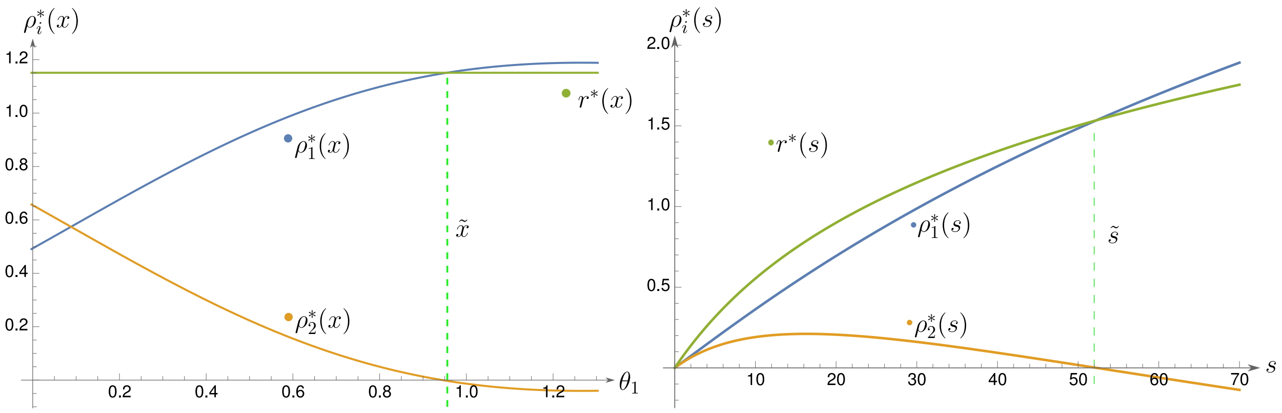


Figure 14: Left: information dissemination ρ_i^* for the noise-based ranking, heterogeneity $s = 35$, as functions of $\theta_1 \in [0, 1]$ for fixed $\theta_2 = 1/10$: $\rho_1^*(x)$ is plotted blue, $\rho_2^*(x)$ is shown gold, and total information $r^*(x) = \rho_1^*(x) + \rho_2^*(x)$ is green. The vertical dashed line indicates the difference $\tilde{x}$ at which $\rho_2^*(x) = 0$. Right: information dissemination $\rho_i^*(s)$ as a function of the consumer heterogeneity s, parameters $d = 10$, $\bar{\theta} = 1$. The dashed green line shows the critical heterogeneity $\tilde{s} \approx 30.0$ at which $\rho_2^* = 0$.

As for the ratio-form, and in contrast to the difference ranking example, the lower-quality firm improves the ranking in this example. The left panel of Figure 14 fixes the heterogeneity at $s = 35$ and shows information emission as a function of x.

As in the other examples, for sufficiently wide type-difference, there exists a point $\tilde{x}$ at which the lower-quality firm turns to obfuscation. A similar effect is illustrated in the right-hand panel of Figure 14 for a sufficiently large heterogeneity s.

4 Concluding remarks

We study a novel class of integrated market interactions in which firms compete on the informational content of a ranking which labels otherwise indistinguishable products. We establish a set of economically meaningful properties and show that equilibrium existence is unproblematic for a wide class of ubiquitous contest success functions. Besides defining a novel interaction on the basis of the precision of observed information and characterizing the ensuing equilibria, our principal results include: 1) The modeled credence markets generally provide useful overall information to the consumers. 2) There are widely applicable conditions under which the lower-quality firm turns to strategic obfuscation (i.e., the release of unhelpful information, or "fake news") if the quality spread between product becomes too large or the heterogeneity of consumers too high. 3) The maximum differentiation principle is weakened if the underlying information can be used by consumers to competitively differentiate between products. 4) Conditions on the circumstances under which credence good labeling enhances welfare over an unlabeled market.

Many further interesting applications match our model's structure besides the examples discussed in the Introduction. One example is the current debate around the Eco-score food label, implemented by the French authorities with a consortium of socio-economic actors. The Eco-score labels products from A (preferred, green) to E (to be avoided), based on an aggregation of several environmental criteria (e.g., water or energy consumption, level of biodiversity conservation, pesticides, etc.). The number of criteria used to calculate the Eco-score is shown on the label but consumers cannot observe the evaluation process or results for each criterion. Consumers only observe the number of criteria underlying the ranking, i.e., the precision of the ranking, and the realization of the ranking, i.e., the Eco-score the product receives. The Eco-score introduction has seen considerable debate between various market players. In particular, food companies have been reported to lobby for the inclusion of criteria that ensure intensive farming is given a good Eco-score (Stiletto et al., 2023). Further applications include political party fundraising with parties informing potential donors of their platforms if elected, and pharmaceutical drug testing, marketing, and advertising. In all these examples, the possibility for differentiation through some form of labeling seems crucial for a firm's ex-ante quality investment incentives, which otherwise may not be valued by the market.

Appendix

Proof of Proposition 1.

The proof is derived from Gabszewicz et al. (1981). Since our setup is slightly different, we state the full proof here which establishes that p_1^*, p_2^* are the unique maximizers of the functions P_1, P_2 defined in (7).

1. Consider firm 1's problem (7) and denote by p_1^* a solution to the corresponding first-order condition

$$\frac{\partial P_1}{\partial p_1} = -\frac{p_1 G'\left(\hat{\mu}_2^1\right)}{\Lambda_1(r) - \Lambda_2(r)} - G\left(\hat{\mu}_2^1\right) + G(s) = 0. \tag{72}$$

Consider the second derivative of P_1 with respect to p_1

$$\frac{\partial P_1}{\partial^2 p_1} = -\frac{p_1 G''\left(\hat{\mu}_2^1\right)}{\left(\Lambda_1(r) - \Lambda_2(r)\right)^2} - \frac{2G'\left(\hat{\mu}_2^1\right)}{\Lambda_1(r) - \Lambda_2(r)}. \tag{73}$$

This is smaller than 0 whenever

$$-\frac{2G'\left(\hat{\mu}_2^1\right)}{\Lambda_1(r) - \Lambda_2(r)} < \frac{p_1 G''\left(\hat{\mu}_2^1\right)}{\left(\Lambda_1(r) - \Lambda_2(r)\right)^2}. \tag{74}$$

Multiplication with $(\Lambda_1(r) - \Lambda_2(r))^2 > 0$ gives

$$-2G'\left(\hat{\mu}_2^1\right)\left(\Lambda_1(r) - \Lambda_2(r)\right) < p_1 G''\left(\hat{\mu}_2^1\right). \tag{75}$$

For distributions G with strictly increasing hazard rate (8) we have

$$\frac{d}{d\mu}\left(\frac{G'(\mu)}{1 - G(\mu)}\right) > 0 \iff -\frac{G'(\mu)^2}{1 - G(\mu)} < G''(\mu). \tag{76}$$

Bounding the right-hand side of (75) and simplification gives

$$2\left(\Lambda_1(r)-\Lambda_2(r)\right) > p_1\frac{G'(\mu)}{1-G(\mu)}. \tag{77}$$

Set $p_1 = p_1^*$. Solving (72) for p_1^* gives

$$p_1^* = \frac{\left(\Lambda_1(r)-\Lambda_2(r)\right)\left(G(s)-G\left(\hat{\mu}_2^1\right)\right)}{G'\left(\hat{\mu}_1\right)}. \tag{78}$$

Inserting this back into (77), together with $G(s)=1$, gives

$$2\left(\Lambda_1(r)-\Lambda_2(r)\right) > \left(\Lambda_1(r)-\Lambda_2(r)\right), \tag{79}$$

which is true since $\left(\Lambda_1(r)-\Lambda_2(r)\right) > 0$. Hence, at every critical point determined by (72), P_1 is strictly concave. Since P_1 is continuous, this establishes the claimed result.

2. Consider firm 2's problem (7) and denote by p_2^* a solution to the corresponding first-order condition

$$\frac{\partial P_2}{\partial p_2} = G\left(\hat{\mu}_2^1\right) - G\left(\hat{\mu}_3^2\right) - p_2\left(\frac{G'\left(\hat{\mu}_2^1\right)}{\Lambda_1(r)-\Lambda_2(r)} + \frac{G'\left(\hat{\mu}_3^2\right)}{\Lambda_2(r)}\right) = 0. \tag{80}$$

The second derivative equals

$$\frac{\partial P_2}{\partial p_2} = p_2\left(\frac{G''\left(\hat{\mu}_2^1\right)}{\left(\Lambda_1(r)-\Lambda_2(r)\right)^2} - \frac{G''\left(\hat{\mu}_3^2\right)}{\Lambda_2(r)^2}\right) - 2\left(\frac{G'\left(\hat{\mu}_2^1\right)}{\Lambda_1(r)-\Lambda_2(r)} + \frac{G'\left(\hat{\mu}_3^2\right)}{\Lambda_2(r)}\right). \tag{81}$$

Define

$$M = G\left(\hat{\mu}_2^1\right) - G\left(\hat{\mu}_3^2\right), \quad \chi_1 = \frac{-1}{\Lambda_1(r)-\Lambda_2(r)}, \quad \chi_2 = \frac{1}{\Lambda_2(r)} \tag{82}$$

and observe that $\chi_1 < 0$ and $\chi_2 > 0$ by Lemma 1. Inserting these definitions into (80), we simplify the first order condition to

$$M + p_2\chi_1 G'\left(\hat{\mu}_2^1\right) - p_2\chi_2 G'\left(\hat{\mu}_3^2\right) = 0. \tag{83}$$

Solving for p_2 establishes

$$p_2^* = \frac{M}{\chi_2 G'\left(\hat{\mu}_3^2\right) - \chi_1 G'\left(\hat{\mu}_2^1\right)} > 0 \tag{84}$$

by the fact that $M > 0, \chi_1 < 0, \chi_2 > 0$ and $G'(\cdot) > 0$. Setting $p_2 = p_2^*$, the second derivative (81) can be represented using (82) by

$$\chi_1\left(2G'\left(\hat{\mu}_2^1\right) + p_2^*\chi_1 G''\left(\hat{\mu}_2^1\right)\right) - \chi_2\left(2G'\left(\hat{\mu}_3^2\right) + p_2^*\chi_2 G''\left(\hat{\mu}_3^2\right)\right). \tag{85}$$

Consider the two expressions in parentheses. We want to establish that:

(a) $(2G'(\hat{\mu}_2^1) + p_2^*\chi_1 G''(\hat{\mu}_2^1)) > 0$. By $\chi_1 < 0$, and (84), we have

$$2G'\left(\hat{\mu}_2^1\right) + p_2^*\chi_1 G''\left(\hat{\mu}_2^1\right) \geq 2G'\left(\hat{\mu}_2^1\right) + p_2^*\chi_1 \left|G''\left(\hat{\mu}_2^1\right)\right| > 0. \tag{86}$$

Solving the first-order condition (83) for χ_1 gives

$$\chi_1 = \frac{p_2^*\chi_2 G'\left(\hat{\mu}_3^2\right) - M}{p_2^* G'\left(\hat{\mu}_2^1\right)}. \tag{87}$$

Plugging this back into (86) gives

$$\begin{aligned} 2G'\left(\hat{\mu}_2^1\right) + p_2^*\left|G''\left(\hat{\mu}_2^1\right)\right| \frac{p_2^*\chi_2 G'\left(\hat{\mu}_3^2\right) - M}{p_2^* G'\left(\hat{\mu}_2^1\right)} = \\ 2G'\left(\hat{\mu}_2^1\right) - M\left|G''\left(\hat{\mu}_2^1\right)\right| \frac{1}{G'\left(\hat{\mu}_2^1\right)} + p_2^*\chi_2\left|G''\left(\hat{\mu}_2^1\right)\right| \frac{G'\left(\hat{\mu}_3^2\right)}{G'\left(\hat{\mu}_2^1\right)} > \\ 2G'\left(\hat{\mu}_2^1\right) - M\left|G''\left(\hat{\mu}_2^1\right)\right| \frac{1}{G'\left(\hat{\mu}_2^1\right)} > 0 \end{aligned} \tag{88}$$

by the fact that $\chi_2 > 0, |G''(\hat{\mu}_2^1)| > 0$, and $G'(\cdot) > 0$ together with (84). Take the last inequality, rearranging terms, multiplying by $G'(\hat{\mu}_2^1) > 0$, and applying the definition of M yields

$$2G'\left(\hat{\mu}_2^1\right)^2 > \left(G\left(\hat{\mu}_2^1\right) - G\left(\hat{\mu}_3^2\right)\right)\left|G''\left(\hat{\mu}_2^1\right)\right|. \tag{89}$$

We simplify to

$$2G'\left(\hat{\mu}_2^1\right)^2 > G\left(\hat{\mu}_2^1\right)\left|G''\left(\hat{\mu}_2^1\right)\right|, \tag{90}$$

since, $G(\hat{\mu}_3^2)|G''(\hat{\mu}_2^1)| > 0$. The above inequality then follows again using (76).

(b) $2G'(\hat{\mu}_3^2) + p_2^*\chi_2 G''(\hat{\mu}_3^2) > 0$. Observe that

$$G'\left(\hat{\mu}_3^2\right) + p_2^*\chi_2 G''\left(\hat{\mu}_3^2\right) > 0 \tag{91}$$

whenever $G''(\hat{\mu}_3^2) \geq 0$, by the fact that $G'(\hat{\mu}_3^2) > 0$, $p_2^* > 0$, and $\chi_2 > 0$. For the case of $G''(\hat{\mu}_3^2) < 0$ take again the first-order condition (83) and solve for χ_2, giving in return

$$\chi_2 = \frac{p_2^*\chi_1 G'\left(\hat{\mu}_2^1\right) + M}{p_2^* G'\left(\hat{\mu}_3^2\right)}. \tag{92}$$

Plugging this back into (91) gives

$$\begin{aligned} G'\left(\hat{\mu}_3^2\right) + p_2^* G''\left(\hat{\mu}_3^2\right) \frac{p_2^*\chi_1 G'\left(\hat{\mu}_2^1\right) + M}{p_2^* G'\left(\hat{\mu}_3^2\right)} > 0 \\ G'\left(\hat{\mu}_3^2\right) + M\frac{G''\left(\hat{\mu}_3^2\right)}{G'\left(\hat{\mu}_3^2\right)} + \frac{p_2^* G''\left(\hat{\mu}_3^2\right)\chi_1 G'\left(\hat{\mu}_2^1\right)}{G'\left(\hat{\mu}_3^2\right)} > 0 \end{aligned} \tag{93}$$

The left-hand side is again greater than

$$G'\left(\hat{\mu}_3^2\right) + G''\left(\hat{\mu}_3^2\right)\frac{M}{G'\left(\hat{\mu}_3^2\right)} > 0 \tag{94}$$

whenever $G''(\hat{\mu}_3^2) < 0$, by the fact that $\chi_1 < 0$ and $G'(\cdot) > 0, p_2^* > 0$. The above inequality then again follows by using the definition M and the increasing hazard rate property (76).

The above properties, together with $\chi_1 < 0, \chi_2 > 0$, establish (85). Thus, at any critical point determined by (83), P_2 is strictly concave. Since P_2 is continuous, this completes the proof. □

Proof of Lemma 1.

1. We start by proving (11), $\Lambda_1(r) + \Lambda_2(r) = \mathbb{E}[\Theta_{(1:2)} + \Theta_{(2:2)}]$. This follows immediately from $q_1(\tilde{\boldsymbol{\theta}}, r) + q_2(\tilde{\boldsymbol{\theta}}, r) = 1$ (Q1), through

$$\begin{aligned}\Lambda_1(r) + \Lambda_2(r) &= \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(q_1(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + (1 - q_1(\tilde{\boldsymbol{\theta}}, r))\tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 \\ &\quad + \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(q_2(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + (1 - q_2(\tilde{\boldsymbol{\theta}}, r))\tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 \\ &= \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(\left(q_1(\tilde{\boldsymbol{\theta}}, r) + q_2(\tilde{\boldsymbol{\theta}}, r)\right) \tilde{\theta}_1 \right. \\ &\quad \left. + \left(q_1(\tilde{\boldsymbol{\theta}}, r) + q_2(\tilde{\boldsymbol{\theta}}, r)\right) \tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 \\ &= \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(\tilde{\theta}_1 + \tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 = \mathbb{E}[\Theta_{(1:2)} + \Theta_{(2:2)}].\end{aligned} \tag{95}$$

Splitting integrals and changing the order of integration gives

$$\begin{aligned}&\int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \tilde{\theta}_1 f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\tilde{\theta}_1} \int_0^{\bar{\theta}} \tilde{\theta}_2 f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_1 d\tilde{\theta}_2 \\ &= \int_0^{\bar{\theta}} \tilde{\theta}_1 \int_0^{\tilde{\theta}_1} f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\tilde{\theta}_1} \tilde{\theta}_2 \int_0^{\bar{\theta}} f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_1 d\tilde{\theta}_2.\end{aligned} \tag{96}$$

The definition of marginal densities gives

$$\begin{aligned}&\int_0^{\bar{\theta}} \tilde{\theta}_1 \int_0^{\tilde{\theta}_1} f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 + \int_0^{\tilde{\theta}_1} \tilde{\theta}_2 \int_0^{\bar{\theta}} f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_1 d\tilde{\theta}_2 \\ &= \int_0^{\bar{\theta}} \tilde{\theta}_1 f_{(1:2)}(\tilde{\theta}_1) d\tilde{\theta}_1 + \int_0^{\tilde{\theta}_1} \tilde{\theta}_2 f_{(2:2)}(\tilde{\theta}_2) d\tilde{\theta}_2 = \mathbb{E}[\Theta_{(1:2)}] + \mathbb{E}[\Theta_{(2:2)}],\end{aligned} \tag{97}$$

where the last equality follows by the definition of expectation and $\tilde{\theta}_1 \geq \tilde{\theta}_2$.

2. Consider (12), $\mathbb{E}[\Theta_{(1:2)}] \geq \Lambda_1(r) \geq \hat{\theta}/2 > \Lambda_2(r) \geq \mathbb{E}[\Theta_{(2:2)}] > 0$:

   - We start with $\mathbb{E}[\Theta_{(1:2)}] \geq \Lambda_1(r)$. Using the joint density (4) to calculate $\mathbb{E}[\Theta_{(1:2)}]$, the first inequality reduces to

$$\begin{aligned}&\int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \tilde{\theta}_1 f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 \\ &\geq \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(q_1(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + \left(1 - q_1(\tilde{\boldsymbol{\theta}}, r)\right) \tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1.\end{aligned} \tag{98}$$

By monotonicity of the integral it is sufficient to show

$$\tilde{\theta}_1 f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) \geq \left(q_1(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + \left(1 - q_1(\tilde{\boldsymbol{\theta}}, r)\right)\tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}). \quad (99)$$

Since $f_{(1,2:2)} > 0$, $q_1(\tilde{\boldsymbol{\theta}}, r) + q_2(\tilde{\boldsymbol{\theta}}, r) = 1$, and $\tilde{\theta}_1 > \tilde{\theta}_2$, this confirms the inequality.

- For $\Lambda_1(r) \geq \hat{\theta}/2$ we use again the joint density (45) to calculate $\hat{\theta}/2$ and, following the same steps as above, we need to confirm

$$\left(q_1(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + \left(1 - q_1(\tilde{\boldsymbol{\theta}}, r)\right)\tilde{\theta}_2\right) \geq \frac{\tilde{\theta}_1 + \tilde{\theta}_2}{2}. \quad (100)$$

Since $q_1(\tilde{\boldsymbol{\theta}}, r) \geq 1/2$ for $\tilde{\theta}_1 \geq \tilde{\theta}_2$ (Q2 & Q3), this confirms the inequality.

- For $\hat{\theta}/2 \geq \Lambda_2(r)$, the same reasoning leads us to confirm

$$\frac{\tilde{\theta}_1 + \tilde{\theta}_2}{2} \geq \left(q_2(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + \left(1 - q_2(\tilde{\boldsymbol{\theta}}, r)\right)\tilde{\theta}_2\right) \quad (101)$$

in which substituting $q_2(\tilde{\boldsymbol{\theta}}, r) = 1 - q_1(\tilde{\boldsymbol{\theta}}, r)$ (Q1) together with $\tilde{\theta}_1 \geq \tilde{\theta}_2$ confirms the inequality.

- $\Lambda^2(r) > 0$ follows again from monotonicity. It suffices to show that

$$q_2(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + \left(1 - q_2(\tilde{\boldsymbol{\theta}}, r)\right)\tilde{\theta}_2 > 0, \quad (102)$$

which obviously holds since $\tilde{\theta}_1 \geq \tilde{\theta}_2 > 0$.

3. Considering (13), the limits

$$\lim_{r \to \infty} \Lambda_1(r) = \mathbb{E}[\Theta_{(1:2)}], \quad \lim_{r \to \infty} \Lambda_2(r) = \mathbb{E}[\Theta_{(2:2)}] \quad (103)$$

follow directly rom (Q1) and (Q3) and application of the monotone convergence theorem.

4. Consider (14),

$$\Lambda_1^{(n)}(r) = \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(\tilde{\theta}_1 - \tilde{\theta}_2\right) \frac{\partial^n q_1(\tilde{\boldsymbol{\theta}}, r)}{\partial r^n} f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\boldsymbol{\theta}}, \quad \forall n \in \mathbb{N}^+. \quad (104)$$

$\Lambda_1'(r)$ is computed using the Leibniz integral rule on fixed integration limits, i.e.,

$$\begin{aligned}
\Lambda_1'(r) &= \frac{\partial}{\partial r}\left(\int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left[q_1(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + (1 - q_1(\tilde{\boldsymbol{\theta}}, r))\tilde{\theta}_2\right] f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1\right) \\
&= \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \frac{\partial}{\partial r}\left(\left[q_1(\tilde{\boldsymbol{\theta}}, r)\tilde{\theta}_1 + (1 - q_1(\tilde{\boldsymbol{\theta}}, r))\tilde{\theta}_2\right] f_{(1,2:2)}(\tilde{\boldsymbol{\theta}})\right) d\tilde{\theta}_2 d\tilde{\theta}_1 \\
&= \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(\frac{\partial q_1(\tilde{\boldsymbol{\theta}}, r)}{\partial r}\tilde{\theta}_1 - \frac{\partial q_1(\tilde{\boldsymbol{\theta}}, r)}{\partial r}\tilde{\theta}_2\right) f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1 \\
&= \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(\tilde{\theta}_1 - \tilde{\theta}_2\right) \frac{\partial q_1(\tilde{\boldsymbol{\theta}}, r)}{\partial r} f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1.
\end{aligned} \quad (105)$$

Any higher derivatives $\Lambda_1^{(n)}(r)$ then follow inductively

$$\Lambda_1^{(n)}(r) = \int_0^{\bar{\theta}} \int_0^{\tilde{\theta}_1} \left(\tilde{\theta}_1 - \tilde{\theta}_2\right) \frac{\partial^n q_1(\tilde{\boldsymbol{\theta}}, r)}{\partial r^n} f_{(1,2:2)}(\tilde{\boldsymbol{\theta}}) d\tilde{\theta}_2 d\tilde{\theta}_1. \tag{106}$$

5. Concerning (15), $\Lambda_1(0) = \Lambda_2(0) = \hat{\theta}/2$ follows directly from (Q3). □

Proof of Lemma 2.
The inequality (16) is an extension of the statement that $\mathbb{E}[\Theta_{i:n}]$ is bounded, whatever the form of its distribution F. By David and Nagaraja (2003, Inequalities 4.2.6), for any distribution exhibiting a finite mean-variance pair (m, σ^2), it is the case that

$$\mathbb{E}[\Theta_{1:n}] \leq m + \frac{(n-1)\,\sigma}{\sqrt{(2n-1)}} \quad \text{and likewise} \quad \mathbb{E}[\Theta_{n:n}] \geq m - \frac{(n-1)\,\sigma}{\sqrt{(2n-1)}.} \tag{107}$$

Since the support of F is bounded, its mean and variance are finite. For the case of $n = 2$, combining inequalities gives the desired property (16). □

Proof of Lemma 3.
We restrict attention to the case of uniformly distributed consumer tastes $G(\mu) = \mu/s$ and write $\Lambda_1'(r) = \partial\Lambda_1(r)/\partial r$ as well as $\Lambda_1''(r) = \partial^2\Lambda_1(r)/\partial r^2$.

1. Consider (22), $P_1(r) > P_2(r) > 0$ and $P_1(0) = P_2(0) = 0$. From (21), we have to show that

$$\begin{aligned} P_1(r) &= 4s \frac{\left(2\Lambda_1(r) - \hat{\theta}\right) \Lambda_1(r)^2}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2} \\ &> s \frac{\left(2\Lambda_1(r) - \hat{\theta}\right) \left(\hat{\theta} - \Lambda_1(r)\right) \Lambda_1(r)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2} = P_2(r) > 0. \end{aligned} \tag{108}$$

   Since $s > 0$, we know that $(2\Lambda_1(r) - \hat{\theta}) > 0$ and $(\hat{\theta} - 5\Lambda_1(r))^2 > 0$ by (11) & (12). Simplifying on both sides yields

$$4\Lambda_1(r)^2 > \left(\hat{\theta} - \Lambda_1(r)\right) \Lambda_1(r) > 0 \tag{109}$$

   which is indeed true by (11) & (12).

   To see that $P_1(0) = P_2(0) = 0$, take again (21) and substitute $\Lambda_1(0) = \hat{\theta}/2$, from Lemma 1, resulting in the denominator $9\hat{\theta}/4 > 0$. Substituting into the first term in the numerator of both prizes, i.e, $(2\Lambda_1(r) - \hat{\theta}) = 0$, yields $P_1(0) = P_2(0) = 0$.

2. Consider (23), $P_1'(r) > 0, P_1''(r) > 0$ as well as $P_1'(r) > P_2'(r)$.

- $P_1'(r) > 0$: The derivative of $P_1(r)$ with respect to r is

$$P_1{}'(r) = 8s\frac{\Lambda_1(r)\left(\hat{\theta}^2 - 3\hat{\theta}\Lambda_1(r) + 5\Lambda_1(r)^2\right)}{\left(5\Lambda_1(r) - \hat{\theta}\right)^3}\Lambda_1'(r). \tag{110}$$

Dividing by $8s > 0$ and multiplying by $(5\Lambda_1(r) - \hat{\theta})^3 > 0$ by (12) gives

$$\Lambda_1(r)\left(\hat{\theta}^2 - 3\hat{\theta}\Lambda_1(r) + 5\Lambda_1(r)^2\right)\Lambda_1'(r) > 0. \tag{111}$$

We have $\Lambda_1(r) > 0$ by (12) and $\Lambda_1'(r) > 0$ for $r > 0$ by (14) together with (Q2). Hence, it is sufficient to show that

$$\left(\hat{\theta}^2 - 3\hat{\theta}\Lambda_1(r) + 5\Lambda_1(r)^2\right) > 0, \tag{112}$$

which holds by (12).

- $P_1''(r) < 0$: Take $P_1(r)$, differentiating twice with respect to r gives

$$\begin{aligned} P_1''(r) = -8s &\left(\frac{\hat{\theta}^2\left(\hat{\theta} + 4\Lambda_1(r)\right)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^4}\Lambda_1'(r)^2\right. \\ &\left.+\frac{\Lambda_1(r)\left(\hat{\theta} - 5\Lambda_1(r)\right)\left(\hat{\theta}^2 - 3\hat{\theta}\Lambda_1(r) + 5\Lambda_1(r)^2\right)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^4}\Lambda_1''(r)\right) < 0. \end{aligned} \tag{113}$$

Dividing by $-8s < 0$ and multiplying by $(\hat{\theta} - 5\Lambda_1(r))^4 > 0$ by (12) gives

$$\begin{aligned} &\hat{\theta}^2\left(\hat{\theta} + 4\Lambda_1(r)\right)\Lambda_1'(r)^2 + \\ &\quad\Lambda_1(r)\left(\hat{\theta} - 5\Lambda_1(r)\right)\left(\hat{\theta}^2 - 3\hat{\theta}\Lambda_1(r) + 5\Lambda_1(r)^2\right)\Lambda_1''(r) > 0. \end{aligned} \tag{114}$$

Clearly, $\hat{\theta}^2(\hat{\theta} + 4\Lambda_1(r))\Lambda_1'(r)^2 > 0$, since $\hat{\theta} > 0, \Lambda_1(r) > 0$ and $\Lambda_1'(r) > 0$ for $r > 0$ by Lemma 1 and (Q1) and (Q2). Hence, it is sufficient to show that

$$\Lambda_1(r)\left(\hat{\theta} - 5\Lambda_1(r)\right)\left(\hat{\theta}^2 - 3\hat{\theta}\Lambda_1(r) + 5\Lambda_1(r)^2\right)\Lambda_1''(r) \geq 0. \tag{115}$$

Observe that $\Lambda_1(r) > 0$ and $(\hat{\theta}^2 - 3\hat{\theta}\Lambda_1(r) + 5\Lambda_1(r)^2) > 0$ again by (12). Thus, we need to show that

$$\left(\hat{\theta} - 5\Lambda_1(r)\right)\Lambda_1''(r) \geq 0. \tag{116}$$

Since $(\hat{\theta} - 5\Lambda_1(r)) < 0$ follows again from (12) and $\Lambda_1''(r) \leq 0$ follows from (14) together with (Q6), this proves the inequality.

- $P_1'(r) > P_2'(r)$: Take $P_1(r)$ and $P_2(r)$, differentiating with respect to r gives

$$\begin{aligned} {P_1}'(r) &= 8s\frac{\Lambda_1(r)\left(\hat\theta^2 - 3\hat\theta\Lambda_1(r) + 5\Lambda_1(r)^2\right)}{\left(5\Lambda_1(r)-\hat\theta\right)^3}\Lambda_1'(r) \\ &> s\frac{\hat\theta^3 - \Lambda_1(r)\left(\hat\theta^2 - 6\hat\theta\Lambda_1(r) + 10\Lambda_1(r)^2\right)}{\left(5\Lambda_1(r)-\hat\theta\right)^3}\Lambda_1'(r) = {P_2}'(r). \end{aligned} \tag{117}$$

  Multiplying both sides by $(5\Lambda_1(r)-\hat\theta)^3/s\Lambda_1'(r) > 0$ (established by Lemma 1) gives

$$\begin{aligned} 8\Lambda_1(r)\left(\hat\theta^2 - 3\hat\theta\Lambda_1(r) + 5\Lambda_1(r)^2\right) > \hat\theta^3 - \Lambda_1(r)\left(\hat\theta^2 - 6\hat\theta\Lambda_1(r)\right. \\ \left. + 10\Lambda_1(r)^2\right). \end{aligned} \tag{118}$$

  Adding $\Lambda_1(r)(\hat\theta^2 - 6\hat\theta\Lambda_1(r) + 10\Lambda_1(r)^2)$ on both hand sides simplifies the above expression to

$$\Lambda_1(r)\left(7\hat\theta^2 - 18\hat\theta\Lambda_1(r) + 30\Lambda_1(r)^2\right) > \hat\theta^3 \tag{119}$$

  which is always satisfied since $\Lambda_1(r) > \hat\theta/2 > 0$ for any $r > 0$, by Lemma 1.

3. Consider (24): $P_2'(r) > 0, P_2''(r) > 0$. First observe that by (A1) and (16) we have

$$\frac{\mathbb{E}[\Theta_{(1:2)}]}{\mathbb{E}[\Theta_{(2:2)}]} \leq 2. \tag{120}$$

   Adding $\mathbb{E}[\Theta_{(1:2)}]$ on both sides gives

$$\frac{3}{2}\,\mathbb{E}[\Theta_{(1:2)}] \leq \hat\theta. \tag{121}$$

   By (12) we substitute $\Lambda_1(r) < \mathbb{E}[\Theta_{(1:2)}]$ on the left-hand side and obtain

$$3\Lambda_1(r) \leq 2\hat\theta. \tag{122}$$

   This fact establishes

$$3\Lambda_1(r) \leq 2\hat\theta \Rightarrow \hat\theta^3 - \hat\theta^2\Lambda_1(r) + 6\hat\theta\Lambda_1(r)^2 - 10\Lambda_1(r)^3 > 0. \tag{123}$$

   - $P_2'(r) > 0$: Differentiating $P_2(r)$ with respect to r gives

$${P_2}'(r) = s\frac{\hat\theta^3 - \Lambda_1(r)\left(\hat\theta^2 - 6\hat\theta\Lambda_1(r) + 10\Lambda_1(r)^2\right)}{\left(5\Lambda_1(r)-\hat\theta\right)^3}\Lambda_1'(r) > 0. \tag{124}$$

     Dividing by $s > 0$ and $\Lambda_1'(r) > 0$ for $r > 0$ by (14) together with (Q2), and multiplying by $(5\Lambda_1(r)-\hat\theta)^3 > 0$ by (12) gives

$$\hat\theta^3 > \Lambda_1(r)\left(\hat\theta^2 - 6\hat\theta\Lambda_1(r) + 10\Lambda_1(r)^2\right). \tag{125}$$

Rearranging and multiplying out gives

$$\hat{\theta}^3 - \hat{\theta}^2\Lambda_1(r) + 6\hat{\theta}\Lambda_1(r)^2 - 10\Lambda_1(r)^3 > 0 \tag{126}$$

which is implied by (123).

- $P_2''(r) < 0$: Take $P_2(r)$, differentiating twice with respect to r gives

$$\begin{aligned} P_2''(r) = & -2s\frac{\hat{\theta}^2\left(7\hat{\theta} + \Lambda_1(r)\right)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^4}\Lambda_1'(r)^2 \\ & -s\frac{\left(\hat{\theta} - 5\Lambda_1(r)\right)\left(\hat{\theta}^3 - \hat{\theta}^2\Lambda_1(r) + 6\hat{\theta}\Lambda_1(r)^2 - 10\Lambda_1(r)^3\right)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^4}\Lambda_1''(r) < 0. \end{aligned} \tag{127}$$

Dividing by $-s < 0$ and multiplying by $(\hat{\theta} - 5\Lambda_1(r))^4 > 0$ by (12) gives

$$\begin{aligned} & 2\hat{\theta}^2\left(7\hat{\theta} + \Lambda_1(r)\right)\Lambda_1'(r)^2 + \left(\hat{\theta} - 5\Lambda_1(r)\right) \\ & \left(\hat{\theta}^3 - \hat{\theta}^2\Lambda_1(r) + 6\hat{\theta}\Lambda_1(r)^2 - 10\Lambda_1(r)^3\right)\Lambda_1''(r) > 0. \end{aligned} \tag{128}$$

For the first term we have $2\hat{\theta}^2(7\hat{\theta} + \Lambda_1(r))\Lambda_1'(r)^2 > 0$, since $\hat{\theta} > 0, \Lambda_1(r) > 0$ and $\Lambda_1'(r) > 0$ for $r > 0$ by Lemma 1, (Q1) and (Q2). It is thus sufficient to show that

$$\left(\hat{\theta} - 5\Lambda_1(r)\right)\left(\hat{\theta}^3 - \hat{\theta}^2\Lambda_1(r) + 6\hat{\theta}\Lambda_1(r)^2 - 10\Lambda_1(r)^3\right)\Lambda_1''(r) \geq 0. \tag{129}$$

Since $(\hat{\theta} - 5\Lambda_1(r)) < 0$ follows again from (12), and $\Lambda_1''(r) \leq 0$ follows from (14) together with (Q6), this gives

$$\hat{\theta}^3 - \hat{\theta}^2\Lambda_1(r) + 6\hat{\theta}\Lambda_1(r)^2 - 10\Lambda_1(r)^3 \geq 0. \tag{130}$$

Implication (123) completes the proof. □

Proof of Proposition 2.

Firm i chooses information ρ_i. We drop the subscripts on rankings, writing $q_1 = q$ and $q_2 = 1 - q$, and abuse notation by writing $q'(\boldsymbol{\theta}, \rho_1 + \rho_2)$ for the partial derivative of $q(\boldsymbol{\theta}, \rho_1 + \rho_2)$ with respect to ρ_1 or ρ_2. The first-order condition of firm 1's objective (1) with respect to ρ_1 is then

$$q(\boldsymbol{\theta}, r){P_1}'(r) + (1 - q(\boldsymbol{\theta}, r))\,{P_2}'(r) + \frac{\partial q(\boldsymbol{\theta}, r)}{\partial \rho_1}P_1(r) - \frac{\partial q(\boldsymbol{\theta}, r)}{\partial \rho_1}P_2(r) - c'(|\rho_1|) = 0 \tag{131}$$

and the same condition for firm 2 is

$$(1 - q(\boldsymbol{\theta}, r))\,{P_1}'(r) + q(\boldsymbol{\theta}, r){P_2}'(r) - \frac{\partial q(\boldsymbol{\theta}, r)}{\partial \rho_2}P_1(r) + \frac{\partial q(\boldsymbol{\theta}, r)}{\partial \rho_2}P_2(r) - c'(|\rho_2|) = 0. \tag{132}$$

Summing the two for $q'(\boldsymbol{\theta}, r) = \partial q(\boldsymbol{\theta}, r)/\partial\rho_1 = \partial q(\boldsymbol{\theta}, r)/\partial\rho_2$ gives (25). □

Proof of Proposition 3.

As in the previous proof, we drop the ranking-subscripts, writing $q_1 = q$ and $q_2 = 1 - q$, and abuse notation by writing $q'(\boldsymbol{\theta}, \rho_1 + \rho_2)$ and $P_i'(r,s) = \partial P_i(r,s)/\partial r$ for the partial derivatives with respect to ρ_1, ρ_2, and r. We first show that firm 1's benefit function, defined as

$$\phi_1(\rho_1) = q(\boldsymbol{\theta}, \rho_1 + \rho_2^*)P_1(\rho_1 + \rho_2^*) + (1 - q(\boldsymbol{\theta}, \rho_1 + \rho_2^*))\, P_2(\rho_1 + \rho_2^*), \tag{133}$$

is strictly increasing in r. Taking the derivative with respect to r gives

$$\begin{aligned}\frac{\partial \phi_1(\rho_1)}{\partial \rho_1} &= q(\boldsymbol{\theta}, \rho_1 + \rho_2^*){P_1}'(\rho_1 + \rho_2^*) + (1 - q(\boldsymbol{\theta}, \rho_1 + \rho_2^*))\, {P_2}'(\rho_1 + \rho_2^*) \\ &\quad + q'(\boldsymbol{\theta}, \rho_1 + \rho_2^*)\,(P_1(r) - P_2(r)) > 0,\end{aligned} \tag{134}$$

in which the inequality follows from ${P_1}'(r) > {P_2}'(r)$ and $P_1(r) > P_2(r)$, for any $r > 0$, as a consequence of Lemma 3, together with assumptions (Q).

Next, we show that player 1's objective (1) is concave in ρ_1, for any given ρ_2^*. Recall that (26) establishes $\rho_1 \geq |\rho_2|$, yielding $r \geq 0$ in equilibrium. By definition, $-c(|\rho_1|)$ is strictly concave. Since the sum of two concave functions is again concave, it is sufficient to establish concavity of firm 1's benefit function (133) on the interval $r \in [0, \infty)$. Arbitrarily fixing $\rho_2^* \in \mathbb{R}$, this is the case iff:

$$\lambda\phi_1(u) + (1-\lambda)\,\phi_1(v) \leq \phi_1(\lambda u + (1-\lambda)\,v), \quad \text{for } u, v \in [0,\infty),\ \lambda \in [0,1]. \tag{135}$$

We use the fact that the consumers' expectations sum up to a constant—Lemma 1, (11)—and replace $\Lambda_2(r) = \hat{\theta} - \Lambda_1(r)$. For any $\theta_1 \geq \theta_2$, (Q6) together with property (14) establishes concavity of $\Lambda_1(r)$, i.e.,

$$\lambda\Lambda_1(u) + (1-\lambda)\,\Lambda_1(v) \leq \Lambda_1(\lambda u + (1-\lambda)\,v). \tag{136}$$

We redefine contest prizes as functions of the consumers' expectation $\Lambda_1(r)$ on the basis of (21):

$$P_1(\Lambda_1(r)) = 4s\frac{\Lambda_1(r)^2\left(2\Lambda_1(r) - \hat{\theta}\right)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2},\ P_2(\Lambda_1(r)) = s\frac{\left(\hat{\theta} - \Lambda_1(r)\right)\Lambda_1(r)\left(2\Lambda_1(r) - \hat{\theta}\right)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2}. \tag{137}$$

As $r > 0$ and Λ_1 is bounded by $\hat{\theta}/2 < \Lambda_1(r) < 2\hat{\theta}/3$, in which the lower bound follows from (12) and the upper bound is a consequence of (A1), obtained by repeating steps (120)–(122). Differentiation with respect to $\Lambda_1(r) \in [\hat{\theta}/2, 2\hat{\theta}/3)$ yields the inequalities:

$$\frac{\partial P_1(\Lambda_1(r))}{\partial \Lambda_1(r)} = 8s\frac{\Lambda_1(r)\left(\hat{\theta}^2 - 3\hat{\theta}\Lambda_1(r) + 5\Lambda_1(r)^2\right)}{\left(5\Lambda_1(r) - \hat{\theta}\right)^3} \geq 0, \tag{138}$$

$$\frac{\partial P_2(\Lambda_1(r))}{\partial \Lambda_1(r)} = s\frac{\hat{\theta}^3 - \Lambda_1(r)\left(\hat{\theta}^2 - 6\hat{\theta}\Lambda_1(r) + 10\Lambda_1(r)^2\right)}{\left(5\Lambda_1(r) - \hat{\theta}\right)^3} \geq 0, \tag{139}$$

$$\frac{\partial P_1(\Lambda_1(r))}{\partial \Lambda_1(r)} > \frac{\partial P_2(\Lambda_1(r))}{\partial \Lambda_1(r)}. \tag{140}$$

Since we have $\Lambda_1'(r) > 0$ by (14) and (Q2), inequality (138) follows directly by inspection of ${P_1}'(r) > 0$ (110), inequality (139) follows by ${P_2}'(r) > 0$ (124), and inequality (140) follows by ${P_1}'(r) > {P_2}'(r)$ (117). Moreover we have

$$\frac{\partial^2 P_1(\Lambda_1(r))}{\partial \Lambda_1(r)^2} = -\frac{8s\hat{\theta}^2\left(\hat{\theta}+4\Lambda_1(r)\right)}{\left(\hat{\theta}-5\Lambda_1(r)\right)^4} < 0 \tag{141}$$

which follows from $s > 0$, and $\hat{\theta} > \Lambda_1(r) > \hat{\theta}/2 > 0$ by Lemma 1. Similarly, the same arguments establish

$$\frac{\partial^2 P_2(\Lambda_1(r))}{\partial \Lambda_1(r)^2} = -\frac{2s\hat{\theta}^2\left(7\hat{\theta}+\Lambda_1(r)^2\right)}{\left(\hat{\theta}-5\Lambda_1(r)\right)^4} < 0 \tag{142}$$

Hence, the inequalities (138)–(142) establish concavity of the form

$$\lambda P_1(\Lambda_1(u)) + (1-\lambda)P_1(\Lambda_1(v)) < P_1\left(\lambda\Lambda_1(u) + (1-\lambda)\Lambda_1(v)\right), \tag{143}$$

$$\lambda P_2(\Lambda_1(u)) + (1-\lambda)P_2(\Lambda_1(v)) < P_2(\lambda\Lambda_1(u) + (1-\lambda)\Lambda_1(v)). \tag{144}$$

We similarly redefine firm 1's benefit function (133) as $\phi_1(r, \Lambda_1(r))$. We use and establish two properties of the benefit functions. Firstly, $\phi_1(r, \Lambda_1(r))$ is strictly increasing in $\Lambda_1(r)$:

$$\frac{\partial \phi_1(r, \Lambda_1(r))}{\partial \Lambda_1(r)} = q(\boldsymbol{\theta}, r)P_1'(\Lambda_1(r)) + (1-q(\boldsymbol{\theta}, r))P_2'(\Lambda_1(r)) > 0 \tag{145}$$

which is confirmed by the fact that $P_1'(\Lambda_1(r)) > P_2'(\Lambda_1(r)) > 0$ and (Q). Secondly, $\phi_1(r, \Lambda_1(r))$ is strictly increasing in r:

$$\begin{aligned}\frac{\partial \phi_1(r, \Lambda_1(r))}{\partial r} &= q'(\boldsymbol{\theta}, r)\left(P_1(\Lambda_1(r)) - P_2(\Lambda_1(r))\right) \\ &\quad + (q(\boldsymbol{\theta}, r)P_1'(\Lambda_1(r)) + (1-q(\boldsymbol{\theta}, r)P_2'(\Lambda_1(r)))\Lambda_1'(r) > 0\end{aligned} \tag{146}$$

which follows from (145) together with $P_1(\Lambda_1(r)) > P_2(\Lambda_1(r)) > 0$, (Q), and the fact that $\Lambda_1'(r) > 0$ as a consequence of Lemma 1. Inserting the definition of $\phi_1(r, \Lambda_1(r))$ into (135) yields on the lhs

$$\begin{aligned}&\lambda q(\boldsymbol{\theta}, u)P_1(\Lambda_1(u)) + (1-\lambda)\, q(\boldsymbol{\theta}, v)P_1(\Lambda_1(v)) \\ &\quad + \lambda\,(1-q(\boldsymbol{\theta}, u))\, P_2(\Lambda_1(u)) + (1-\lambda)\,(1-q(\boldsymbol{\theta}, v))\, P_2(\Lambda_1(v))\end{aligned} \tag{147}$$

and on the right-hand side

$$\begin{aligned}&q(\boldsymbol{\theta}, \lambda u + (1-\lambda)\, v)P_1(\Lambda_1(\lambda u + (1-\lambda)\, v)) \\ &\quad + (1-q(\boldsymbol{\theta}, \lambda u + (1-\lambda)\, v))\, P_2(\Lambda_1(\lambda u + (1-\lambda)\, v)).\end{aligned} \tag{148}$$

We start on the right-hand side; since $q(\boldsymbol{\theta}, r)$ is weakly concave for $\theta_1 \geq \theta_2$, we have

$$\lambda q(\boldsymbol{\theta}, u) + (1-\lambda)\, q(\boldsymbol{\theta}, v) \leq q(\boldsymbol{\theta}, (\lambda u + (1-\lambda)\, v)). \tag{149}$$

Combined with (146), the right-hand side is thus greater than

$$\begin{aligned}&(\lambda q(\boldsymbol{\theta}, u) + (1-\lambda)\, q(\boldsymbol{\theta}, v))\, P_1(\Lambda_1(\lambda u + (1-\lambda)\, v)) \\ &\quad + (1-(\lambda q(\boldsymbol{\theta}, u)\,(1-\lambda)\, q(\boldsymbol{\theta}, v)))\, P_2(\Lambda_1(\lambda u + (1-\lambda)\, v)).\end{aligned} \tag{150}$$

Rearranging terms yields

$$\begin{aligned}&\lambda q(\boldsymbol{\theta},u)P_1(\Lambda_1(\lambda u+(1-\lambda)\,v))+(1-\lambda)\,q(\boldsymbol{\theta},v)P_1(\Lambda_1(\lambda u+(1-\lambda)\,v))\\&\quad+\lambda\,(1-q(\boldsymbol{\theta},u))\,P_2(\Lambda_1(\lambda u+(1-\lambda)v))+(1-\lambda)\,(1-q(\boldsymbol{\theta},v))\,P_2(\Lambda_1(\lambda u+(1-\lambda)\,v)).\end{aligned} \tag{151}$$

Using (143) and (144) again on the right-hand side, substitution and simplification yields

$$\begin{aligned}&(\lambda q(\boldsymbol{\theta},u)+q(\boldsymbol{\theta},v)-\lambda q(\boldsymbol{\theta},v))\,(\lambda P_1(\Lambda_1(u))+(1-\lambda)\,P_1(\Lambda_1(v)))\\&\quad+(1-\lambda q(\boldsymbol{\theta},u)-q(\boldsymbol{\theta},v)+\lambda q(\boldsymbol{\theta},v))\,(\lambda P_2(\Lambda_1(u))+(1-\lambda)\,P_2(\Lambda_1(v)))\,.\end{aligned} \tag{152}$$

Simplifying both sides of the modified inequality (147) $\leq$ (152) gives

$$\lambda\,(1-\lambda)\,(q(\boldsymbol{\theta},u)-q(\boldsymbol{\theta},v))\,(P_1(\Lambda_1(u))-P_1(\Lambda_1(v))+P_2(\Lambda_1(v))-P_2(\Lambda_1(u)))\leq 0. \tag{153}$$

Assume $u<v$. Since $q(r,\theta)$ is strictly increasing in r by (Q), we have $q(\boldsymbol{\theta},u)<q(\boldsymbol{\theta},v)$. Using $\lambda\in[0,1]$ and inserting definitions (137) yields

$$\begin{aligned}&4s\frac{\Lambda_1(u)^2\left(2\Lambda_1(u)-\hat{\theta}\right)}{\left(\hat{\theta}-5\Lambda_1(u)\right)^2}+s\frac{\left(\hat{\theta}-\Lambda_1(v)\right)\Lambda_1(v)\left(2\Lambda_1(v)-\hat{\theta}\right)}{\left(\hat{\theta}-5\Lambda_1(v)\right)^2}\\&\geq 4s\frac{\Lambda_1(v)^2\left(2\Lambda_1(v)-\hat{\theta}\right)}{\left(\hat{\theta}-5\Lambda_1(v)\right)^2}+s\frac{\left(\hat{\theta}-\Lambda_1(u)\right)\Lambda_1(u)\left(2\Lambda_1(u)-\hat{\theta}\right)}{\left(\hat{\theta}-5\Lambda_1(u)\right)^2}.\end{aligned} \tag{154}$$

Converting terms to a common denominator and factoring in the numerator yields

$$\frac{s\left(2\Lambda_1(u)\left(5\Lambda_1(v)-\hat{\theta}\right)+\hat{\theta}\left(\hat{\theta}-2\Lambda_1(v)\right)\right)(\Lambda_1(u)-\Lambda_1(v))}{\left(\hat{\theta}-5\Lambda_1(u)\right)\left(\hat{\theta}-5\Lambda_1(v)\right)}\geq 0. \tag{155}$$

Since $u>v\Rightarrow\Lambda_1(u)>\Lambda_1(v)$ and $s>0$, the above expression simplifies to

$$\frac{\left(2\Lambda_1(u)\left(5\Lambda_1(v)-\hat{\theta}\right)+\hat{\theta}\left(\hat{\theta}-2\Lambda_1(v)\right)\right)}{\left(\hat{\theta}-5\Lambda_1(u)\right)\left(\hat{\theta}-5\Lambda_1(v)\right)}\geq 0. \tag{156}$$

Splitting the fraction and simplification gives

$$\frac{2}{5}+\frac{3\hat{\theta}^2}{5\left(\hat{\theta}-5\Lambda_1(u)\right)\left(\hat{\theta}-5\Lambda_1(v)\right)}\geq 0. \tag{157}$$

Using $\hat{\theta}>\Lambda_1(u)\geq\hat{\theta}/2$ gives

$$2\Lambda_1(u)\geq\frac{\hat{\theta}\left(\hat{\theta}-2\Lambda_1(v)\right)}{\left(\hat{\theta}-5\Lambda_1(v)\right)} \tag{158}$$

which is confirmed by the fact that

$$2\Lambda_1(u)\geq\frac{2\hat{\theta}}{3}\geq\frac{\hat{\theta}\left(\hat{\theta}-2\Lambda_1(v)\right)}{\left(\hat{\theta}-5\Lambda_1(v)\right)}, \tag{159}$$

follows from $\hat{\theta}>\Lambda_1(v)\geq\hat{\theta}/2$. The case of $v<u$ is entirely symmetric and the special case of $u=v$ is immediate since $u=v\Rightarrow(q(\boldsymbol{\theta},u)-q(\boldsymbol{\theta},v))=0$ in (153), completing the argument. ☐

Proof of Proposition 4.

As in the previous proofs, we drop the subscripts on rankings, writing $q_1 = q$ and $q_2 = 1 - q$, and abuse notation by writing $q'(\boldsymbol{\theta}, \rho_1 + \rho_2)$ for the partial derivatives with respect to ρ_1, ρ_2, or r. Firm 2's objective is given by:

$$u_2(\boldsymbol{\theta}, \rho_1^* + \rho_2) = (1 - q(\boldsymbol{\theta}, \rho_1^* + \rho_2))P_1(\rho_1^* + \rho_2) + q(\boldsymbol{\theta}, \rho_1^* + \rho_2))P_2(\rho_1^* + \rho_2) \quad - c(|\rho_2|) \tag{160}$$

in which ρ_1^* denotes the optimal response of firm 1 from Proposition 3. Following Quah and Strulovici (2012), global strict quasi-concavity of firm 2's objective is ensured whenever

$$\begin{aligned} -u_2'(\boldsymbol{\theta}, \rho_1^* + \rho_2) &= -\frac{\partial u_2(\boldsymbol{\theta}, \rho_1^* + \rho_2)}{\partial \rho_2} \\ &= -(1 - q(\boldsymbol{\theta}, r))\, {P_1}'(r) - q(\boldsymbol{\theta}, r){P_2}'(r) \\ &\quad + q'(\boldsymbol{\theta}, r)P_1(r) - q'(\boldsymbol{\theta}, r)P_2(r) + c'(|\rho_2|) \end{aligned} \tag{161}$$

satisfies the strict single crossing condition in the first derivative

$$-u_2'(\boldsymbol{\theta}, \rho_1^* + \rho_2') > 0 \Rightarrow -u'(\boldsymbol{\theta}, \rho_1^* + \rho_2'') > 0 \text{ whenever } \rho_2'' > \rho_2'. \tag{162}$$

Denote by

$$\xi(\rho_2) = q'(\boldsymbol{\theta}, \rho_1^* + \rho_2)P_1(\rho_1^* + \rho_2) - (1 - q(\boldsymbol{\theta}, \rho_1^* + \rho_2)){P_1}'(\rho_1^* + \rho_2) \tag{163}$$

$$-\psi(\rho_2) = -q(\boldsymbol{\theta}, \rho_1^* + \rho_2){P_2}'(\rho_1^* + \rho_2) - q'(\boldsymbol{\theta}, \rho_1^* + \rho_2)P_2(\rho_1^* + \rho_2) \tag{164}$$

$$-{\phi_2}'(\rho_2) = \xi(\rho_2) - \psi(\rho_2) \tag{165}$$

and thus $-u'(\boldsymbol{\theta}, \rho_1^* + \rho_2) = -{\phi_2}'(\rho_2) + c'(|\rho_2|)$. In a first step we want to establish single crossing for the component function $-{\phi_2}'(\rho_2) = \xi(\rho_2) - \psi(\rho_2)$. Recall again that (26) establishes $\rho_1^* \geq |\rho_2|$, yielding $r \geq 0$ in equilibrium. It is thus sufficient to establish single crossing on the domain $r > 0$. Observe that $-\psi(\rho_2) < 0$ by definition (Q) and the fact that $P_2(r) > 0, {P_2}'(r) > 0$, and thus trivially satisfies strict single crossing. For $\xi(r)$, denote by

$$\xi_1(\rho_2) = q'(\boldsymbol{\theta}, r)P_1(r) \tag{166}$$

$$\xi_2(\rho_2) = -(1 - q(\boldsymbol{\theta}, r)){P_1}'(r). \tag{167}$$

Observe that $\xi_1(\rho_2) > 0$ and $\xi_2(\rho_2) < 0$ for any $r > 0$, by definition (Q) and Lemma 3. Hence, $\xi_1(\rho_2), \xi_2(\rho_2)$ both trivially satisfy single crossing. Applying Proposition 1 of Quah and Strulovici (2012), the sum $\xi(\rho_2) = \xi_1(\rho_2) + \xi_2(\rho_2)$ satisfies strict single crossing, whenever for $\xi_1(\rho_2) > 0$ and $\xi_2(\rho_2) < 0$

$$-\frac{\xi_2(\rho_2)}{\xi_1(\rho_2)} = \frac{(1 - q(\boldsymbol{\theta}, r))}{q'(\boldsymbol{\theta}, r)} \frac{{P_1}'(r)}{P_1(r)} \text{ is decreasing.} \tag{168}$$

Observe that ${P_1}'(r)/P_1(r)$ is positive and strictly decreasing due to Lemma 3, and $(1-q(\boldsymbol{\theta}, r))/q'(\boldsymbol{\theta}, r)$ is positive and strictly decreasing whenever q satisfies the increasing hazard rate property (28), which we assume. Since single crossing of the component functions $\xi(\rho_2), -\psi(\rho_2)$ is thus established, we can again apply Proposition 1 in Quah and Strulovici (2012), which states that the sum $-{\phi_2}'(\rho_2) = \xi(\rho_2) - \psi(\rho_2)$ satisfies strict single crossing if

$$-\frac{-\psi(\rho_2)}{\xi(\rho_2)} \text{ is decreasing, whenever } \xi(\rho_2) > 0 \text{ and } -\psi(\rho_2) < 0. \tag{169}$$

This is equivalent to

$$\frac{\xi(\rho_2)}{\psi(\rho_2)} \text{ is increasing, whenever } \xi(\rho_2) > 0 \text{ and } \psi(\rho_2) > 0. \tag{170}$$

Taking derivatives with respect to ρ_2 gives

$$\frac{\xi'(\rho_2)}{\psi(\rho_2)} \geq \frac{\xi(\rho_2)\psi'(\rho_2)}{\psi(\rho_2)^2} \text{ whenever } \xi(\rho_2) > 0 \text{ and } \psi(\rho_2) > 0 \tag{171}$$

and simplification yields

$$\frac{\xi'(\rho_2)}{\xi(\rho_2)} \geq \frac{\psi'(\rho_2)}{\psi(\rho_2)} \text{ whenever } \xi(\rho_2) > 0 \text{ and } \psi(\rho_2) > 0. \tag{172}$$

This is implied by taking definite integrals on both sides of the following form

$$\begin{aligned} \int_{-\rho_1^*}^{\rho_2} \frac{\xi'(x)}{\xi(x)} dx &\geq \int_{-\rho_1^*}^{\rho_2} \frac{\psi'(x)}{\psi(x)} dx \\ \log(\xi(\rho_2)) - \log(\xi(-\rho_1^*)) &\geq \log(\psi(\rho_2)) - \log(\psi(-\rho_1^*)) \\ \log(\xi(\rho_2)) - \log(\psi(\rho_2)) &\geq \log(\xi(-\rho_1^*)) - \log(\psi(-\rho_1^*)) \end{aligned} \tag{173}$$

whenever $\xi(\rho_2) > 0$ and $\psi(\rho_2) > 0$. Taking the exponential on both sides gives

$$\frac{\xi(\rho_2)}{\psi(\rho_2)} \geq \frac{\xi(-\rho_1^*)}{\psi(-\rho_1^*)} \text{ whenever } \xi(\rho_2) > 0 \text{ and } \psi(\rho_2) > 0. \tag{174}$$

Observe that on the right-hand side we have

$$\xi(-\rho_1^*) = q'(\boldsymbol{\theta}, 0)P_1(0) - (1 - q(\boldsymbol{\theta}, 0))P_1{}'(0) = -1/2P_1{}'(0) \tag{175}$$
$$\psi(-\rho_1^*) = q(\boldsymbol{\theta}, r)P_2{}'(0) + q'(\boldsymbol{\theta}, 0)P_2(0) = 1/2P_2{}'(0). \tag{176}$$

Plugging in the values on the right-hand side gives

$$\frac{\xi(\rho_2)}{\psi(\rho_2)} \geq -1/2\frac{P_1{}'(0)}{P_2{}'(0} \text{ whenever } \xi(\rho_2) > 0 \text{ and } \psi(\rho_2) > 0 \tag{177}$$

which is always true by the fact that $P_i(0) > 0$ as a consequence of Lemma 3 as well as $\xi(r) > 0$ and $\psi(r) > 0$.
With these two intermediate results in hand and the fact that $c'(|\rho_2|) > 0$ also trivially satisfies strict single crossing, we establish strict single crossing of $-u'(\boldsymbol{\theta}, \rho_2) = -\phi_2{}'(\rho_2) + c'(|\rho_2|)$ by applying again Proposition 1 in Quah and Strulovici (2012). This is the case if

$$\frac{\phi_2'(\rho_2)}{c'(|\rho_2|)} \text{ is decreasing, whenever } \phi_2{}'(\rho_2) > 0. \tag{178}$$

Taking derivatives with respect to ρ_2 gives

$$\frac{\phi_2{}''(\rho_2)}{c'(|\rho_2|)} \leq \frac{\phi_2{}'(\rho_2)|\rho_2|c''(|\rho_2|)}{c'(|\rho_2|)^2} \text{ whenever } \phi_2{}'(\rho_2) > 0. \tag{179}$$

Simplification yields

$$\frac{\phi_2{}''(\rho_2)}{\phi_2{}'(\rho_2)} \leq \frac{|\rho_2|c''(|\rho_2|)}{c'(|\rho_2|)} \text{ whenever } \phi_2{}'(\rho_2) > 0. \tag{180}$$

Observe that the right-hand side is strictly greater than 0 by convexity of $c(|\cdot|)$. (180) is implied by taking definite integrals on both sides

$$\begin{aligned} \int_{-\rho_1^*}^{\rho_2} \frac{{\phi_2}''(x)}{{\phi_2}'(x)}dx &\leq \int_{-\rho_1^*}^{\rho_2} \frac{|x|c''(|x|)}{c'(|x|)}dx \\ \log({\phi_2}'(\rho_2) - \log({\phi_2}'(-\rho_1^*)) &\leq \log(c'(|\rho_2|)) - \log(c'(|-\rho_1^*|)) \\ \log({\phi_2}'(\rho_2)) - \log(c'(|\rho_2|)) &\leq \log({\phi_2}'(-\rho_1^*)) - \log(c'(|\rho_1^*|)) \end{aligned} \tag{181}$$

whenever ${\phi_2}'(\rho_2) > 0$. Taking the exponential on both sides gives

$$\frac{{\phi_2}'(\rho_2)}{c'(|\rho_2|)} \leq \frac{{\phi_2}'(-\rho_1^*)}{c'(|\rho_1^*|)} \tag{182}$$

whenever ${\phi_2}'(\rho_2) > 0$. Moreover, observe that at any possible critical point we must have

$${\phi_2}'(\rho_2) = c'(|\rho_2|) \tag{183}$$

by inspection of firm 2's first-order condition. Hence, the left-hand side of (182) equals 1 at any possible equilibrium candidate. Furthermore, observe that

$$\begin{aligned} {\phi_2}'(-\rho_1^*) &= (1 - q(\boldsymbol{\theta}, 0)){P_1}'(0) + q(\boldsymbol{\theta}, 0){P_2}'(0) - q'(\boldsymbol{\theta}, 0)(P_1(0) - P_2(0)) \\ &= 1/2({P_1}'(0) + {P_2}'(0)). \end{aligned} \tag{184}$$

Since $c'(|\rho_1^*|) > 0$, we simplify to

$$c'(|\rho_1^*|) \leq 1/2({P_1}'(0) + {P_2}'(0)) \text{ whenever } {\phi_2}'(\rho_2) > 0. \tag{185}$$

By inspection of firm 1's first-order condition we have

$$\begin{aligned} c'(|\rho_1^*|) &= q(\boldsymbol{\theta}, \rho_1^* + \rho_2){P_1}'(\rho_1^* + \rho_2) + (1 - q(\boldsymbol{\theta}, \rho_1^* + \rho_2)){P_2}'(\rho_1^* + \rho_2) \\ &+ q'(\boldsymbol{\theta}, \rho_1^* + \rho_2)(P_1(\rho_1^* + \rho_2) - P_2(\rho_1^* + \rho_2)) = {\phi_1}'(\rho_1^* + \rho_2) \end{aligned} \tag{186}$$

which is strictly decreasing by concavity of firm 1's benefits $\phi_1(\rho_1^*, \rho_2)$ (as established in Proposition 3) and bounded above by $1/2({P_1}'(0)+{P_2}'(0))$, in the case of $-\rho_2 = \rho_1^*$, again by $q(\boldsymbol{\theta}, 0) = 1/2$ and $P_i(0) = 0$. Thus, at any possible critical point determined trough (183), firm 2's objective is strictly quasi-concave, and thus any critical point is a strict local maximum. From continuity of $u_2(\cdot)$ as a consequence of (Q6) we can therefore conclude that firm 2's utility function is globally strictly quasi-concave. An equilibrium with $r > 0$ determined trough (183) and (186) therefore exists. □

Proof of Proposition 5.

Notice that prizes (21) depend linearly on s while none of the other utility-components depend directly on s. Similarly, $P_1'(r)$ and $P_2'(r)$ (explicitly determined in Lemma 3, (110) and (124), respectively) are linear in s. Hence, firm 2's necessary condition (132) can be rewritten as a function of consumer heterogeneity s at the point of interest $r^* = \rho_1^* > 0$ with $\rho_2^* = 0$ (yielding $c'(|\rho_2^*|) = 0$, by assumption) as:

$$(1 - q(\boldsymbol{\theta}, \rho_1^*))\frac{{P_1}'(\rho_1^*)}{s} + q(\boldsymbol{\theta}, \rho_1^*)\frac{{P_2}'(\rho_1^*)}{s} - q'(\boldsymbol{\theta}, \rho_1^*)\frac{P_1(\rho_1^*)}{s} + q'(\boldsymbol{\theta}, \rho_1^*)\frac{P_2(\rho_1^*)}{s} = 0. \tag{187}$$

Define

$$\phi(\rho_1^*) = (1 - q(\boldsymbol{\theta}, \rho_1^*))\frac{P_1(\rho_1^*)}{s} \text{ and } \psi(\rho_1^*) = q(\boldsymbol{\theta}, \rho_1^*)\frac{P_2(\rho_1^*)}{s}. \tag{188}$$

Inserting these definitions, (187) holds whenever

$$\frac{\phi'(\rho_1^*)}{\psi'(\rho_1^*)} = -1. \tag{189}$$

At this point we apply Cauchy's mean value theorem (see, e.g., Apostol, 1974, Theorem 5.12), which states that, for functions ϕ and ψ, both continuous on some closed interval $[a, b]$ and differentiable on the open interval (a, b) with $a < b$, there exists some $\rho_1^* \in (a, b)$, such that

$$\frac{\phi'(\rho_1^*)}{\psi'(\rho_1^*)} = \frac{\phi(b) - \phi(a)}{\psi(b) - \psi(a)}, \tag{190}$$

provided that $\psi(b) \neq \psi(a)$ and $\psi'(\rho_1^*) \neq 0$. Note that ψ is strictly increasing and positive by Lemma 3 and Assumptions Q. Moreover, both functions are continuous and differentiable on any closed interval $[a, b]$, with $0 \leq a < b$. For (187) to hold for some ρ_1^*, it is thus sufficient to provide some interval $[a, b]$, with $a < b$, such that

$$\xi(b) = \frac{\phi(a) - \phi(b)}{\psi(b) - \psi(a)} = 1. \tag{191}$$

As next step, note that $\phi(\rho_1^*)$ attains a maximum since $P_1(\rho_1^*)/s$ is concave and increasing, as a consequence of Lemma 3 and the fact that $(1 - q(\boldsymbol{\theta}, \rho_1^*))$ is strictly decreasing and converges to 0 as ρ_1^* increases (by Assumptions Q). Denoting the maximizers of ϕ by $\hat{\rho}_1 = \arg\max_{\rho_1^*} \phi(\rho_1^*)$, we establish the following claims for $\hat{\rho}_1(\theta_2)$ and $\phi(\hat{\rho}_1(\theta_2), \theta_2)$ for arbitrarily fixed θ_1 with $\theta_1 > \theta_2$:

(C1) The value function $v(\theta_2) = \phi(\hat{\rho}_1(\theta_2), \theta_2)$ is strictly increasing in θ_2. Applying the envelope theorem (Mas-Colell et al., 1995, Theorem M.L.1) gives

$$\frac{\mathrm{d}v(\theta_2)}{\mathrm{d}\theta_2} = \left.\frac{\partial\phi(\rho_1, \theta_2)}{\partial\theta_2}\right|_{\rho_1 = \hat{\rho}_1(\theta_2)} = -\left.\frac{\partial q(\boldsymbol{\theta}, \rho_1)}{\partial\theta_2}\frac{P_1(\rho_1)}{s}\right|_{\rho_1 = \hat{\rho}_1(\theta_2)} \tag{192}$$

which is strictly increasing for $\theta_2 < \theta_1$, by the fact that $\partial q(\boldsymbol{\theta}, \rho_1)/\partial\theta_2 < 0$ by Assumptions Q, and $P_1(\rho_1) > 0$ by Lemma 3.

(C2) We claim that

$$\lim_{\theta_2 \to \theta_1} \hat{\rho}_1(\theta_2) = \infty \tag{193}$$

which follows from the fact that

$$\lim_{\theta_2 \to \theta_1} \phi(\hat{\rho}_1(\theta_2), \theta_2) = \frac{1}{2}\frac{P_1(\hat{\rho}_1)}{s}, \tag{194}$$

which is strictly increasing in $\hat{\rho}_1$ by Lemma 3.

(C3) We claim that

$$\lim_{\theta_2 \to \theta_1} \phi(\hat{\rho}_1(\theta_2), \theta_2) = \lim_{\rho_1 \to \infty} \frac{1}{2} \frac{P_1(\rho_1)}{s} \tag{195}$$

which follows as a consequence of (C1) and (C2), by the fact that the maximizer $\hat{\rho}_1(\theta_2)$ converges to infinity.

Now set $a = \hat{\rho}_1$. By continuity of $\phi(b)$ and $\psi(b)$, respectively, we apply the intermediate value theorem (see, e.g., Apostol, 1974, Theorem 4.33). It is sufficient for $\xi(b) = 1$ to establish that for an adequately chosen parameter pair $\theta_1 > \theta_2$ within $[0, \bar{\theta}]$

$$\exists b : \xi(b) < 1, \tag{196}$$
$$\exists b : \xi(b) > 1. \tag{197}$$

For the case of (196), we take the limit of $b \to \hat{\rho}_1$ in (191), resulting in

$$\lim_{b \to \hat{\rho}_1} \frac{\phi(\hat{\rho}_1) - \phi(b)}{\psi(b) - \psi(\hat{\rho}_1)} = 0, \tag{198}$$

directly following from the application of L'Hôpital's rule with

$$\lim_{b \to \hat{\rho}_1} \phi'(b) = 0,$$

by definition of the maximum in the numerator, and

$$\lim_{b \to \hat{\rho}_1} \psi'(b) > 0$$

in the denominator, by the fact that ψ is strictly increasing by (Q) together with Lemma 3.
For the case of (197), we take the limit of $b \to \infty$. Note that

$$\lim_{b \to \infty} \frac{\phi(\hat{\rho}_1) - \phi(b)}{\psi(b) - \psi(\hat{\rho}_1)} > \lim_{b \to \infty} \frac{\phi(\hat{\rho}_1) - \phi(b)}{\psi(b)}, \tag{199}$$

by the fact that ψ is strictly increasing. Hence for (197), we only need to establish

$$\lim_{b \to \infty} \frac{\phi(\hat{\rho}_1) - \phi(b)}{\psi(b)} \geq 1. \tag{200}$$

Note that

$$\lim_{b \to \infty} \phi(b) = 0,$$

by the fact that $P_1(r)/s$ is bounded and $(1 - q(\boldsymbol{\theta}, r))$ converges to zero. Furthermore, note that

$$\lim_{b \to \infty} \psi(b) = \lim_{b \to \infty} \frac{P_2(b)}{s}.$$

Hence, (200) simplifies to

$$\phi(\hat{\rho}_1) \geq \lim_{b \to \infty} \frac{P_2(b)}{s}. \tag{201}$$

But since $\phi(\hat{\rho}_1(\theta_2), \theta_2)$ converges to

$$\lim_{\rho_1 \to \infty} \frac{1}{2} \frac{P_1(\rho_1)}{s}$$

for diminishing type spreads $\boldsymbol{\theta}$ by (C3), it is sufficient to show that

$$\lim_{\rho_1 \to \infty} \frac{1}{2} \frac{P_1(\rho_1)}{s} \geq \lim_{b \to \infty} \frac{P_2(b)}{s}. \tag{202}$$

Taking this limit for the demand-side expectations (3) gives

$$\lim_{\rho_1 \to \infty} \Lambda_1(\rho_1) = \mathbb{E}[\Theta_{(1:2)}] \text{ and } \lim_{\rho_1 \to \infty} \Lambda_2(\rho_1) = \mathbb{E}[\Theta_{(2:2)}], \tag{203}$$

by Assumptions (Q) and the definition of order statistics. Inserting these into prizes (20) simplifies (202) to

$$\frac{2\left(\mathbb{E}[\Theta_{(1:2)}]\right)^2 \left(\mathbb{E}[\Theta_{(1:2)}] - \mathbb{E}[\Theta_{(2:2)}]\right)}{\left(\mathbb{E}[\Theta_{(2:2)}] - 4\,\mathbb{E}[\Theta_{(1:2)}]\right)^2} \geq \frac{\mathbb{E}[\Theta_{(1:2)}]\,\mathbb{E}[\Theta_{(2:2)}] \left(\mathbb{E}[\Theta_{(1:2)}] - \mathbb{E}[\Theta_{(2:2)}]\right)}{\left(\mathbb{E}[\Theta_{(2:2)}] - 4\,\mathbb{E}[\Theta_{(1:2)}]\right)^2}, \tag{204}$$

which is true by the fact that $\mathbb{E}[\Theta_{(1:2)}] > \mathbb{E}[\Theta_{(2:2)}] > 0$, completing the proof. □

Proof of Proposition 6.

The maximization of joint firm (cartel) benefits over costs solves

$$\max_{\rho_1, \rho_2} q_1(\boldsymbol{\theta}, r)\left(P_1(r) + P_2(r)\right) + \left(1 - q_1(\boldsymbol{\theta}, r)\right)\left(P_1(r) + P_2(r)\right) - c(|\rho_1|) - c(|\rho_2|) \tag{205}$$

which simplifies, using (Q1), to

$$\max_{\rho_1, \rho_2} P_1(r) + P_2(r) - c(|\rho_1|) - c(|\rho_2|). \tag{206}$$

The first-order conditions of this problem are

$$P_1{}'(r) + P_2{}'(r) = c'(|\rho_1|), \;\; P_1{}'(r) + P_2{}'(r) = c'(|\rho_2|). \tag{207}$$

The implied symmetry in costs simplifies the analysis since in a joint producer utility maximum (206) it must be the case that $\rho_1 = \rho_2$. We therefore rewrite (206) as the simplified symmetric problem

$$\max_{r} W_P(r) = P_1(r) + P_2(r) - 2c\left(|r/2|\right) \tag{208}$$

with the first-order condition

$$P_1{}'(r) + P_2{}'(r) = c'\left(|r/2|\right). \tag{209}$$

Since information emissions differ in equilibrium, (26), strict convexity of $c(\cdot)$ implies that the right-hand side of (209) is strictly smaller than the right-hand side of (25) for non-zero r. □

Proof of Proposition 7.

Consider the welfare of served consumer segments $W_H(r), W_L(r)$ in (29). We first show that equilibrium prices p_i^* are both strictly increasing in r. To see this, take equilibrium prices (18) and apply Lemma 1 to obtain

$$p_1^*(r) = 2s\frac{\Lambda_1(r)\left(2\Lambda_1(r) - \hat{\theta}\right)}{5\Lambda_1(r) - \hat{\theta}}, \;\; p_2^*(r) = s\frac{\left(2\Lambda_1(r) - \hat{\theta}\right)\left(\hat{\theta} - \Lambda_1(r)\right)}{5\Lambda_1(r) - \hat{\theta}}. \tag{210}$$

Taking the derivative of $p_1^*(r)$ with respect to r gives

$$\frac{\partial p_1^*(r)}{\partial r} = 2s \frac{\left(\hat{\theta}^2 - 4\hat{\theta}\Lambda_1(r) + 10\Lambda_1(r)^2\right)\Lambda_1'(r)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2} \tag{211}$$

which is positive by the fact that $\Lambda_1'(r) > 0$ and $2\Lambda_1(r) > \hat{\theta}$, as a consequence of Lemma 1. Taking the derivative of $p_2^*(r)$ with respect to r gives

$$\frac{\partial p_2^*(r)}{\partial r} = 2s \frac{\left(\hat{\theta}^2 + 2\hat{\theta}\Lambda_1(r) - 5\Lambda_1(r)^2\right)\Lambda_1'(r)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2}. \tag{212}$$

The same reasoning as above establishes the inequality

$$\hat{\theta}\left(\hat{\theta} + 2\Lambda_1(r)\right) > 5\Lambda_1(r)^2 \tag{213}$$

which is true by the fact that Assumption (A1) together with Lemma 2 ensure $3\Lambda_1(r) \leq \hat{\theta}$.
We proceed to show that the served consumer segments' welfare, defined over the equilibrium cutoff vector (19), also decrease in r. For the high type segment, $W_H(r)$ from (29), the argument follows by the fact that the upper bound, $\hat{\mu}_1^0(r) = s$, is constant and the lower bound $\hat{\mu}_1^2(r)$ increases in r. Differentiating $\hat{\mu}_1^2(r)$ with respect to r gives

$$\frac{\partial \hat{\mu}_1^2(r)}{\partial r} = 2s \frac{\hat{\theta}\Lambda_1'(r)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2} \tag{214}$$

which is positive by the fact that $\Lambda_1'(r) > 0$ and $2\Lambda_1(r) > \hat{\theta}$, as a consequence of Lemma 1. The argument for $W_L(r)$ follows by the fact that $\hat{\mu}_1^{2\prime}(r) < \hat{\mu}_2^{3\prime}(r)$. Taking derivatives of $\hat{\mu}_1^2(r)$, $\hat{\mu}_2^3(r)$ with respect to r gives:

$$2s \frac{\hat{\theta}\Lambda_1'(r)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2} < 3s \frac{\hat{\theta}\Lambda_1'(r)}{\left(\hat{\theta} - 5\Lambda_1(r)\right)^2} \tag{215}$$

which is always true. □

Proof of Proposition 8.
Computing the integrals of served consumer welfare (30) results in

$$W_H(r) + W_L(r) = \frac{s\Lambda_1(r)^2\left(4\Lambda_1(r) + 5\Lambda_2(r)\right)}{2\left(\Lambda_2(r) - 4\Lambda_1(r)\right)^2}. \tag{216}$$

Using (Q), the firms' utilities $u_1 + u_2$ simplify to

$$P_1(r) + P_2(r) - c(|\rho_1|) - c(|\rho_2|). \tag{217}$$

Adding the above two expressions to obtain total welfare (32), inserting contest prizes (20), and substituting $\Lambda_2 = \hat{\theta} - \Lambda_1$ yields

$$W(r) = \frac{s\Lambda_1(r)\left(11\Lambda_1(r)^2 + 3\Lambda_1(r)\hat{\theta} - 2\hat{\theta}^2\right)}{2\left(\hat{\theta} - 5\Lambda_1(r)\right)^2} - c(|\rho_1|) - c(|\rho_2|). \tag{218}$$

We now demonstrate that both partial derivatives of (218), evaluated at $\rho_1 = \rho_2 = 0 = r$, are positive. Evaluating the first derivative

$$\frac{\partial W(r)}{\partial \rho_1} = -s\frac{\left(2\hat{\theta}^3 + 4\hat{\theta}^2\Lambda_1(r) - 33\hat{\theta}\Lambda_1(r)^2 + 55\Lambda_1(r)\right)}{2\left(\hat{\theta} - 5\Lambda_1(r)\right)^3} - c'(\rho_1) \tag{219}$$

at $\rho_1 = \rho_2 = 0 = r$, using the fact that $\Lambda_1(0) = \hat{\theta}/2$ by Lemma 1, simplifies the derivative to

$$\frac{7}{18}s\Lambda_1{}'(0) - c'(0) \tag{220}$$

which is positive by the fact that $s > 0$, $c'(0) = 0$, and $\Lambda_1{}'(0) > 0$ by (14). Since the derivative with respect to ρ_2 enters (218) in exactly the same way as that for ρ_1, the same argument applies to firm 2, completing the proof. □

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