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Ambiguous Business Cycles, Recessions and Uncertainty: A Quantitative Analysis

Abstract

This paper investigates the effects of uncertainty on the macro economy by replicating its micro effects on individual subjective beliefs. In our model, the representative household has smooth ambiguity preferences and is uncertain about which scenario the economy will be in the next period: normal growth or recession. We anchor the ratio of expected utilities between the two scenarios through the empirical macroeconomic uncertainty index. The higher the macroeconomic uncertainty is, the deeper the recession that the household is expecting. Our estimations demonstrate that the smooth ambiguity model with an appropriate level of ambiguity aversion outperforms the benchmark model with no uncertainty in fitting the US output growth rate, especially during recessions. This holds true even when tested with out-of-sample forecasts. Our analyses show that the effect of uncertainty on the representative household's beliefs aligns with the corresponding empirical literature. Moreover, the Global Financial Crisis was associated with an increase in both risk aversion and ambiguity aversion, while the Dot-com Crisis only affected risk aversion.

JEL-Codes: E700, D800, D900, E100, E300.

Keywords: behavioural macro, uncertainty, estimated DSGE models.

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1 Introduction

Since the Global Financial Crisis, there has been increased interest in the effects of macroeconomic uncertainty[1] on the economy. However, macroeconomic models have had difficulty in capturing this effect, particularly during recessions (Born and Pfeifer, 2021; Ng and Wright, 2013; Wieland and Wolters, 2011). This can be attributed, in part, to a complex relationship between macroeconomic uncertainty and people's beliefs. For a starting point of the analysis, we focus on three empirical stylized facts: (1) macroeconomic uncertainty makes people more pessimistic (Bhandari et al., 2019; Bianchi et al., 2020; Born et al., 2018), (2) it can have both positive and negative effects on individual subjective uncertainty[2] (Glas, 2020; Piccillo and Poonpakdee, 2021), and (3) its effects are nonlinear, with a disproportionately stronger effect in the recessions (Jackson et al., 2020; Lhuissier and Tripier, 2021; Ng and Wright, 2013). Drawing upon these stylized facts, we study a real business cycle model featuring smooth ambiguity preferences. Our model is able to replicate the micro effects of macroeconomic uncertainty, which are characterized by the three stylized facts, and capture its macro effects on the economy, particularly in times of recession.

Our model is an extension of the smooth ambiguity model in Altug, Collard, Çakmaklı, Mukerji, and Özsöylev (2020), where risk aversion and ambiguity aversion are differentiated. Risk aversion implies that a higher standard deviation of stochastic shocks yields a lower expected utility, while ambiguity aversion results in a lower expected utility when a probability distribution cannot be assigned to future outcomes and the agent's expectations regarding the future outcomes cannot be reconciled (Ilut and Schneider, 2022; Klibanoff et al., 2005; Marinacci, 2015). The novel feature we add is the introduction of a macroeconomic uncertainty index as a variable in the model, enabling us to trace the effects of uncertainty with a greater precision. The estimation reveals that our models significantly perform better in fitting output growth rates when compared to the benchmark model with no uncertainty and no ambiguity aversion. Furthermore, the model's out-of-sample forecasts of US output growth are comparable to those of US professional forecasters in both normal growths and recession periods. Finally, we find that the Dot-com crisis might contribute to an increase in risk aversion, but had no impact on ambiguity aversion, while the Global Financial Crisis associated with a structural increase in both risk aversion and ambiguity aversion.

We contribute to the existing literature in three main ways. First, a large strand of macro models studies uncertainty as a time-varying volatility or risk, assuming that the likelihood is known (Born and Pfeifer, 2021; Fernández-Villaverde and Guerrón-Quintana, 2020; Lhuissier and Tripier, 2021). However, we incorporate uncertainty in as form of ambiguity, assuming a lack of knowledge about the likelihood of future events (Knight, 1921). This notion has been employed in business cycle models by, for example, Altug et al. (2020); Bianchi et al. (2018); Ilut and Schneider (2014, 2022). Our uncertainty or ambiguity is contingent upon two conditions: (1) the household believes that the next-period economy

[1] Various indices have been developed to measure this uncertainty, such as the Economic Policy Uncertainty index (Baker et al., 2016), the 1-month macroeconomic uncertainty index (Jurado et al., 2015), and implied volatility indices.

[2] Subjective uncertainty reflects how much an individual is uncertain about his or her own beliefs.

could enter either a normal growth period or a recession, and (2) that its expectations for the two scenarios have a wider spread as macroeconomic uncertainty increases.

Second, literature has extensively studied the transmission channels of uncertainty, such as financial frictions (Chatterjee and Milani, 2020; Christiano et al., 2018; Fernández-Villaverde and Guerrón-Quintana, 2020; Lhuissier and Tripier, 2021), price and wage mark-up (Born and Pfeifer, 2021), investment adjustment costs (Bloom, 2009), and agents' expectations (Altug et al., 2020; Bhandari et al., 2019; Ilut and Schneider, 2014). In our model, the transmission occurs through expected utilities and belief distortions. An increase in macroeconomic uncertainty leads to a larger pessimistic belief distortion toward the recession scenario, meaning that the household assigns a greater probability to this scenario than a Bayesian probability. In this way, the effect of macroeconomic uncertainty is nonlinear and becomes more pronounced as it increases.

Third, many solution methods have been utilized as an alternative to loglinear solutions in order to account for the nonlinear effects of uncertainty, such as higher-order perturbations (Born and Pfeifer, 2021; Fernández-Villaverde and Guerrón-Quintana, 2020) and nonlinear or markov-switching VARs (Bianchi et al., 2018; Jackson et al., 2020; Lhuissier and Tripier, 2021). The smooth ambiguity models, for which there is no closed-form solution, are generally solved by projection methods (Collard et al., 2018; Ju and Miao, 2010) or value function iterations (Altug et al., 2020; Jahan-Parvar and Liu, 2012). To solve our model, we apply the parameterized expectations algorithm which is a projection method that can preserve the nonlinearity in the transmission mechanism. Furthermore, we estimate the model to minimize the distance between the model-generated and actual output growth rates. Our model's estimation uses three empirical time series: Economic policy uncertainty index (Baker et al., 2016), the recession probability computed from the survey of professional forecasters, and the utilization-adjusted technological process (Fernald, 2014). To the best of our knowledge, this is the first study to measure the level of ambiguity aversion using macroeconomic data.

Our paper is organized as follows. To begin, Section 2 summarizes recent literature regarding uncertainty in macroeconomic models and discusses how we incorporate uncertainty in our model. In Section 3, we describe our model, discuss the implications of belief distortions caused by uncertainty, and show how the model can replicate the three stylized facts. Section 4 presents the estimation result for US output growth which includes estimated parameters, steady states, data fitness and out-of-sample forecasts. Section 5 discusses the estimation results regarding the aforementioned three stylized facts, as well as the evolution of ambiguity and risk aversions over time. Finally, Section 6 concludes our paper.

2 Uncertainty in macroeconomic models

In this section, we survey the macroeconomic literature on uncertainty and present our key assumption regarding the relationship between uncertainty and expected utility.

2.1 Uncertainty in macroeconomic literature

The literature is vast and uses various - not always transferable - definitions of the concept of uncertainty. In order to avoid any confusion in the jargon, we organize the literature according to the specific definition of uncertainty used. Knight (1921) defines the components of uncertainty by distinguishing between risk, where the likelihood of an event is known, and ambiguity, where the likelihood is unknown. Ambiguity is often referred to as Knightian uncertainty. To show the difference between these two modelling approaches, we first study models of risk, and then models of ambiguity.

A large strand of literature studies uncertainty as a time-varying volatility assuming that the likelihood is known but its variance is changing overtime. According to the distinction above, and in line with Fernández-Villaverde and Guerrón-Quintana (2020), uncertainty in this case is more similar to the modern concept of risk. In this vein, Born and Pfeifer (2021) studies the effects of uncertainty through markup channels, in which uncertainties are time-varying volatility of the TPF and government spending processes. They find that a two S.D. uncertainty shock can generate only a 0.0035% decrease in output unless employing more extreme and less common parameters such as a risk aversion of 20.[3] Fernández-Villaverde and Guerrón-Quintana (2020) introduces uncertainty in TFP, financial frictions and preference processes. In their estimation, this time-varying volatility explains a significant part of economic fluctuations; for example, financial frictions uncertainty can account for 63% of output volatility. Lhuissier and Tripier (2021) creates a Markov-switching model with two economic regimes: tranquil and distress periods, in which uncertainty is the volatility of TFP. In their estimation, the monitoring cost in the distress period is higher than the tranquil period, so the risk premium is higher in periods of distress, which derails investment. This mechanism amplifies the negative effect of uncertainty in the distress period by four times.

Another way to model uncertainty is to impose multiple potential scenarios where the true scenario is unknown until time t, when it becomes observable. In this version of uncertainty, before the true scenario is known, the likelihood of the events is unknown, so it is closer to ambiguity in Knight (1921)'s framework. An explicit preference to ambiguity is necessary to model the agent's behavior which ambiguity aversion implies that the agent is worse-off when exposed to uncertainty. In multiple priors preferences (Gilboa and Schmeidler, 1989), and robust preferences (Hansen and Sargent, 2011), the ambiguity averse agents behave as if they are in the worst-case scenario. Ilut and Schneider (2014) adopts multiple priors preferences in which uncertainty means the increased range of priors which is proxied by SPFs' disagreement. Uncertainty in their model can explain 70% of output volatility. Using robust preferences, Bhandari et al. (2019) derives household's belief wedges of inflation and unemployment,[4] which increase during uncertain periods. The belief wedges make the household's worst-case belief more pessimistic, and, with this mechanism, the model can match the volatility of output, inflation, and unemployment. Although agents in these two models are ambiguity averse, their attitude toward

[3] According to standard models such as Slobodyan and Wouters (2012), the standard value of risk aversion is around 2.

[4] Bhandari et al. (2019) measures the belief wedge as a difference between expectations of consumers and professional forecasters.

ambiguity is not adjustable and cannot be distinguished from the attitude toward risk.

Smooth ambiguity preferences (Klibanoff et al., 2005) differentiate between risk and ambiguity aversions and only extremely ambiguity averse agents will always adopt the worst-case scenario (Klibanoff et al., 2005; Marinacci, 2015). Altug et al. (2020) uses smooth ambiguity preferences with two scenarios: high and low persistent technological progresses, where the true scenario is unknown. In their model, the agent is ambiguity averse and learns about the probability of the true scenario, using Bayes' rule. Due to ambiguity aversion, the agent always puts more weight on the low-utility scenario than Bayes' rule. The authors label this behavior a pessimistic belief distortion because the agent's belief is more pessimistic than the Bayesian benchmark. Higher uncertainty means a larger variance of a Bayesian prior, and the simulations show that uncertainty increases the volatility of the economy with a small magnitude.

Our model is an extension of the smooth ambiguity model by Altug et al. (2020). We choose the smooth ambiguity model for three reasons. First, it nests the properties of multiple priors preferences and robust preferences as special cases and distinguishes between the attitudes toward ambiguity and risk (Ju and Miao, 2010; Marinacci, 2015). In this way, we will focus on the result of changes to ambiguity. Second, the properties of the smooth ambiguity model are in line with a growing micro finance literature (Guidolin and Liu, 2016; Nowzohour and Stracca, 2020; Pulford, 2009). For example, uncertainty impacts the economy through pessimistic beliefs, and the magnitude of pessimism is conditional on individual ambiguity attitudes. Finally, the smooth ambiguity model can be estimated in a macro setting using available variables from macroeconomic and survey data.

2.2 Uncertainty in our model

This section introduces our key assumption about the relationship between macroeconomic uncertainty and expected utilities. We propose that the ratio of expected utilities is time varying. There are periods when expected utilities in the good and bad scenarios are relatively similar, and periods when a deep crisis is feared, meaning that the ratio of the expected utilities in the two possible outcomes is more relevant[5]. To pin down this ratio, we use an empirical macroeconomic uncertainty index. In this section, we discuss this assumption in detail and provide empirical evidence to support it.

Uncertainty affects the dynamics of the model when two conditions are satisfied. First, when the household believes that the economy could at least potentially fall into a recession - i.e. if the household believes that there is 0% chance of recession, this condition is not satisfied. Second, the household expects that the utilities of the two scenarios are different. Let μ_t be a Bayesian belief of the recession probability, $E_t(V_{t+1}^R)$ be the expected utility at time t for the economy to be in recession at time $t+1$, and $E_t(V_{t+1}^{NR})$ be the expected utility when the economy is in the period of normal growth at $t+1$. V indicates the utility, and superscripts R and NR indicate recession and normal growth scenarios,

[5]This ratio is not the same as the agent's probability belief in the likelihood that one scenario will be realized over the other. A description of the roles of the two concepts in the dynamics is given in Section 3.2.

respectively. Therefore, uncertainty is relevant when:

$$\mu_t > 0 \text{ and } E_t(V_{t+1}^{NR}) > E_t(V_{t+1}^{R}) \tag{1}$$

We assume that macroeconomic uncertainty affects the expected utilities of the two scenarios asymmetrically and illustrate this through an example. Stefanie currently has a permanent position in a large firm, and thus would not be severely affected in the case of a recession. Consequently, her expected utility in the recession scenario is close to that of the normal growth scenario. In contrast, twenty years ago, Stefanie was employed at an entry-level job in a start-up, and thus would have been significantly impacted by a recession. Therefore, her expected utility in the recession scenario in the past was much lower than in the normal growth scenario. In this example, the young Stefanie is more vulnerable to recessions than the current Stefanie. It implies that macroeconomic uncertainty increases the spread of the expected utilities between the two scenarios.

Because in a representative agent model the expected utilities of Stefanie are the average utilities in the whole economy, we anchor the ratio of $E_t(V_{t+1}^{R})$ in relation to $E_t(V_{t+1}^{NR})$ to the empirical series of macroeconomic uncertainty M_t:

$$M_t = \frac{E_t(V_{t+1}^{NR})}{E_t(V_{t+1}^{R})} \text{ where } M_t > 1 \tag{2}$$

Our concept of uncertainty is consistent with that by Ilut and Schneider (2014, 2022). They purpose that an increase in uncertainty (or ambiguity in their paper) decreases the utility of the worst case scenario due to a larger set of beliefs.

Empirical evidence. We use empirical evidence to motivate the assumption that the ratio of expected utilities could be anchored to a macroeconomic uncertainty index. As a proxy for expected utilities, we use GDP growth expectations from US professional forecasters. Ceteris paribus, we assume that:

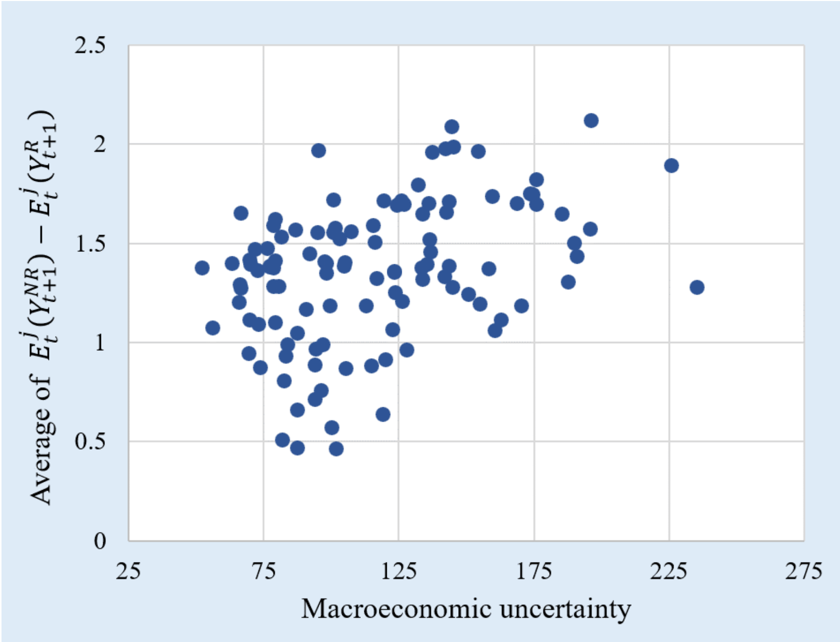
$$\frac{E_t(V_{t+1}^{NR})}{E_t(V_{t+1}^{R})} \propto \frac{E_t^j(Y_{t+1}^{NR})}{E_t^j(Y_{t+1}^{R})}$$

where $E_t^j(Y_{t+1}^{NR})$ is forecaster j's next-year GDP growth expectation if GDP growth will be positive, and $E_t^j(Y_{t+1}^{R})$ is forecaster j's next-year GDP growth expectation if GDP growth will be negative (in a recession).

The survey of US professional forecasters provides an individual subjective histogram of next-year GDP growth expectations. Here each forecaster fills in his or her subjective probabilities that GDP growth will be within a given bin. In this survey, the bins range from ($-\infty$, -3%), [x%, x+0.9%] for $x \in \{-3, -2, \ldots, 5\}$, and [6%, ∞) [6]. We use this subjective histogram to calculate the expected GDP growth in each scenario.

[6] During 1985 - 2019, our sample period, the size of the bin was changed once at the beginning of the Global Financial crisis.

Figure 1: Difference of GDP growth expectations and macro uncertainty



Note: Y-axis is the average difference between point estimates of GDP growth forecasts in the normal and recession scenarios. X-axis is the US Economic Policy Uncertainty index (Baker et al., 2016).

Figure 1 and Table 1 support our assumption that the difference of expected GDP growth rates between normal growth and recession periods is positively correlated with macroeconomic uncertainty. The Y axis of Figure 1 shows the cross-sectional average of the difference between $E_t^j(Y_{t+1}^{NR})$ and $E_t^j(Y_{t+1}^R)$[7], plotted against macroeconomic uncertainty on the X axis, measured by the US Economic Policy Uncertainty index. The positive relationship is visible to the naked eye, and the significant result is shown more precisely in Table 1, which indicates that when macroeconomic uncertainty increases by 1%, the difference between the two expected GDP growths increases by 0.18%.

Throughout this paper, the term 'uncertainty' will refer to the two conditions in Equation 1 and the term 'macroeconomic uncertainty' will be specific to Equation 2.

3 Model

This section describes the representative-agent model with smooth ambiguity preferences based on Altug et al. (2020). Our model differs from theirs in two main ways. First, the two scenarios in Altug et al. (2020) are the periods of high and low persistent technological process, where as ours are the periods of normal growth and recession. Second, in Altug et al. (2020), higher uncertainty or ambiguity is measured by a larger variance of the Bayesian prior. Here, we use a macroeconomic uncertainty index M_t to proxy

[7]Using the difference rather than the ratio to compare expected growth rates better reflects the deviation between two expected utilities. For example, when macroeconomic uncertainty is high, the expected growth rates are -5% and 5%, yielding a ratio of -1 and a difference of 10. When macroeconomic uncertainty is low, the expected growth rates are -1% and 1%, resulting in a ratio of -1 and a difference of 2. Therefore, the difference is the more appropriate measure in this situation.

Table 1: Effect of uncertainty on the difference between expected GDP growths

	$E_t^j(Y_{t+1}^{NR}) - E_t^j(Y_{t+1}^{R})$
Macro uncertainty growth$_t$	0.1818
	(0.0641)
GDP growth$_t$	-0.0321
	(0.0140)
$E_{t-1}^j(Y_t^{NR}) - E_{t-1}^j(Y_t^{R})$	0.4646
	(0.0570)
Constant	Y
Quarter FE	Y
Individual FE	Y
Observations	3,259
R-squared	0.5663

Note: Macroeconomic uncertainty is the US Economic Policy Uncertainty index by Baker et al. (2016). The dependent variable is the difference between point estimates of GDP growth forecasts for normal and recession scenarios. The GDP forecasts are from the survey of US professional forecasters. The model is a fixed-effect regression that controls for heteroskedasticity.

for the level of macroeconomic uncertainty and anchor it to the spread of expected utilities between the two scenarios. A summary of Altug et al. (2020) is provided in Appendix A.

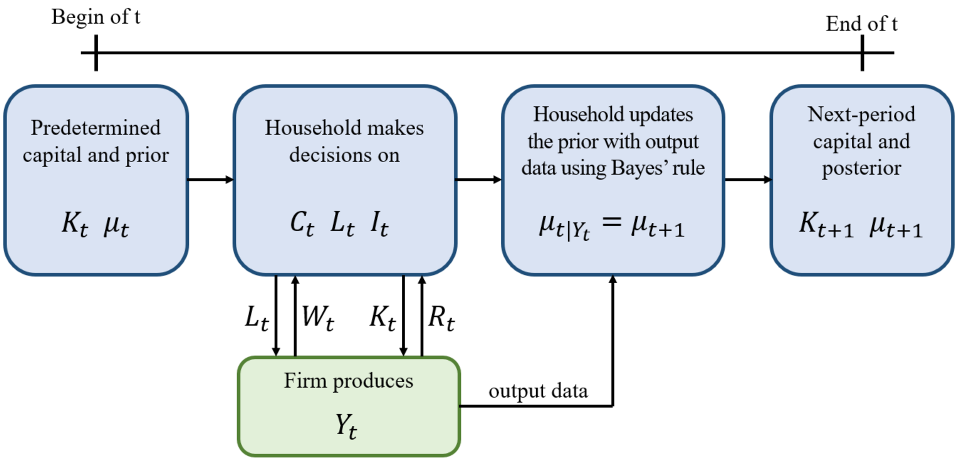
In our economy, the household has smooth ambiguity preferences, is a Bayesian learner and is uncertain whether there will be a recession in the next period. The smooth ambiguity preferences imply that the household tries to smooth out its expected utilities across the two scenarios. This is conceptually analogous to a consumption smoothing in a sense that the consumption is smoothed out overtime. When consumption smoothing is perfect, the expected consumption remains constant over time; similarly, when ambiguity smoothing is perfect, the expected utility is same for both recession and normal growth scenarios. The firm however is not directly subject to uncertainty since it makes decisions based on the currently observable information. Uncertainty indirectly affects the firm only through the household's decisions.

Figure 2 illustrates the timeline of decision-makings and Bayesian updating. At each period, the household chooses how much to consume, work and invest given a predetermined capital and Bayesian prior of recession (i.e. the probability of recession). The household provides labor and capital to the firm. The firm provides wage and a rental fee on capital to the household and produces output. The profit from the production will be transferred to the household. The household uses the observed output data to update the Bayesian prior of recession, which will be used in the next period.

3.1 Household

This section describes the household's objective function and shows how we incorporate uncertainty into the model. Our household forms the expected utilities of the two scenarios: recession and a normal

Figure 2: Decision-making and Bayesian updating



where K_t is capital, C_t is consumption, L_t is labor, I_t is investment, Y_t is output, W_t is the labor wage, R_t is the rental fee on capital, μ_t is the Bayesian prior of recession, μ_{t+1} is the Bayesian posterior of recession.

growth period. The expected utilities will be evaluated with the following smooth ambiguity function: $\phi(E_t(V_{t+1})) = \frac{[E_t(V_{t+1})]^{1-\gamma}}{1-\gamma}$, where $\gamma \geq 0$ is the ambiguity aversion parameter and $E_t(V_{t+1})$ is the expected utility of period $t+1$. The concavity of the function ϕ captures the reaction to ambiguity, which can be interpreted as aversion to mean-preserving spreads. When the spread of expected utilities increases, the mean expected utility decreases, implying that the ambiguity averse household are better off when the spread between expected utilities of the two scenarios is smaller. The combination of expected utilities, ambiguity aversion, and Bayesian beliefs plays an important role in the household's decision-making process. The household's objective function is the following:

$$\begin{aligned}
\max_{C_t,L_t,I_t} V(C_t, L_t) &= \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\nu}}{1+\nu} \\
&\quad + \beta\phi^{-1}\left[\left(\mu_t\phi(E_t(V(C_{t+1}^R, L_{t+1}^R))) + (1-\mu_t)\phi(E_t(V(C_{t+1}^{NR}, L_{t+1}^{NR})))\right)\right] \\
\text{subject to } & C_t + I_t = W_t L_t + R_t K_t + \Pi_t
\end{aligned}$$

where $\phi(E_t(V_{t+1})) = \frac{[E_t(V_{t+1})]^{1-\gamma}}{1-\gamma}$ is the smooth ambiguity function, $\gamma \geq 0$ is ambiguity aversion, C_t is consumption, I_t is investment, L_t is labor, K_t is capital, R_t is the rental price of capital, W_t is the wage rate, Π_t is the firm's profit distributed to the household, β is the discount factor and μ_t is the Bayesian belief of the recession at t+1. $\sigma > 0$ is risk aversion, $\nu > 0$ is the disutility of labor.

The Lagrangian equation is:

$$\begin{aligned}
\max_{C_t,L_t,I_t} & \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\nu}}{1+\nu} + \beta\phi^{-1}\left[\left(\mu_t\phi(E_t(V(C_{t+1}^R, L_{t+1}^R))) + (1-\mu_t)\phi(E_t(V(C_{t+1}^{NR}, L_{t+1}^{NR})))\right)\right] \\
& - \Lambda_t\left(C_t + I_t - W_t L_t - R_t K_t - \Pi_t\right)
\end{aligned}$$

The first order optimality conditions for C_t and L_t are:

$$\Lambda_t = C_t^{-\sigma} \tag{3}$$

$$\Lambda_t = \frac{L_t^{\nu}}{W_t} \tag{4}$$

Equations 3 and 4 imply that the substitution rate between consumption and labor is proportional to the wage rate. The first order optimality condition for I_t is:

$$\Lambda_t = \beta \Upsilon_t \left(\mu_t \xi_t^R \frac{\partial E_t(V_{t+1}^R)}{\partial I_t} + (1-\mu_t)\xi_t^{NR} \frac{\partial E_t(V_{t+1}^{NR})}{\partial I_t} \right) \tag{5}$$

$$\text{where } \Upsilon_t = \frac{\mu_t E_t(V_{t+1}^R)^{-\gamma} + (1-\mu_t)E_t(V_{t+1}^{NR})^{-\gamma}}{\left(\mu_t E_t(V_{t+1}^R)^{1-\gamma} + (1-\mu_t)E_t(V_{t+1}^{NR})^{1-\gamma}\right)^{\frac{-\gamma}{1-\gamma}}} \tag{6}$$

$$\xi_t^k = \frac{E_t(V_{t+1}^k)^{-\gamma}}{\mu_t E_t(V_{t+1}^R)^{-\gamma} + (1-\mu_t)E_t(V_{t+1}^{NR})^{-\gamma}} \tag{7}$$

$$k \in \{R, NR\}$$

Equation 5 is the smooth ambiguity Euler equation which contains the marginal expected utilities of investment $\frac{\partial E_t(V_{t+1}^k)}{\partial I_t}$ for the two scenarios, the Bayesian beliefs μ_t, the scaling factor Υ_t and belief distortions ξ_t^k. The weights attached to the recession scenario and the normal growth scenario are $\mu_t \xi_t^R$ and $(1-\mu_t)\xi_t^{NR}$ respectively. We call these weights subjective beliefs because they consist of a nonbehavioral part, Bayesian beliefs, and a behavioral part, belief distortions [8]. If the household is ambiguity neutral $\gamma = 0$ ξ_t^k will equal one, so the household's subjective belief is the Bayesian belief. If the household is ambiguity averse $\gamma > 0$, ξ_t^R is greater than ξ_t^{NR} because $E_t(V_{t+1}^R)$ is smaller than $E_t(V_{t+1}^{NR})$. Thus, the ambiguity averse household's subjective belief is biased toward the recession scenario compared to the Bayesian belief. Since the recession scenario has a lower expected utility, we refer to this weighting scheme as pessimistic belief distortions, following Altug et al. (2020), Collard et al. (2018) and Ju and Miao (2010).

According to Section 2, we assume that the ratio between expected utilities of normal and recession scenarios can be approximated by a macroeconomic uncertainty index, $M_t = \frac{E_t(V_{t+1}^{NR})}{E_t(V_{t+1}^R)}$. We substitute $E_t(V_{t+1}^R) = \frac{E_t(V_{t+1}^{NR})}{M_t}$ into the Euler equation (Eq. 5) and solve the partial derivatives. The capital accumulation process is: $K_{t+1} = (1-\delta)K_t + I_t$ where δ is the capital depreciation rate. We

[8] We write Υ_t and ξ_t^k in the forms of Equations 6 and 7 because we formulate ξ_t^k as a Radon-Nikodym derivative that effectively distorts from the Bayesian belief to the subjective belief. Marinacci (2015) and Klibanoff et al. (2009) define the Radon-Nikodym derivative of the smooth ambiguity function as $\frac{\phi'(E_t(V_{t+1}))}{E_{\mu_t}(\phi'(E_t(V_{t+1})))}$, which we use for ξ_t^k. As ξ_t^k is defined, Υ_t naturally follows, and does not affect the subjective beliefs related to the two scenarios.

obtain:[9]

$$\Lambda_t = \beta E_t(\Lambda_{t+1}^{NR}(R_{t+1}^{NR} + 1 - \delta))\Upsilon_t \left(\frac{\mu_t \xi_t^R}{M_t} + (1-\mu_t)\xi_t^{NR} \right) \tag{8}$$

$$\text{where } \Upsilon_t = \frac{\mu_t M_t^\gamma + (1-\mu_t)}{\left(\mu_t M_t^{\gamma-1} + (1-\mu_t)\right)^{\frac{-\gamma}{1-\gamma}}}$$

$$\xi_t^R = \frac{M_t^\gamma}{\mu_t M_t^\gamma + (1-\mu_t)}$$

$$\xi_t^{NR} = \frac{1}{\mu_t M_t^\gamma + (1-\mu_t)}$$

Λ_{t+1}^{NR} is the marginal utility of consumption in the normal scenario.

R_{t+1}^{NR} is the rental price of capital in the normal scenario.

Now, the belief distortions ξ_t^k and the scaling factor Υ_t become the function of macroeconomic uncertainty M_t, the Bayesian belief μ_t, and ambiguity aversion γ. In Section 3.2, we discuss how these variables affects the belief distortions.

Firm. In this economy, we define a firm as simply as possible: a single representative firm producing one good. This firm is not subject to uncertainty as its decisions are based on current information. The first-order optimality conditions with respect to labor and capital are:

$$W_t = (1-\alpha)\frac{Y_t}{L_t} \tag{9}$$

$$R_t = \alpha \frac{Y_t}{K_t} \tag{10}$$

subject to the following conditions:

$$Y_t = Z_t K_t^\alpha L_t^{1-\alpha}$$

$$K_t = (1-\delta)K_{t-1} + I_t$$

$$Z_t = \exp(a_t)$$

$$a_t = (1-\rho)\bar{a} + \rho a_{t-1} + \sigma_a \epsilon_t^a \text{ where } \epsilon_t^a \sim \mathcal{N}(0,1)$$

where Y_t is output, K_t is capital, L_t is labor, I_t is investment, W_t is the wage rate, and R_t is the rental price of capital. α is the capital share in production and δ is the depreciation rate of capital. Finally, Z_t is the total productivity factor (TFP) which is developing as an AR(1) process, around mean $\bar{a}$.

3.2 Belief distortions

This section discusses the dynamics of the belief distortions ξ_t^k. In order to illustrate this analytically, we devide our analyses into thre cases. First we discuss Benchmark case when $M_t = 1$ or $\mu_t = 0$ and

[9] We assume that the second-order effect of capital on uncertainty is very small and can be ignored. The derivation is in Appendix B.

therefore uncertainty is not relevant. Then we compare it with the cases when uncertainty is relevant in the model $M_t > 1$ and $\mu_t > 0$ by analyzing the ambiguity neutral and ambiguity averse cases.

Benchmark. In this benchmark economy, the household behaves as if it will be surely in a normal growth period, so the Euler equation is reduced to one scenario as follows:

$$\Lambda_t = \beta E_t(\Lambda_{t+1}^{NR}(R_{t+1}^{NR} + 1 - \delta)) \tag{11}$$

Ambiguity neutral. When there is uncertainty $M_t > 1$ and $\mu_t > 0$ the household will take the recession scenario into account. If the household is ambiguity neutral $\gamma = 0$, uncertainty will have some impacts through the average expectation of the household (since a recession is also taken into account) but there will be no belief distortion $\xi_t^k = 1$ and no scaling factor $\Upsilon_t = 1$. Thus, the household is purely Bayesian, and the ambiguity neutral Euler equation is:

$$\begin{aligned}\Lambda_t &= \beta\left(\mu_t \frac{\partial E_t(V_{t+1}^R)}{\partial I_t} + (1-\mu_t)\frac{\partial E_t(V_{t+1}^{NR})}{\partial I_t}\right)\\ &= \beta E_t(\Lambda_{t+1}^{NR}(R_{t+1}^{NR} + 1 - \delta))\left(\frac{\mu_t}{M_t} + (1-\mu_t)\right)\end{aligned} \tag{12}$$

The ambiguity neutral Euler equation is the linear combination of the marginal expected utilities of investment, weighted by Bayesian beliefs. Once we disentangle these expectations to compare them to the benchmark model, we obtain Equation 12 where $\frac{\mu_t}{M_t} + (1-\mu_t)$ is a ratio of the expected marginal utilities to the benchmark model. Since $M_t > 1$ and $\mu_t > 0$, the ratio is smaller than one. Thus the marginal expected utility of investment in the benchmark model is greater than that in the ambiguity neutral model. This implies that the household's expectation becomes lower when uncertainty exists although it is ambiguity neutral.

Ambiguity averse. If the household is ambiguity averse $\gamma > 0$, the belief distortions will be different from 1 and the scaling factor will be greater than 1. Thus, the ambiguity averse Euler equation is:

$$\begin{aligned}\Lambda_t &= \beta\Upsilon_t\left(\mu_t\xi_t^R \frac{\partial E_t(V_{t+1}^R)}{\partial I_t} + (1-\mu_t)\xi_t^{NR}\frac{\partial E_t(V_{t+1}^{NR})}{\partial I_t}\right)\\ &= \beta E_t(\Lambda_{t+1}^{NR}(R_{t+1}^{NR} + 1 - \delta))\Upsilon_t\left(\frac{\mu_t\xi_t^R}{M_t} + (1-\mu_t)\xi_t^{NR}\right)\end{aligned} \tag{13}$$

In the ambiguity averse Euler equation, the marginal expected utilities of investment are weighted by the Bayesian beliefs, the belief distortions and the scaling factor. $\Upsilon_t\left(\frac{\mu_t\xi_t^R}{M_t} + (1-\mu_t)\xi_t^{NR}\right)$ indicates a ratio of the expected marginal utility of the ambiguity averse model to the benchmark model. Since $\xi_t^R > \xi_t^{NR}$, the ambiguity averse household is always biased toward the recession scenario compared to the Bayesian belief, regardless of the scaling factor. The scaling factor only increases the difference between ξ_t^R and ξ_t^{NR} given everything equals. This is because $\Upsilon_t > 1$ so $\Upsilon_t(\xi_t^R - \xi_t^{NR}) > \xi_t^R - \xi_t^{NR}$, implying that the pessimistic belief distortions with Υ_t is larger than the pessimistic belief distortions without Υ_t. Overall, the marginal expected utility of investment in the ambiguity averse model is less

Table 2: Dynamics of belief distortions and total effect on the expected utility

	Belief distortions		Scaling factor	Ratio to the benchmark model
When the variable increases	ξ_t^R	ξ_t^{NR}	Υ_t	$\Upsilon_t\left(\frac{\mu_t \xi_t^R}{M_t}+(1-\mu_t)\xi_t^{NR}\right)$
Ambiguity aversion (γ)				
$\gamma=0$ (ambiguity neutral)	1	1	1	$\frac{\mu_t}{M_t}+(1-\mu_t)$
$0<\gamma<1$	↑	↓	↑	↓
$\gamma>1$	↑	↓	↓	↓
$\gamma\to\infty$	$\frac{1}{\mu_t}$	0	1	$\frac{1}{M_t}$
Bayesian belief (μ_t)				
$\mu_t=0$ (no uncertainty)	M_t^{γ}	1	1	1
$0<\mu_t<1$	↓fast	↓slow	↑then↓	↓
$\mu_t=1$	1	$\mathrm{M}_t^{-\gamma}$	1	$\frac{1}{M_t}$
Macro uncertainty (M_t)				
$M_t=1$ (no uncertainty)	1	1	1	1
$M_t>1$	↑	↓	↑	↓
$M\to\infty$	$\frac{1}{\mu_t}$	0	a constant	0

Note: ↓: decrease, ↑: increase

than that in the ambiguity neutral model and the benchmark model. This means that the investment in future capital becomes less attractive for the ambiguity averse household than for the ambiguity neutral household.

As we can see, in addition to its direct effect on expected utilities, macroeconomic uncertainty indirectly impacts the decision-making process through the scaling factor and the belief distortions. Both factors then affect the ratio to the benchmark model. Table 2 summarizes how the belief distortions and the scaling factor respond to ambiguity aversion γ, Bayesian belief μ_t, and macroeconomic uncertainty M_t. The downward arrow (upward arrow) means decrease (increase) when these three variables increase. The last column shows the ratio to the benchmark model. As the ratio in this column decreases, the marginal expected utility of investment becomes smaller compared to the benchmark.

According to Table 2, we can draw three implications. First, ambiguity aversion increases pessimistic belief distortions. When ambiguity aversion increases, the belief distortions are more biased toward the recession scenario as ξ_t^R increases while ξ_t^{NR} decreases. Υ_t increases until $\gamma=1$ and then decreases. As a result, the total weight on the recession scenario increases more than the total weight on the normal growth scenario. Therefore, the marginal expected utility of investment decreases. If the household is extremely ambiguity averse $\gamma\to\infty$, the belief distortion toward the recession will be $\frac{1}{\mu_t}$ such that the total weight of recession is one and the total weight of the normal growth is zero, so the household will become a Maxmin optimizer and acts as if it will be in the recession. This implication is in line with Altug et al. (2020), Marinacci (2015) and Ju and Miao (2010).

Second, the Bayesian beliefs have a hedging effect against the belief distortions. When μ_t increases, ξ_t^R decreases faster than ξ_t^{NR} does, implying that the belief distortion toward recession is smaller when the Bayesian belief of recession is larger. This can be interpreted as the ambiguity averse household avoiding the extreme expectation to minimize the loss when the situation turns out unexpected. Baliga et al. (2013) show that the hedging effect can cause the polarization of beliefs when there is ambiguous information and heterogeneous agents. When the information is ambiguous, the ambiguity averse agents prefer not to extremely deviate from their Bayesian priors to hedge against the forecast error loss. If the agents hold heterogeneous prior beliefs their posterior beliefs will polarize toward their prior beliefs.

Lastly, macroeconomic uncertainty increases pessimistic belief distortions. When macroeconomic uncertainty increases, the belief distortions are more biased toward the recession scenario as ξ_t^R increases while ξ_t^{NR} decreases. Moreover, when macroeconomic uncertainty increases the scaling factor rises, which further amplifies the deviating effects of ξ_t^R and ξ_t^{NR}. As a result, the ambiguity averse household puts more weight on the recession scenario so the ratio decreases. This negative effect of macroeconomic uncertainty on average expected utilities is in line with the findings in Piccillo and Poonpakdee (2021).

To summarize, ambiguity aversion γ and macroeconomic uncertainty M_t increase the pessimistic belief distortions while Bayesian beliefs of recession μ_t have a hedging effect against the belief distortions. The pessimistic belief distortions lead to a lower average marginal expected utility of investment, which makes investment into future capital less attractive.

3.3 Replication of three stylized facts

Using the subjective belief of recession derived in Section 3.1, we discuss how our model can replicate the three empirical stylized facts. As the subjective belief of the normal growth scenario is reciprocal to the subjective belief of recession, the discussion of the recession scenario also covers the normal growth scenario.

First stylized fact. Macroeconomic uncertainty makes people more pessimistic (Bhandari et al., 2019; Bianchi et al., 2020; Piccillo and Poonpakdee, 2021). We relate pessimism with the household's subjective belief. An increase in the subjective belief of recession means that the household believes the economy will be more likely to be in recession. In the smooth ambiguity model, the household knows that the occurrence of the next-period recession follows a Bernoulli distribution where the outcome is either one or zero. We define the household's subjective belief as the first moment of the Bernoulli distribution or the probability of the recession according to the household as follows:

$$\text{Subjective belief}_t = \mu_t \xi_t^R \tag{14}$$
$$\text{where } \xi_t^R = \frac{M_t^\gamma}{\mu_t M_t^\gamma + (1-\mu_t)}$$

We derive the subjective belief with respect to macroeconomic uncertainty to mathematically show the effect of macroeconomic uncertainty:

$$\frac{\partial \mu_t \xi_t^R}{\partial M_t} = (1-\mu_t \xi_t^R)\mu_t \xi_t^R \frac{\gamma}{M_t}$$

Since subjective belief $\mu_t \xi_t^R$ is always between zero and one, $\frac{\partial \mu_t \xi_t^R}{\partial M_t}$ is always greater or equals zero. Therefore, macroeconomic uncertainty positively impacts the subjective belief of recession. This is in line with what discussed in Section 3.2. When macroeconomic uncertainty increases, the belief distortion toward the recession ξ_t^R increases, so the subjective belief of recession $\mu_t \xi_t^R$ increases.

Second stylized fact. Macroeconomic uncertainty can have both positive and negative effects on subjective uncertainty (Glas, 2020; Piccillo and Poonpakdee, 2021). Intuitively, this implies that households can be more or less uncertain about their subjective beliefs when faced with an increased macroeconomic uncertainty. Subjective uncertainty is defined as the second moment of the subjective beliefs, which reflects how confident the household is in their first-moment belief (Altig et al., 2019; Ben-David et al., 2018; Piccillo and Poonpakdee, 2021). From Equation 14, the second moment of the subjective belief can be expressed as follows:

$$\text{Subjective uncertainty}_t = \sqrt{\mu_t \xi_t^R \times (1-\mu_t \xi_t^R)} \tag{15}$$
$$\text{where } \xi_t^R = \frac{M_t^\gamma}{\mu_t M_t^\gamma + (1-\mu_t)}$$

The derivative of subjective uncertainty with respect to macroeconomic uncertainty is as follows:

$$\frac{\partial \sqrt{\mu_t \xi_t^R (1-\mu_t \xi_t^R)}}{\partial M_t} = \sqrt{\mu_t \xi_t^R (1-\mu_t \xi_t^R)} \frac{1-2\mu_t \xi_t^R}{2} \frac{\gamma}{M_t}$$

The sign of this derivative depends on the sign of $\frac{1-2\mu_t \xi_t^R}{2}$ since other terms are always positive. If $\mu_t \xi_t^R$ is less than 0.5, the derivative is greater than zero or vice versa. This implies that the effect of macroeconomic uncertainty on subjective uncertainty is positive when subjective belief is between 0 and 0.5. When subjective belief is between 0.5 and 1, the effect of macroeconomic uncertainty becomes negative. On the top of that, macroeconomic uncertainty increases subjective belief and ambiguity aversion strengthens this effect. Therefore, the relationship between macroeconomic uncertainty and subjective uncertainty also depends on the level of ambiguity aversion.

Third stylized fact. The effect of macroeconomic uncertainty on the economy is nonlinear and is stronger when macroeconomic uncertainty is higher (Jackson et al., 2020; Lhuissier and Tripier, 2021; Ng and Wright, 2013). To replicate this, we focus on the ratio of the marginal expected utility of investment (henceforth the ratio) to the benchmark model in the smooth ambiguity Euler equation:

$$\Lambda_t = \beta E_t(\Lambda_{t+1}^{NR}(R_{t+1}^{NR} + 1 - \delta))\Upsilon_t\left(\frac{\mu_t \xi_t^R}{M_t} + (1-\mu_t)\xi_t^{NR}\right) \tag{16}$$

$$\text{where } \Upsilon_t\left(\frac{\mu_t \xi_t^R}{M_t} + (1-\mu_t)\xi_t^{NR}\right) : \text{the ratio to the benchmark model} \tag{17}$$

The ratio to the benchmark model (Eq. 17) shows that macroeconomic uncertainty enters the model nonlinearly. It affects the household's average expected utility directly through an increase in the spread of expected utilities between the two scenarios and indirectly through subjective beliefs. When macroeconomic uncertainty increases, the recession scenario's expected utility is relatively lower than the normal growth scenario. Moreover, the subjective belief of recession rises due to the increased macroeconomic uncertainty. As a result, the expected utility of the recession scenario, while decreasing, becomes more relevant to the household's average expected utility. This mechanism creates a nonlinear effect of macroeconomic uncertainty on the economy.

The nonlinear effect of macroeconomic uncertainty on the economy is bounded by Bayesian beliefs. When Bayesian belief is closed to zero, the ratio to the benchmark model converges to one regardless of the levels of macroeconomic uncertainty and ambiguity aversion (Table 2). This implies that the household's pessimism is tightly bounded when its Bayesian belief describes the recession as very unlikely.

4 Estimation results

This section briefly explains our solution and estimation approaches. Then we present the estimation results by comparing the benchmark, ambiguity neutral and ambiguity averse models.

Solution and estimation approaches. We numerically solve the smooth ambiguity model using parameterized expectations algorithm (PEA) and estimate it with nonlinear least squares (NLS) method. The PEA approximates the household's conditional expectations with a parametric function that includes an interacting component, which captures the concavity of the household's Euler equation. To include a friction into the model, we assume that the household's conditional expectation is a mixture of one-period lagged expectations and state variables, with ρ_λ representing the weight on the lagged expectation. The NLS estimation minimizes the distance between the model-implied and observed output growth rates. This approach is also used in Carroll et al. (2019). We estimate six parameters using three macroeconomic data from 1985Q1 to 2019Q4: utilization-adjusted total factor productivity (Fernald, 2014) as a proxy for z_t, the Economics Policy Uncertainty index (Baker et al., 2016) for M_t and the next-quarter recession probabilities computed from the survey of professional forecasters for μ_t. The initial values and bounds of estimated parameters are summarized in Table 3. The full solution and estimation methods are described in Appendix C, and the dataset are listed in Appendix D.

Table 3: Initial values and bounds of estimated parameters

Parameter	Description	Initial value	Bound [lower, upper]
α	capital share	0.3	[0,1]
ν	labor disutility	1.5	[0,20]
σ	risk aversion	2	[0,20]
ρ_λ	weight on the lagged expectations	0.5	[0,1]
μ_s	steady-state Bayesian belief of recession	average of data	[0,1]
γ	ambiguity aversion	0,5,10,20	[0,40]

We compare the estimations of US GDP growths across three models: the benchmark model (BM), the ambiguity neutral model (AN), and the ambiguity averse model (AA). The BM allows only the total factor productivity shocks and ignores uncertainty. The AN incorporates two shocks from TFP and macroeconomic uncertainty, while assuming that the household is ambiguity neutral $\gamma = 0$. The AA allows for the same shocks as the AN, and the household is allowed to be ambiguity averse $\gamma \geq 0$; the parameter of ambiguity aversion is estimated in the AA. Table 4 presents the estimated parameters, the values of steady state and the root mean square errors. The asymptotic standard error (ASE) is shown in the parentheses[10]. Our estimation analysis is divided into four parts: estimated parameters, steady states, model fit, and out-of-sample forecast.

Parameters. The estimated capital share ranges between 0.29 to 0.34 which is closed to the capital share income in the US (0.36 - 0.41). However, the estimated risk aversion is between 0.42 to 0.49, significantly lower than the standard values (1 - 2) found in the literature of business cycle model. This discrepancy could be due to the fact that we minimized the RMSEs, as opposed to maximizing the likelihood as done in other studies. In Appendix E, we show that the maximum likelihood estimation with standard Bayesian technique obtains a value of risk aversion that is closer to the standard value.

When comparing the estimated parameters across three models, we find that uncertainty and ambiguity aversion have a substantial impact on the weight of lagged expectations and labor disutility. In the benchmark model (BM), the parameter for the weight of lagged expectations is 86%, whereas it is 54% in the ambiguity neutral model (AN). This implies that the expectations of these households are mainly driven by past information, resulting in large frictions in the economies. Conversely, the weight of lagged expectations in the ambiguity averse model (AA) is only 7%, implying that the ambiguity averse household mainly uses current information to form its expectations. This does not mean that the ambiguity averse model has less friction than the ambiguity neutral model. The labor disutility parameter in the ambiguity averse model is larger than in the ambiguity neutral model, indicating that the ambiguity averse household is more sensitive to changes in labor supply. This heightened sensitivity leads to smoother labor supply dynamics and consequently more friction in the economy.

The ambiguity aversion of the US representative household is estimated to be 3.3481, indicating

[10] A small standard error implies that a change in the parameter around its estimated values leads to a large increase in the RMSE, indicating that the objective function is highly convex around the estimated value.

Table 4: Estimation results

Estimated parameter	Description	BM	AN	AA
α	capital share	0.34	0.29	0.34
		(0.08)	(0.06)	(0.12)
ν	labor disutility	7.14	4.006	6.44
		(4.85)	(0.84)	(3.43)
σ	risk aversion	0.42	0.49	0.43
		(0.41)	(0.17)	(0.11)
ρ_λ	weight of the lagged expectations	0.86	0.54	0.08
		(0.06)	(0.22)	(0.59)
μ_s	SS Bayesian belief of recession		0.0007	1.00
			(1.18)	(0.54)
γ	ambiguity aversion			3.35
				(2.28)
Steady state	Description	BM	AN	AA
i_s/y_s	share of investment in output	0.24	0.20	0.22
c_s/l_s	ratio of consumption to labor	2.49	1.85	2.38
M_s	macroeconomic uncertainty		1.00	1.00
$\mu_s \xi_s^R$	subjective belief of recession		0.0007	1.00
RMSE	Periods	BM	AN	AA
	all periods	0.53%	0.42%	0.41%
	recession periods	1.15%	0.42%	0.41%
	normal growth periods	0.43%	0.42%	0.41%

Note: All models were estimated using the parameterized expectations algorithm and pattern search algorithm described in Appendix C. BM stands for Benchmark model, AN is Ambiguity neutral model where γ is fixed to 0. AA is the ambiguity averse model where γ is estimated. RMSE stands for root mean square error. The standard error of the estimated parameter is in (...).

that the household has a pessimistic belief distortion towards a recession scenario $\xi_t^R \geq 1$. This results in a subjective belief of recession probability that is higher than the Bayesian belief as presented in Equation 14. For instance, in the fourth quarter of 2008, the Bayesian belief of recession μ_t was 75% and macroeconomic uncertainty M_t was 1.42, resulting in a model-implied subjective belief $\mu_t \xi_t^R$ of 90%, given that ambiguity aversion $\gamma = 3.35$. This can be interpreted that the household believes there is a 75% chance of recession occurring in the next quarter, however, due to ambiguity aversion, it behaves as if the probability is 90%. Note that, the asymptotic standard error of ambiguity aversion is 2.28, implying that the level of ambiguity aversion is not significantly different from zero. This is in line with the small difference of RMSEs between AN and AA models.

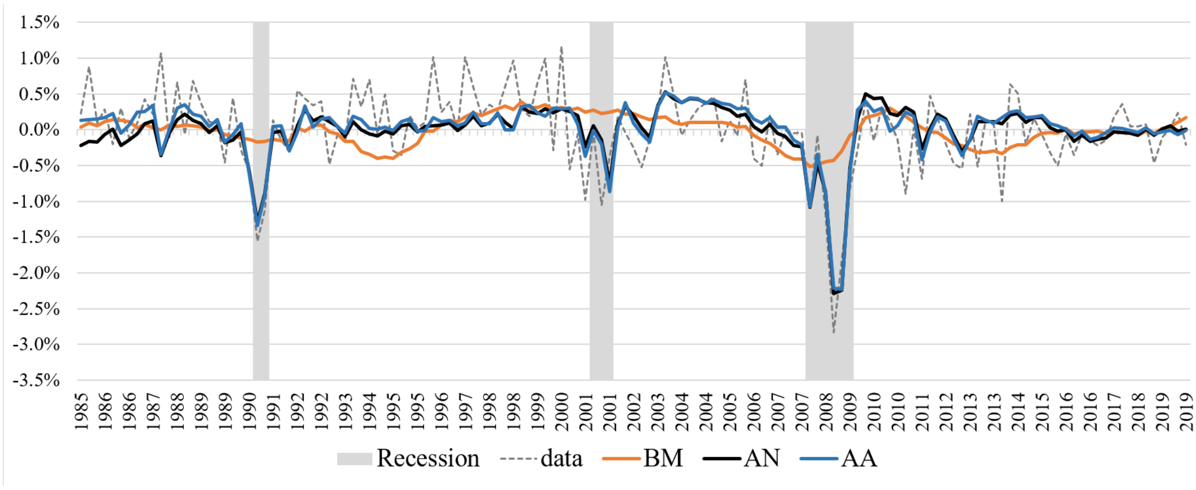
Steady state. Surprisingly, all smooth ambiguity models have one-scenario steady states but have different implications. The ambiguity neutral model has a steady state Bayesian belief of 0.07%, and a steady state macroeconomic uncertainty of 1.00. This indicates that the household believes there is only a 0.07% chance of recession when a shock hits the steady state economy. Furthermore, this small probability does not have any effect, as the household is almost indifferent between the two scenarios $M_s = 1.00$. In contrast, the ambiguity averse model has a worst-case steady state where Bayesian belief and macroeconomic uncertainty are 1. This implies that the household is certain a recession will occur when a shock hits the steady state but macroeconomic uncertainty does not impact the steady state decision, as the household is indifferent between the two scenarios $M_s = 1.00$. These results suggest that the US representative household makes decisions such that its expected utility is robust to uncertainty at the steady state.

The asymptotic standard errors of steady state Bayesian belief are large for both AN and AA models, indicating that RMSE is insensitive to changes in the estimated μ_s. The Bayesian belief of recession in the ambiguity averse model is only weakly significant at a 10% confidence level. This implies that the household's steady state belief does not have a significant effect on economic fluctuation outside of the steady state.

Uncertainty and ambiguity aversion can have indirect effects on the steady state through other parameters. For instance, uncertainty can reduce the expected return from investment, thus discouraging the household from investing. As Table 4 shows, the steady state share of investment in output in the benchmark model (0.24) is larger than that in the smooth ambiguity models (0.20-0.22). Additionally, ambiguity aversion can lead the household to prioritize its current utility (consuming more and working less). Consequently, the ratio of consumption to labor in the ambiguity averse model is 2.38, which is higher than the ratio of 1.8538 observed in the ambiguity neutral model.

Model fit. The smooth ambiguity models clearly outperform the benchmark model in terms of data fitting. The RMSEs of the smooth ambiguity models are markedly lower than the BM, particularly in recession periods. Moreover, the RMSE of the ambiguity averse model is marginally better than that of the ambiguity neutral model in both recession and normal growth periods. These results suggest that adding uncertainty helps significantly improve data fitting for the US. We also confirm this result with the log likelihood from the standard Bayesian estimation in Appendix E. Further, Figure 3 clearly demonstrates the distinction between the benchmark model (orange line) and smooth ambiguity models (black and blue lines).

Figure 3: Fitted quarterly real output growth



Note: The results of Table 4 are illustrated through the solid lines, which represent the fitted real GDP growth. The dashed line represents the actual quarterly output growth minus the mean. The models are as follows: BM - Benchmark model; AN - Ambiguity Neutral model; AA - Ambiguity Averse model.

Out-of-sample forecast. The good fit of our model is reflected in the relatively accurate out-of-sample forecast. Table 5 reports the RMSEs and Figure 4 depicts the out-of-sample forecasts of US output growth, generated from our model and from the survey of US professional forecasters[11]. To forecast the output growth at time t, we estimated the model up until time $t-1$ and used the Economic Policy Uncertainty Index and the SPF's recession probability at time t to simulate the output growth at time t. We excluded the Fernald (2014)'s technological progress data from the predictions as this time series is constructed ex-post the release of GDP. We use the first 10 years (1985Q1 -1994Q4) as a calibrating period and start the forecast from 1995Q1 until 2019Q4.

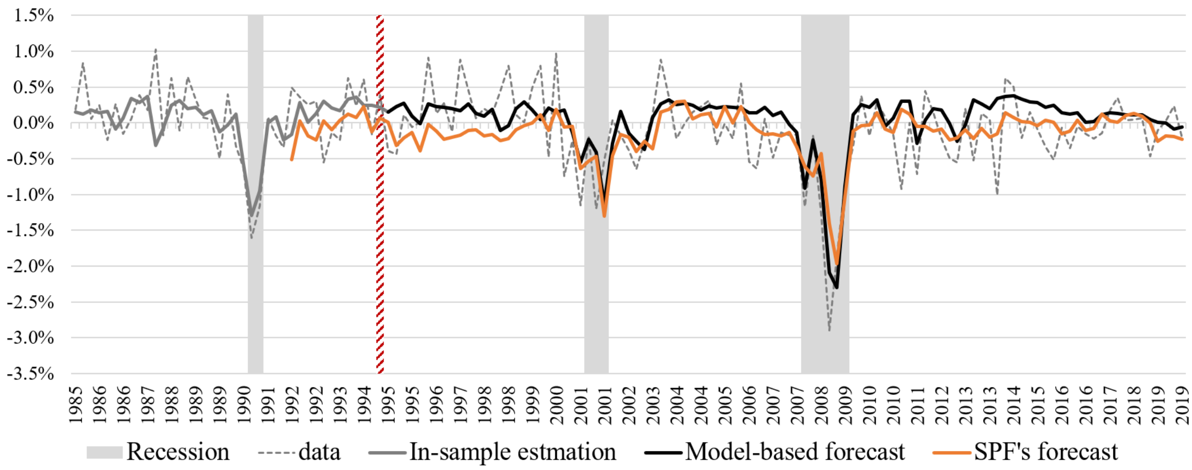
Our model-based forecast was comparable to the SPF's forecast, with an overall RMSE of 0.4439% for the SPF's forecast and 0.4381% for the model-based forecast as reported in Table 5. During the periods of normal growth, the SPF's forecast slightly outperformed our model (0.40% vs 0.43%). Interestingly, however, the greatest discrepancy was observed in the forecasts during recessions, with the SPF's RMSE being 0.68% and our model-based forecast performing better at 0.49%. This result is surprising, as most model-based forecasts have been unable to accurately capture the Great Recession and its turning point due to fixed parameters and a mean-reverting property. On average, US professional forecasters tend to outperform economic models as they are able to adjust to the new information faster (Ng and Wright, 2013; Wieland and Wolters, 2012). However, it is important to note that the out-of-sample forecasts of the smooth ambiguity model are based on revised GDP data, and therefore may be subject to potential biases.

[11] To measure the quarterly GDP growth forecasts from the Survey of Professional Forecasters (SPF), we calculate the log difference between the average GDP level forecast for the current quarter and the actual GDP level of the previous quarter that was available to the forecasters when making the forecasts. This method is also employed in the Federal Reserve Bank of Philadelphia's report on the SPF. Subtracting the SFPs' forecast from the average GDP growth rate allows us to fit the forecast to the zero-growth model. The US professional forecasters' real GDP forecasts have been available since 1992.

Table 5: Out-of-sample forecast performance

RMSEs	SPF	Model
All periods	0.44%	0.44%
Recession periods	0.68%	0.49%
Normal growth periods	0.40%	0.43%

Figure 4: Out-of-sample forecast of the US quarterly real output growth



Note: The forecast period is 1995Q1 - 2019Q2 as indicated by the dashed vertical red line. We estimate the model until time $t-1$ and forecast the output growth at t utilizing the US Economic policy uncertainty index and the Survey of US Professional Forecasters' recession probability at time t.

5 Discussions

In this section, we provide a theoretical analysis of the estimations results in relation to the model structure. We first investigate the viability of the three stylized facts within our estimations and then examine ambiguity and risk aversion over time.

5.1 Three stylized facts in the estimation

We compare our estimation results to three stylized facts introduced in Section 3.3. We present an empirical demonstration of how our model's estimation fits with the three stylized facts.

First stylized fact. Macroeconomic uncertainty makes people more pessimistic (Bhandari et al., 2019; Bianchi et al., 2020; Piccillo and Poonpakdee, 2021). We relate pessimism with subjective belief of recession. To calculate subjective belief, we use the formula in Equation 14 and the estimated coefficient of ambiguity aversion $\gamma = 3.35$.

Figure 5a displays the model-implied subjective beliefs of recession and the Bayesian beliefs computed from the SPF's recession probability. The level of subjective belief is consistently higher Bayesian belief, reflecting that the household in the AA model is more pessimistic than those in the

AN model[12]. Furthermore, the spread between subjective belief and Bayesian belief becomes larger during periods of high macroeconomic uncertainty as evidenced by the correlation of 0.88 between this spread and macroeconomic uncertainty. Therefore, macroeconomic uncertainty makes the representative ambiguity averse household more pessimistic as it believes that the next-period recession probability is higher when macroeconomic uncertainty rises.

Second stylized fact. Macroeconomic uncertainty can increase and decrease subjective uncertainty (Glas, 2020; Piccillo and Poonpakdee, 2021). To calculate subjective uncertainty, we use the formula in Equation 15 and the model-implied subjective beliefs.

Figure 5b depicts the model-implied subjective uncertainty with the dashed line representing the average subjective uncertainty in normal growth periods between each recession. Subjective uncertainty indicates how much the household is uncertain about its subjective beliefs. The correlation between macroeconomic uncertainty and subjective uncertainty is 0.60. The periods between 1991 and 2006 had a lower subjective uncertainty than the other periods on average, implying the household was certain that a recession would not occur, given that the subjective belief of recession was low. Looking at the three recession episodes, subjective uncertainty was high in the 2000 Dot-com crisis while it was low in the 1990 recession and the 2008 Global Financial crisis (GFC). In particular, the subjective uncertainty in GFC was the lowest among all recessions while macroeconomic uncertainty was at its peak in the GFC. This implies that the US representative household was certain that a recession would occur, given the high subjective belief of recession. Therefore, we can see that the relationship between macroeconomic uncertainty and subjective uncertainty is non monotonic.

Third stylized fact. The effect of macroeconomic uncertainty on the economy is nonlinear and is stronger when macroeconomic uncertainty is higher (Jackson et al., 2020; Lhuissier and Tripier, 2021; Ng and Wright, 2013). To illustrate the effect of macroeconomic uncertainty, we use the ratio to the benchmark model (Eq. 17) which indicates the deviation of the marginal expected utility of investment in the ambiguity averse model from the benchmark model (no uncertainty). The ratio lies between zero and one where one means that there is no deviation.

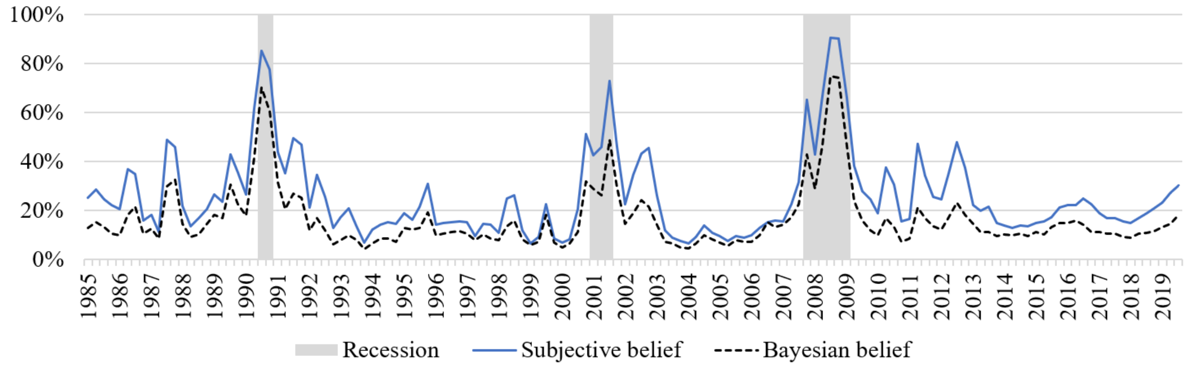
Figure 5c shows the model-implied ratio to the benchmark model. The ratio is close to one in normal growth periods, indicating that the ambiguity averse model is comparable to the benchmark model (no uncertainty). Furthermore, there is a substantial downward deviation in the ratio during recession periods. This is attributed to a higher macroeconomic uncertainty which causes pessimistic belief distortions and the increased spread in the expected utilities between normal growth and recession scenarios. For example, in the GFC, macroeconomic uncertainty was 1.42, implying that the representative household expected its utility to be 30% ($\frac{1.42-1}{1.42}$) lower during the recession as compared to normal growth. In combination with the heightened subjective belief of 90%, the ratio to the benchmark model decreased to 0.75, implying that the expected return of the investment (in term of utility) was 25% lower than if the GFC had never occurred. As a result, the investment was delayed and the output

[12]The ambiguity neutral household holds only the Bayesian belief of recession.

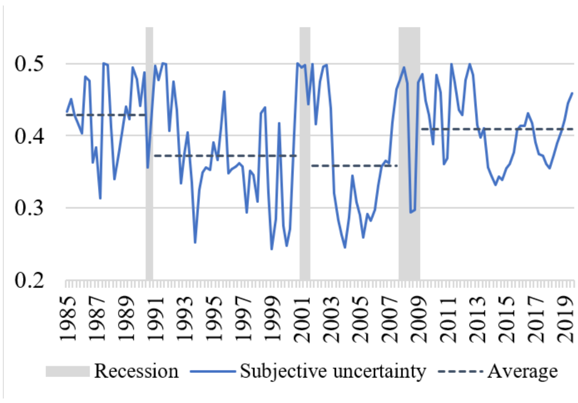
growth diminished. This highlights that subjective beliefs of the ambiguity averse household, consisting of Bayesian beliefs and pessimistic belief distortions, strengthen the nonlinear effect of macroeconomic uncertainty during extreme situations like economic crises.

Figure 5: Replication of the three stylized facts

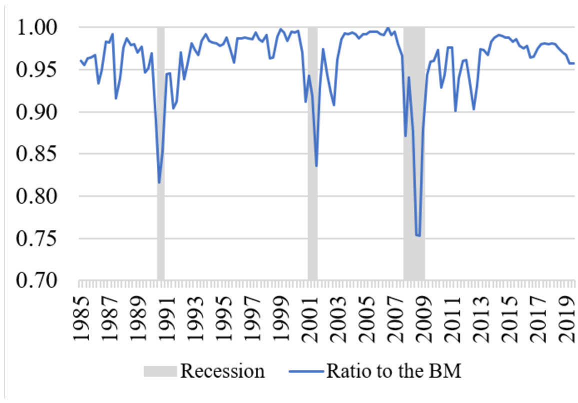
(a) Bayesian belief and model-implied subjective belief of recessions



(b) Subjective uncertainty



(c) Ratio to the benchmark model



Note: All blue solid lines are calculated using the formulae in Section 3.3 with ambiguity aversion $\gamma = 3.35$.

5.2 Time variation of the attitudes toward risk and ambiguity

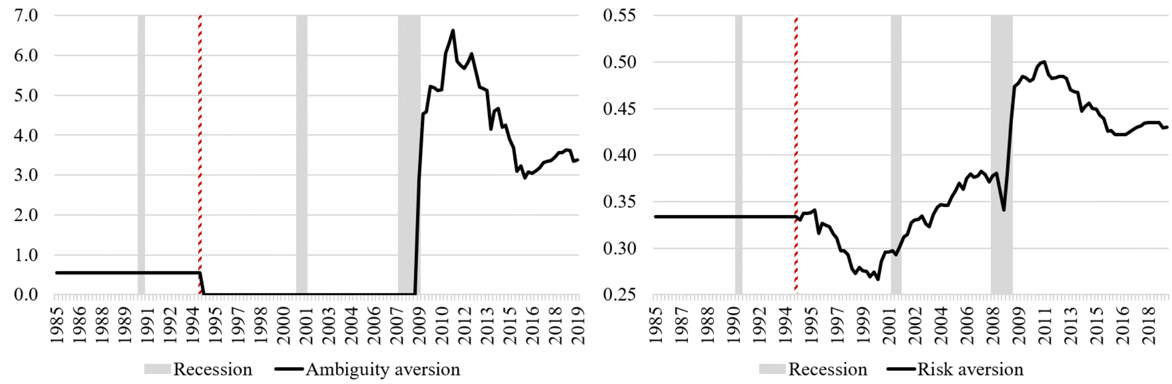
We observe the time variation of risk and ambiguity aversion as measured by recursive estimations. This result highlights the impact of the two major economic crises, the Dot-com crisis and the Global Financial crisis (GFC), on households' attitudes towards risk and ambiguity.

Figure 6 demonstrates the time variations of ambiguity and risk aversions from 1995Q1 to 2019Q4. Prior to the Dot-com crisis and following its conclusion, the level of ambiguity aversion remained at zero, indicating that the household was ambiguity neutral. However, risk aversion decreased from 0.34 to 0.27 prior to the Dot-Com crisis, which is consistent with the risk-taking behaviour and low risk premium that contributed to the financial market bubble of that time. When the bubble burst in 2000Q2, risk aversion began to increase and continued rising even after the Dot-com crisis concluded. During the GFC, ambiguity aversion experienced a sharp increase, peaking at 6.63 in 2010Q3. Subsequently, it decreased and stabilized around 3 since 2016Q4. Risk aversion displays a similar movement but it briefly decreased in the middle of GFC. After the GFC, risk aversion stabilized around 0.43, almost twice the level of pre Dot-com crisis.

Our results emphasize the different impacts of crises on both ambiguity and risk aversions. The dot-com crisis seems to contribute to the increased risk aversion, but had no effect on ambiguity aversion. In contrast, the Global Financial Crisis led to a structural rise in both parameters. The increase in risk aversion indicates that the marginal utility of consumption decreases, which in turn reduces consumption. Moreover, the increase in ambiguity aversion results in a lower expected marginal utility of investment, discouraging investment. This could explain the slower US economic recovery from the GFC when compared to the Dot-com crisis.

Figure 6: Dynamics of ambiguity and risk aversion in the US

(a) Ambiguity aversion **(b)** Risk aversion



Note: The recursive estimation period begins at 1995Q1 as indicated by the dashed vertical red line.

6 Conclusion

In this paper, we develop and estimate a business cycle model with smooth ambiguity preferences based on Altug et al. (2020). We use the macroeconomic uncertainty index to anchor uncertainty in our model, which equals the ratio between the expected utilities of normal growth and recession scenarios. With this assumption, we study the transmission mechanism of macroeconomic uncertainty and its effects on the household's beliefs and the economy. By employing mathematical analysis and estimations, we demonstrate that the smooth ambiguity model can replicate three empirical stylized facts: the households' pessimistic beliefs, the nonmonotonic responses of subjective uncertainty to macroeconomic uncertainty, and the nonlinear effects of macroeconomic uncertainty.

The estimation suggests that the smooth ambiguity model outperforms the benchmark model in terms of data fitness for the US output growth. With a relevant level of ambiguity aversion, the model is able to capture the large output drop during recession periods. Our out-of-sample forecast further supports this notion, implying that ambiguity aversion and pessimistic belief distortions could be important determinants of the severity of the crisis. Moreover, the Global Financial Crisis led to a structural increase in ambiguity aversion, whereas it remained unchanged throughout the Dot-com crisis. This may explain why the recovery from the GFC was slower than the Dot-com crisis.

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A Summary of Altug et al. (2020)'s model

Altug et al. (2020) present a social planner maximization model in which the agent holds smooth ambiguity preference of Klibanoff et al. (2005). The source of uncertainty and belief is in the total production factor (TFP). The growth of TPF consists of 2 components which are long-run and temporary. To the agent or social planner, the long-run component ($\bar{g}$) is known but the temporary component (x_t) is ambiguous.

The data generating processes of TFP growth, temporary component and TFP are defined as following:

$$g_{A,t+1} = \bar{g} + x_{t+1} + \sigma_A \epsilon_{A,t+1}$$
$$x_{t+1} = \rho x_t + \sigma_x \epsilon_{x,t+1}$$
$$A_{t+1} = A_t \exp(g_{A,t+1})$$

Social planner tries to forecast the temporary component. She or he knows that, at a time, the temporary component is either in high persistent or low persistent stage. Therefore, the agent have two forecasts which are:

- $\hat{x}_{k,t}$: the temporary TFP component for state k (high/low persistence) following Kalman filter
- η: the belief of probability that the economy is in low persistent stage following Bayesian rule

Production function is:

$$y_t = k_t^a (A_t n_t)^{1-a}$$
$$k_{t+1} = (1-\delta)k_t + i_t$$

The social planner has the following indirect value function:

$$\hat{J}(\hat{k}_t, \mu_t) = \max_{\hat{c}_t, n_t, \hat{i}_t} \left\{ \frac{(\hat{c}_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \beta \left[E_{\mu_t} \left(E_{x_t} \left[\hat{J}(\hat{k}_{t+1}, \mu_{t+1}) \exp(\gamma(1-\nu) g_{A,t+1}) \right] \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\}$$

subject to

$$
\begin{aligned}
\hat{c}_t + \hat{i}_t &\leq \hat{k}_t^a n_t^{1-a} \\
\exp(g_{A,t+1})\hat{k}_{t+1} &= (1-\delta)\hat{k}_t + \hat{i}_t \\
l_t + n_t &\leq 1 \\
\hat{i}_t &\geq 0 \\
\mu_t &= (\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) \\
\hat{x}_{k,t} &\sim \text{ Kalman filter} \\
\eta_t &\sim \text{ Bayesian updating}
\end{aligned}
$$

B Derivation for section 3.1

We show the derivation of Euler equation with uncertainty. First, we substitute $E_t(V_{t+1}^R) = \frac{E_t(V_{t+1}^{NR})}{M_t}$ into the belief distortions and obtain:

$$
\begin{aligned}
\zeta_t^R &= \frac{E_t(V_{t+1}^R)^{-\gamma}}{\left(\mu_t E_t(V_{t+1}^R)^{1-\gamma} + (1-\mu_t)E_t(V_{t+1}^{NR})^{1-\gamma}\right)^{\frac{-\gamma}{1-\gamma}}} \\
&= \frac{\left(\frac{E_t(V_{t+1}^{NR})}{M_t}\right)^{-\gamma}}{\left(\mu_t\left(\frac{E_t(V_{t+1}^{NR})}{M_t}\right)^{1-\gamma} + (1-\mu_t)E_t(V_{t+1}^{NR})^{1-\gamma}\right)^{\frac{-\gamma}{1-\gamma}}} \\
&= \frac{\left(\frac{E_t(V_{t+1}^{NR})}{M_t}\right)^{-\gamma}}{\left(E_t(V_{t+1}^{NR})^{1-\gamma}(\frac{\mu_t}{M_t^{1-\gamma}} + (1-\mu_t))\right)^{\frac{-\gamma}{1-\gamma}}} \\
\zeta_t^R &= \frac{M_t^\gamma}{\left(\mu_t M_t^{\gamma-1} + (1-\mu_t)\right)^{\frac{-\gamma}{1-\gamma}}} \\
\zeta_t^{NR} &= \frac{1}{\left(\mu_t M_t^{\gamma-1} + (1-\mu_t)\right)^{\frac{-\gamma}{1-\gamma}}}
\end{aligned}
$$

Then, substituting $E_t(V^R_{t+1}) = \frac{E_t(V^{NR}_{t+1})}{M_t}$ into the Euler equation, we have:

$$\begin{aligned}
\Lambda_t &= \beta\Upsilon_t \left(\mu_t \xi^R_t \frac{\partial \frac{E_t(V^{NR}_{t+1})}{M_t}}{\partial K_{t+1}} + (1-\mu_t)\xi^{NR}_t \frac{\partial E_t(V^{NR}_{t+1})}{\partial I_t} \right) \\
&= \beta\Upsilon_t \left(\mu_t \xi^R_t \left(\frac{1}{M_t} \frac{\partial E_t(V^{NR}_{t+1})}{\partial I_t} + E_t(V^{NR}_{t+1}) \frac{\partial M_t^{-1}}{\partial I_t} \right) + (1-\mu_t)\xi^{NR}_t \frac{\partial E_t(V^{NR}_{t+1})}{\partial I_t} \right) \\
&= \beta\Upsilon_t \left(\frac{\mu_t \xi^R_t}{M_t} E_t(\Lambda^{NR}_{t+1}(R^{NR}_{t+1} + 1 - \delta)) + (1-\mu_t)\xi^{NR}_t E_t(\Lambda^{NR}_{t+1}(R^{NR}_{t+1} + 1 - \delta)) \right) \because \frac{\partial M_t^{-1}}{\partial I_t} \approx 0 \\
&=_t (\Lambda^{NR}_{t+1}(R^{NR}_{t+1} + 1 - \delta))\Upsilon_t \left(\frac{\mu_t \xi^R_t}{M_t} + (1-\mu_t)\xi^{NR}_t \right)
\end{aligned}$$

$$\begin{aligned}
\text{where } \Upsilon_t &= \frac{\mu_t M_t^\gamma + (1-\mu_t)}{\left(\mu_t M_t^{\gamma-1} + (1-\mu_t)\right)^{\frac{-\gamma}{1-\gamma}}} \\
\xi^R_t &= \frac{M_t^\gamma}{\mu_t M_t^\gamma + (1-\mu_t)} \\
\xi^{NR}_t &= \frac{1}{\mu_t M_t^\gamma + (1-\mu_t)}
\end{aligned}$$

Λ^{NR}_{t+1} is the marginal utility of consumption in the normal scenario.

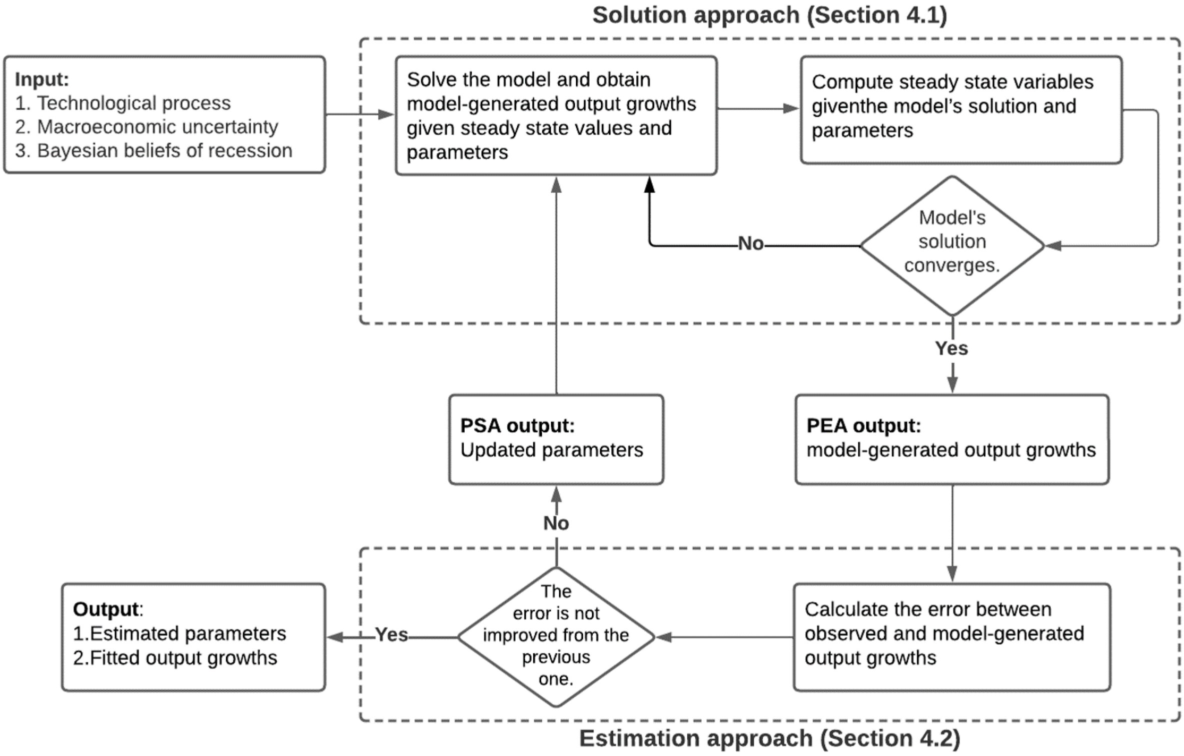
R^{NR}_{t+1} is the rental price of capital in the normal scenario.

The investment I_t could have a very small or zero second-order effect on the current uncertainty M_t. Therefore, we assume that $\frac{\partial M_t^{-1}}{\partial I_t} \approx 0$.

C Solution and estimation methods

We provide in-depth discussion of our solution and estimation methods, which are one of the main contributions. The numerical algorithms used to solve and estimate our model are the parameterized expectation algorithm (PEA) and the pattern search algorithm (PSA) respectively. Figure 7 illustrates the flowcharts of our solution and estimation approaches. The PEA is used to solve the model and to generate model-implied output growths which are then used as an input for the PSA. The PSA then searches for a set of parameters that minimizes the difference between the model-generated and observed output growths. This procedure provides an estimated set of parameters and corresponding fitted output growths.

Figure 7: Solution and estimation flowcharts



Note: PEA refers to parameterized expectations algorithm and PSA refers to pattern search algorithm.

To solve the model, the system of equations is transformed into the zero-growth steady state by normalizing the variables as $y_t = \frac{Y_t}{Z_t}$, $i_t = \frac{I_t}{Z_t}$, $k_{t+1} = \frac{K_{t+1}}{Z_t}$, $c_t = \frac{C_t}{Z_t}$, $i_t = \frac{I_t}{Z_t}$, $w_t = \frac{W_t}{Z_t}$, $\lambda_t = \frac{\Lambda_t}{Z_t^{-\sigma}}$, and $z_t = \frac{Z_t}{\exp(\bar{a})}$, where R_t, L_t, M_t, and μ_t are assumed to be stationary. The equilibrium conditions are following.

Household:

$$\lambda_t = \beta E_t \left[\lambda_{t+1}^{NR} \left(\frac{z_{t+1}}{z_t} \right)^{-\sigma} (R_{t+1}^{NR} + 1 - \delta) \right] \Upsilon_t \left[\mu_t \xi_t^R \frac{1}{M_t} + (1 - \mu_t) \xi_t^{NR} \right]$$

$$\lambda_t = c_t^{-\sigma}$$

$$\lambda_t = \frac{\exp((1-\sigma)\bar{a}) L_t^{\nu}}{z_t^{1-\sigma} w_t}$$

Firm:

$$k_{t+1} = (1-\delta)k_t \frac{z_{t-1}}{z_t} + i_t$$
$$y_t = \exp(\alpha\bar{a}) z_{t-1}^{\alpha} k_t^{\alpha} L_t^{1-\alpha}$$
$$w_t = (1-\alpha)\frac{y_t}{L_t}$$
$$R_t = \alpha \frac{y_t}{k_t} \frac{z_t}{z_{t-1}}$$
$$z_t = \exp(a_t - \bar{a})$$

Market clearing:

$$y_t = c_t + i_t$$

Other processes:

$$a_t = (1-\rho_a)\bar{a} + \rho_a a_{t-1} + e_t^a; e_t^a \sim \mathcal{N}(0, \sigma_a^2)$$
$$M_t = (1-\rho_M)\bar{M} + \rho_M M_{t-1} + e_t^M; e_t^M \sim \mathcal{N}(0, \sigma_M^2)$$
$$\mu_t = B(\mu_t^{prior}, \mathcal{I}_t)$$

C.1 Solution approach

As illustrated in Figure 7, the solution procedure involves solving the model with given steady state values and parameters. Then the model solution is again used to calculate the steady state values. We repeat these steps until the solution reaches a desired level of accuracy. This section describes parameterized expectation algorithm that we use to solve to model and the calculation of steady state values. Then we analyze the PEA solutions to gain insights into the transmission mechanisms of macroeconomic uncertainty and ambiguity aversion since we do not have an analytical solution.

The standard linearization method is not suitable to solve the model due to the concave property of the right side of the smooth ambiguity Euler equation. This concavity can be eliminated by the first-order approximation, which then eliminates the pessimistic belief distortions, the core mechanism of the model. Bhandari et al. (2019) proposed a perturbation technique to solve a business cycle model with robust preferences, which has similarly disappearing belief distortions with linearization. However, their approach is not used here since we have another challenge of the underdetermined system with 12 equilibrium equations and 14 variables: $\{\Lambda_t, C_t, I_t, L_t, K_t, Y_t, W_t, R_t, Z_t, a_t, M_t, \mu_t\}$ and $\{\Lambda_{t+1}^{NR}, R_{t+1}^{NR}\}$.

To solve the model, we first transform the system of equations into the zero-growth steady state by normalizing the variables: $y_t = \frac{Y_t}{Z_t}$, $i_t = \frac{I_t}{Z_t}$, $k_{t+1} = \frac{K_{t+1}}{Z_t}$, $c_t = \frac{C_t}{Z_t}$, $i_t = \frac{I_t}{Z_t}$, $w_t = \frac{W_t}{Z_t}$, $\lambda_t = \frac{\Lambda_t}{Z_t^{-\sigma}}$, and $z_t = \frac{Z_t}{\exp(\bar{a})}$, where R_t, L_t, M_t, and μ_t are assumed to be stationary. Then we use the parameterized expectation algorithm (PEA) since it can address the two aforementioned problems: concavity and an underdetermined system of equations. PEA approximates the household's conditional expectations with a parametric function that includes an interaction component, which captures the concavity of the household's Euler equation. According to Barañano et al. (2002), this approach reproduces the effect of the utility function's curvature more accurately than a log linear approach. We can also determine $\{\lambda_{t+1}^{NR}, R_{t+1}^{NR}\}$ by solving for the household's expectations conditional on the normal growth scenario. We then use this solution to solve the household's expectations with respect to both scenarios. Notably, the PEA solution for the normal growth scenario is identical to that of the benchmark model, which assumes the household makes its decisions with no uncertainty.

Parameterized expectations algorithm

Parameterized expectations algorithm is based on the intuition that the household makes decisions that are consistent with its expectations. PEA learns the decision rule in each iteration and finds a solution that is in accordance with the household's expectations.

We assume that the household's conditional expectation is a mixture of one-period lagged expectations and state variables, with ρ_λ representing the weight on the lagged expectation. The state variables are represented by a parametric function $P(k_t, z_t, M_t, \mu_t; \theta)$, where θ is a set of coefficients. Therefore, the parameterized Euler equation can be written as:

$$\begin{aligned}\lambda_t &= \beta E_t \left[\lambda_{t+1}^{NR} \left(\frac{z_{t+1}}{z_t}\right)^{-\sigma} (R_{t+1}^{NR} + 1 - \delta)\right] \Upsilon_t \left[\mu_t \xi_t^R \frac{1}{M_t} + (1-\mu_t)\xi_t^{NR}\right] \\ &= \rho_\lambda \lambda_{t-1} + (1-\rho_\lambda) E_t(P(k_t, z_t, M_t, \mu_t; \theta)) \end{aligned} \tag{18}$$

Define the parametric function as a combination of state variables z_t, M_t and predetermined variables k_t, μ_t. We include the interaction term $M_t\mu_t$ to capture the nonlinear effect of macroeconomic uncertainty.

$$\begin{aligned}&P(k_t, z_t, M_t, \mu_t; \theta) = \theta^c + \theta^k k_t + \theta^z z_t + \theta^M M_t + \theta^\mu \mu_t + \theta^{M\mu} M_t \mu_t + u_t \\ &\text{where } \theta = \{\theta^c, \theta^k, \theta^z, \theta^M, \theta^\mu, \theta^{M\mu}\}, E(u_t) = 0\end{aligned}$$

To solve for θ, we use the parameterized expectations algorithm adapted from Collard (2015) and incorporate the moving bound technique of Maliar and Maliar (2003) to avoid explosive solutions. First we address the underdetermined system by solving the benchmark model. This is to find a parametric function that represents how the household forms expectations in periods of no uncertainty, i.e., the normal growth scenario and pin down the decision's rule for $\{\lambda_t^{NR}, R_t^{NR}\}$. To do so, we use only the

Box C.1: Parameterized expectations algorithm to solve for θ^{NR}

1. Set an initial guesses for $\theta^{NR} = \{1, 0, 0\}$ and let k_1^{NR}, λ_0^{NR} be the steady state value of the benchmark model, and $S = \{a_t\}_{t=1}^{T}$ is observed from data. Consequently, $\{z_t\}_{t=1}^{T}$ is given.
2. At iteration i and for the given θ_i^{NR}, generate (1) $\{\lambda_t^{NR}\}_{t=1}^{T}$ using $\lambda_t^{NR} = \rho_\lambda \lambda_{t-1}^{NR} + (1-\rho_\lambda)P(k_t^{NR}, z_t; \theta_i^{NR})$ where ρ_λ is the weight on the lagged expectation, and (2) $\{c_t^{NR}, i_t^{NR}, k_{t+1}^{NR}, w_t^{NR}, R_t^{NR}, L_t^{NR}\}_{t=1}^{T}$ using the equilibrium conditions except the Euler equation
3. Let $X(\theta_i^{NR}) = \{\lambda_t^{NR}, c_t^{NR}, i_t^{NR}, k_{t+1}^{NR}, w_t^{NR}, R_t^{NR}, L_t^{NR}; \theta_i^{NR}\}_{t=1}^{T}$ and for given upper and lower bounds, $\bar{X}_i$ and $\underline{X}_i$,
   - Set $X(\theta_i^{NR}) = \bar{X}_i$ if any element in $X(\theta_i^{NR}) > \bar{X}_i$ and
   - Set $X(\theta_i^{NR}) = \underline{X}_i$ if any element in $X_i(\theta_i^{NR}) < \underline{X}_i$
4. Generate $\{\hat{\lambda}_t^{NR}\}_{t=1}^{T-1}$ using $\hat{\lambda}_t^{NR} = \beta\left(\lambda_{t+1}^{NR}\left(\frac{z_{t+1}}{z_t}\right)^{-\sigma}(R_{t+1}^{NR} + 1 - \delta)\right)$
5. Obtain $\hat{\theta}_{i+1}^{NR}$ by regressing $\{\frac{\hat{\lambda}_t^{NR} - \rho_\lambda \lambda_{t-1}^{NR}}{1-\rho_\lambda}\}_{t=1}^{T}$ against $\{1, k_t^{NR}, z_t\}_{t=1}^{T}$ such that:
$$\frac{\hat{\lambda}_t^{NR} - \rho_\lambda \lambda_{t-1}^{NR}}{1-\rho_\lambda} = \theta^{NR,c} + \theta^{NR,k}k_t + \theta^{NR,z}z_t + u_t^{NR} \text{ where } E(u_t^{NR}) = 0$$
6. Update $\theta_{i+1}^{NR} = \omega\hat{\theta}_{i+1}^{NR} + (1-\omega)\theta_i^{NR}; \omega = 0.5$ and if any variable hits the bounds in step 3, expand the bounds for the next iteration according to the following formula:
$$\bar{X}_{i+1} = X_s^{NR}(1+\Delta_i)$$
$$\underline{X}_{i+1} = X_s^{NR}(1-\Delta_i)$$
where $\Delta_i = 0.05 + 0.01i, X_s^{NR} =$ steady state values of variables in X
7. Go back to step 2 and iterate until $\left|\frac{\theta_i^{NR} - \theta_{i-1}^{NR}}{\theta_{i-1}^{NR}}\right| < 10^{-6}$ and no variable hits the bounds

time series of technological process and define the parametric function of the benchmark model as:

$$P(k_t, z_t; \theta^{NR}) = \theta^{NR,c} + \theta^{NR,k}k_t^{NR} + \theta^{NR,z}z_t + u_t^{NR}$$
$$\text{where } \theta^{NR} = \{\theta^{NR,c}, \theta^{NR,k}, \theta^{NR,z}\}, E(u_t^{NR}) = 0$$

We use the algorithm in Box C.1 to solve for the parametric function θ^{NR} of the expectation in the normal growth scenario. Given this result, we then use the algorithm in Box C.2 to solve for θ of the smooth ambiguity model. This model takes three time series as its input: an exogenous technological process, macroeconomic uncertainty, and Bayesian beliefs of recession. The technological process in the benchmark and smooth ambiguity models is same. It is reasonable to assume that the technological process is independent of Bayesian beliefs and macroeconomic uncertainty, implying that the technology does not associate with uncertainty. Furthermore, the works of Fernald (2014) and Basu et al. (2006) suggest that the pure technological process is exogenous to the firm and household's decision-makings regarding the utilization of capital and labor. Consequently, we can assume that the technological process is the same in both the benchmark and smooth ambiguity models.

Box C.2: Parameterized expectations algorithm to solve for θ

1. Set initial guesses for $\theta = \{1, 0, 0, 0, 0, 0\}$ and let k_1, λ_0 be the steady state value of the smooth ambiguity model, and $S = \{a_t, M_t, \mu_t\}_{t=1}^T$ is observed from data. Consequently, $\{z_t\}_{t=1}^T$ is given.
2. At iteration i, for the given θ_i, generate (1) $\{\lambda_t\}_{t=1}^T$ using $\lambda_t = \rho_\lambda \lambda_{t-1} + (1-\rho_\lambda)P(k_t, z_t, M_t, \mu_t; \theta_i)$ where ρ_λ is the weight on the lagged expectation, and (2) $\{c_t, i_t, k_{t+1}, w_t, R_t, L_t, \xi_t^R, \xi_t^{NR}, \Upsilon_t\}_{t=1}^T$ using the equilibrium conditions
3. Let $X(\theta_i) = \{\lambda_t, c_t, i_t, k_{t+1}, w_t, R_t, L_t, \xi_t^R, \xi_t^{NR}, \Upsilon_t; \theta_i\}_{t=1}^T$ and for given upper and lower bounds, $\bar{X}_i$ and $\underline{X}_i$,
   - Set $X(\theta_i) = \bar{X}_i$ if any element in $X(\theta_i) > \bar{X}_i$ and
   - Set $X(\theta_i) = \underline{X}_i$ if any element in $X_i(\theta_i) < \underline{X}_i$
4. Given the θ^{NR} obtained from the PEA in the benchmark model, generate $\{\lambda_t^{NR}, R_t^{NR}\}_{t=1}^T$ using $P(k_t, z_t; \theta^{NR})$.
   We use k_t instead of k_t^{NR} because the household forms an expectation given that capital is predetermined. Thus for the smooth ambiguity household, k_t^{NR} is not a predetermined variable but rather the expected capital in the normal growth scenario.
5. Generate $\{\hat{\lambda}_t\}_{t=1}^{T-1}$ where $\hat{\lambda}_t = \beta \left[\lambda_{t+1}^{NR} \left(\frac{z_{t+1}}{z_t}\right)^{-\sigma} (R_{t+1}^{NR} + 1 - \delta)\right] \Upsilon_t \left[\mu_t \xi_t^R \frac{1}{M_t} + (1-\mu_t)\xi_t^{NR}\right]$
6. Obtain $\hat{\theta}_{i+1}$ by regressing $\{\frac{\hat{\lambda}_t - \rho_\lambda \lambda_{t-1}}{1-\rho_\lambda}\}_{t=1}^T$ against $\{1, k_t, z_t, M_t, \mu_t, M_t\mu_t\}_{t=T_{\text{begin}}}^{T-1}$ such that:
   $$\frac{\hat{\lambda}_t - \rho_\lambda \lambda_{t-1}}{1-\rho_\lambda} = \theta^c + \theta^k k_t + \theta^z z_t + \theta^M M_t + \theta^\mu \mu_t + \theta^{M\mu} M_t \mu_t + u_t \text{ where } E(u_t) = 0$$
7. Update $\theta_{i+1} = \omega \hat{\theta}_{i+1} + (1-\omega)\theta_i; \omega = 0.5$ and if any variable hits the bounds in step 3, expand the bounds for the next iteration according to the following formula:
   $$\begin{aligned} &\bar{X}_{i+1} = X_s(1+\Delta_i) \\ &\underline{X}_{i+1} = X_s(1-\Delta_i) \\ &\text{where } \Delta_i = 0.05 + 0.01i, X_s = \text{ steady state values of variables in } X \end{aligned}$$
8. Go back to step 2 and iterate until $\left|\frac{\theta_i - \theta_{i-1}}{\theta_{i-1}}\right| < 10^{-6}$ and no variable hits the bounds

Determining the ambiguous steady state using PEA solution

In order to determine the ambiguous steady state, we initially calculate the steady-state Bayesian belief μ_s from an external source, such as empirical data. We then compute the steady-state macroeconomic uncertainty M_s by ensuring the original Euler equation (Eq. ??) and the parameterized Euler equation (Eq. 18) are satisfied.

$$\begin{aligned} \lambda_s &= \rho_\lambda \lambda_s + (1-\rho_\lambda)(\theta^c + \theta^k k_s + \theta^z z_s + \theta^M M_s + \theta^\mu \mu_t + \theta^{M\mu} M_s \mu_s) \\ \lambda_s &= \beta \left[\lambda_s^{NR}(R_s^{NR} + 1 - \delta)\right] \Upsilon_s \left[\mu_s \xi_s^R \frac{1}{M_s} + (1-\mu_s)\xi_s^{NR}\right] \\ \text{where } \Upsilon_s &= \frac{\mu_s M_s^\gamma + (1-\mu_s)}{\left(\mu_s M_s^{\gamma-1} + (1-\mu_s)\right)^{\frac{-\gamma}{1-\gamma}}} \\ \xi_s^R &= \frac{M_s^\gamma}{\mu_s M_s^\gamma + (1-\mu_s)} \\ \xi_s^{NR} &= \frac{1}{\mu_s M_s^\gamma + (1-\mu_s)} \end{aligned}$$

λ_s^{NR} and R_s^{NR} are the steady state values in the benchmark model. Other steady state variables can be solved using the following equilibrium equations.

$$\begin{aligned}
\lambda_s &= \frac{\exp((1-\sigma)\bar{a})L_s^{\nu}}{w_s} \\
\lambda_s &= c_s^{-\sigma} \\
y_s &= \exp(\alpha\bar{a})k_s^{\alpha}L_s^{1-\alpha} \\
w_s &= (1-\alpha)\frac{y_s}{L_s} \\
R_s &= \alpha\frac{y_s}{k_s} \\
i_s &= \delta k_s \\
y_s &= c_s + i_s \\
z_s &= 1
\end{aligned}$$

C.2 Estimation method

In this section, we describe the estimation method and data used for the estimations. Our estimation is a nonlinear least squares method (NLS) which minimizes the distance between the model-implied and empirical output growth rates. We define the model-implied output growth as $\dot{y}(S_t;\Theta) = \log\left(\frac{y(S_t;\Theta)}{y(S_{t-1};\Theta)}\right)$, where the observable variables at time t are $S_t = \{a_t, M_t, \mu_t\}$ and the estimated parameters are $\Theta = \{\alpha, \nu, \sigma, \rho_\lambda, \mu_s, \gamma\}$. The model is fitted with $\dot{y}_t^{\text{obs}}$, the observed output growth rate. We use a pattern search algorithm to find the set of parameters $\hat{\Theta}$ that minimizes the root mean square errors (RMSE) between these two variables:

$$\text{RMSE}(\hat{\Theta}) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\dot{y}(S_t;\hat{\Theta}) - \dot{y}_t^{\text{obs}}\right)^2}$$

Additionally, we employ the delta method to compute the asymptotic standard errors of the estimated parameters.

$$\begin{aligned}
\dot{y}(S_t;\hat{\Theta}) &\approx \dot{y}(S_t;\Theta) + \nabla\dot{y}(S_t;\Theta)(\hat{\Theta}-\Theta) \\
\hat{\Theta}-\Theta &\approx (\dot{y}(S_t;\hat{\Theta}) - \dot{y}(S_t;\Theta))\left(\nabla\dot{y}(S_t;\hat{\Theta})\right)^{-1} \\
\hat{\Theta}-\Theta &\longrightarrow_d \mathcal{N}(0, \hat{\sigma}^2 F'F)
\end{aligned}$$

where $\hat{\sigma} = RMSE(\hat{\Theta})$, $F = (\nabla\dot{y}(S_t;\Theta))^{-1}$ and $\nabla\dot{y}(S_t;\Theta)$ is a gradient matrix with respect to $\hat{\Theta}$[13].

The NLS approach, which is also used in Carroll et al. (2019), has the advantage of allowing us to use the observable data as the target which the estimated model seeks to fit. Thus, any failure to match the target can be used to study the limitations of the model and derive a useful economic explanation.

[13] We compute the gradient matrix numerically using $\nabla\dot{y}(S_t;\Theta) = \frac{\dot{y}(S_t;\hat{\Theta}+h)-\dot{y}(S_t;\hat{\Theta}-h)}{2h}$ where $h = \min\{10^{-7}, \hat{\Theta}10^{-7}\}$

In contrast, maximum likelihood estimation used in most macroeconomic models relies on linearization around a steady state in combination with Bayesian estimation to maximize data density. Since the true likelihood is unknown, it is difficult to interpret the gap between the estimated model and the target. Other estimation's targets are methods of moments (Duffie and Singleton, 1993; Kim and Ruge-Murcia, 2009) and indirect inference (Guerron-Quintana et al., 2017; Theodoridis, 2011). The methods of moment estimator seeks to minimize the gap between model-implied and data-implied moments, while indirect inference focuses on fitting the impulse response computed from the data. For robustness check, we provide the results of these estimators in Section **??**.

In the context of the nonlinear least squares method, we briefly discuss the econometric issues associated with estimating DSGE models: weak identification, stochastic singularity, and small-sample distortion as pointed out by Ruge-Murcia (2007). NLS generally suffers from weak identification less than other estimators since it does not require selection of moments or likelihood to estimate the model. Furthermore, stochastic singularity is relevant to linearized DSGE models (Ruge-Murcia, 2007) thus it does not apply in our estimation as we do not linearize the model. Lastly, small-sample distortion leads to a discrepancy between the asymptotic standard errors and the actual variability of the estimated parameters, which affects the significance of the estimated parameters. Ruge-Murcia (2007) demonstrates that all three estimations methods (maximum likelihood, methods of moments, indirect inference) suffer from small-sample distortion to varying degrees. In particular, maximum likelihood estimation tends to produce asymptotic standard errors that are larger than the actual standard errors for all estimated parameters. This is expected to be the case for NLS as well, given the similarity in asymptotic properties between NLS and ML under the Gaussian distribution.

We use the pattern search algorithm[14] to estimate the parameters that minimize the distance between the model-implied and observed output growth rate. To illustrate how the pattern search algorithm works, we provide an example of the estimation with two parameters: $\Theta = \{\gamma, \sigma\}$ in Box C.3.

C.3 Alternative estimators

This section describes three alternative estimators, maximum likelihood (ML), simulated methods of moments (SMM), and indirect inference (II), which were used in Section E.1. The nonlinear least squares (NLS) estimator is the main estimator, and its results are presented in Section **??**. Each estimator has a different objective function but uses the same pattern search algorithm described in Section **??**. NLS minimizes the distance between model-generated and observed output growths. ML maximizes the sum of the log likelihood such that the observed output growths are most probable in the model. SMM minimizes the weighted distance of selected moments, which are implied from the model and observed data. Lastly, II fits the impulse response of the model to the observed data. Let Θ be a set of parameter, $\dot{\mathbf{y}}(\Theta)$ be the model-implied output growths and $\dot{\mathbf{y}}^{\text{obs}}$ be the observed output growths. Table 6 summarizes

[14] This is *patternsearch* function in *MATLAB*.

Box C.3: Pattern search algorithm - an example

Let $S_t = \{a_t, M_t, \mu_t\}$ be observed variables.

1. Set an initial guess of the parameter to $\Theta_0 = \{\gamma_0, \sigma_0\}$ and the initial mesh size to $m_1 = 0.2$. The upper bound is set to $\bar{\Theta} = \{\bar{\gamma}, \bar{\sigma}\}$ and the lower bound is set to $\underline{\Theta} = \{\underline{\gamma}, \underline{\sigma}\}$. Rescale the parameters using $x(\Theta_0) = \frac{\Theta_0 - \underline{\Theta}}{\bar{\Theta} - \underline{\Theta}}$.
2. At iteration i, compute the four mesh pairs as following:
$$\begin{pmatrix} \mathrm{x}(\gamma_{i-1}) & \mathrm{x}(\sigma_{i-1}) \\ \mathrm{x}(\gamma_{i-1}) & \mathrm{x}(\sigma_{i-1}) \\ \mathrm{x}(\gamma_{i-1}) & \mathrm{x}(\sigma_{i-1}) \\ \mathrm{x}(\gamma_{i-1}) & \mathrm{x}(\sigma_{i-1}) \end{pmatrix} + \begin{pmatrix} \mathrm{m}_i & 0 \\ 0 & \mathrm{m}_i \\ \text{-}\mathrm{m}_i & 0 \\ 0 & \text{-}\mathrm{m}_i \end{pmatrix} = \begin{pmatrix} \mathrm{x}_1(\Theta_i) \\ \mathrm{x}_2(\Theta_i) \\ \mathrm{x}_3(\Theta_i) \\ \mathrm{x}_4(\Theta_i) \end{pmatrix}$$
If any mesh element is less than 0, we set it to 0 or if any mesh point is more than 1, we set it to 1.
3. For each pair, revert the $x(\Theta_i)$ parameters back to their original scales and solve the model with PEA. Then, compute model-implied output growth $\dot{y}(S_t; \Theta_i)$ rate and RMSE(Θ_i) using:
$$\dot{y}(S_t; \Theta_i) = \log y(S_t; \Theta_i) - \log y(S_{t-1}; \Theta_i)$$
$$\text{RMSE}(\Theta_i) = \sqrt{\frac{1}{T} \sum_{t=1}^{T} \left(\dot{y}(S_t; \Theta) - \dot{y}_t^{\text{obs}}\right)^2}$$
4. If any mesh pair $x(\Theta_i)$ yields RMSE that is lower than or equals the RMSE(Θ_{i-1}):
   - Set the new parameter $x(\Theta_i)$ to the mesh pair that generates the lowest RMSE
   - Expand the mesh size by setting $m_{i+1} = m_i \times 2$

   If no mesh pair $x(\Theta_i)$ yields RMSE that is lower than RMSE(Θ_{i-1}):
   - Set the new parameter $x(\Theta_i)$ to $x(\Theta_{i-1})$
   - Shrink the mesh size by setting $m_{i+1} = m_i \times 0.5$
5. Go back to Step 2 and iterate until $|x(\Theta_i) - x(\Theta_{i-1})| < 10^{-6}$ or $m_i < 10^{-6}$
6. After the patternsearch algorithm, we run Nelder-Mead simplex algorithm[a] used in Carroll et al. (2019), to ensure if we have obtained the local minimum.
7. Set the solution $\hat{\Theta}$ as a new initial value of the parameter and iterate until the parameter values converge, with a tolerance of 10^{-6}.

[a]This is *fminsearch* function in Matlab. See Lagarias et al. (1998) for detail.

the objective functions of all estimators.

D Data

We use three empirical time series as the model inputs. The TFP growth a_t is utilization-adjusted technological growth from Fernald (2014). This TFP series is suitable for our model since it is assumed to measure 'pure technology' and is thus exogenous to the business cycle. Macroeconomic uncertainty M_t is the Economics Policy Uncertainty index of respective countries from Baker et al. (2016), log-scaled to reduce volatility and divided by its minimum such that it is greater than one. Bayesian belief of recession μ_t is the recession probabilities computed from the survey of professional forecasters. We use it as a proxy for the Bayesian belief of recession[15], assuming that professional forecasters are Bayesian

[15]Since it is out of the scope of this paper, we do not precisely define or estimate the parameters of the Bayesian updating process.

Table 6: **Objective function of each estimator**

Estimator	Objective function
Nonlinear least squares (NLS)	$\min(\dot{\mathbf{y}}(\Theta) - \dot{\mathbf{y}}^{\text{obs}})'(\dot{\mathbf{y}}(\Theta) - \dot{\mathbf{y}}^{\text{obs}})$
	where $\dot{\mathbf{y}}$: output growth column matrix
Maximum likelihood (ML)	$\max \sum_t \log L(\dot{y}_t^{\text{obs}}; \Theta)$
	where L: normal density function
	The normal density function is obtained from
	Matlab code: fitdist($\dot{\mathbf{y}}(\Theta)$, 'Normal')
Simulated methods of moments (SMM)	$\min(\mathbf{m}(\Theta) - \mathbf{m}^{\text{obs}})\mathbf{W}(\mathbf{m}(\Theta) - \mathbf{m}^{\text{obs}})'$
	where a row matrix $\mathbf{m} = [\dot{y}_t, \dot{y}_t^2, \dot{y}_t^3, \dot{y}_t\dot{y}_{t-1}]$,
	$\mathbf{W}$: identity weighting matrix
Indirect inference (II)	$\min(\mathbf{p}(\Theta) - \mathbf{p}^{\text{obs}})'\mathbf{W}(\mathbf{p}(\Theta) - \mathbf{p}^{\text{obs}})$
	where a row matrix $\mathbf{p}$ contains parameters of the AR(1) process
	AR(1) process is estimated from
	Matlab code: estimate(varm(4,1), [$\dot{\mathbf{y}}(\Theta)$,$\mathbf{M}$, $\boldsymbol{\mu}$, $\mathbf{z}$])
	$\mathbf{W}$: identity weighting matrix

on average. The recession probability of the US is for the next quarter. The estimation is conducted over the period of 1985Q1 to 2019Q4. A list of data sources used is provided below.

- **Real quarterly output growth**: U.S. Bureau of Economic Analysis, retrieved from FRED, https://fred.stlouisfed.org/series/GDPC1; 21 January 2023.
- **US Economic Policy Uncertainty index**: Baker et al. (2016) retrieved from https://www.policyuncertainty.com/us_monthly.html, 21 January 2023.
- **Recession probability**: Survey of Professional forecasters, Federal Reserve Bank of Philadelphia, retrieved from https://www.philadelphiafed.org/surveys-and-data/real-time-data-research/anxious-index, 21 January 2023.
- **Utilization-adjusted technological process**: Fernald (2014) retrieved from https://www.frbsf.org/economic-research/indicators-data/total-factor-productivity-tfp, 21 January 2023.
- **Recession dates**: NBER's US Business Cycle Expansions and Contractions

E Robustness

The purpose of robustness tests is to show the extent to which our main results are affected by the estimation method. We first compare the estimations of nonlinear models using nonlinear least square (our main result), maximum likelihood, simulated methods of moments, and indirect inference. We then report the results of the linearized model using the standard maximum likelihood with Bayesian techniques that are commonly used in macroeconomic models.

E.1 Nonlinear models with different estimators

In this section, we compare the results of ambiguity averse models estimated with maximum likelihood (ML), simulated methods of moments (SMM), and indirect inference (II), to those estimated with nonlinear least squares (NLS). We use the NLS parameters as the initial values of each estimator to determine if the NLS result is the optimum for the other estimators. Table 7 presents the results of fitting US output growth from these estimators in which the first column is the AA result from Table 4. We do not report the standard errors as they are not comparable across estimators due to different computational methods. Our findings indicate that the estimated parameters vary across estimators, which diminishes the model's performance especially during recessions.

Table 7: Estimation results

Parameter	Description	NLS	ML	SMM	II
α	capital share	0.3369	0.3369	0.4086	0.4959
ν	labor disutility	6.4369	5.3742	2.5073	1.7054
σ	risk aversion	0.4297	0.4297	0.6112	0.6659
ρ_λ	weight of the lagged expectations	0.0764	0.4929	0.2348	0.0000
μ_s	Bayesian belief of recession	1.0000	0.9960	0.9751	1.0000
γ	ambiguity aversion	3.3481	11.3474	3.0439	0.0000
Steady state	Description	NLS	ML	SMM	II
M_s	macroeconomic uncertainty	1.0000	1.0001	1.0001	1.0001
$\mu_s\xi_s^R$	subjective belief of recession	1.0000	0.9960	0.9751	1.0000
RMSE	Periods	NLS	ML	SMM	II
	all periods	0.4052%	0.4752%	0.5193%	0.6365%
	recession periods	0.4081%	0.5940%	0.8594%	1.3502%
	normal growth periods	0.4050%	0.4636%	0.4789%	0.5322%

Note: All models were estimated using the parameterized expectations algorithm and pattern search algorithm described in Appendix C. The objective functions of the respective estimators are summarized in Appendix C.3.

Estimates of capital share, risk aversion, and Bayesian belief of recession are found to be similar across estimators. Capital share is estimated to be between 0.3369 and 0.4959, while risk aversion is estimated to be between 0.4297 and 0.6659. The steady state Bayesian belief is estimated to be close to one, indicating that there is one-scenario steady state. However, other parameters vary greatly across estimators, for example, the levels of ambiguity aversion range from 0 to 11.3474. This discrepancy across estimators indicates that different objective functions can yield different results.

The RMSEs of maximum likelihood, simulated methods of moments, and indirect inference are 0.4752%, 0.5193%, and 0.6365%, respectively, higher than that of NLS. The ML and NLS estimations are theoretically supposed to yield the same results under normality assumptions but the output growth rate is not normally distributed (Fagiolo et al., 2008), resulting in different outcomes. The high RMSEs of SSM and II are mainly attributed to the recession periods as the recession RMSEs are more than twofold higher than those of the NLS estimation[16].

E.2 Linearized model with maximum likelihood and Bayesian technique

This section compares our main results to those from standard maximum likelihood estimation used by most macroeconomic models[17]. This method employs Bayesian techniques to run the maximum likelihood estimation, which requires the linearization around steady state of the model and the assumption of prior distributions for each parameter to be estimated. Further details of the estimation are provided in Appendix F. For simplicity, we refer to this estimator as Linear-ML. Table 8 reports the posterior mean, standard deviation, log likelihood and RMSEs of the Linear-ML estimation.

The estimated parameters of Linear-ML differ from those obtained by the NLS estimator, though the model performance is consistent across both estimators. The log likelihood of the ambiguity neutral and ambiguity averse models are 1324.7881 and 1338.3135, respectively, performing better than the benchmark model with a log likelihood of 1010.9854. This result is in line with the main finding in Section 4 although most estimated parameters are significantly different. The posterior means are closer to their priors, such as the posterior risk aversion ranges between 1.32 and 1.51 (prior of 2) and the steady state Bayesian beliefs are 18.40% and 15.70% (prior of 16.18%). Note that, despite these Bayesian beliefs, the steady-state decision remains robust to uncertainty due to a low macroeconomic uncertainty between 1.0001 and 1.0044.

We find that linearization reduces the transmission of uncertainty, requiring a higher level of ambiguity aversion to compensate. The Linear-ML posterior of ambiguity aversion is 17.7670, significantly higher than 11.3474 of the ML estimator in the nonlinear ambiguity averse model (Table 4. Table 8 further reveals that the recession RMSE is only slightly improved from the BM to AN models (1.3432% to 1.1682%) but is markedly reduced from AN to AA models (1.1682% to 0.7401%). This is inconsistent with our main estimation of nonlinear models in Section **??** which shows a great improvement from BM to AN. Thus, we can infer that linearization largely diminishes the effects of macroeconomic uncertainty

[16]The results of SMM and II are highly sensitive to the choice of moments and Vector Autoregression models used for estimation. For SMM estimator, we fit the model to four moments: mean, variance, skewness and correlation with the lagged component. The II estimation is fitted with AR(1) process. Moreover, an identity weighting matrix is employed for a simple computational process. Ruge-Murcia (2012, 2020) demonstrates that the identity weight results in a larger asymptotic standard error than the optimal weight. However, these issues are beyond the scope of this paper.

[17]We solve the model in *Matlab* and estimate the model in *Dynare*.

Table 8: Posterior estimations

Parameter	Description	BM	AN	AA
α	capital share	0.1149	0.1311	0.1438
		(0.0318)	(0.0261)	(0.0292)
ν	labor disutility	2.0824	3.2944	7.1775
		(0.7163)	(0.52336)	(0.9281)
σ	risk aversion	1.3206	1.3748	1.5178
		(0.2211)	(0.2109)	(0.2612)
ρ_λ	weight of lagged expectations	0.2121	0.2313	0.7393
		(0.2161)	(0.1067)	(0.0633)
μ_s	Bayesian belief of recession		0.1840	0.1570
			(0.0240)	(0.0221)
γ	ambiguity aversion			17.7670
				(1.6582)
Steady state	Description	BM	AN	AA
M_s	macroeconomic uncertainty		1.0001	1.0044
$\mu_s \xi_s^R$	Subjective belief of recession		0.1840	0.1676
Log likelihood		1010.9854	1324.7881	1338.3135
RMSE	Periods	BM	AN	AA
	all periods	0.6040%	0.5595%	0.4827%
	recession periods	1.3432%	1.1682%	0.7401%
	normal growth periods	0.4493%	0.4591%	0.4470%

Note: BM is benchmark model, AN is ambiguity neutral model ($\gamma = 0$). AA is ambiguity averse model where the prior of ambiguity aversion is uniform distribution with a range from 0 to 40. The standard deviation is stated in (...). Macroeconomic uncertainty and subjective belief of recession are not estimated but implied from the model thus the standard deviation is not available. Log likelihood of the model is measured by the modified harmonic mean method.

on the spread of the expected utility between the two scenarios, which is the only transmission mechanism in the ambiguity neutral model. However, the effect due to pessimistic belief distortions caused by ambiguity aversion is still preserved, albeit to a lesser degree and at a cost of a high level of ambiguity aversion.

We find that linearization reduces the transmission of uncertainty, requiring a higher level of ambiguity aversion to compensate. The Linear-ML posterior of ambiguity aversion is 17.7670, significantly higher than 11.3474 of the ML estimator in the nonlinear ambiguity averse model (Table 7). Table 8 further reveals that the recession RMSE is only slightly improved from the BM to AN models (1.3432% to 1.1682%) but is markedly reduced from AN to AA models (1.1682% to 0.7401%). This is inconsistent with our main estimation of nonlinear models in Section 4 which shows a great improvement from BM to AN. Thus, we can infer that linearization largely diminishes the effects of macroeconomic uncertainty on the spread of the expected utility between the two scenarios, which is the only transmission mechanism in the ambiguity neutral model. However, the effect due to pessimistic belief distortions caused by ambiguity aversion is still preserved, albeit to a lesser degree and at a cost of a high level of ambiguity aversion.

F Bayesian estimation method

This section describes Bayesian estimation method. To estimate the model, we substitute the original Euler equation with the parameterized Euler equation in the equilibrium conditions and linearize the model around steady states. Table 9 shows the equilibrium conditions in original and linear forms. The structural parameters were estimated using Bayesian estimation and Monte Carlo Markov Chain (MCMC). A sample of 20,000 draws was created and the first 10,000 draws were used as burnt-in. We used the prior variance as the MCMC jumping covariance defining the transition probability function to the next draw. A step size was chosen such that an acceptance rate is between 0.2 and 0.4. Given the structural parameters in each draw, the PEA coefficients (θ) and steady states were computed using the methods described in Section C.1.

Table 9: Equilibrium conditions

Original model	Linearized model
$\lambda_t = \rho_\lambda \lambda_{t-1} + (1-\rho_\lambda)(\theta^c + \theta^k k_t + \theta^z z_t + \theta^M M_t + \theta^\mu \mu_t + \theta^{M\mu}(M_t \mu_t))$	$\lambda_s \tilde{\lambda}_t = \rho_\lambda \lambda_s \tilde{\lambda}_{t-1} + (1-\rho_\lambda)(\theta^k k_s \tilde{k}_t + \theta^z \tilde{z}_t + \theta^M M_s \tilde{M}_t + \theta^\mu \mu_s \tilde{\mu}_t + \theta^{M\mu} M_s \mu_s (\tilde{M}_t + \tilde{\mu}_t))$
$\lambda_t = c_t^{-\sigma}$	$\tilde{\lambda}_t = -\sigma \tilde{c}_t$
$\lambda_t = \frac{\exp((1-\sigma)\bar{a}) L_t^\nu}{z_t^{1-\sigma} w_t}$	$\tilde{\lambda}_t = \nu \tilde{L}_t - (1-\sigma)\tilde{z}_t - \tilde{w}_t$
$k_{t+1} = (1-\delta) k_t \frac{z_{t-1}}{z_t} + i_t$	$k_s \tilde{k}_{t+1} = (1-\delta)(k_s \tilde{k}_t + \tilde{z}_{t-1} - \tilde{z}_t) + i_s \tilde{i}_t$
$y_t = \exp(\alpha \bar{a}) z_{t-1}^\alpha k_t^\alpha L_t^{1-\alpha}$	$\tilde{y}_t = \alpha \tilde{z}_{t-1} + \alpha \tilde{k}_t + (1-\alpha)\tilde{L}_t$
$w_t = (1-\alpha)\frac{y_t}{L_t}$	$\tilde{w}_t = \tilde{y}_t - \tilde{L}_t$
$R_t = \alpha \frac{y_t}{k_t} \frac{z_t}{z_{t-1}}$	$\tilde{R}_t = \tilde{y}_t - \tilde{k}_t + \tilde{z}_t - \tilde{z}_{t-1}$
$y_t = c_t + i_t$	$y_s \tilde{y}_t = c_s \tilde{c}_t + i_s \tilde{i}_t$

Note: $\tilde{x}_t = \frac{x_t - x_s}{x_s}$ where x_s is the steady state.

We do not directly estimate the Bayesian process but use the next-quarter recession probability from the survey of US professional forecasters as a observed data for μ_t. For simplicity, we assume that the Bayesian belief follows an AR(1) process:

$$\tilde{\mu}_t = \rho_\mu \tilde{\mu}_{t-1} + \sigma_\mu \epsilon_t^\mu$$

The technological process and macroeconomic uncertainty are also obtained from the data. We assume that they follow AR(1) processes:

$$\tilde{z}_t = \rho_a \tilde{z}_{t-1} + \sigma_a \epsilon_t^z$$
$$\tilde{M}_t = \rho_M \tilde{M}_{t-1} + \sigma_M \epsilon_t^M$$

In total, we use four time-series data for the estimation: the quarterly GDP per capita (GDP), the utilization-adjusted technological progress by Fernald (2014) (TFP), the US Economic Policy Uncertainty index by Baker et al. (2016) (EPU) [18], and the next-quarter recession probability from the US professional forecasters (SPF). The last three data are same as those used for the PEA simulations in Section ??. The measurement equations are:

$$d \log GDP_t = \tilde{y}_t - \tilde{y}_{t-1} + ME_y$$
$$d \log TFP_t = \tilde{z}_t - \tilde{z}_{t-1}$$
$$EPU_t = M_s \tilde{M}_t + M_s$$
$$SPF_t = \mu_s \tilde{\mu}_t + \mu_s$$

where $d \log$ is log difference and ME_y is measurement error of output. $d \log GDP_t$ and $d \log TFP_t$ are demeaned and the model is estimated during the sample period of 1985Q1 - 2019Q4. The estimation of benchmark model has y_t and z_t as observables. The ambiguity neutral and ambiguity averse models use all four observables. The Bayesian priors are summarized in Table 10.

Table 10: Priors of main parameters

Parameter	Description	Type	Mean	S.D.
Parameters related to ambiguity				
γ	ambiguity aversion	U	between 0 and 40	
ρ_λ	weight on lagged expectation	U	between 0 and 1	
μ_s	steady-state Bayesian belief of recession	U	between 0 and 1	
Other structural parameters				
σ	risk aversion	IG	2	0.5
ν	labor disutility	IG	1.5	0.5
α	capital share	B	0.3	0.01
Bayesian beliefs parameters				
ρ_μ	persistence of Bayesian belief	B	0.7	0.1
σ_μ	volatility of Bayesian beliefs	IG	0.2	0.001
Macro uncertainty parameters				
ρ_M	persistence of macro uncertainty	B	0.7	0.01
σ_M	volatility of macro uncertainty	IG	0.05	0.01
Technological progress parameters				
ρ_a	persistence of technology growth	B	0.95	0.01
σ_a	volatility of technology	IG	0.008	0.001
Measurement error				
ME_y	measurement error of output	IG	0.006	0.0001

Note: B: Beta distribution, IG: Inverse gamma distribution. U:Uniform distribution

[18]For the US Economic Policy Uncertainty index, we take the log scale to reduce the volatility and divide the index its minimum value, so it is always bigger than or equal to one.