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Abstract

We study a growth model with two types of agents who are heterogeneous in their degree of family altruism. We prove that every equilibrium path converges to a unique steady state, and study the effect of altruism on the properties of steady-state equilibrium. We show that aggregate income is positively related to both level of altruism and altruism heterogeneity. When altruism heterogeneity is low, income inequality follows an inverse U-shaped pattern relative to the level of altruism, which is consistent with the cross-country Kuznets curve. When altruism heterogeneity is high, income inequality monotonically decreases with the level of altruism. Our results suggest that heterogeneous altruism is an important mechanism linking economic growth and income inequality.

JEL-Codes: D150, D640, E210, O400.

Keywords: economic growth, heterogeneous agents, altruism, bequests, inequality, Kuznets curve.

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1 Introduction

Why income inequality differs between countries? Whereas inequality is a complex and multidimensional phenomenon, there are various answers to this question. Inequality is often linked to differences in productivity (levels of economic and technological development), government policies (tax rates and social welfare programs), human capital (access to education and opportunities for skills development), political and institutional factors (demand for redistribution). In this paper, we highlight the role of socio-cultural norms in shaping between-country inequality and study the role of heterogeneous altruism in economic growth and income inequality.

It is generally observed that high-income countries tend to have lower levels of income inequality compared to low-income countries (see, for instance, OECD, 2011). Table 1 illustrates this observation using the World Bank 2019 open data on the Gini index for 105 countries divided into three income groups.[1] The relationship between income and inequality is not absolute, and there are variations among individual countries – some high-income countries are very unequal, while some low-income countries are relatively equal.[2] Nevertheless, it is clearly seen that high-income countries have much lower median and mean Gini indices compared to middle-income and low-income countries.

	Low income	**Middle income**	**High income**
Median	0.374	0.377	0.319
Mean	0.374	0.379	0.324
Maximum	0.513 (Angola)	0.535 (Brazil)	0.498 (Panama)
Minimum	0.260 (Moldova)	0.253 (Belarus)	0.232 (Slovakia)
N. of countries	37	30	38

Table 1: Gini indices for different income groups of countries in 2019. *Source*: Authors' calculations based on the World Bank data.

Moreover, Table 1 also implies that cross-country inequality is slightly higher for middle-income countries than for low-income countries. This pattern is consistent with the *cross-country Kuznets curve*: an inverted U-shaped relationship between inequality and income in a cross-section of countries, confirmed in a large number of empirical studies (among others, Campano and Salvatore, 1988; Bourguignon and Morrison, 1990; Jha, 1996; Milanovic, 2000; Savvides

[1] We construct income groups based on the World Bank classification in 2019. High-income countries are World Bank's high-income economies (GNI per capita greater than 12375 \$); middle-income countries are upper middle-income economies (GNI per capita between 3995 and 12375 \$); low-income countries correspond to World Bank's low-income economies and lower middle-income economies combined (GNI per capita less than 3995 \$). Gini indices refer mainly to 2019, but also to 2017 and 2018 where data for 2019 are not available.

[2] For instance, in 2018, the Guinea Gini index was 0.296, while, in 2019, the US Gini index was 0.415.

and Stengos, 2000). At the same time, the recent rise in inequality in developed countries seems to be incompatible with the originally proposed *within-country Kuznets curve*.[3] The fact that there is no within-country Kuznets curve but ample evidence of the cross-country Kuznets curve suggests that there are country-specific characteristics which affect both the level of income and the level of inequality in each country and lead to the observed cross-sectional pattern.

A natural determinant of the joint evolution of growth and inequality is *parental altruism*, a concern for the well-being of children as opposed to pure self-interest. It is generally acknowledged that altruism has a positive impact on economic development.

For instance, Hatcher and Pourpourides (2018) report a positive correlation between country-level parental altruism and economic growth in a sample of 48 countries. At the same time, there is substantial heterogeneity in the degree of altruism: some people and societies are more altruistic than others. Falk et al. (2018) find that the within-country altruism variation is much larger than the between-country variation: the former amounts to 87.7% in the total individual-level variation in altruism, while the latter explains only the remaining 12.3%.[4]

Furthermore, altruistic bequests are a crucial driver of wealth accumulation, and difference in altruism is an important factor for inequality (see also Mankiw, 2000, for a discussion). In particular, Laitner (2002) argues that calibrated models with altruistic bequests are able to account for the empirical distribution of wealth in the US. In this paper, we develop and study a simple growth model where agents differ in their degree of altruism. We show that the empirically relevant assumption of within-country altruism heterogeneity might contribute to the explanation of the cross-country Kuznets curve.

Specifically, we consider a successive generations economy in which agents are motivated by *family altruism*, that is, they *care about the disposable income of their offsprings*. There are two types of agents who are heterogeneous in their degree of family altruism: agents of the first type are *less altruistic*, while agents of the second type are *more altruistic*. Altruistic transfer is the only savings motive, and bequests left by agents become the capital involved in the production. We prove that when instantaneous utility functions are logarithmic and production technology is Cobb-Douglas, every equilibrium path of consumption, bequests and capital converges to a unique steady-state equilibrium. We characterize the properties of a steady-state equilibrium and study the impact of the level of altruism and altruism heterogeneity on the steady state.

To study the impact of the level of altruism on the steady state, we employ the share of the more altruistic agents in total population as a measure of

[3]Kuznets (1955) analyzed the evolution of the US and UK income distributions in the first half of the 20th century, and suggested that in the process of development within a single country, income inequality increases with the shift of labor force from traditional agricultural sector to modern industrial sector, but eventually declines as industrialization progresses.

[4]Formally speaking, Falk et al. (2018) define altruism as a willingness to give to good causes without expecting anything in return. However, their measure of altruism is also a good proxy for parental altruism.

altruism. Given both degrees of altruism, the higher is the share of the more altruistic agents, the more altruistic is society as a whole. We show that a higher level of altruism in society always leads to a higher capital stock and a higher aggregate income. Intuitively, the more altruistic agents leave higher bequests than the less altruistic agents, and an increase in the share of the more altruistic agents leads to higher capital accumulation. This result confirms the empirical observation that altruism positively affects aggregate income at the country level.

The impact of the level of altruism on the steady-state income inequality (measured by the Gini index) depends on the difference between the more and the less altruistic agents. If both types have rather similar degrees of altruism, then an inverted U-shaped relationship is observed. An increase in the share of the more altruistic agents would first increase inequality due to the concentration of income among the more altruistic agents who leave higher bequests. However, when the share of the more altruistic agents continues to increase beyond a certain point, making a larger share of population richer, inequality will decrease as income spreads more equally throughout the more altruistic society. Therefore, this case is consistent with the *cross-country Kuznets curve* linking different steady-state levels of income to the steady-state Gini indices. An increase in aggregate income caused by increased share of the more altruistic agents at first is associated with rising inequality, but beyond a certain point a further increase in aggregate income is accompanied with falling inequality.

If the difference in the degrees of altruism between the two types is sufficiently high, then, as the share of the more altruistic agents increases, the steady-state level of inequality decreases. Hence, for sufficiently heterogeneous altruistic societies there is no trade-off between economic growth and income inequality. This result suggests that high-income countries may indeed have lower levels of inequality because of altruism heterogeneity effects.

To study the impact of altruism heterogeneity on the steady state, we employ the variance of the degrees of altruism as a measure of heterogeneity. Given the mean and the share of the more altruistic agents, the higher is the variance, the more heterogeneous is society as a whole. We show that a higher altruism heterogeneity always leads to a higher capital stock and a higher aggregate income. Intuitively, a mean-preserving shift in the degree of altruism increases the degree of altruism of the more altruistic agents who in response would increase their bequests proportionally more than the less altruistic agents would reduce their bequests. This result points out that not only the average level of altruism, but also the diversity in the degree of altruism plays a significant role in economic development.

The impact of altruism heterogeneity on the steady-state income inequality also depends on the difference between the more and the less altruistic agents. If both types have similar degrees of altruism, then increasing altruism heterogeneity increases inequality.[5] Intuitively, a mean-preserving shift in the degree

[5]See also Krusell and Smith (1998) and Hendricks (2007), who find similar effect of discount rate heterogeneity on the wealth Gini index in stochastic growth models.

of altruism makes more altruistic agents richer while less altruistic agents become poorer. However, if the difference in the degrees of altruism between the two types is sufficiently high, then an increase in altruism heterogeneity does not affect inequality. In this case, the less altruistic agents leave no bequests, and a further decrease in their degree of altruism would not affect their relative position. This result can be interpreted as the existence of a maximum possible steady-state level of income inequality (in terms of altruism heterogeneity).

Furthermore, we analyze the impact of the level of altruism on the steady-state welfare of both types of agents. We show that the steady-state utility of the more altruistic agents is strictly decreasing in the level of altruism. Intuitively, the disposable income of the more altruistic agents is determined mainly by their bequests. An increase in the share of the more altruistic agents would lower the interest rate and incentives to save, which would in turn lower the disposable income of the more altruistic agents, decreasing their consumption and utility.

At the same time, the steady-state utility of the less altruistic agents has the U-shape: it is decreasing in the level of altruism for low levels of altruism and is increasing for high levels of altruism. Intuitively, in the disposable income of the less altruistic agents, the importance is gradually shifting from bequests to the wage bill. For low levels of altruism, their steady-state levels of consumption and utility decrease in the level of altruism, because the wages are low and bequests are decreasing. However, for high levels of altruism, the disposable income of the less altruistic agents is determined mainly by the wage rate which is increasing in the level of altruism, increasing their consumption and utility.

Our paper is related to a large theoretical literature on the links between parental altruism, growth and inequality. First, this paper contributes to the discussion of the role of altruism in economic growth. The existing literature typically follows Barro (1974) and explores overlapping generations models with *dynastic altruism* where agents care about their offspring's welfare: each agent derives utility from her own consumption and the utility of her offspring. Barro (1974) shows that when the degree of altruism is sufficiently strong (so that bequest motive is operative), the dynamics of an OLG model are analogous to the dynamics of the infinite-horizon Ramsey model, and Ricardian equivalence holds (government debt does not influence the steady-state capital stock).

Another strand of literature studies *paternalistic altruism* where agents care about their offspring's consumption: each agent derives utility from her own consumption and the consumption level of her offspring.[6] Since each agent has a limited altruism towards only immediate successor, there is a conflict of interests among different dynasty members about consumption schedule. Kohlberg (1976), Leininger (1986) and Bernheim and Ray (1987) study this conflict from a game-theoretic point of view, establish the existence of equilibria in a game between different altruistic dynasty members and characterize their properties.

[6] The term "paternalistic" emphasizes that the altruist values the consumption of the others, irrespective of their preferences. However, sometimes *paternalistic altruism* also refers to the situation where bequests are treated as a consumption good, and the altruistic agent derives pleasure directly from the act of giving ("joy-of-giving" or "warm glow giving").

Our paper is different, as we assume that agents exhibit *family altruism*. In our setting, each agent derives utility from her own consumption and the disposable income of her offspring. This approach has two advantages: there is no conflict about consumption at different dates, while at the same time all dynasty members are not equivalent to a single infinitely lived agent. Thus, the assumption of family altruism allows one to gain new perspectives and understanding.[7] Our contribution here is to clarify the mechanisms by which family altruism is positively related to economic growth.

Second, this paper contributes to the analysis of growth models with agents who differ in their degree of altruism. Michel and Pestieau (1998; 1999), Smetters (1999) and Mankiw (2000) study the effectiveness of fiscal policy under heterogeneous dynastic altruism. The very general result is that Ricardian equivalence also holds in heterogeneous agents models in the long run. However, government policies typically lead to a redistribution of income from the less altruistic agents (poor) to the more altruistic agents (rich) and an increase in inequality within society, which is not observed in representative agent models. Palivos (2005) shows that monetary policy under heterogeneous altruism also leads to substantial distributional effects. Reichlin (2020) considers the OLG model with heterogeneous dynastic altruism and highlights the difficulties with standard social welfare criteria in this setting.[8]

Our paper differs from previous studies in two important respects. To the best of our knowledge, this is the first paper to study heterogeneous family altruism. Furthermore, we focus on inequality across societies, distinguishing between the case where a cross-country Kuznets curve holds and the case where there is no trade-off between growth and inequality.

The paper is organized as follows. In Section 2 we present the model and define equilibria. Section 3 provides main results and their discussion. Section 4 concludes. All the proofs are relegated to the Appendix.

2 The model

We consider a closed market economy with households and firms. As usual, their fundamentals are given by preferences, technology and endowments. In this section, we describe the consumer's and the producer's programs (individual level) and define dynamic general equilibrium (aggregate level).

2.1 Households

Time is discrete and runs from $t = 0$ to infinity. The economy is populated by successive generations of agents. Each agent lives for one period, gives birth to one offspring and supplies one unit of labor. Population is constant over time, and the population size is normalized to 1.

[7]For the analysis of fiscal policy in models with family altruism, see, among others, Lambrecht et al. (2006) and Borissov and Kalk (2020).

[8]See also Pakhnin (2023) for the discussion of similar problems with social welfare under heterogeneous time preferences.

Population consists of two types of agents indexed by $i = 1, 2$. The share of type i in population is π_i, with $\pi_1 + \pi_2 = 1$. Agents are identical within each type. The agent and her offspring are always of same type, so population shares are constant over time. A disposable income of type i agent is defined as a sum of the wage bill, w_t, identical across types, and the current value of bequest left by her parent, b_t^i. Out of this, an agent consumes $c_t^i \geq 0$ and leaves $b_{t+1}^i \geq 0$ to her offspring as a bequest. Since the case of negative bequests is hard to justify on either a juridical or empirical ground, we assume that bequests are non-negative.[9] Formally, the budget constraint of the type i agent has the form

$$c_t^i + b_{t+1}^i \leq R_t b_t^i + w_t$$

where R_t is the interest factor.

Each agent cares about her consumption and the disposable income of her offspring. The relative preference for the offspring's disposable income with respect to own consumption is naturally interpreted as a degree of altruism. Formally, the preferences of type i agent are represented by the following utility function:

$$\ln c_t^i + \beta_i \ln \left(R_{t+1} b_{t+1}^i + w_{t+1} \right)$$

where $\beta_i > 0$ is the degree of altruism of type i agent.

Throughout the paper, we assume that agents are heterogeneous in terms of altruism: type 1 agents are less altruistic, while type 2 agents are more altruistic.

Assumption 1 $\beta_2 > \beta_1$.

Note that our approach to model heterogeneous altruism significantly differs from the previous studies (e.g., Michel and Pestieau, 1998, or Reichlin, 2020). The existing literature considers dynastic altruism and assumes that agents weigh the offspring's utility in their own utility function. The long-run properties of the models with dynastic altruism are analogous to that of the many-agent Ramsey model in the spirit of Becker (1980), where the intertemporal utility function of a single dynasty is an infinite-horizon discounted sum of instantaneous utilities. The advantage of our approach where agents exhibit family altruism is twofold. First, an agent can ignore the unknown preferences of her unborn offspring when making her decisions. Second, the long-run dynamics of our model are different from those of a many-agent Ramsey model and better fit some evidence.

Thus, a type i agent living in period t solves the maximization problem:

$$\max_{c_t^i, b_{t+1}^i} \left[\ln c_t^i + \beta_i \ln \left(R_{t+1} b_{t+1}^i + w_{t+1} \right) \right] \tag{1}$$
$$c_t^i + b_{t+1}^i \leq R_t b_t^i + w_t$$

with $c_t^i \geq 0$ and $b_{t+1}^i \geq 0$. The following lemma characterizes the necessary and sufficient conditions for the solution to the utility maximization problem.

[9]Negative bequests would mean that offsprings have to pay parents' debts.

Lemma 1 (Utility maximization) *A non-negative pair* $\left(c_t^i, b_{t+1}^i\right)$ *is a solution to problem (1) if and only if there exists* $\mu_t^i \geq 0$ *such that* $\mu_t^i b_{t+1}^i = 0$, *and the following conditions hold:*

$$\frac{1}{c_t^i} = \frac{\beta_i R_{t+1}}{R_{t+1} b_{t+1}^i + w_{t+1}} + \mu_t^i \tag{2}$$

$$c_t^i + b_{t+1}^i = R_t b_t^i + w_t \tag{3}$$

2.2 Firms

In every period, the economy produces a unique good which is either consumed or invested. There is a large number of identical small price-taking firms. Technology of every firm is given by the neoclassical production function $F(K, N)$, where K is the stock of capital and N is the labor input. Capital fully depreciates each period, which is justified by the length of the period (the life span). Throughout the paper, we assume that $F(K, N)$ satisfies the standard assumptions.

Assumption 2 *The production function is continuous, concave and homogeneous of degree one.*

Firm j in period t maximizes the profit:

$$\max_{K_{jt}, N_{jt}} \left[F\left(K_{jt}, N_{jt}\right) - R_t K_{jt} - w_t N_{jt}\right]$$

where K_{jt} and N_{jt} are the demands for capital and labor, and the interest factor R_t coincides with the interest rate (because of the complete capital depreciation). The next lemma characterizes the necessary and sufficient conditions for the solution to the profit maximization problem.

Lemma 2 (Profit maximization) *The following conditions are necessary and sufficient to profit maximization:*

$$f'\left(k_t\right) = R_t \tag{4}$$

$$f\left(k_t\right) - k_t f'\left(k_t\right) = w_t \tag{5}$$

where $k_t = k_{jt} \equiv K_{jt}/N_{jt}$ *is the capital intensity and* $f\left(k_t\right) \equiv F\left(k_t, 1\right)$ *is the production function in intensive form common to every firm.*

Since the size of population is normalized to one, the aggregate capital K_t coincides with capital per capita k_t:

$$K_t = \sum_j \frac{K_{jt}}{N_{jt}} N_{jt} = k_t \sum_j N_{jt} = k_t$$

We introduce the price functions:

$$R\left(k_t\right) \equiv f'\left(k_t\right) \tag{6}$$

$$w\left(k_t\right) \equiv f\left(k_t\right) - k_t f'\left(k_t\right) \tag{7}$$

and the income ratio (labor income over capital income):

$$\gamma(k_t) \equiv \frac{w(k_t)}{k_t R(k_t)} = \frac{1-\alpha(k_t)}{\alpha(k_t)}$$

where

$$\alpha(k_t) \equiv \frac{k_t f'(k_t)}{f(k_t)}$$

is the capital share in total income.

2.3 Temporary, intertemporal and steady-state equilibria

The definitions of equilibria in our model are fairly standard. First, we define a temporary equilibrium where each agent maximizes her utility, each firm maximizes its profit, and the capital market clears, meaning that bequests become the capital involved in the production.

Definition 3 (Temporary equilibrium) *Given the bequests $b_t^i \geq 0$ left by agents in period $t-1$, and the capital stock $k_t = \pi_1 b_t^1 + \pi_2 b_t^2$, a vector $\left(\left(c_t^i, b_{t+1}^i\right)_{i=1}^2, k_{t+1}\right)$ is a time-t temporary equilibrium if:*

(i) for any i, $\left(c_t^i, b_{t+1}^i\right)$ is a solution to the utility maximization problem (1) where $(R_t, w_t) = (R(k_t), w(k_t))$ and $(R_{t+1}, w_{t+1}) = (R(k_{t+1}), w(k_{t+1}))$, and the functions R and w are given by (6) and (7);

(ii) $k_{t+1} = \pi_1 b_{t+1}^1 + \pi_2 b_{t+1}^2$.

Second, we define an intertemporal equilibrium as a sequence of temporary equilibria.

Definition 4 (Intertemporal equilibrium) *Given the initial bequests $b_0^i \geq 0$ and the capital stock $k_0 = \pi_1 b_0^1 + \pi_2 b_0^2$, a sequence $\left(\left(c_t^i, b_{t+1}^i\right)_{i=1}^2, k_{t+1}\right)_{t=0}^{\infty}$ is an intertemporal equilibrium if $\left(\left(c_t^i, b_{t+1}^i\right)_{i=1}^2, k_{t+1}\right)$ is a time-t temporary equilibrium for any $t \geq 0$.*

Lemmas 1–2 and Definition 4 allow us to obtain the dynamic system representing an intertemporal equilibrium.

Proposition 5 (Dynamic system) *The dynamics of bequests in an intertemporal equilibrium are given by*

$$b_{t+1}^i = \frac{1}{1+\beta_i} \max\left\{0, \beta_i R(k_t)\left[b_t^i + k_t \gamma(k_t)\right] - k_{t+1}\gamma(k_{t+1})\right\} \tag{8}$$

for $i = 1, 2$, with $k_t = \pi_1 b_t^1 + \pi_2 b_t^2$, and the initial condition $\left(b_0^1, b_0^2\right)$.

Equation (8) is a two-dimensional dynamic system in the variables $\left(b_t^1, b_t^2\right)$. We observe that these variables are predetermined because bequests b_0^1 and b_0^2 are given. Since $k_{t+1} \equiv \sum_{i=1}^2 \pi_i b_{t+1}^i$, we also have the dynamics of capital stock:

$$k_{t+1} = \sum_{i=1}^{2} \frac{\pi_i}{1+\beta_i} \max\left\{0, \beta_i R\left(k_t\right)\left[b_t^i + k_t \gamma\left(k_t\right)\right] - k_{t+1}\gamma\left(k_{t+1}\right)\right\}$$

A steady-state equilibrium is naturally defined.

Definition 6 (Steady state) *A vector* $\left(\left(c^i, b^i\right)_{i=1}^2, k\right)$ *is a steady-state equilibrium if* $k > 0$ *and the sequence* $\left(\left(c_t^i, b_{t+1}^i\right)_{i=1}^2, k_{t+1}\right)_{t=0}^{\infty}$ *with* $\left(c_t^i, b_{t+1}^i\right) = \left(c^i, b^i\right)$ *and* $k_{t+1} = k$ *for any* $i = 1, 2$ *and any* $t \geq 0$ *is an intertemporal equilibrium starting from* $\left(b^1, b^2\right)$.

The following proposition determines the steady state.

Proposition 7 (Steady-state equilibrium) *Assume that* $k > 0$. *The steady-state bequests* b^i *are given by*

$$b^i = k\gamma\left(k\right)\max\left\{0, \frac{\beta_i R\left(k\right) - 1}{1+\beta_i - \beta_i R\left(k\right)}\right\} \tag{9}$$

where the steady-state capital stock k *is a solution to the following equation:*

$$\gamma\left(k\right)\sum_{i=1}^{2}\pi_i \max\left\{0, \frac{\beta_i R\left(k\right) - 1}{1+\beta_i - \beta_i R\left(k\right)}\right\} = 1 \tag{10}$$

Note that, for any k, b^i is increasing, possibly not strictly, in β_i. Therefore, if a steady state exists, the more altruistic agents leave higher steady-state bequests than the less altruistic agents, $b^2 > b^1$.

2.4 Local dynamics

Let k be the steady-state capital stock. Consider the local dynamics of bequests in a neighborhood of a steady state. By (9), the steady-state bequests for agents of type i are positive if and only if

$$\frac{1}{\beta_i} < R\left(k\right) < 1 + \frac{1}{\beta_i} \tag{11}$$

Because of Assumption 1, we have $1/\beta_2 < 1/\beta_1$. Then it follows from (11) that there are two possible cases: (1) $1/\beta_1 < R\left(k\right) < 1 + 1/\beta_2$, and (2) $1/\beta_2 < R\left(k\right) < \min\left\{1/\beta_1, 1 + 1/\beta_2\right\}$.

Case (1) If $1/\beta_1 < R\left(k\right) < 1 + 1/\beta_2$, then both the more and the less altruistic agents leave bequests. Local dynamics are given by

$$b_{t+1}^1 = \frac{\beta_1 R\left(\pi_1 b_t^1 + \pi_2 b_t^2\right)\left[b_t^1 + \left(\pi_1 b_t^1 + \pi_2 b_t^2\right)\gamma_t\right] - \left(\pi_1 b_{t+1}^1 + \pi_2 b_{t+1}^2\right)\gamma_{t+1}}{1+\beta_1} \tag{12}$$

$$b_{t+1}^2 = \frac{\beta_2 R\left(\pi_1 b_t^1 + \pi_2 b_t^2\right)\left[b_t^2 + \left(\pi_1 b_t^1 + \pi_2 b_t^2\right)\gamma_t\right] - \left(\pi_1 b_{t+1}^1 + \pi_2 b_{t+1}^2\right)\gamma_{t+1}}{1+\beta_2} \tag{13}$$

where $\gamma_t = \gamma\left(\pi_1 b_t^1 + \pi_2 b_t^2\right)$.

Case (2) If $1/\beta_2 < R(k) < \min\{1/\beta_1, 1 + 1/\beta_2\}$, then, according to (11), $b^1 = 0$, and only the more altruistic agents leave bequests. Local dynamics are given by $b_{t+1}^1 = 0$ and

$$b_{t+1}^2 = \frac{\beta_2 R\left(\pi_2 b_t^2\right)\left[b_t^2 + \pi_2 b_t^2 \gamma\left(\pi_2 b_t^2\right)\right] - \pi_2 b_{t+1}^2 \gamma\left(\pi_2 b_{t+1}^2\right)}{1 + \beta_2} \tag{14}$$

3 Main results

In this section, we focus on the case of a Cobb-Douglas technology. Suppose that the production function is given by

$$F(K, N) = AK^\alpha N^{1-\alpha}$$

Then $f(k_t) = Ak_t^\alpha$, and the price functions (6) and (7) take the form

$$R(k_t) = \alpha A k_t^{\alpha-1} \tag{15}$$

$$w(k_t) = (1-\alpha) A k_t^\alpha \tag{16}$$

The capital share in total income and the income ratio are constant:

$$\alpha(k_t) = \alpha \text{ and } \gamma(k_t) = \gamma = \frac{1-\alpha}{\alpha} \tag{17}$$

and the dynamic system (8) becomes

$$b_{t+1}^i = \frac{1}{1+\beta_i} \max\left\{0, \beta_i R(k_t)\left(b_t^i + \gamma k_t\right) - \gamma k_{t+1}\right\}$$

with $k_t = \pi_1 b_t^1 + \pi_2 b_t^2$. The transition dynamics of capital stock are given by

$$k_{t+1} = \sum_{i=1}^{2} \frac{\pi_i}{1+\beta_i} \max\left\{0, \beta_i R(k_t)\left(b_t^i + \gamma k_t\right) - \gamma k_{t+1}\right\}$$

3.1 Steady state and convergence

Let $\pi \equiv \pi_2$ be the share of the more altruistic agents in total population, which is our measure of altruism. Let also

$$\delta \equiv \frac{1}{\beta_1} - \frac{1}{\beta_2}$$

be the altruism gap (the inverse of β_i captures the selfishness), which is our measure of altruism heterogeneity.

We introduce two critical interest rates:

$$R_1 \equiv \frac{1}{2}\left(1 + \alpha + \frac{1}{\beta_1} + \frac{1}{\beta_2} - \sqrt{(\delta + \alpha - 1)^2 + 4\delta\pi(1-\alpha)}\right) \tag{18}$$

$$R_2 \equiv \frac{1}{\beta_2} + \frac{1}{1+\gamma\pi} \tag{19}$$

Denote by

$$\delta^* \equiv \frac{1}{1+\gamma\pi} \tag{20}$$

a threshold value of altruism heterogeneity which will play an important role below. The following proposition characterizes the steady-state equilibrium.

Proposition 8 (Steady-state equilibrium) *(1) Suppose that $\delta < \delta^*$. Then there exists a unique steady-state equilibrium characterized by the interest rate R_1. The steady-state equilibrium is given by $\left(c_1^1, b_1^1, c_1^2, b_1^2, k_1\right)$, where*

$$\begin{aligned}
c_1^1 &= \frac{\gamma k_1}{\beta_1 + 1 - \beta_1 R_1} \quad \text{and} \quad c_1^2 = \frac{\gamma k_1}{\beta_2 + 1 - \beta_2 R_1} \\
b_1^1 &= \gamma k_1 \frac{\beta_1 R_1 - 1}{\beta_1 + 1 - \beta_1 R_1} \quad \text{and} \quad b_1^2 = \gamma k_1 \frac{\beta_2 R_1 - 1}{\beta_2 + 1 - \beta_2 R_1} \\
k_1 &= \left(\frac{\alpha A}{R_1}\right)^{\frac{1}{1-\alpha}}
\end{aligned} \tag{21}$$

(2) Suppose that $\delta \geq \delta^*$. Then there exists a unique steady-state equilibrium characterized by the interest rate R_2. The steady-state equilibrium is given by $\left(c_2^1, b_2^1, c_2^2, b_2^2, k_2\right)$, where

$$\begin{aligned}
c_2^1 &= \frac{\gamma k_2}{\beta_2}\left(1 + \frac{\beta_2}{1+\gamma\pi}\right) \quad \text{and} \quad c_2^2 = \frac{k_2}{\beta_2}\frac{1+\gamma\pi}{\pi} \\
b_2^1 &= 0 \quad \text{and} \quad b_2^2 = \frac{k_2}{\pi} \\
k_2 &= \left(\frac{\alpha A}{R_2}\right)^{\frac{1}{1-\alpha}}
\end{aligned} \tag{22}$$

Proposition 8 implies that our model admits two types of steady-state equilibria which depend on the difference between the more and the less altruistic agents. First, if agents of both types have rather similar degrees of altruism (the measure of altruism heterogeneity δ does not exceed the threshold value δ^*), then both types of agents make positive bequests in the steady-state equilibrium. Intuitively, in this case, the more altruistic agents, who are the primary savers in the model, do not leave very high bequests, and the resulting interest rate is sufficiently high to allow the less altruistic agents to also leave bequests.

Second, if the altruism gap between the more and the less altruistic agents is sufficiently large ($\delta \geq \delta^*$), then only the more altruistic agents leave positive bequests in the steady-state equilibrium: $b_2^2 > 0 = b_2^1$. Intuitively, in this case the more altruistic agents are so altruistic that they leave a substantial amount of bequests which drives the interest rate down. The interest rate becomes too low and induce the less altruistic agents to leave no bequests.

Note that in both cases, in the steady-state equilibrium, the more altruistic agents leave higher bequests and have higher income. However, they do not necessarily have a higher consumption level. If the steady-state interest rate is greater than 1, then the more altruistic agents consume more than the less altruistic agents. However, if the steady-state interest rate is lower than 1, which holds, in particular, when the degree of altruism of the more altruistic agents is large enough, then the less altruistic agents enjoy a higher level of consumption.

The role of a steady-state equilibrium is highlighted by the following important result which demonstrates that every intertemporal equilibrium converges to the steady state.

Proposition 9 (Global convergence) *Let* $\left(\left(c_t^i, b_{t+1}^i\right)_{i=1}^2, k_{t+1}\right)_{t=0}^\infty$ *be an intertemporal equilibrium.*

(1) Suppose that $\delta < \delta^*$*. Then,* $b_t^1 > 0$ *and* $b_t^2 > 0$ *for all* $t \geq 1$*, and the intertemporal equilibrium converges to the steady-state equilibrium* $\left(c_1^1, c_1^2, b_1^1, b_1^2, k_1\right)$ *defined in part (1) of Proposition 8.*

(2) Suppose that $\delta \geq \delta^*$*. Then intertemporal equilibrium converges to the steady-state equilibrium* $\left(c_2^1, c_2^2, b_2^1, b_2^2, k_2\right)$ *defined in part (2) of Proposition 8. If* $\delta = \delta^*$*, then either* $b_t^1 = 0$ *for all* $t \geq 1$ *or* b_t^1 *converges to* 0*. If* $\delta > \delta^*$*, then there exists* t_0 *such that* $b_t^1 = 0$ *for all* $t \geq t_0$*.*

According to Proposition 9, the steady-state equilibrium is globally stable. To provide a numerical illustration of this property, we compute the speed of convergence which depends on the modulus of eigenvalues of the linearized dynamic system.

Case (1) of Propositions 8 and 9.

Since $1/\beta_1 < R_1 < 1 + 1/\beta_2$, both the more and the less altruistic agents leave bequests and dynamics follow (12)-(13) with constant γ_t given by (17):

$$b_{t+1}^1 = \frac{\beta_1 R\left(k_t\right)\left(b_t^1 + \gamma k_t\right) - \gamma k_{t+1}}{1+\beta_1} \tag{23}$$

$$b_{t+1}^2 = \frac{\beta_2 R\left(k_t\right)\left(b_t^2 + \gamma k_t\right) - \gamma k_{t+1}}{1+\beta_2} \tag{24}$$

where $k_t = \pi_1 b_t^1 + \pi_2 b_t^2$.

We define the saving shares:

$$z_i \equiv \frac{\pi_i b_1^i}{\pi_1 b_1^1 + \pi_2 b_1^2} = \frac{\pi_i b_1^i}{k_1} \tag{25}$$

with $i = 1, 2$.

Lemma 10 (Local dynamics) *Dynamics (23)-(24) are locally approximated by the following system*

$$\left[\begin{array}{c} \frac{db_{t+1}^1}{b_1^1} \\ \frac{db_{t+1}^2}{b_1^2} \end{array}\right] = J \left[\begin{array}{c} \frac{db_t^1}{b_1^1} \\ \frac{db_t^2}{b_1^2} \end{array}\right] \tag{26}$$

where J is the Jacobian matrix:

$$J = \begin{bmatrix} (1+\beta_1+\gamma\pi_1)\,z_1 & \gamma\pi_1 z_2 \\ \gamma\pi_2 z_1 & (1+\beta_2+\gamma\pi_2)\,z_2 \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 R_1 [1+(1-\alpha)(\pi_1 - z_1)]\,z_1 & \beta_1 R_1 (1-\alpha)(\pi_1 - z_1)\,z_2 \\ \beta_2 R_1 (1-\alpha)(\pi_2 - z_2)\,z_1 & \beta_2 R_1 [1+(1-\alpha)(\pi_2 - z_2)]\,z_2 \end{bmatrix}$$

and

$$z \equiv z_2 = \pi \frac{1-\alpha}{\alpha} \frac{\beta_2 R_1 - 1}{1+\beta_2 - \beta_2 R_1}$$

is the saving share of the more altruistic agents. The trace and the determinant of J are given by

$$T = R_1 \left[1 - \frac{1-\alpha\beta_1\beta_2 - \pi(1-\alpha)(\beta_2-\beta_1)\frac{1+\alpha\beta_2-\beta_2 R_1}{1+\beta_2-\beta_2 R_1}}{(1+\alpha\beta_1)(1+\beta_2) - \pi(1-\alpha)(\beta_2-\beta_1)}\right] \quad (27)$$

$$D = R_1^2 \frac{\alpha\beta_1\beta_2}{(1+\alpha\beta_1)(1+\beta_2) - \pi(1-\alpha)(\beta_2-\beta_1)} \quad (28)$$

and its eigenvalues by

$$\lambda_1 = \left(T - \sqrt{T^2 - 4D}\right)/2 \quad \textit{and} \quad \lambda_2 = \left(T + \sqrt{T^2 - 4D}\right)/2 \quad (29)$$

Global convergence implies local convergence (both the eigenvalues are inside the unit circle in the Argand-Gauss plane). A constructive proof of local convergence can be also provided noticing that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if the pair (T, D) lie in the interior of the triangle defined by $D > -T - 1$, $D > T - 1$ and $D < 1$. Using expressions (27)-(28) for trace and determinant, it is possible to prove that when both types of agents leave bequests, these three inequalities are always jointly verified.

Proposition 11 (Local stability when less altruistic agents leave bequests) *The steady state (21) (part (1) of Proposition 8) is locally stable.*

Since the initial condition $\left(b_0^1, b_0^2\right)$ is given (predetermined), local stability means that there is a unique equilibrium trajectory starting from this initial condition towards the steady state, provided that $\left(b_0^1, b_0^2\right)$ lies in a neighborhood of the steady state. In other terms, the equilibrium (transition) is locally unique.

In order to illustrate convergence in this case, we fix the parameter values as follows: $\alpha = 0.33$, $\pi = 0.5$, $\beta_1 = 0.5$, $\beta_2 = 0.6$. Using (18), we obtain $R_1 = 2.1273$ and, thus, $2 = 1/\beta_1 < R_1 < 1 + 1/\beta_2 = 2.667$. Using (27) and (28), we obtain $T = 1.0739$ and $D = 0.24685$. The eigenvalues given by (29), $\lambda_1 = 0.33333$ and $\lambda_2 = 0.74054$, are both inside the unit circle (sink). The smaller their modulus, the faster the convergence to the steady state, coherently with Proposition 9.

Case (2) of Propositions 8 and 9. Since $1/\beta_2 < R_2 < 1/\beta_1$, only the more altruistic agents leave bequests.

Proposition 12 (Local stability when less altruistic agents leave no bequests) *Dynamics are given by*

$$b_{t+1}^2 = b_t^2 \frac{R\left(\pi b_t^2\right)}{R_2} \tag{30}$$

and are locally approximated by the following equation

$$\frac{db_{t+1}^2}{b_2^2} = \alpha \frac{db_t^2}{b_2^2} \tag{31}$$

The steady state (22) (part (2) of Proposition 8) is locally stable.

The eigenvalue is given by $\lambda = \alpha$. The trajectory locally converges to the steady state because $0 < \alpha < 1$. Moreover, the lower is the capital share in total income, the faster is the convergence.

3.2 Altruism and economic growth

Proposition 9 allows us to focus on the properties of the steady-state equilibrium. We now study the effect of altruism on economic growth in our model. For this, we analyze how the steady-state capital stock depends on the level of altruism and on altruism heterogeneity.

To study the impact of the level of altruism, we employ the share of the more altruistic agents in total population, π, as a measure of altruism. Given β_1 and β_2, the higher is π, the more altruistic is society as a whole.

Let us introduce the threshold value of altruism which corresponds to the threshold value of altruism heterogeneity (20):

$$\pi^* \equiv \frac{\alpha}{1-\alpha} \frac{1-\delta}{\delta} \tag{32}$$

The impact of π on the steady-state capital stock k^* is characterized as follows.

Proposition 13 (Capital stock and altruism) *The steady-state capital stock k^* is continuous and monotonically increasing in π. For $\pi < \pi^*$, we have $k^* = k_1$, while for $\pi \geq \pi^*$, $k^* = k_2$.*

This result shows that the more altruistic is society as a whole, the higher is the steady-state capital stock. Since the more altruistic agents leave higher bequests in the steady state, an increase in π, which reduces the share of the less altruistic agents with low bequests, reduces the steady-state interest rate and increases capital accumulation and output. Thus, more altruism implies higher level of aggregate income.

Proposition 13 is illustrated in Figure 1 with $\alpha = 0.33$, $\beta_1 = 0.5$, $\beta_2 = 0.75$, $A = 1$. For each π, we have $k^* = \max\{k_1, k_2\}$.

To study the impact of the altruism heterogeneity, we employ social heterogeneity in terms of altruism as a measure. Given π and the average of $1/\beta_i$, the standard deviation of $1/\beta_i$ is a relevant measure of altruism heterogeneity.

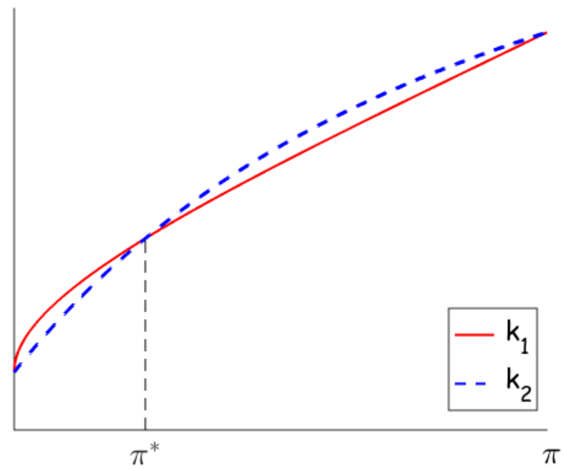

Figure 1: Steady-state capital stock and altruism

Let $\bar{\beta}$ be the weighted harmonic mean of β_1 and β_2:

$$\frac{1}{\bar{\beta}} = \frac{1-\pi}{\beta_1} + \frac{\pi}{\beta_2}$$

Then, the variance of $1/\beta_1$ and $1/\beta_2$ is given by:

$$\sigma^2 = (1-\pi)\left(\frac{1}{\beta_1} - \frac{1}{\bar{\beta}}\right)^2 + \pi\left(\frac{1}{\beta_2} - \frac{1}{\bar{\beta}}\right)^2 = \pi(1-\pi)\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)^2 = \pi(1-\pi)\delta^2$$

Therefore, σ, the standard deviation of $1/\beta_i$, is directly proportional to the altruism gap δ, which can be used as a relevant indicator of altruism heterogeneity.

The impact of δ on the steady-state capital stock k^* is characterized as follows.

Proposition 14 (Capital stock and heterogeneity) *The steady-state capital stock k^* is continuous and is monotonically increasing in δ. For $\delta < \delta^*$, $k^* = k_1$, while for $\delta \geq \delta^*$, $k^* = k_2$.*

Thus, increasing altruism heterogeneity, that is, the standard deviation σ of $1/\beta_i$ around their mean $1/\bar{\beta}$, increases the steady-state capital stock. The reason is that a mean-preserving shift implies a higher degree of altruism for the more altruistic agents, and a lower degree of altruism for the less altruistic agents. When primary savers become even more altruistic, they leave much higher bequests, pulling the steady-state interest rate down and promoting capital accumulation. This effect is more pronounced for $\delta \geq \delta^*$, when the less altruistic agents does not leave bequests. However, even when the less altruistic agents reduce their bequests as a result of lower degree of altruism, the overall effect remains positive. In other words, higher altruism heterogeneity always leads to a higher level of aggregate income.

Proposition 14 is illustrated in Fig. 2 with $\alpha = 0.33$, $\bar{\beta} = 0.75$, $\pi = 0.4$, $A = 1$. For each δ, we have $k^* = \max\{k_1, k_2\}$.

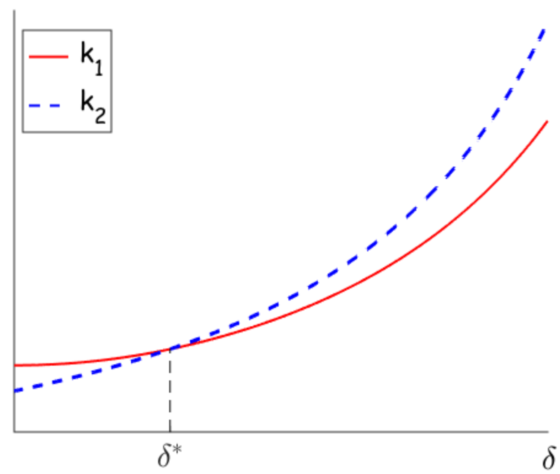

Figure 2: Steady-state capital stock and altruism heterogeneity

3.3 Altruism and income inequality

Consider now the effect of the level of altruism and of the altruism heterogeneity on the steady-state income inequality.

It is natural to represent the level of social inequality by the Gini index applied to income distribution. The following proposition characterizes the Gini index of income inequality in the steady-state equilibrium.

Proposition 15 (Gini index) *(1) Suppose that $\delta < \delta^*$. The Gini index in the steady-state equilibrium $\left(c_1^1, b_1^1, c_1^2, b_1^2, k_1\right)$ is given by*

$$G_1 = \frac{2\delta\pi(1-\pi)}{1-\alpha+(2\pi-1)\,\delta+\sqrt{(\delta+\alpha-1)^2+4\delta\pi\,(1-\alpha)}} \tag{33}$$

(2) Suppose that $\delta \geq \delta^*$. The Gini index in the steady-state equilibrium $\left(c_2^1, b_2^1, c_2^2, b_2^2, k_2\right)$ is given by $G_2 = \alpha\,(1-\pi)$.

The impact of the level of altruism π on the steady-state Gini index G^*, given β_1 and β_2, is characterized as follows.

Proposition 16 (Gini index and altruism) *The steady-state Gini index* G^* *is continuous in* π*. For* $\pi < \pi^*$, $G^* = G_1$*, while, for* $\pi \geq \pi^*$, $G^* = G_2$*.*

(1) Suppose that $\delta < \min\{1, 2(1-\alpha)\}$*. Then there exists a threshold share of the more altruistic agents,* $\tilde{\pi} \leq \pi^*$*, such that for* $\pi < \tilde{\pi}$, G^* *is increasing in* π*, while for* $\pi \geq \tilde{\pi}$, G^* *is decreasing in* π*. When* $\delta < 1-\alpha$, $G^*(0) = 0$*, while, when* $\delta > 1-\alpha$, $G^*(0) = 1-(1-\alpha)/\delta > 0$*.*

(2) Suppose that $\delta \geq \min\{1, 2(1-\alpha)\}$*. Then,* G^* *is decreasing in* π*.*

Proposition 16 suggests that there are three regimes of the steady-state Gini index, which are determined by the interplay between the level of altruism and altruism heterogeneity. The first regime occurs when altruism heterogeneity is low, $\delta < 1-\alpha$. In this case, when π is very low or very high, the population is almost constituted by the same type of agents (less altruistic or more altruistic, respectively). Since the population is almost homogeneous up to a small minority of different agents, the social inequalities are close to zero. When π takes values in the middle of the range, the shares of the rich (the more altruistic agents who leave higher bequests) and the poor (the less altruistic agents) are similar, which drives up social inequality.

Thus, in this regime, the dependence of the steady-state level of inequality on the average level of altruism has a rather symmetric inverted U-shape. The first regime is illustrated in Figure 3, where we set $\alpha = 0.33$, $\beta_1 = 0.5$, $\beta_2 = 0.75$. For each π, we have $G^* = \min\{G_1, G_2\}$.

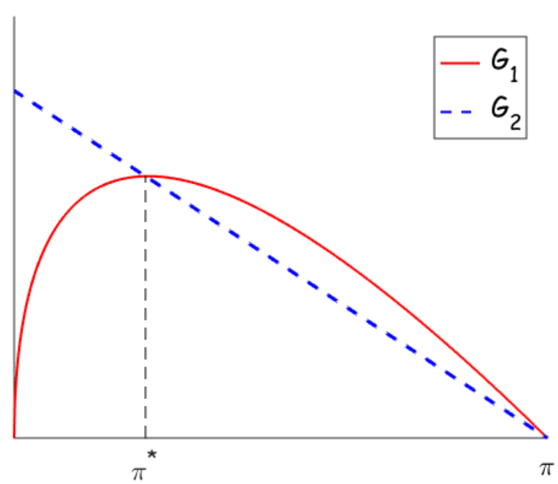

Figure 3: Steady-state Gini index and altruism: Low heterogeneity

The second regime occurs when altruism heterogeneity is moderate, $1-\alpha < \delta < \min\{1, 2(1-\alpha)\}$. In this case, when π is very low, the level of inequality is positive. Due to the altruism gap, the population is not homogeneous. Even though the share of the more altruistic agents is small, their income is large enough to significantly affect inequality. Similarly to the first regime, an increase in π increases the level of inequality. When π is already high, then, irrespective of whether the less altruistic agents leave bequests or not, a further increase in π would make a larger share of population richer, which reduces social inequalities.

Therefore, in this regime, the dependence of the steady-state Gini index on the level of altruism has an asymmetric inverted U-shape which is shifted upwards for low levels of altruism. The second regime is illustrated in Figure 4, where we set $\alpha = 0.33$, $\beta_1 = 0.45$, $\beta_2 = 0.75$. For each π, we have $G^* = \min\{G_1, G_2\}$.

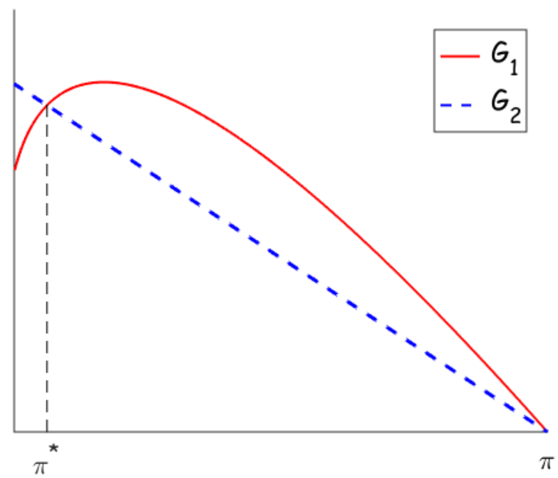

Figure 4: Steady-state Gini index and altruism: Moderate heterogeneity

The third regime occurs when altruism heterogeneity is sufficiently high, $\delta \geq \min\{1, 2(1-\alpha)\}$. When the difference between the more and the less altruistic agents is very high, then the highest possible level of inequality is observed in societies consisting of only the less altruistic agents, and inequality is decreasing with the level of altruism. Intuitively, in this regime, even if the less altruistic agents leave some bequests, the amount of these bequests is very low and has almost no impact on their income. An increase in the share of the more altruistic agents unambiguously decreases inequality, as a larger share of population becomes rich.

The third regime of the steady-state Gini index is illustrated in Figure 5, where we set $\alpha = 0.6$, $\beta_1 = 0.45$, $\beta_2 = 0.75$. For each π, we have $G^* = \min\{G_1, G_2\}$.

Comparing Proposition 13 and part (1) of Proposition 16, we observe that if altruism heterogeneity is low or moderate, then the resulting dependence of the steady-state level of inequality on the steady-state level of income is non-monotonic, which is consistent with the cross-country Kuznets curve.

Consider three countries (A, B, C) which differ in their average levels of

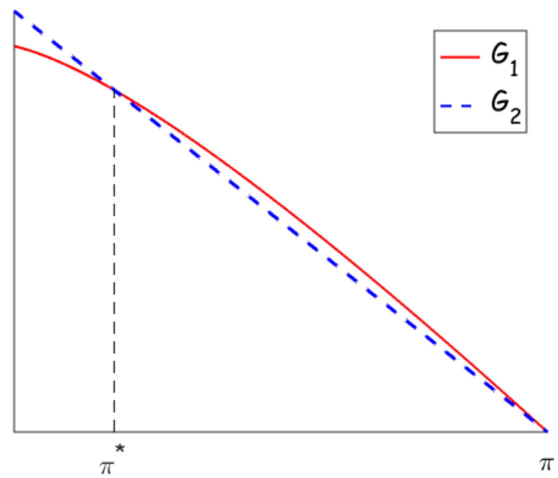

Figure 5: Steady-state Gini index and altruism: High heterogeneity

altruism ($\pi_A > \pi_B > \pi_C$) but are identical in every other respect. Then country A would have the highest aggregate income in the steady-state, and country C the lowest ($k_A^* > k_B^* > k_C^*$). At the same time, high-income country A would have the lowest income inequality in the steady state, while middle-income country B the highest ($G_B^* > G_C^* > G_A^*$).

Moreover, for moderate altruism heterogeneity, according to Figure 4, the low-income country C and middle-income country B would have very close Gini indices (because of the asymmetric inverted U-shape), which is consistent with empirical evidence (see Table 1).

Comparing Proposition 13 and part (2) of Proposition 16, we can see that if altruism heterogeneity is high, there is no trade-off between economic growth and social inequality. The more altruistic is society as a whole, the higher is the steady-state capital stock, and the lower is the steady-state Gini index. Higher aggregate income is accompanied with lower level of inequality.

These results suggest that heterogeneous altruism is an important mechanism contributing to the tendency of high-income countries to have lower levels of income inequality.

Consider now the impact of the altruism heterogeneity on the steady-state Gini index. Fix π and $\bar{\beta}$. The following proposition characterizes the dependence of G^* on δ.

Proposition 17 (Gini index and heterogeneity) *The steady-state Gini index G^* is continuous and non-decreasing in δ. For $\delta < \delta^*$, $G^* = G_1$ and G^* is strictly increasing in δ. For $\delta \geq \delta^*$, $G^* = G_2$ and is independent of δ.*

Comparing Propositions 14 and 17, we draw the following conclusions. When agents of both types have similar degrees of altruism, there is the growth-inequality trade-off in terms of altruism heterogeneity. When δ is sufficiently low, the higher is altruism heterogeneity (or, equivalently, the higher is the standard deviation σ), the higher is inequality, and the higher is aggregate income. However, when δ is sufficiently high, this trade-off disappears: a further increase in altruism heterogeneity would increase the steady-state capital stock, but would not affect the steady-state Gini index. In terms of altruism heterogeneity, there exists a maximum possible steady-state Gini index which is equal to $\alpha(1-\pi)$.

Proposition 17 is illustrated in Fig. 6 with $\alpha = 0.33$, $\pi = 0.65$. For each δ, we have $G^* = \min\{G_1, G_2\}$.

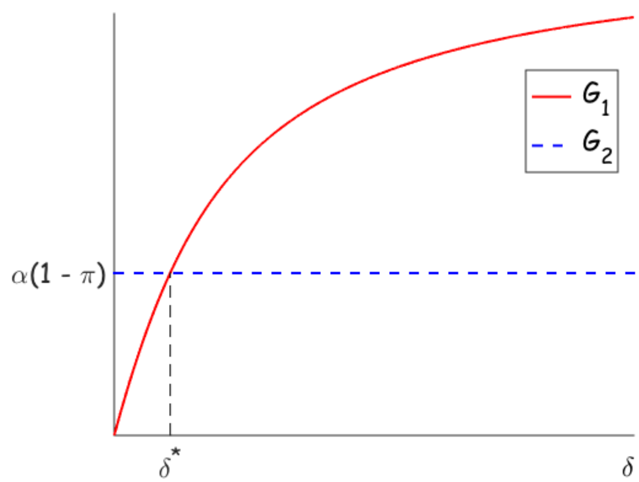

Figure 6: Steady-state Gini index and altruism heterogeneity

3.4 Altruism and welfare

Finally, we consider the effect of altruism on welfare in the steady state. We analyze how the steady-state utilities of both types of agents depend on the level of altruism. We focus on individual utilities instead of a social welfare function for two reasons. First, in this paper, we have a positive instead of a normative approach. We do not need to compare welfare along alternative equilibrium paths. Second, when agents have heterogeneous degrees of altruism, social welfare criteria are problematic (see Reichlin, 2020, among others). For these reasons, we study separately the steady-state utility levels of the more and the less altruistic agents.

According to (1), in the steady-state equilibrium $\left(c_j^1, b_j^1, c_j^2, b_j^2, k_j\right)$ with $j = 1, 2$, the utility of type i agent is given by

$$U_j^i = \ln c_j^i + \beta_i \ln\left(c_j^i + b_j^i\right)$$

Propositions 8 and 13 imply that the steady-state utility of type i agent, U^i, is continuous, and for $\pi < \pi^*$ we have $U^i = U_1^i$, while for $\pi \geq \pi^*$ we have $U^i = U_2^i$. The following proposition shows how the steady-state utility levels depend on the level of altruism.

Proposition 18 (Agents' utilities and altruism) *(1) The steady-state utility of the less altruistic agents U^1 is continuous in π, strictly decreasing for any $\pi < \breve{\pi}$ and strictly increasing for any $\pi > \breve{\pi}$ where $\breve{\pi} \leq \pi^*$ is a threshold.*

(2) The utility of the more altruistic agents U^2 is continuous and strictly decreasing in π.

Therefore, the steady-state utility of the less altruistic agents is U-shaped: U^1 is decreasing in π for low levels of altruism and increasing in π for high levels of altruism. On the contrary, the steady-state utility of the more altruistic agents is always decreasing in π.

Intuitively, the higher is the share of the more altruistic agents π, the lower is the amount of bequest left by a single agent. Indeed, the higher is π, the lower is the interest rate, which reduces individual's incentive to save.[10] The disposable income of the more altruistic agents is determined mainly by bequests. Therefore, their consumption also tends to decrease, which lowers their steady-state utility. For the more altruistic agents, the optimal level of altruism is always $\pi = 0$.

Differently, the disposable income of the less altruistic agents is determined mainly by the wage bill which is increasing in π together with output. For low levels of altruism, the wages are low and bequests play some role for these agents, so their steady-state levels of consumption and utility decrease in π, similar to those of the more altruistic agents. However, after a certain threshold level, the role of bequests for the less altruistic agents becomes negligible, and a further increase in π would increase their steady-state levels of consumption and utility. This effect is evident for very high levels of altruism, $\pi \geq \pi^*$, when the less altruistic agents leave no bequests and consume their wages, but it could also be observed already for moderate levels of π. Thus, for the less altruistic agents the optimal level of altruism is either $\pi = 0$ or $\pi = 1$.

Proposition 18 is illustrated in Fig. 7 for the less altruistic agents and in Fig. 8 for the more altruistic agents with $\alpha = 0.64$, $\beta_1 = 0.495$, $\beta_2 = 0.75$, $A = 1$.

4 Concluding remarks

In this paper, we argue that altruism heterogeneity is a possible mechanism that provides an additional explanation of why income inequality differs between countries. We develop and analyze a simple growth model with agents who differ in their degree of altruism. The novelty of our approach rests on combining the

[10] Recall, however, that since the more altruistic agents leave higher bequests than the less altruistic agents, higher π leads to an increase in aggregate steady-state capital stock. Even though agents leave lower bequests, the total amount of bequests is growing because of the higher share of those who save more.

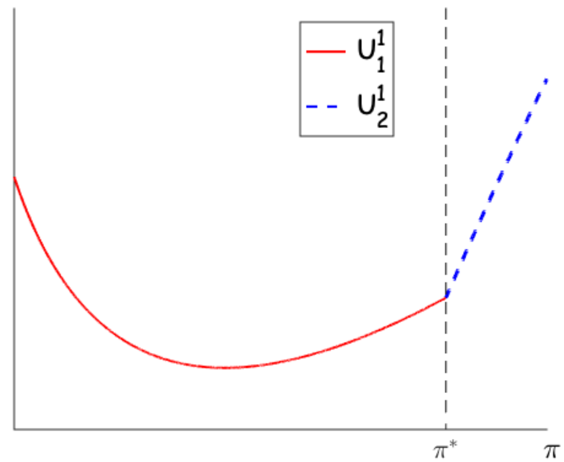

Figure 7: Less altruistic agents' welfare and altruism

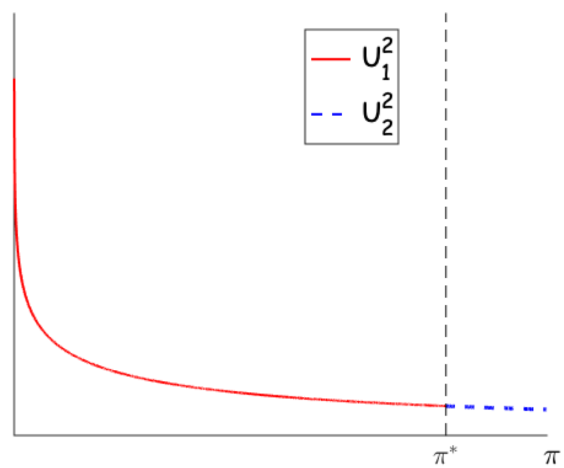

Figure 8: More altruistic agents' welfare and altruism

assumption of family altruism (we consider agents who leave bequests taking care of the disposable income of their offsprings) with the assumption of agents' heterogeneity (we consider two types of agents: the one being more altruistic, the other less).

We prove that every path of bequests and capital converges to a unique steady-state equilibrium and study the properties of a steady state. Our results suggest that the effects of the level of altruism and altruism heterogeneity depend on the difference between the more and the less altruistic agents.

When altruism heterogeneity is low, we observe a Kuznets curve linking different steady-state levels of income to the steady-state levels of inequality. An increase in the average level of altruism which implies an increase in aggregate income, at first is associated with growing inequality, as the share of the more altruistic agents who leave higher bequests increases. However, after some threshold level of altruism, further increase in aggregate income is accompanied with falling inequality, as the larger share of population becomes rich. Also, when altruism heterogeneity is low, an increase in altruism heterogeneity leads to both higher aggregate income and higher income inequality.

However, when altruism heterogeneity is sufficiently high, any trade-off between growth and inequality disappears. An increase in the level of altruism would increase aggregate income and decrease the level of inequality. Furthermore, an increase in altruism heterogeneity leads to a higher steady-state capital stock and does not affect the level of steady-state inequality.

There are also several opportunities for further theoretical research. First, altruism heterogeneity is an important factor for policy implications. When designing policies related to income redistribution or social welfare programs, one should take into account the empirically relevant fact that individuals have different degrees of altruism and hence respond differently to different incentives. Future research could introduce redistributive fiscal policies through bequest taxation, public debt and social security or more general social welfare programs.

Second, it is natural to assume that agents' degrees of altruism are not constant but change over time depending on the relative wealth of agents. This case of endogenous altruism has lately received considerable attention (see, among others, Das, 2007; Rapoport and Vidal, 2007). It is also interesting to understand the consequences of endogenous altruism in our framework. Overall, we believe that our approach and results contribute to the understanding of the role of heterogeneous altruism in economic growth and income inequality.

5 Appendix

Proof of Lemma 1

Let ν_t^i and μ_t^i be the Lagrange multipliers of the budget constraint and non-negativity bequest constraint respectively. Maximizing the Lagrangian function of the Kuhn-Tucker program

$$\ln c_t^i + \beta_i \ln \left(R_{t+1} b_{t+1}^i + w_{t+1}\right) + \nu_t^i \left(R_t b_t^i + w_t - c_t^i - b_{t+1}^i\right) + \mu_t^i b_{t+1}^i$$

we find a system of first-order conditions:

$$\frac{1}{c_t^i} = \nu_t^i = \frac{\beta_i R_{t+1}}{R_{t+1} b_{t+1}^i + w_{t+1}} + \mu_t^i$$

jointly with $\nu_t^i \geq 0$, $R_t b_t^i + w_t - c_t^i - b_{t+1}^i \geq 0$, $\nu_t^i \left(R_t b_t^i + w_t - c_t^i - b_{t+1}^i\right) = 0$, and $\mu_t^i \geq 0$, $b_{t+1}^i \geq 0$, $\mu_t^i b_{t+1}^i = 0$. Since $\nu_t^i = 1/c_t^i > 0$, we obtain (2) and (3).

The reduced utility function $v\left(c_t^i\right) \equiv \ln c_t^i + \beta_i \ln\left[R_{t+1}\left(R_t b_t^i + w_t - c_t^i\right) + w_{t+1}\right]$ is strictly concave: $v''\left(c_t^i\right) = -(c_t^i)^{-2} - \beta_i\left(b_{t+1}^i + w_{t+1}/R_{t+1}\right)^{-2} < 0$. Then the first-order conditions are necessary and sufficient to utility maximization. ■

Proof of Lemma 2

The first-order conditions are: $f'(k_{jt}) = R_t$ and $f(k_{jt}) - k_{jt} f'(k_{jt}) = w_t$, where $f(k_{jt}) \equiv F(k_{jt}, 1)$ is the average productivity. Since all firms share the same technology, $f'(k_{jt}) = R_t$ implies that the capital intensity is the same for any firm: $k_{jt} = k_t$. Then profit maximization entails conditions (4) and (5). ■

Proof of Proposition 5

Consider equations (2)-(3) (with $b_{t+1}^i \geq 0$, $\mu_t^i \geq 0$ and $\mu_t^i b_{t+1}^i = 0$) together with equations (6)-(7) where $k_t = \pi_1 b_t^1 + \pi_2 b_t^2$.

If $\mu_t^i > 0$, then $b_{t+1}^i = 0$. If $b_{t+1}^i > 0$, then $\mu_t^i = 0$, and

$$\frac{\beta_i R_{t+1}}{R_{t+1} b_{t+1}^i + w_{t+1}} = \frac{1}{c_t^i} = \frac{1}{R_t b_t^i + w_t - b_{t+1}^i}$$

that is

$$\begin{aligned}
(1+\beta_i)\, b_{t+1}^i &= \beta_i R(k_t)\left[b_t^i + k_t \frac{w(k_t)}{k_t R(k_t)}\right] - k_{t+1}\frac{w(k_{t+1})}{k_{t+1} R(k_{t+1})} \\
b_{t+1}^i &= \frac{1}{1+\beta_i}\left(\beta_i R(k_t)\left[b_t^i + k_t \gamma(k_t)\right] - k_{t+1}\gamma(k_{t+1})\right)
\end{aligned}$$

■

Proof of Proposition 7

We observe that at the steady state,

$$(1+\beta_i)\, b^i = \max\left\{0, \beta_i R(k)\, b^i + \left[\beta_i R(k) - 1\right] k\gamma(k)\right\}$$

with $i = 1, 2$ and $k = \pi_1 b^1 + \pi_2 b^2$.

In order to have positive bequests for type i, we need (11). In order to have positive bequests for both types, we need

$$\frac{1}{\beta_1} < R(k) < 1 + \frac{1}{\beta_2}$$

Then the steady-state capital stock is a solution to the following equation in the unknown k:

$$k = \sum_{i=1}^{2} \pi_i b^i = k\gamma(k) \sum_{i=1}^{2} \pi_i \max\left\{0, \frac{\beta_i R(k) - 1}{1 + \beta_i - \beta_i R(k)}\right\}$$

that is (10). ■

Proof of Proposition 8

Let $R(k) = R$ be the steady-state interest rate and $b^i \geq 0$ be the steady-state bequests, which are solutions to the following equation:

$$(1+\beta_i)\, b^i = \max\left\{0, \beta_i R b^i + (\beta_i R - 1)\gamma k\right\} \tag{34}$$

If $R \leq 1/\beta_i$, then equation (34) has a unique solution: $b^i = 0$.

If $R > 1/\beta_i$, then b^i is positive, and we have

$$b^i = \gamma k \frac{\beta_i R - 1}{1 + \beta_i - \beta_i R} \tag{35}$$

Therefore, if $1/\beta_i < R < 1 + 1/\beta_i$, equation (34) has a unique solution given by (35). If $R \geq 1 + 1/\beta_i$, equation (34) has no solutions.

For $1/\beta_2 < R < 1 + 1/\beta_2$, according to (9), we have

$$\frac{b^i}{k} = \gamma \max\left\{0, \frac{\beta_i R - 1}{1 + \beta_i - \beta_i R}\right\}$$

Observing that $\pi_1 b^1/k + \pi_2 b^2/k = 1$, we have that the steady-state interest rate R is a solution to the following equation:

$$\rho(R) \equiv \pi_1 \max\left\{0, \frac{\beta_1 R - 1}{1 + \beta_1 - \beta_1 R}\right\} + \pi_2 \max\left\{0, \frac{\beta_2 R - 1}{1 + \beta_2 - \beta_2 R}\right\} = \frac{1}{\gamma}$$

Note that $\rho(R)$ is a continuous function which is increasing in R with $\rho(1/\beta_2) = 0$ and $\lim_{R \to (1+1/\beta_2)^-} \rho(R) = +\infty$. Then there exists a solution R to the equation $\rho(R) = 1/\gamma$, and this solution is such that $1/\beta_2 < R < 1 + 1/\beta_2$. Three cases are possible: (1) $1/\beta_1 < R < 1 + 1/\beta_2$; (2) $1/\beta_2 < R \leq 1/\beta_1 < 1 + 1/\beta_2$; and (3) $1/\beta_2 < R < 1 + 1/\beta_2 \leq 1/\beta_1$.

Consider first Cases (2) and (3). Since in both cases $R \leq 1/\beta_1$, we have

$$\rho(R) = \pi \frac{\beta_2 R - 1}{1 + \beta_2 - \beta_2 R} \equiv \rho_2(R)$$

The solution to the equation $\rho_2(R) = 1/\gamma$ is given by

$$R = \frac{1}{\beta_2} + \frac{1}{1 + \gamma\pi} \equiv R_2$$

This solution is the steady-state interest rate if and only if $R_2 \leq 1/\beta_1$, which is equivalent to $\delta \geq \delta^*$. Since $f'(k) = R$, the capital stock corresponding to this steady state is

$$k_2 \equiv \left(\frac{\alpha A}{R_2}\right)^{\frac{1}{1-\alpha}}$$

In this case we also have $b_2^1 = 0$ and $b_2^2 = k_2/\pi$. Further,

$$c_2^1 = (1-\alpha) A k_2^{\alpha} = k_2 (1-\alpha) A k_2^{\alpha - 1} = \gamma k_2 R_2$$

while

$$c_2^2 = (1-\alpha) A k_2^{\alpha} + \frac{(R_2 - 1) k_2}{\pi} = \frac{k_2}{\pi}(\gamma\pi R_2 + R_2 - 1) = \frac{k_2}{\pi} \frac{1 + \gamma\pi}{\beta_2}$$

When $\rho\,(1/\beta_1) < 1/\gamma$, which is equivalent to $\delta < \delta^*$, we are in the conditions of Case (1). We have

$$\rho\,(R) = (1-\pi)\,\frac{\beta_1 R - 1}{1+\beta_1-\beta_1 R} + \pi\frac{\beta_2 R - 1}{1+\beta_2-\beta_2 R} \equiv \rho_1\,(R)$$

The steady-state interest rate R is a solution to the equation $\rho_1\,(R) = 1/\gamma$, which can be written as

$$(1-\pi)\,\frac{\beta_1 R - 1}{1+\beta_1-\beta_1 R} + \pi\frac{\beta_2 R - 1}{1+\beta_2-\beta_2 R} = \frac{1}{\gamma}$$

or, equivalently, as

$$\begin{aligned}&\beta_1\beta_2\,(1+\gamma)\,R^2 - [(1+\gamma)\,(\beta_1+\beta_2) + (2+\gamma)\,\beta_1\beta_2]\,R\\&+(1+\gamma+\beta_1+\beta_2+\beta_1\beta_2+\gamma\,[(1-\pi)\,\beta_2+\pi\beta_1]) = 0\end{aligned}$$

Noticing that $\gamma = (1-\alpha)\,/\alpha$, we obtain

$$R_1^{\pm} \;=\; \frac{1}{2}\left(1+\alpha+\frac{1}{\beta_1}+\frac{1}{\beta_2}\pm\sqrt{(\delta+\alpha-1)^2+4\delta\pi\,(1-\alpha)}\right)$$

Let $D\,(\pi) \equiv (\delta+\alpha-1)^2 + 4\delta\pi\,(1-\alpha)$. It is easily seen that $D(\pi)$ is increasing in π, and

$$D\,(0) \;=\; (\delta+\alpha-1)^2 \tag{36}$$
$$D\,(\pi^*) \;=\; (1-\delta+\alpha)^2 \tag{37}$$
$$D\,(1) \;=\; (\delta+1-\alpha)^2 \tag{38}$$

where π^* is given by (32).

Let us show that

$$R_1^- \;\leq\; 1+\frac{1}{\beta_2} \;\leq R_1^+$$

Indeed, the first inequality is true whenever $\sqrt{D\,(\pi)} \geq \delta+\alpha-1$, while the second inequality is true whenever $\sqrt{D\,(\pi)} \geq 1-\alpha-\delta$. Both these inequalities hold when $D\,(\pi) \geq (\delta+\alpha-1)^2$, which follows from (36).

Denote $R_1 \equiv R_1^-$. Then the corresponding steady-state capital stock is

$$k_1 = \left(\frac{\alpha A}{R_1}\right)^{\frac{1}{1-\alpha}}$$

Since $1/\beta_1 < R_1 < 1+1/\beta_2$, at the steady state we have for $i = 1, 2$,

$$b_1^i = \gamma k_1 \frac{\beta_i R_1 - 1}{1+\beta_i-\beta_1 R_1}$$

and

$$\begin{aligned} c_1^i &= (1-\alpha) A k_1^\alpha + (R_1 - 1) b_1^i = \gamma k_1 \alpha A k_1^{\alpha-1} + \gamma k_1 \frac{(R_1-1)(\beta_i R_1 - 1)}{\beta_i + 1 - \beta_i R_1} \\ &= \gamma k_1 \left[R_1 + \frac{(R_1-1)(\beta_i R_1 - 1)}{\beta_i + 1 - \beta_i R_1} \right] = \frac{\gamma k_1}{\beta_i + 1 - \beta_i R_1} \end{aligned}$$

■

Proof of Proposition 9

(0) Consider a sequence $\left(b_t^1, b_t^2, k_t\right)_{t=0}^\infty$. Fix t. Let $\lambda_t^i \equiv b_t^i / k_t$. Let $\lambda_{t+1}^1, \lambda_{t+1}^2$ be such that for $i = 1, 2$,

$$\lambda_{t+1}^i k_{t+1} = \frac{\beta_i R(k_t) k_t \left(\lambda_t^i + \gamma\right) - \gamma k_{t+1}}{1 + \beta_i}$$

We observe that, at this stage of the proof, λ_{t+1}^i may lie outside of the interval $(0, 1/\pi_i)$. We have

$$\left(\lambda_{t+1}^i + \frac{\gamma}{1+\beta_i}\right) k_{t+1} = \frac{\beta_i R(k_t) k_t \left(\lambda_t^i + \gamma\right)}{1+\beta_i}$$

and, therefore,

$$\frac{\lambda_{t+1}^2 + \frac{\gamma}{1+\beta_2}}{\lambda_{t+1}^1 + \frac{\gamma}{1+\beta_1}} = \frac{\beta_2}{\beta_1} \frac{1+\beta_1}{1+\beta_2} \frac{\lambda_t^2 + \gamma}{\lambda_t^1 + \gamma}$$

Since we need $\pi_1 \lambda_t^1 + \pi_2 \lambda_t^2 = 1$, we study the following equation

$$\zeta\left(\lambda_{t+1}^2\right) \equiv \frac{\lambda_{t+1}^2 + \frac{\gamma}{1+\beta_2}}{\frac{1-\pi_2\lambda_{t+1}^2}{\pi_1} + \frac{\gamma}{1+\beta_1}} = \frac{\beta_2}{\beta_1} \frac{1+\beta_1}{1+\beta_2} \frac{\lambda_t^2 + \gamma}{\frac{1-\pi_2\lambda_t^2}{\pi_1} + \gamma} \equiv \xi\left(\lambda_t^2\right) \tag{39}$$

Functions ζ and ξ are increasing in the interval $(0, 1/\pi_2)$.

Notice that, given λ_t^2, equation (39) has a solution $\lambda_{t+1}^2 \in (0, 1/\pi_2)$ if and only if $b_{t+1}^1, b_{t+1}^2 > 0$. More precisely, let λ_{t+1}^2 be solution to (39) in $(0, 1/\pi_2)$, $b_{t+1}^i = \lambda_{t+1}^i k_{t+1}$ and

$$\lambda_{t+1}^1 = \frac{1}{\pi_1} - \frac{\pi_2}{\pi_1}\lambda_{t+1}^2$$

with

$$\begin{aligned} k_{t+1} &= \sum_{i=1}^2 \frac{\pi_i \left[\beta_i R(k_t)\left(b_t^i + \gamma k_t\right) - \gamma k_{t+1}\right]}{1+\beta_i} \\ &= k_t R(k_t) \sum_{i=1}^2 \frac{\pi_i \beta_i \left(\frac{b_t^i}{k_t} + \gamma\right)}{1+\beta_i} - \gamma k_{t+1} \sum_{i=1}^2 \frac{\pi_i}{1+\beta_i} \end{aligned}$$

that is

$$k_{t+1} = k_t R(k_t) \frac{\sum_{i=1}^2 \frac{\pi_i \beta_i \left(\lambda_t^i + \gamma\right)}{1+\beta_i}}{1 + \gamma \sum_{i=1}^2 \frac{\pi_i}{1+\beta_i}} \tag{40}$$

Let us prove that each solution to the equation $\zeta(\lambda) = \xi(\lambda)$ in $(0, 1/\pi_2)$ corresponds to a steady state where both bequests are strictly positive. Indeed, let λ^{2*} be a solution to equation $\zeta(\lambda) = \xi(\lambda)$ in this interval and

$$\lambda^{1*} = \frac{1}{\pi_1} - \frac{\pi_2}{\pi_1}\lambda^{2*}$$

Let the sequence $\left(\hat{b}_t^1, \hat{b}_t^2, \hat{k}_t\right)$ be such that $\hat{b}_0^1 = \lambda^{1*}k_0$ and $\hat{b}_0^1 = \lambda^{2*}k_0$. Let $\lambda_t^i = \hat{b}_t^i/\hat{k}_t$, for every i and $t \geq 0$. By induction, we obtain $\lambda_t^i = \lambda^{i*}$ and $\hat{b}_t^i = \lambda^{i*}\hat{k}_t$, for every $t \geq 0$.

From (40), we have

$$\left(1 + \gamma \sum_{i=1}^{2} \frac{\pi_i}{1+\beta_i}\right)\hat{k}_{t+1} = \alpha f\left(\hat{k}_t\right) \sum_{i=1}^{2} \frac{\pi_i\beta_i\left(\lambda^{i*}+\gamma\right)}{1+\beta_i}$$

since $\hat{k}_t R\left(\hat{k}_t\right) = \alpha A\hat{k}_t^{\alpha} = \alpha f\left(\hat{k}_t\right)$.

Therefore,

$$\hat{k}_{t+1} = \alpha f\left(\hat{k}_t\right) \frac{\sum_{i=1}^{2} \frac{\pi_i\beta_i\left(\lambda^{i*}+\gamma\right)}{1+\beta_i}}{1+\gamma\sum_{i=1}^{2}\frac{\pi_i}{1+\beta_i}} = \alpha S f\left(\hat{k}_t\right)$$

where

$$S \equiv \frac{\sum_{i=1}^{2} \frac{\pi_i\beta_i\left(\lambda^{i*}+\gamma\right)}{1+\beta_i}}{1+\gamma\sum_{i=1}^{2}\frac{\pi_i}{1+\beta_i}}$$

is a constant.

Since $f\left(\hat{k}_t\right) = A\hat{k}_t^{\alpha}$ with $\alpha \in (0,1)$ and $\hat{k}_{t+1} = \alpha S f\left(\hat{k}_t\right)$ with αS constant, it is known that $\hat{k}_t$ monotonically converges to some capital level k^*.

(1) Consider part (1) of Proposition 8. We want to prove that:

(1.1) For any $0 \leq \lambda_t^2 \leq 1/\pi_2$, there exists a unique $\lambda_{t+1}^2 \in (0, 1/\pi_2)$ such that $\zeta\left(\lambda_{t+1}^2\right) = \xi\left(\lambda_t^2\right)$.

(1.2) There exists a unique $\lambda^{2*} \in (0, 1/\pi_2)$ which solves $\zeta(\lambda) = \xi(\lambda)$.

(1.1) To prove the first claim, we show that $\zeta(0) < \xi(0) \leq \xi\left(\lambda_t^2\right)$, and $\zeta(1/\pi_2) > \xi(1/\pi_2) \geq \xi\left(\lambda_t^2\right)$.

We have $\zeta(0) < \xi(0)$ is equivalent to

$$\frac{\frac{\gamma}{1+\beta_2}}{\frac{1}{\pi_1} + \frac{\gamma}{1+\beta_1}} < \frac{\beta_2}{\beta_1}\frac{1+\beta_1}{1+\beta_2}\frac{\gamma}{\frac{1}{\pi_1}+\gamma}$$

that is to

$$1 + \gamma\pi_1 < \frac{\beta_2}{\beta_1}\left(1 + \beta_1 + \gamma\pi_1\right)$$

which is always true since $\beta_2 > \beta_1 > 0$.

The inequality $\zeta(1/\pi_2) > \xi(1/\pi_2)$ is equivalent to

$$\frac{\frac{1}{\pi_2}+\frac{\gamma}{1+\beta_2}}{\frac{\gamma}{1+\beta_1}} > \frac{\beta_2}{\beta_1}\frac{1+\beta_1}{1+\beta_2}\frac{\frac{1}{\pi_2}+\gamma}{\gamma}$$

that is to

$$\frac{1}{1+\gamma\pi_2} > \frac{1}{\beta_1}-\frac{1}{\beta_2}$$

which is true because, in this case, $\delta < \delta^*$.

Therefore, $\zeta(0) < \xi(0) \le \xi\left(\lambda_t^2\right)$ and $\zeta(1/\pi_2) > \xi(1/\pi_2) \ge \xi\left(\lambda_t^2\right)$.

Hence, there exists $0 < \lambda_{t+1}^2 < 1/\pi_2$ such that $\zeta\left(\lambda_{t+1}^2\right) = \xi\left(\lambda_t^2\right)$. The strict monotonicity of function ζ ensures the uniqueness. Let $\lambda_{t+1}^2 = \varphi\left(\lambda_t^2\right)$ be the unique solution to $\zeta\left(\lambda_{t+1}^2\right) = \xi\left(\lambda_t^2\right)$. The function φ is continuous in the interval $(0, 1/\pi_2)$ and strictly increasing, with $\varphi(0) > 0$ and $\varphi(1/\pi_2) < 1/\pi_2$. This means that starting from any initial pair $\left(b_0^1, b_0^2\right)$ with at least one positive bequest, bequests b_t^1 and b_t^2 are both strictly positive for any $t \ge 1$.

(1.2) Let us focus on the second claim, which is determinant in the proof of convergence. As a preliminary step, we observe that any solution in $(0, 1/\pi_2)$ to equation $\lambda = \varphi(\lambda)$ corresponds to a steady state, which, according to Proposition 8, is unique. We obtain also the uniqueness of λ^{2*}, a solution to $\lambda = \varphi(\lambda)$.

The uniqueness of the solution ensures that we have $\varphi(\lambda) > \lambda$ on $\left(0, \lambda^{2*}\right)$ and $\varphi(\lambda) < \lambda$ on $\left(\lambda^{2*}, 1/\pi_2\right)$. Then, if $0 \le \lambda_0^2 < \lambda^{2*}$, the sequence $\left(\lambda_t^2\right)_{t=0}^{\infty}$ is increasing and converges to λ^{2*}, and, in the opposite case $\lambda_0^2 > \lambda^{2*}$, this sequence is decreasing and converges to λ^{2*}. We can therefore ensure the convergence of $\left(\lambda_t^1, \lambda_t^2\right)$ to $\left(\lambda^{1*}, \lambda^{2*}\right)$.

This implies also the convergence of $\left(b_t^1, b_t^2\right)$ to $\left(b_1^1, b_1^2\right)$. Indeed, we observe that

$$k_{t+1} = k_t R(k_t) \frac{\sum_{i=1}^2 \frac{\pi_i\beta_i\left(\lambda_t^i+\gamma\right)}{1+\beta_i}}{1+\gamma\sum_{i=1}^2 \frac{\pi_i}{1+\beta_i}} = \alpha f(k_t) \frac{\sum_{i=1}^2 \frac{\pi_i\beta_i\left(\lambda_t^i+\gamma\right)}{1+\beta_i}}{1+\gamma\sum_{i=1}^2 \frac{\pi_i}{1+\beta_i}} = \alpha S_t f(k_t)$$

where

$$S_t \equiv \frac{\sum_{i=1}^2 \frac{\pi_i\beta_i\left(\lambda_t^i+\gamma\right)}{1+\beta_i}}{1+\gamma\sum_{i=1}^2 \frac{\pi_i}{1+\beta_i}}$$

By the convergence of λ_t^i to λ^{i*}, we obtain

$$S_t \to S \equiv \frac{\sum_{i=1}^2 \frac{\pi_i\beta_i\left(\lambda^{i*}+\gamma\right)}{1+\beta_i}}{1+\gamma\sum_{i=1}^2 \frac{\pi_i}{1+\beta_i}}$$

Fix any $0 < \varepsilon < S$. There exists T such that $S - \varepsilon < S_t < S + \varepsilon$ for any $t \ge T$.

Let $\left(\bar{k}_t\right)_{t=T}^{\infty}$ and $\left(\underline{k}_t\right)_{t=T}^{\infty}$ be defined as

$$\begin{aligned}
\bar{k}_T &= \underline{k}_T = k_T \\
\bar{k}_{t+1} &= \alpha\left(S+\varepsilon\right)f\left(\bar{k}_t\right) = \alpha A\left(S+\varepsilon\right)\bar{k}_t^{\alpha} \\
\underline{k}_{t+1} &= \alpha\left(S-\varepsilon\right)f\left(\underline{k}_t\right) = \alpha A\left(S-\varepsilon\right)\underline{k}_t^{\alpha}
\end{aligned}$$

By induction, we have $\underline{k}_t \leq k_t \leq \bar{k}_t$, for every $t \geq T$. Clearly,

$$\lim_{t\to\infty} \bar{k}_t = \left[\alpha A\left(S+\varepsilon\right)\right]^{\frac{1}{1-\alpha}} \text{ and } \lim_{t\to\infty} \underline{k}_t = \left[\alpha A\left(S-\varepsilon\right)\right]^{\frac{1}{1-\alpha}}$$

Hence we obtain

$$\lim_{t\to\infty} \sup k_t \leq \left[\alpha A\left(S+\varepsilon\right)\right]^{\frac{1}{1-\alpha}} \text{ and } \lim_{t\to\infty} \inf k_t \geq \left[\alpha A\left(S-\varepsilon\right)\right]^{\frac{1}{1-\alpha}}$$

Since ε is arbitrary, we have

$$\lim_{t\to\infty} k_t = \left(\alpha A S\right)^{\frac{1}{1-\alpha}}$$

The convergence of k_t implies the convergence of b_t^i and c_t^i. Therefore, the sequence $\left(c_t^1, c_t^2, b_t^1, b_t^2, k_t\right)_{t=0}^{\infty}$ converges to the values defined in part (1) of Proposition 8.

(2) Consider part (2) of Proposition 8 and suppose that $\delta > \delta^*$.

(2.1) First, we prove the existence of some t such that $b_t^2 \geq b_t^1$.

Assume the contrary: $b_t^1 > b_t^2$ for every $t \geq 0$.

(2.1.1) We prove that $b_{t+1}^2 > 0$ for every $t \geq 0$. Indeed, assume the contrary, $b_{t+1}^2 = 0$ for some t. Then, from $b_{t+1}^1 > b_{t+1}^2 = 0$, we have

$$\beta_1 R(k_t)(b_t^1 + \gamma k_t) \geq \gamma k_{t+1} \geq \beta_2 R(k_t)(b_t^2 + \gamma k_t)$$

for some t, that is

$$b_t^2 + \gamma k_t \leq \frac{\beta_1}{\beta_2}\left(b_t^1 + \gamma k_t\right)$$

or, equivalently,

$$\frac{\beta_1}{\beta_2} > \frac{\lambda_t^2 + \gamma}{\lambda_t^1 + \gamma}$$

We know that $b_{t+1}^2 > 0$ if the equation $\zeta\left(\lambda_{t+1}^2\right) = \xi\left(\lambda_t^2\right)$ has a solution in the interval $(0, 1/\pi_2)$. We already have $\zeta(0) < \xi(0) < \xi\left(\lambda_t^2\right)$. We will verify that $\zeta(1/\pi_2) > \xi\left(\lambda_t^2\right)$. Indeed, we have

$$\begin{aligned}
\zeta\left(\frac{1}{\pi_2}\right) &= \frac{\frac{1}{\pi_2} + \frac{\gamma}{1+\beta_2}}{\frac{\gamma}{1+\beta_1}} > \frac{1+\beta_1}{1+\beta_2} = \frac{\beta_1}{\beta_2}\frac{\beta_2}{\beta_1}\frac{1+\beta_1}{1+\beta_2} \\
&> \frac{\lambda_t^2 + \gamma}{\lambda_t^1 + \gamma}\frac{\beta_2}{\beta_1}\frac{1+\beta_1}{1+\beta_2} = \xi\left(\lambda_t^2\right)
\end{aligned}$$

Hence, equation $\zeta\left(\lambda_{t+1}^2\right) = \xi\left(\lambda_t^2\right)$ has a solution in $(0, 1/\pi_2)$. Therefore, $\lambda_{t+1}^2 > 0$ and $b_{t+1}^2 > 0$, a contradiction. Then, $b_{t+1}^2 > 0$ for every $t \geq 0$ under the assumption that $b_t^1 > b_t^2$ for every $t \geq 0$.

(2.1.2) Since $b_{t+1}^1 > b_{t+1}^2 > 0$ for every $t \geq 0$, using the same arguments as in the preliminary part of the proof, we have $0 < \lambda_t^i < 1/\pi_2$ for any $t \geq 0$. Moreover, this sequence is monotonic and converges to a solution to equation $\zeta(\lambda) = \xi(\lambda)$. Hence, equation $\zeta(\lambda) = \xi(\lambda)$ has a solution in the interval $(0, 1/\pi_2)$. As proven in the preliminary part, this implies the existence of a steady state with positive bequests: a contradiction with the second part of Proposition 8.

Hence, there exists some t_0 such that $b_{t_0}^2 \geq b_{t_0}^1$.

(2.2) We will prove that $b_t^2 \geq b_t^1$ for every $t \geq t_0$. Indeed, since $\beta/(1+\beta)$ is increasing in β and $1/(1+\beta)$ is decreasing, we have

$$\begin{aligned}
\frac{\beta_2 R(k_{t_0})\left(b_{t_0}^2 + \gamma k_{t_0}\right) - \gamma k_{t_0+1}}{1+\beta_2} &\geq \frac{\beta_2 R(k_{t_0})\left(b_{t_0}^1 + \gamma k_{t_0}\right) - \gamma k_{t_0+1}}{1+\beta_2} \\
&\geq \frac{\beta_1 R(k_{t_0})\left(b_{t_0}^1 + \gamma k_{t_0}\right) - \gamma k_{t_0+1}}{1+\beta_1}
\end{aligned}$$

This implies $b_{t_0+1}^2 \geq b_{t_0+1}^1$. By induction, we have $b_t^2 \geq b_t^1$ for every $t \geq t_0$.

(2.3) We prove the existence of some $t_1 \geq t_0$ such that $b_{t_1}^1 = 0$. Assume the contrary: $b_t^2 \geq b_t^1 > 0$ for any $t \geq t_0$. This implies the existence of a steady state with strictly positives bequests: a contradiction. Therefore, there exists $t_1 \geq t_0$ such that $b_{t_1} = 0$.

(2.4) Now, we prove that $b_t^1 = 0$ for every $t \geq t_1$. Assume the contrary: there is some $t \geq t_1$ such that $b_t^1 = 0$ and $b_{t+1}^1 > 0$. In this case, both b_{t+1}^1 and b_{t+1}^2 are strictly positive. Since $b_t^1 = 0$, we have $\lambda_t^2 = 1/\pi_2$. Using the same arguments as in part (1) with $b_{t+1}^1, b_{t+1}^2 > 0$, we find that $\lambda_{t+1}^2 = \varphi\left(\lambda_t^2\right) > \lambda_t^2 = 1/\pi_2$, a contradiction.

Hence, $b_t^1 = 0$ for every $t \geq t_1$. Therefore, the sequence $\left(b_t^1, b_t^2\right)$ converges to $\left(0, b_2^2\right)$ in Proposition 8.

(3) Consider the cutting-edge case, where $\delta = \delta^*$. Consider functions ζ and ξ defined as in part (1) of the proof. We observe that, for any $\lambda \in [0, 1/\pi_2]$, we have $\zeta(\lambda) \leq \xi(\lambda)$, with equality if and only if $\lambda = 1/\pi_2$. Using the same arguments as in part (2) of the proof, we have $b_t^2 > 0$ for every $t \geq 1$. Now, we consider two cases: either $b_0^1 = 0$ or $b_0^1 > 0$.

In the first case, following the same line of arguments as in part (2) of the proof, we have $b_t^1 = 0$ for any $t \geq 1$ and the solution converges to the one described in Proposition 8.

In the second case, we have $0 < \lambda_0^2 < 1/\pi_2$. Using arguments in part (1) of the proof, we have $0 < \lambda_0^1 < \lambda_2^1 < 1/\pi_2$. By induction we obtain that the sequence $\left(\lambda_t^2\right)_{t\geq 1}$ is strictly increasing and converges to the unique solution to $\zeta(\lambda) = \xi(\lambda)$, that is $1/\pi_2$. A direct consequence of this is that λ_t^1 converges to 0. As in part (2), the convergence of λ_t^1 to 0 and of λ_t^2 to $1/\pi_2$ implies the convergence of $\left(b_t^1, b_t^2, k_t\right)$. It is easy to compute that they converge to the values defined in part (1) of Proposition 8. ■

Proof of Lemma 10

We observe that, in this case, according to Proposition 8, the steady-state interest rate is given by (18).

We linearize the dynamic system (23)–(24) where $k_t = \pi_i b_t^i + \pi_j b_t^j$ around the steady state $\left(b_1^1, b_1^2\right)$ to obtain

$$\begin{aligned} & \left(1+\beta_i+\gamma\pi_i\right) db_{t+1}^i + \gamma\pi_j db_{t+1}^j \\ = & \left(\frac{\pi_i}{k_1}\varepsilon_R\left(k_1\right)\beta_i R\left(k_1\right)\left[\left(1+\gamma\pi_i\right)b_1^i + \gamma\pi_j b_1^j\right] + \beta_i R\left(k_1\right)\left(1+\gamma\pi_i\right)\right) db_t^i \\ & + \left(\frac{\pi_j}{k_1}\varepsilon_R\left(k_1\right)\beta_i R\left(k_1\right)\left[\left(1+\gamma\pi_i\right)b_1^i + \gamma\pi_j b_1^j\right] + \beta_i R\left(k_1\right)\gamma\pi_j\right) db_t^j \end{aligned}$$

where $k_1 = \pi_1 b_1^1 + \pi_2 b_1^2$ is the steady-state capital and

$$\varepsilon_R\left(k_t\right) \equiv \frac{k_t R'\left(k_t\right)}{R\left(k_t\right)} = \alpha - 1 \tag{41}$$

is the elasticity of the interest rate.

Setting $R_1 = R\left(k_1\right)$ and noticing that, at the steady state, the saving share of type i agents (25) is given by

$$z_i = \gamma\pi_i \frac{\beta_i R_1 - 1}{1+\beta_i-\beta_i R_1}$$

with, clearly, $z_1 + z_2 = 1$, we find

$$\begin{aligned} & \left(1+\beta_i+\gamma\pi_i\right) z_i \frac{db_{t+1}^i}{b_1^i} + \gamma\pi_i z_j \frac{db_{t+1}^j}{b_1^j} \\ = & \beta_i R_1 \left[1+\left(1-\alpha\right)\left(\pi_i - z_i\right)\right] z_i \frac{db_t^i}{b_1^i} + \beta_i R_1 \left(1-\alpha\right)\left(\pi_i - z_i\right) z_j \frac{db_t^j}{b_1^j} \end{aligned}$$

with $i = 1, 2$ and $j \neq i$. In matrix terms, we get (26). Linearizing this system around the steady state (21), we obtain the eigenvalues (29), where T and D are given by expressions (27) and (28). ■

Proof of Proposition 11

Denote the saving share of the more altruistic agents by $z \equiv z_2$. Note that, in this case, the steady-state interest rate is such that

$$\frac{1}{\beta_1} < R_1 < 1 + \frac{1}{\beta_2} \tag{42}$$

The steady state is a sink (two eigenvalues inside the unit circle) if and only if the pair (T, D) lies in the stability triangle defined by the inequalities $D > -T-1$, $D > T-1$ and $D < 1$.

Let us show that (1.1) $D > 0$ and $T > 0$ (implying $D > -T - 1$); (1.2) $D > T - 1$; (1.3) $D < 1$.

(1.1) Notice that $D > 0$ because $(1 + \alpha\beta_1)(1 + \beta_2) - \pi(1 - \alpha)(\beta_2 - \beta_1) > 0$ and

$$\begin{aligned}
T &= R_1 \frac{(1 + \alpha\beta_1)(1 + \beta_2) - 1 - [\alpha z + (1 - \alpha)\pi](1 - \alpha)(\beta_2 - \beta_1)}{(1 + \alpha\beta_1)(1 + \beta_2) - \pi(1 - \alpha)(\beta_2 - \beta_1)} \\
&\quad + R_1 \frac{\alpha\beta_1\beta_2}{(1 + \alpha\beta_1)(1 + \beta_2) - \pi(1 - \alpha)(\beta_2 - \beta_1)} \\
&\geq R_1 \left[\frac{(1 + \alpha\beta_1)(1 + \beta_2) - 1 - (1 - \alpha)(\beta_2 - \beta_1)}{(1 + \alpha\beta_1)(1 + \beta_2) - \pi(1 - \alpha)(\beta_2 - \beta_1)} + \frac{\alpha\beta_1\beta_2}{(1 + \alpha\beta_1)(1 + \beta_2) - \pi(1 - \alpha)(\beta_2 - \beta_1)} \right] \\
&= R_1 \left[\frac{\beta_1 + \alpha\beta_2 + 2\alpha\beta_1\beta_2}{(1 + \alpha\beta_1)(1 + \beta_2) - \pi(1 - \alpha)(\beta_2 - \beta_1)} \right] > 0
\end{aligned}$$

because $\alpha z + (1 - \alpha)\pi \leq 1$.

(1.2) We want to prove that $D > T - 1$.

According to (27) and (28), $D > T - 1$ is equivalent to

$$(R_1 - 1)[(1 + \alpha\beta_1)(1 + \beta_2) - \alpha\beta_1\beta_2 R_1] - R_1 + (\pi - R_1[\alpha z + (1 - \alpha)\pi])(1 - \alpha)(\beta_2 - \beta_1) < 0$$

Using the expression for z, we obtain

$$\begin{aligned}
&\alpha\beta_1\beta_2 R_1^2 - (\beta_2 + \alpha\beta_1 + 2\alpha\beta_1\beta_2) R_1 + (1 + \alpha\beta_1)(1 + \beta_2) \\
&+\pi(1 - \alpha)(\beta_2 - \beta_1) \left[(1 - \alpha) \frac{\beta_2 R_1}{1 + \beta_2 - \beta_2 R_1} - 1 \right] \\
> \; & 0 \qquad (43)
\end{aligned}$$

The steady state R_1 is solution to

$$(1 - \pi) \frac{\beta_1 R_1 - 1}{1 + \beta_1 - \beta_1 R_1} + \pi \frac{\beta_2 R_1 - 1}{1 + \beta_2 - \beta_2 R_1} = \frac{\alpha}{1 - \alpha}$$

that is to

$$\beta_1\beta_2 R_1^2 = (\beta_1 + \beta_2 + \beta_1\beta_2 + \alpha\beta_1\beta_2) R_1 + \pi(1 - \alpha)(\beta_2 - \beta_1) - (1 + \beta_2)(1 + \alpha\beta_1)$$

Thus, at the steady state $R = R_1$, (43) becomes

$$\begin{aligned}
&\alpha[(\beta_1 + \beta_2 + \beta_1\beta_2 + \alpha\beta_1\beta_2) R_1 + \pi(1 - \alpha)(\beta_2 - \beta_1) - (1 + \beta_2)(1 + \alpha\beta_1)] \\
&- (\beta_2 + \alpha\beta_1 + 2\alpha\beta_1\beta_2) R_1 + (1 + \alpha\beta_1)(1 + \beta_2) \\
&+\pi(1 - \alpha)(\beta_2 - \beta_1) \left[(1 - \alpha) \frac{\beta_2 R_1}{1 + \beta_2 - \beta_2 R_1} - 1 \right] > 0
\end{aligned}$$

or, equivalently, $X^2 - 2PX + (1 + \beta_2) P > 0$, where $X \equiv \beta_2 R_1$ and

$$P \equiv 1 + \beta_2 + \pi(\beta_1 - \beta_2) \frac{1 - \alpha}{1 + \alpha\beta_1}$$

The roots are $X_{\pm} = P \pm \sqrt{P^2 - (1+\beta_2)P}$. Thus, $X^2 - 2PX + (1+\beta_2)P > 0$ if $P^2 - (1+\beta_2)P < 0$.

We observe that $P < 1 + \beta_2$ because of Assumption 1. Moreover, $P > 0$ if and only if

$$\pi < \frac{1+\alpha\beta_1}{1-\alpha}\frac{1+\beta_2}{\beta_2-\beta_1}$$

According to (42),

$$\beta_1 > \frac{\beta_2}{1+\beta_2}$$

and

$$\frac{1+\alpha\beta_1}{1-\alpha}\frac{1+\beta_2}{\beta_2-\beta_1} > \frac{1+\alpha\beta_1}{1-\alpha}\frac{1+\beta_2}{\beta_2-\frac{\beta_2}{1+\beta_2}} = \frac{1+\alpha\beta_1}{1-\alpha}\left(\frac{1+\beta_2}{\beta_2}\right)^2 > 1$$

Hence,

$$\pi \leq 1 < \frac{1+\alpha\beta_1}{1-\alpha}\frac{1+\beta_2}{\beta_2-\beta_1}$$

and $P > 0$.

Summing up, we have $0 < P < 1 + \beta_2$, which implies $P^2 - (1+\beta_2)P < 0$, that is $X^2 - 2PX + (1+\beta_2)P > 0$ or, equivalently, (43), that is $D > T - 1$.

(1.3) Finally, let us prove that $D < 1$.

According to (28), $D < 1$ is equivalent to

$$R_1^2 < \frac{(1+\alpha\beta_1)(1+\beta_2)}{\alpha\beta_1\beta_2} - \pi\frac{1-\alpha}{\alpha}\left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right) = \frac{(1+\alpha\beta_1)(1+\beta_2)}{\alpha\beta_1\beta_2} - \delta\pi\frac{1-\alpha}{\alpha}$$

that is to

$$\alpha\beta_1\beta_2R_1^2 < (1+\alpha\beta_1)(1+\beta_2) - \delta\pi\beta_1\beta_2(1-\alpha) \tag{44}$$

According to (42),

$$R_1^2 < \left(1+\frac{1}{\beta_2}\right)^2 < \left(1+\frac{1}{\beta_1}\right)\left(1+\frac{1}{\beta_2}\right)$$

Thus, (44) is verified if

$$\alpha\beta_1\beta_2\left(1+\frac{1}{\beta_1}\right)\left(1+\frac{1}{\beta_2}\right) \leq (1+\alpha\beta_1)(1+\beta_2) - \delta\pi\beta_1\beta_2(1-\alpha)$$

or, equivalently, if $\psi(\alpha) \equiv (1+\alpha\beta_1)(1+\beta_2) - \delta\pi\beta_1\beta_2(1-\alpha) - \alpha(1+\beta_1)(1+\beta_2) \geq 0$.

We observe that $\psi(1) = 0$ and $\psi(0) > 0$ is equivalent to $\pi < (1+\beta_2)/(\beta_2-\beta_1)$, which always holds since $\pi \in [0,1]$.

Since the function ψ is affine, we have also $\psi(\alpha) \geq 0$ for any $\alpha \in [0,1]$. Then, $D < 1$. ■

Proof of Proposition 12
Dynamics are given by

$$b_{t+1}^2 = \frac{\beta_2 R\left(\pi_2 b_t^2\right)\left(b_t^2 + \pi_2 b_t^2 \gamma\right) - \pi_2 b_{t+1}^2 \gamma}{1+\beta_2}$$

that is by

$$b_{t+1}^2 = \frac{\beta_2\left(1+\pi\gamma\right)}{1+\beta_2+\pi\gamma} R\left(\pi b_t^2\right) b_t^2$$

since $\pi \equiv \pi_2$.

We observe that, in this case, according to Proposition 8, the steady-state interest rate is given by

$$R_2 \equiv \frac{1}{\beta_2} + \frac{1}{1+\gamma\pi}$$

Thus, dynamics reduce to (30).

Linearizing the one-dimensional dynamics (30) around the steady state $\left(b_1^2, R_2\right)$, we obtain the eigenvalue:

$$\frac{db_{t+1}^2}{b_2^2} = \left[\frac{R\left(\pi b_2^2\right)}{R_2} + \frac{\pi b_2^2 R'\left(\pi b_2^2\right)}{R_2}\right]\frac{db_t^2}{b_2^2} = \left[1+\varepsilon_R\left(k_2\right)\right]\frac{db_t^2}{b_2^2}$$

since $k_2 = \pi b_2^2$ and $R_2 = R\left(k_2\right)$. The elasticity of the interest rate $\varepsilon_R\left(k_2\right) = \alpha - 1$ is given by (41).

Therefore, (31) holds and, since $\alpha \in (0,1)$, dynamics locally converge to the steady state b_2^2 from the initial condition b_0^2 around the steady state. The steady state is locally stable. ■

Proof of Proposition 13
Denote

$$\begin{aligned}
R_1\left(\pi\right) &\equiv \frac{1+\alpha}{2} + \frac{1}{2}\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) - \frac{1}{2}\sqrt{\left(\delta+\alpha-1\right)^2 + 4\delta\pi\left(1-\alpha\right)} \\
R_2\left(\pi\right) &\equiv \frac{1}{\beta_2} + \frac{1}{1+\gamma\pi}
\end{aligned}$$

and consider the impact of π on the steady-state interest rate.

Condition $\delta < \delta^*$ is equivalent to $\pi < \pi^*$. Thus, there are three cases: (1) if $\pi^* > 1$, which is equivalent to $\delta < \alpha$, then $R^*\left(\pi\right) = R_1\left(\pi\right)$; (2) if $0 < \pi^* \leq 1$, then $R^*\left(\pi\right) = R_1\left(\pi\right)$ for $\pi < \pi^*$ and $R^*\left(\pi\right) = R_2\left(\pi\right)$ for $\pi \geq \pi^*$; and (3) if $\pi^* \leq 0$, which is equivalent to $\delta \geq 1$, then $R^*\left(\pi\right) = R_2\left(\pi\right)$.

When $0 < \pi^* \leq 1$ and $\pi = \pi^*$, by (37) and the fact that $\delta < 1$ we have

$$\begin{aligned}
R_1\left(\pi^*\right) &= \frac{1}{2}\left[1+\alpha+\frac{1}{\beta_1}+\frac{1}{\beta_2}-\sqrt{D(\pi^*)}\right] = \frac{1}{2}\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}+\delta\right) = \frac{1}{\beta_1} \\
R_2\left(\pi^*\right) &= \frac{1}{\beta_2} + \frac{1}{1+\frac{1-\delta}{\delta}} = \frac{1}{\beta_2}+\delta = \frac{1}{\beta_1}
\end{aligned}$$

Therefore, $R^*(\pi)$ is continuous. It follows that k^* is also continuous in π. Since both $R_1(\pi)$ and $R_2(\pi)$ are decreasing in π, k^* is increasing in π. ■

Proof of Proposition 14

Let $D(\delta) \equiv (\delta + \alpha - 1)^2 + 4\delta\pi(1-\alpha)$. Denote

$$\begin{aligned} R_1(\delta) &\equiv \frac{1+\alpha}{2} + \frac{1}{\overline{\beta}} + \frac{2\pi - 1}{2}\delta - \frac{1}{2}\sqrt{D(\delta)} \\ R_2(\delta) &\equiv \frac{1}{\overline{\beta}} - (1-\pi)\delta + \frac{1}{1+\gamma\pi} \end{aligned}$$

and consider the impact of δ on the steady-state interest rate.

Since δ^* given by (20) corresponds to π^* given by (32), it is easily seen from (37) that

$$D(\delta^*) = \left(1 + \alpha - \frac{1}{1+\gamma\pi}\right)^2$$

Therefore, $R^*(\delta)$ is continuous, since

$$\begin{aligned} R_1(\delta^*) &= \frac{1}{\overline{\beta}} + \frac{1}{2}\left[1 + \alpha + \frac{2\pi - 1}{1+\gamma\pi} - \sqrt{D(\delta^*)}\right] = \frac{1}{\overline{\beta}} + \frac{\pi}{1+\gamma\pi} \\ R_2(\delta^*) &= \frac{1}{\overline{\beta}} + \frac{\pi}{1+\gamma\pi} \end{aligned}$$

Notice that

$$\begin{aligned} R_2'(\delta) &\equiv \pi - 1 < 0 \\ R_1'(\delta) &\equiv \frac{2\pi - 1}{2} - \frac{\delta + \alpha - 1 + 2\pi(1-\alpha)}{2\sqrt{D(\delta)}} < 0 \end{aligned}$$

Indeed, $R_1'(\delta) < 0$ if and only if

$$(2\pi - 1)\sqrt{D(\delta)} < (1-\alpha)(2\pi - 1) + \delta \quad (45)$$

When $2\pi - 1 < 0$, inequality (45) becomes

$$(1 - 2\pi)\sqrt{\delta^2 + (1-\alpha)^2 - 2\delta(1-\alpha)(1-2\pi)} > (1-\alpha)(1-2\pi) - \delta$$

If $\delta > (1-\alpha)(1-2\pi)$, the inequality holds. If $\delta < (1-\alpha)(1-2\pi)$, it is equivalent to

$$(1-2\pi)^2\left[\delta^2 + (1-\alpha)^2 - 2\delta(1-\alpha)(1-2\pi)\right] > [(1-\alpha)(1-2\pi) - \delta]^2$$

that is to $(1-2\pi)^2[\delta - 2(1-\alpha)(1-2\pi)] > \delta - 2(1-\alpha)(1-2\pi)$, which holds since $1 - 2\pi < 1$ and $\delta < (1-\alpha)(1-2\pi) < 2(1-\alpha)(1-2\pi)$.

When $2\pi - 1 > 0$, inequality (45) becomes

$$(2\pi - 1)^2\left[\delta^2 + (1-\alpha)^2 + 2\delta(1-\alpha)(2\pi - 1)\right] < [\delta + (1-\alpha)(2\pi - 1)]^2$$

that is $(2\pi-1)^2\left[\delta^2+2\delta(1-\alpha)(2\pi-1)\right] < \delta^2+2\delta(1-\alpha)(2\pi-1)$, which is true since $2\pi-1<1$.

Since k^* is inversely related to R^*, it follows that k^* is increasing in δ. ■

Proof of Proposition 15

The level of social inequality in terms of income in the steady-state equilibrium is represented by the Gini index:

$$G = 2\int_0^1 [x-g(x)]\,dx$$

where $g:[0,1]\to[0,1]$ is the Lorenz curve.

Let us introduce the Lorenz curve by considering a population of n individuals.

The individual i earns a revenue y^i. These individuals are ordered: $y^i \le y^{i+1}$ for $i=1,\ldots,n$. We have a finite sequence of points $\left(x^i,z^i\right)_{i=0}^n$ with $\left(x^0,z^0\right)\equiv(0,0)$ and

$$\left(x^i,z^i\right)\equiv\left(\frac{i}{n},\frac{\sum_{j=1}^i y^j}{\sum_{j=1}^n y^j}\right)$$

Clearly, $(x^n,z^n)=(1,1)$.

We connect any pair of successive points by a segment. The union of this segment is the Lorenz curve. The Gini index is the ratio between the area A_1 between the linear sequence $(i/n,i/n)_{i=0}^n$ and the Lorenz sequence $\left(x^i,z^i\right)_{i=0}^n$, and the area $A_2=1/2$ under the linear sequence $(i/n,i/n)_{i=0}^n$.

In our case, there are n_1 individuals with revenue y^1 and n_2 individuals with revenue equal to y^2 with $n_1+n_2=n$ and

$$\pi_1\equiv\frac{n_1}{n_1+n_2}$$

is the share of agents of type 1 in total population.

Thus, the Gini index is given by

$$G\equiv\frac{A_1}{A_2}=\frac{\int_0^1[x-g(x)]\,dx}{1/2}$$

where g is a continuous Lorenz curve:

$$\begin{aligned}
g(x) &= \frac{\frac{n_1y^1}{n_1y^1+n_2y^2}}{\frac{n_1}{n_1+n_2}}x \text{ if } 0\le x\le\frac{n_1}{n_1+n_2}\\
g(x) &= \frac{n_1y^1}{n_1y^1+n_2y^2}+\frac{1-\frac{n_1y^1}{n_1y^1+n_2y^2}}{1-\frac{n_1}{n_1+n_2}}\left(x-\frac{n_1}{n_1+n_2}\right) \text{ if } \pi_1<x\le1
\end{aligned}$$

that is

$$\begin{aligned}
g(x) &= \frac{y^1}{\pi_1y^1+\pi_2y^2}x \text{ if } 0\le x\le\pi_1\\
g(x) &= \frac{\pi_1y^1+(x-\pi_1)y^2}{\pi_1y^1+\pi_2y^2} \text{ if } \pi_1<x\le1
\end{aligned}$$

We obtain

$$\begin{aligned} G &= 2\int_0^1 [x-g(x)]\,dx = 2\left(\int_0^{\pi_1} [x-g(x)]\,dx + \int_{\pi_1}^1 [x-g(x)]\,dx\right) \\ &= 2\int_0^1 x dx - 2\left[\int_0^{\pi_1} g(x)\,dx + \int_{\pi_1}^1 g(x)\,dx\right] \end{aligned} \tag{46}$$

Since

$$\begin{aligned} \int_0^1 x dx &= \frac{1}{2} \\ \int_0^{\pi_1} g(x)\,dx &= \frac{y^1}{\pi_1 y^1 + \pi_2 y^2}\frac{\pi_1^2}{2} \\ \int_{\pi_1}^1 g(x)\,dx &= \frac{\pi_1 y^1 - \pi_1 y^2}{\pi_1 y^1 + \pi_2 y^2}(1-\pi_1) + \frac{y^2}{\pi_1 y^1 + \pi_2 y^2}\frac{1}{2}\left(1-\pi_1^2\right) \end{aligned}$$

using (46), we find

$$G = \frac{\pi_1 \pi_2 \left(y^2 - y^1\right)}{\pi_1 y^1 + \pi_2 y^2}$$

The income of type i agent is given by $y^i = Rb^i + w(k) = Rb^i + (1-\alpha)Ak^\alpha$. Consider the two parts of Proposition 8.

(1) We have $y_1^i = R_1 b_1^i + (1-\alpha)Ak_1^\alpha$ and

$$G_1 = \frac{\pi_1 \pi_2 \left(y_1^2 - y_1^1\right)}{\pi_1 y_1^1 + \pi_2 y_1^2}$$

Observing that $R_1 = \alpha A k_1^{\alpha-1}$ and

$$\frac{b_1^i}{\gamma k_1} = \frac{R_1 - \frac{1}{\beta_i}}{1 + \frac{1}{\beta_i} - R_1}$$

with $i = 1, 2$, and using $y_1^i = R_1 b_1^i + (1-\alpha)Ak_1^\alpha$, we obtain

$$G_1 = \pi_1\pi_2 \frac{R_1 \frac{b_1^2}{\gamma k_1} - R_1 \frac{b_1^1}{\gamma k_1}}{\alpha A k_1^{\alpha-1} + \pi_1 R_1 \frac{b_1^1}{\gamma k_1} + \pi_2 R_1 \frac{b_1^2}{\gamma k_1}} = \pi_1\pi_2 \frac{\frac{1}{\beta_1} - \frac{1}{\beta_2}}{1 - R_1 + \frac{\pi_2}{\beta_1} + \frac{\pi_1}{\beta_2}}$$

that is (33).

(2) We have

$$\begin{aligned} y_2^1 &= R_2 b_2^1 + (1-\alpha)Ak_2^\alpha = (1-\alpha)Ak_2^\alpha \\ y_2^2 &= R_2 b_2^2 + (1-\alpha)Ak_2^\alpha = R_2\frac{k_2}{\pi_2} + (1-\alpha)Ak_2^\alpha \end{aligned}$$

and

$$G_2 = \frac{\pi_1\pi_2\left(y_2^2 - y_2^1\right)}{\pi_1 y_2^1 + \pi_2 y_2^2} = \pi_1 \frac{R_2 k_2}{(1-\alpha)\, A k_2^{\alpha} + R_2 k_2}$$

Since $R_2 = \alpha A k_2^{\alpha-1}$, we find

$$G_2 = \pi_1 \frac{\alpha A k_2^{\alpha-1} k_2}{(1-\alpha)\, A k_2^{\alpha} + \alpha A k_2^{\alpha-1} k_2} = \alpha \pi_1$$

■

Proof of Proposition 16

Let

$$\begin{aligned} G_1(\pi) &\equiv \frac{2\delta\pi(1-\pi)}{1-\alpha+(2\pi-1)\,\delta+\sqrt{(\delta+\alpha-1)^2+4\delta\pi\,(1-\alpha)}} \\ G_2(\pi) &\equiv \alpha\,(1-\pi) \end{aligned}$$

We have $G_2(\pi^*) = \alpha(1-\pi^*)$, and, by (37),

$$\begin{aligned} G_1(\pi^*) &= \frac{2\delta\pi^*(1-\pi^*)}{1-\alpha+(2\pi^*-1)\,\delta+\sqrt{D(\pi^*)}} = \frac{2\delta\pi^*(1-\pi^*)}{1-\alpha+(2\pi^*-1)\,\delta+1-\delta+\alpha} \\ &= \frac{\delta\pi^*(1-\pi^*)}{1-\delta+\delta\pi^*} = \frac{1}{1+\gamma}(1-\pi^*) = \alpha\,(1-\pi^*) \end{aligned}$$

since $\delta\pi^* = (1-\delta)/\gamma$. Therefore, $G^*(\pi)$ is continuous.

To analyze the shape of $G_1(\pi)$, denote $G_1(\pi) = n(\pi)/d(\pi)$, where

$$\begin{aligned} n(\pi) &\equiv 2\delta\pi\,(1-\pi) \\ d(\pi) &\equiv 1-\alpha+\delta\,(2\pi-1)+\sqrt{D(\pi)} \end{aligned}$$

Since

$$\begin{aligned} n'(\pi) &= 2\delta\,(1-2\pi) \\ d'(\pi) &= 2\delta\left[1+\frac{1-\alpha}{\sqrt{D(\pi)}}\right] \end{aligned} \tag{47}$$

we have

$$G_1'(\pi) = 2\delta\frac{(1-2\pi)\,d(\pi)-2\delta\pi\,(1-\pi)\left[1+\frac{1-\alpha}{\sqrt{D(\pi)}}\right]}{d(\pi)^2} \tag{48}$$

Now we have to distinguish between two cases.

(1) Suppose that $\delta < 1-\alpha$. Then $G_1(0) = G_1(1) = 0$. We show that there exists a unique $\hat{\pi}$ such that for $\pi < \hat{\pi}$, $G^1(\pi)$ is monotonically increasing in π, while for $\pi \geq \hat{\pi}$, $G^1(\pi)$ is monotonically decreasing in π.

Indeed, after some algebra, equation $G_1'(\pi) = 0$ can be written as $L(\pi) = R(\pi)$, where

$$\begin{aligned} L(\pi) &\equiv (1-\alpha-\delta)\frac{1-2\pi}{2\delta\pi} \\ R(\pi) &\equiv \frac{\pi\sqrt{D(\pi)}+(1-\alpha)(3\pi-1)}{\sqrt{D(\pi)}+1-\alpha-\delta} = \frac{\sqrt{D(\pi)}+(1-\alpha)\left(3-\frac{1}{\pi}\right)}{\frac{\sqrt{D(\pi)}}{\pi}+\frac{1-\alpha-\delta}{\pi}} \end{aligned}$$

Since $\delta < 1-\alpha$, we have $1-\alpha-\delta > 0$. Then, $L(\pi)$ is convex and decreasing with $L(0^+) = +\infty$, $L(1/2) = 0$ and $L(1) = -(1-\delta-\alpha)/(2\delta) < 0$.

Further, $R(\pi)$ is increasing. Indeed, $\sqrt{D(\pi)}+(1-\alpha)(3-1/\pi)$ is increasing in π, while $\sqrt{D(\pi)}/\pi + (1-\alpha-\delta)/\pi$ is decreasing in π. By (36),

$$R(0) = -\frac{1}{2}\frac{1-\alpha}{1-\alpha-\delta} < 0 \text{ and } R\left(\frac{1}{2}\right) = \frac{1}{2}\frac{\sqrt{D(1/2)}+1-\alpha}{\sqrt{D(1/2)}+1-\alpha-\delta} > 0$$

Therefore, in this case there exists a unique value $0 < \hat{\pi} < 1/2$ such that $L(\hat{\pi}) = R(\hat{\pi})$, or $G_1'(\hat{\pi}) = 0$. Moreover, for $\pi < \hat{\pi}$, $L(\pi) > R(\pi)$, so that $G_1(\pi)$ is increasing, while for $\pi > \hat{\pi}$, $L(\pi) < R(\pi)$, and $G_1(\pi)$ is decreasing.

When $\delta < \alpha$, we have $\pi^* > 1$, and hence, as in the proof of Proposition 13, we have $G^*(\pi) = G_1(\pi)$. In this case the threshold level of π, up to which G^* is increasing and after which G^* is decreasing, is $\hat{\pi}$. Similarly, when $\alpha < \delta \leq 1$, for $\pi < \pi^*$ we have $G^*(\pi) = G_1(\pi)$, while for $\pi \geq \pi^*$, we have $G^*(\pi) = G_2(\pi)$. In this case, the threshold level of π is $\tilde{\pi} \equiv \min\{\hat{\pi}, \pi^*\}$.

(2) Suppose that $\delta > 1-\alpha$. By Bernoulli's rule,

$$G_1(0) = \frac{n'(0)}{d'(0)} = \frac{\delta+\alpha-1}{\delta} > 0 = G_1(1)$$

Moreover, in this case $G_1(0) < \alpha = G_2(0)$ if and only if $\delta < 1$.

The second case have two subcases.

(2.1) Suppose that $1-\alpha < \delta < 2(1-\alpha)$. Then we show that $G_1'(0^+) > 0$ and $G_1'(1^-) < 0$, so there exists an interior $\hat{\pi} = \arg\max_{0\leq\pi\leq1} G_1(\pi)$.

Indeed, using (38) and (48), we obtain

$$G_1'(1) = -\frac{\delta}{1-\alpha+\delta} < 0$$

Applying the Bernoulli's rule to (48), we get

$$\begin{aligned} G_1'\left(0^+\right) &= 2\delta \lim_{\pi\to0^+} \frac{-2d(\pi)+(1-2\pi)d'(\pi)-2\delta(1-2\pi)\left[1+\frac{1-\alpha}{\sqrt{D(\pi)}}\right]+4\delta^2(1-\alpha)^2\frac{\pi(1-\pi)}{D(\pi)^{\frac{3}{2}}}}{2d(\pi)d'(\pi)} \\ &= \lim_{\pi\to0^+} \frac{-d(\pi)\sqrt{D(\pi)}+2\delta^2(1-\alpha)^2\frac{\pi(1-\pi)}{D(\pi)}}{d(\pi)\left[1-\alpha+\sqrt{D(\pi)}\right]} \end{aligned}$$

In this case, by (36), $\sqrt{D(0)} = \delta + \alpha - 1 > 0$. Applying again the Bernoulli's rule, we have

$$\begin{aligned} G_1'(0^+) &= \lim_{\pi \to 0^+} \frac{-d'(\pi)\sqrt{D(\pi)} - d(\pi)\frac{D'(\pi)}{2\sqrt{D(\pi)}} + 2\delta^2(1-\alpha)^2 \frac{(1-2\pi)D(\pi) - \pi(1-\pi)D'(\pi)}{D(\pi)^2}}{d'(\pi)\left[1-\alpha+\sqrt{D(\pi)}\right] + d(\pi)\frac{D'(\pi)}{2\sqrt{D(\pi)}}} \\ &= \frac{2(1-\alpha)-\delta}{\delta+\alpha-1} \end{aligned}$$

Thus, for $1-\alpha < \delta < 2(1-\alpha)$, $G_1'(0^+) > 0$.

(2.2) Suppose that $\delta \geq 2(1-\alpha)$. We have just seen that in this case $G_1'(0^+) \leq 0$. Let us show that $G_1'(\pi) < 0$ for all $\pi > 0$. By (48), this is equivalent to

$$\pi(1-\pi)d'(\pi) > (1-2\pi)d(\pi)$$

Since $d(0) = 0$, it is sufficient to check that for all $\pi > 0$,

$$[\pi(1-\pi)d'(\pi)]' > [(1-2\pi)d(\pi)]'$$

that is $2d(\pi) > -\pi(1-\pi)d''(\pi)$ or $2d(\pi) > \pi(1-\pi)|d''(\pi)|$, since $d''(\pi) < 0$. Or, equivalently, it is sufficient to show that

$$\left[\sqrt{D(\pi)} - (\delta+\alpha-1) + 2\delta\pi\right][D(\pi)]^{\frac{3}{2}} > 2\pi(1-\pi)\delta^2(1-\alpha)^2$$

It is easy to see that

$$\sqrt{D(\pi)} - (\delta+\alpha-1) > \frac{2\delta\pi(1-\alpha)}{\sqrt{D(\pi)}}$$

and, since $\delta \geq 2(1-\alpha) > 1-\alpha$,

$$\sqrt{D(\pi)} = \sqrt{(\delta+\alpha-1)^2 + 4\delta\pi(1-\alpha)} > \delta - (1-\alpha) \geq 1-\alpha$$

Therefore,

$$\begin{aligned} &\left[\sqrt{D(\pi)} - (\delta+\alpha-1) + 2\delta\pi\right] D(\pi)^{\frac{3}{2}} \\ = &\left[\sqrt{D(\pi)} - (\delta+\alpha-1)\right] D(\pi)^{\frac{3}{2}} + 2\delta\pi D(\pi)^{\frac{3}{2}} > \frac{2\delta\pi(1-\alpha)}{\sqrt{D(\pi)}} D(\pi)^{\frac{3}{2}} + 2\delta\pi D(\pi)^{\frac{3}{2}} \\ = &\ 2\delta\pi D(\pi)\left[1-\alpha+\sqrt{D(\pi)}\right] > 2\delta\pi(1-\alpha)^2[1-\alpha+\delta-(1-\alpha)] = 2\pi\delta^2(1-\alpha)^2 \\ \geq &\ 2\pi(1-\pi)\delta^2(1-\alpha)^2 \end{aligned}$$

Again, as in the proof of Proposition 13, in both cases (2.1) and (2.2), when $\delta < \alpha$, we have $G^*(\pi) = G_1(\pi)$. When $\alpha < \delta \leq 1$, for $\pi < \pi^*$ we have $G^*(\pi) = G_1(\pi)$, while for $\pi \geq \pi^*$, we have $G^*(\pi) = G_2(\pi)$. When $\delta > 1$, we have $\pi^* \leq 0$, and hence $G^*(\pi) = G_2(\pi)$. Thus, if $1-\alpha < \delta < \min\{1, 2(1-\alpha)\}$, then $G^*(\pi)$ has an inverted-U shape. If $\delta \geq \min\{1, 2(1-\alpha)\}$, then $G^*(\pi)$ is decreasing in π. ■

Proof of Proposition 17

Let

$$G_1(\delta) = \frac{2\delta\pi(1-\pi)}{1-\alpha+(2\pi-1)\delta+\sqrt{(\delta+\alpha-1)^2+4\delta\pi(1-\alpha)}}$$

We have $G_2(\delta^*) = \alpha(1-\pi)$, and

$$\begin{aligned} G_1(\delta^*) &= \frac{2\delta^*\pi(1-\pi)}{1-\alpha+(2\pi-1)\delta^*+\sqrt{D(\delta^*)}} = \frac{2\delta^*\pi(1-\pi)}{1-\alpha+(2\pi-1)\delta^*+1+\alpha-\delta^*} \\ &= \frac{\delta^*\pi(1-\pi)}{1-(1-\pi)\delta^*} = \frac{\pi(1-\pi)}{1+\gamma\pi-1+\pi} = \alpha(1-\pi) \end{aligned}$$

since $\gamma = (1-\alpha)/\alpha$. Therefore, $G^*(\delta)$ is continuous.

Let us show that $G_1'(\delta) > 0$. Indeed,

$$G_1'(\delta) = \frac{2\pi(1-\pi)}{\left[1-\alpha+\delta(2\pi-1)+\sqrt{D(\delta)}\right]^2}\left[1-\alpha+\sqrt{D(\delta)}-\frac{\delta D'(\delta)}{2\sqrt{D(\delta)}}\right]$$

and

$$\begin{aligned} 1-\alpha+\sqrt{D(\delta)}-\frac{\delta D'(\delta)}{2\sqrt{D(\delta)}} &= \frac{1-\alpha}{\sqrt{D(\delta)}}\left[\sqrt{D(\delta)}+\frac{D(\delta)-\delta(\delta+\alpha-1)}{1-\alpha}-2\delta\pi\right] \\ &= \frac{1-\alpha}{\sqrt{D(\delta)}}\left[2\delta\pi+\sqrt{D(\delta)}-(\delta+\alpha-1)\right] > 0 \end{aligned}$$

since, by (36), $\sqrt{D(\delta)} \geq \delta+\alpha-1$. ■

Proof of Proposition 18

In the steady state $j = 1, 2$, for agent $i = 1, 2$, we obtain

$$U_j^{i\prime}(\pi) = \frac{c_j^{i\prime}(\pi)}{c_j^i(\pi)} + \beta_i\frac{c_j^{i\prime}(\pi)+b_j^{i\prime}(\pi)}{c_j^i(\pi)+b_j^i(\pi)} \tag{49}$$

In the following, for simplicity, we omit the argument π. We have $U_j^{i\prime} = 0$ if and only if

$$\frac{c_j^{i\prime}}{c_j^i}\left[1+(1+\beta_i)\frac{c_j^i}{b_j^i}\right]+\beta_i\frac{b_j^{i\prime}}{b_j^i} = 0 \tag{50}$$

(1) Focus on the first steady state. For $\pi < \pi^*$, the utilities of the more and the less altruistic agents are given by U_1^2 and U_1^1 respectively. Consider the steady-state equilibrium $\left(c_1^1, b_1^1, c_1^2, b_1^2, k_1\right)$. Recall that $R_1'(\pi) < 0$ and

$$\frac{k_1'(\pi)}{k_1(\pi)} = -\frac{1}{1-\alpha}\frac{R_1'(\pi)}{R_1(\pi)}$$

Furthermore, by part (1) of Proposition 8,

$$\frac{c_1^i}{b_1^i} = \frac{1}{\beta_i R_1 - 1} \tag{51}$$

and hence

$$\frac{b_1^{i\prime}}{b_1^i} = \frac{c_1^{i\prime}}{c_1^i} + \frac{\beta_i R_1'}{\beta_i R_1 - 1} \tag{52}$$

On the other hand,

$$\frac{c_1^{i\prime}}{c_1^i} = \frac{k_1'}{k_1} + \frac{\beta_i R_1'}{\beta_i + 1 - \beta_i R_1} = \frac{R_1'}{R_1}\left(\frac{\beta_i R_1}{\beta_i + 1 - \beta_i R_1} - \frac{1}{1-\alpha}\right) \tag{53}$$

Using (49), (50), (51), (52) and (53), after some algebra, we have

$$U_1^{i\prime} = \frac{R_1'}{R_1}\left[\frac{\beta_i\left(1+\beta_i\right)R_1}{1+\beta_i-\beta_i R_1} - \frac{1+\alpha\beta_i}{1-\alpha}\right]$$

Since $R_1'(\pi) < 0$, we obtain that $U_1^{i\prime}(\pi) < 0$ if and only if

$$\frac{\beta_i\left(1+\beta_i\right)R_1}{1+\beta_i-\beta_i R_1} - \frac{1+\alpha\beta_i}{1-\alpha} > 0$$

or, equivalently,

$$R_1(\pi) > \frac{1+\beta_i}{\beta_i}\frac{1+\alpha\beta_i}{2-\alpha+\beta_i} \tag{54}$$

Note that the right-hand side of this inequality depends on β_i and hence is different for different types of agents.

(1.1) Consider the utility of the less altruistic agents. We have

$$U_1^{1\prime}(\pi) < 0 \Leftrightarrow R_1(\pi) > \frac{1+\beta_1}{\beta_1}\frac{1+\alpha\beta_1}{2-\alpha+\beta_1}$$

Define a critical value of the level of altruism $\hat{\pi}$ as a solution to the equation

$$R_1(\pi) = \frac{1+\beta_1}{\beta_1}\frac{1+\alpha\beta_1}{2-\alpha+\beta_1}$$

Since $R_1(\pi)$ is strictly decreasing, this solution is unique.

Define also a critical value for the capital share in total income:

$$\alpha^* \equiv \frac{1}{1+\beta_1+\beta_1^2}$$

There are two cases.

(1.1.1) $\pi^* < \hat{\pi}$. This case holds when

$$R_1(\pi^*) > R_1(\hat{\pi}) \Leftrightarrow \frac{1}{\beta_1} > \frac{1+\beta_1}{\beta_1}\frac{1+\alpha\beta_1}{2-\alpha+\beta_1}$$

or, equivalently, $\alpha < \alpha^*$. Since $R_1(\pi)$ is strictly decreasing, for all $\pi < \pi^*$ we have

$$R_1(\pi) > R_1(\pi^*) > R_1(\hat{\pi}) = \frac{1+\beta_1}{\beta_1}\frac{1+\alpha\beta_1}{2-\alpha+\beta_1}$$

and hence $U_1^{1\prime}(\pi) < 0$ for all $\pi < \pi^*$.

(1.1.2) $\hat{\pi} < \pi^*$. This case holds if and only if

$$R_1(\pi^*) = \frac{1}{\beta_1} < R_1(\hat{\pi}) = \frac{1+\beta_1}{\beta_1}\frac{1+\alpha\beta_1}{2-\alpha+\beta_1}$$

or, equivalently, $\alpha > \alpha^*$.

In this case, for all $\pi < \hat{\pi}$, we have

$$R_1(\pi) > R_1(\hat{\pi}) = \frac{1+\beta_1}{\beta_1}\frac{1+\alpha\beta_1}{2-\alpha+\beta_1}$$

and, according to (54), $U_1^{1\prime}(\pi) < 0$ for all $\pi < \hat{\pi}$.

Conversely, for all $\pi > \hat{\pi}$ we have

$$R_1(\pi) < R_1(\hat{\pi}) = \frac{1+\beta_1}{\beta_1}\frac{1+\alpha\beta_1}{2-\alpha+\beta_1}$$

and hence $U_1^{1\prime}(\pi) > 0$ for all $\hat{\pi} < \pi < \pi^*$.

(1.2) Consider now the more altruistic agents. We have

$$R_1(\pi^*) \ \geq \ R_1(1) = \alpha + \frac{1}{\beta_2} = \frac{1+\alpha\beta_2}{\beta_2} > \frac{1+\alpha\beta_2}{\beta_2}\frac{1+\beta_2}{1+\beta_2+1-\alpha} = \frac{1+\beta_2}{\beta_2}\frac{1+\alpha\beta_2}{2+\beta_2-\alpha}$$

Since $R_1(\pi)$ is strictly decreasing in π, it follows that $U_1^{2\prime}(\pi) < 0$ for all $\pi \leq \pi^*$.

(2) Focus now on the second steady state. For $\pi \geq \pi^*$, the utilities of the less and the more altruistic agents are given by U_2^1 and U_2^2 respectively. Consider the steady-state equilibrium $\left(c_2^1, b_2^1, c_2^2, b_2^2, k_2\right)$.

Recall that

$$\begin{aligned} R_2'(\pi) &= -\frac{\gamma}{(1+\gamma\pi)^2} \\ \frac{k_2'(\pi)}{k_2(\pi)} &= \frac{1}{(1+\gamma\pi)^2}\frac{1}{\alpha R_2(\pi)} > 0 \end{aligned}$$

(2.1) For the less altruistic agents,

$$U_2^1 = \ln c_t^1 + \beta_1 \ln\left(R_2 b_2^1 + w_2\right) = \ln c_2^1 + \beta_1 \ln w_2 = (1+\beta_1)\ln w_2 = (1+\beta_1)\ln\left[(1-\alpha)Ak_2^{\alpha}\right]$$

since $c_2^1 = \gamma R_2 k_2 = (1-\alpha)Ak_2^{\alpha} = w_2$. Since k_2 is strictly increasing in π, $U_2^1(\pi)$ is strictly increasing as well, that is $U_2^{1\prime}(\pi) > 0$ for all $\pi \geq \pi^*$.

(2.2) For the more altruistic agents, we have

$$\frac{b_2^{2\prime}}{b_2^2} = \frac{k_2'}{k_2} - \frac{1}{\pi} = -\frac{1}{\pi R_2}\left[R_2 - \frac{\pi}{\alpha(1+\gamma\pi)^2}\right] \tag{55}$$

Note that b_2^2 is strictly decreasing in π. Indeed, $b_2^{2\prime} < 0$ because

$$R_2 - \frac{\pi}{\alpha (1+\gamma\pi)^2} = \frac{1}{\beta_2} + \frac{1}{1+\gamma\pi} - \frac{\pi (1+\gamma)}{(1+\gamma\pi)^2} = \frac{1}{\beta_2} + \frac{1-\pi}{(1+\gamma\pi)^2} > 0$$

Moreover,

$$\frac{c_2^2}{b_2^2} = \frac{1+\gamma\pi}{\beta_2} \tag{56}$$

and hence

$$\frac{c_2^{2\prime}}{c_2^2} = \frac{b_2^{2\prime}}{b_2^2} + \frac{\gamma}{1+\gamma\pi} = -\frac{1}{\pi (1+\gamma\pi) R_2} \left[R_2 - \frac{\pi}{\alpha (1+\gamma\pi)} \right] \tag{57}$$

Therefore, using (49), (55), (56) and (57), and taking into account (19), we obtain

$$\begin{aligned} U_2^{2\prime} &= \frac{c_2^{2\prime}}{c_2^2} + \beta_2 \frac{c_2^{2\prime} + b_2^{2\prime}}{c_2^2 + b_2^2} = \frac{b_2^2}{c_2^2 + b_2^2} \left(\frac{c_2^{2\prime}}{c_2^2} \left[1 + (1+\beta_2) \frac{c_2^2}{b_2^2} \right] + \beta_2 \frac{b_2^{2\prime}}{b_2^2} \right) \\ &= -\frac{1}{\pi} \frac{b_2^2}{c_2^2 + b_2^2} \left[\frac{1}{\beta_2} + \frac{1 + (1-\pi)(1+\beta_2)}{1+\pi\gamma} \right] \end{aligned}$$

Hence, $U_2^{2\prime}(\pi) < 0$ for any $\pi \geq \pi^*$.

Summing up the results for both steady states, we conclude the following.

Less altruistic agents.

(1) Case $\pi < \pi^*$, where the utility is given by $U^1 = U_1^1$. If $\alpha < \alpha^*$, then $U^{1\prime}(\pi) < 0$. If $\alpha > \alpha^*$, then $U^{1\prime}(\pi) < 0$ for $0 < \pi < \hat{\pi}$ and $U^{1\prime}(\pi) > 0$ for $\hat{\pi} < \pi < \pi^*$ with $U^1(\hat{\pi}) = \min U^1(\pi)$.

(2) Case $\pi \geq \pi^*$, where the utility is given by $U^1 = U_2^1$. Here $U_2^{1\prime}(\pi) > 0$.

Therefore, there is a threshold $\breve{\pi}$ defined as follows:

$$\begin{aligned} \breve{\pi} &\equiv \pi^* \text{ if } \alpha \leq \alpha^* \\ \breve{\pi} &\equiv \hat{\pi} < \pi^* \text{ if } \alpha > \alpha^* \end{aligned}$$

such that U^1 is continuous at $\pi = \breve{\pi}$, strictly decreasing for $\pi < \breve{\pi}$, strictly increasing for $\pi > \breve{\pi}$.

More altruistic agents.

(1) Case $\pi < \pi^*$, where the utility is given by $U^2 = U_1^2$. Here, $U_1^{2\prime}(\pi) < 0$.

(2) Case $\pi \geq \pi^*$, where the utility is given by $U^2 = U_2^2$. Here, $U_2^{2\prime}(\pi) < 0$. Therefore, U^2 is strictly decreasing for all π. ■

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