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# Exponential Time Trends in a Fractional Integration Model

## Abstract

This paper introduces a new modelling approach that incorporates nonlinear, exponential deterministic terms into a fractional integration model. The proposed model is based on a specific version of Robinson's (1994) tests and is more general that standard time series models, which only allow for linear trends. Montecarlo simulations show that it performs well in finite sample. Three empirical examples confirm that the suggested specification captures the properties of the data adequately.

JEL-Codes: C220, C150.

Keywords: exponential time trends, fractional integration, Montecarlo simulations.

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## 1. Introduction

It is common practice in applied work to allow for simple linear deterministic trends when modelling standard economic and financial series (Bhargava, 1986; Stock and Watson, 1988; Schmidt and Phillips, 1992). However, some of them appear to be characterised by exponential growth as in the case of compound interest. An exponential growth trend can be captured by taking logs of the series of interest and regressing it against a constant and a linear trend. However, fitting a linear trend with a constant growth rate is in most cases too restrictive. Alternatively, the raw data can be used to run a regression including an exponential time trend as well as a constant. The present paper takes the latter approach based on exponential trends and develops an appropriate modelling and testing framework in the context of fractional integration. It uses simulation techniques to evaluate the properties of the proposed test and also carries out three empirical applications to show that the advocated framework captures well the behaviour of the data. Modelling exponential trends in a fractional integration setup is a novel contribution and the suggested approach is a useful tool to use in the case of economic and financial series, possibly exhibiting long memory as well as exponential deterministic trends.

The structure of the paper is as follows. Section 2 presents the proposed framework and testing procedure. Section 3 reports some Montecarlo simulation results to assess the finite sample behaviour of the suggested test. Section 4 discusses three empirical applications. Section 5 offers some concluding remarks.

## 2. The Model

We consider a time series $\{y_t, t = 1, 2, \ldots \}$ for which the following regression model is specified:

$$y_t = \alpha + \beta t^{\gamma} + x_t, \qquad t = 1, 2, \ldots, \tag{1}$$

where α, β and γ are unknown parameters (the intercept, the time trend coefficient and its exponent respectively); in addition, $x_t$ is assumed to be an integrated process of order d, i.e.,

$$(1 - B)^d x_t = u_t, \qquad t = 1, 2, ..., \tag{2}$$

where d can be any real value, B is the backshift operator, i.e., $B^k x_t = x_{t-k}$, and thus $u_t$ is an I(0) process, more precisely a covariance-stationary one with a spectral density function that is positive and bounded at the zero frequency.

We test the null hypothesis:

$$H_o : d = d_o, \tag{3}$$

for any real value $d_0$ in the model given by (1) and (2) by choosing specific values for γ, for example between 0 and 2, with 0.01 increments. Under the null hypothesis (3), the model given by (1) and (2) becomes:

$$\tilde{y}_t = \alpha \tilde{1}_t + \beta(\tilde{t}_t) + u_t, t = 1,2, ..., \tag{4}$$

where

$$\tilde{y}_t = (1 - B)^{d_o} y_t,$$

and

$$\tilde{1}_t = (1 - B)^{d_o} 1, \quad \text{and} \quad \tilde{t}_t = (1 - B)^{d_o} (t)^{\gamma}.$$

Since the value of γ is set, one can follow the same strategy as in Robinson (1994) and therefore the test statistic is given by:

$$\hat{R} = \frac{T}{\hat{\sigma}^4} \hat{a}' \hat{A}^{-1} \hat{a}, \tag{5}$$

where T is the sample size, and

$$\hat{a} = \frac{-2\pi}{T} \sum_j^* \psi(\lambda_j) g_u(\lambda_j; \hat{\tau})^{-1} I(\lambda_j); \qquad \hat{\sigma}^2 = \sigma^2(\hat{\tau}) = \frac{2\pi}{T} \sum_{j=1}^{T-1} g_u(\lambda_j; \hat{\tau})^{-1} I(\lambda_j),$$

$$\hat{A} = \frac{2}{T}\left(\sum_j^{*}\psi(\lambda_j)\psi(\lambda_j)' - \sum_j^{*}\psi(\lambda_j)\hat{\varepsilon}(\lambda_j)'\left(\sum_j^{*}\hat{\varepsilon}(\lambda_j)\hat{\varepsilon}(\lambda_j)'\right)^{-1}\sum_j^{*}\hat{\varepsilon}(\lambda_j)\psi(\lambda_j)'\right);$$

$$\psi(\lambda_j) = \log\left|2\sin\frac{\lambda_j}{2}\right|; \qquad \hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial\tau}\log g_u(\lambda_j;\hat{\tau}),$$

where $\lambda_j = 2\pi j/T$, and the summation in * in the above equations is over all frequencies which are bounded in the spectrum.[1] $I(\lambda_j)$ is the periodogram of $\hat{u}_t$, where

$$\hat{u}_t = \tilde{y}_t - \hat{\alpha}\tilde{1}_t - \hat{\beta}(\tilde{t}_t),\, t = 1,2,\ldots,$$

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} = \left(\sum_{t=1}^{T}\tilde{z}_t^T\tilde{z}_t\right)^{-1}\sum_{t=1}^{T}\tilde{z}_t\tilde{y}_t, \quad \text{and} \quad \hat{\tau} = \arg\min_{\tau\in T^*}\sigma^2(\tau), \text{with } T^* \text{ as a}$$

suitable subset of the $R^q$ Euclidean space. Finally, $g_u$ is a known function coming from the spectral density of $u_t$:

$$f_u(\lambda) = \frac{\sigma^2}{2\pi}g_u(\lambda;\tau), \qquad -\pi < \lambda \leq \pi.$$

Note that this test is parametric and, therefore, it requires specific modelling assumptions about the short-memory specification of $u_t$. In particular, if $u_t$ is a white noise, $g_u \equiv 1$, whilst if it is an AR process of the form $\phi(L)u_t = \varepsilon_t$, then $g_u = |\phi(e^{i\lambda})|^{-2}$, with $\sigma^2 = V(\varepsilon_t)$ and the AR coefficients being a function of $\tau$.

In this context Robinson (1994) showed that for $\gamma = 1$:

$$\hat{R} \rightarrow_d \chi_1^2, as T \rightarrow \infty, \qquad (6)$$

where "$\rightarrow_d$" stands for convergence in distribution. Therefore, unlike in the case of other (unit root / fractional) procedures, this is a classical large-sample testing situation. On the

[1] For this particular version of the tests of Robinson (1994), the spectrum has a singularity at the zero frequency, therefore j runs from 1 to T-1.

basis of (5), the null $H_o$ (3) will be rejected against the alternative $H_a$: $d \neq d_o$ if $\hat{R} > \chi^2_{1,\alpha}$,

with Prob ($\chi^2_1 > \chi^2_{1,\alpha}$) = α. It is easy to see that this result holds for any value of γ in the interval (0, 1). Specifically, Robinson (1994) used the following regression model

$$y_t \; = \; \beta \, z_t \; + \; x_t, \qquad t = 1,2,\ldots,$$

where $z_t$ is a (kx1) observable vector whose elements are assumed to be non-stochastic, such as polynomials in t, for example, to include the null hypothesis of a unit root with drift if $d_o = 1$ and $z_t = (1,t)^T$. According to him: "The limiting null and local distributions of our test statistic are unaffected by the presence of such regressors. For simplicity, we treat only linear regression, but undoubtedly a nonlinear regression will also leave our limit distributions unchanged, under standard regularity conditions". These regularity conditions are described in his definition of the class G provided in the Appendix to the paper: " G is the class of k X 1 vector sequences { $z_t$, t = 0, +1, . . } such that $z_t = 0$, t < 0 and D defined as

$$D \;=\; \sum_{t=1}^{T} \widetilde{w}_t \; \widetilde{w}_t^T \;, \quad \text{and} \quad \widetilde{w}_t^T \;=\; (\tilde{1}_t, \; \tilde{t}_t)$$

is positive definite for sufficiently large T.". G imposes no rate of increase on D; different elements can in- crease at different rates, and indeed D need not tend to infinity as T → ∞. If D is positive definite for T = $T_o$, then it is positive definite for all T > $T_o$. In the empirical applications carried out in Section 4 we set values of γ = 0, 0.10, 0.20, … (0.10), …, 1.40 and 1.50, and in each case we estimate the differencing parameter by choosing the test statistic (based on Robinson, 1994) with the lowest value. The estimate of d is virtually identical to the Whittle one based on the frequency domain used in Robinson (1994). Then, for each value of γ and the associated d we compute the residual sum of the squares and choose the pair producing the lowest statistic, $\hat{R}$, in (6).

Figures 1 to 4 display some realisations of the model given by equations (1) and (2). More specifically, we first generate a white noise process with sample size T = 1000, and produce time series for $\widetilde{I}_t$ and $\widetilde{t}_t$ by setting $d_0$ and $\gamma$ equal to 0.25, 0.50, 0.75 and 1 respectively. Then $\widetilde{y}_t$ is obtained from equation (4) with $\alpha = 0.2$ and $\beta = 0.4$ and first differences of $d_0$ are taken after removing the first 100 observations.

**FIGURES 1 – 4 ABOUT HERE**

Figures 1 to 4 correspond to $d_o = 0.25$, 0.50, 0.75 and 1 respectively, and each of them includes plots of the series (i.e., $y_t$ in (1) and (2)) for $\gamma = 0.25$, 0.50, 0.75, 1, 1.25 and 1.50. It can be seen that when $\gamma = 0.25$ the trend is almost unnoticeable; however, as $\gamma$ increases the series exhibits a clear trend characterised by convexity, whilst $\gamma = 1$ corresponds to a linear trend, and $\gamma > 1$ to one exhibiting concavity.

## 3. Simulation Results

In this section we examine the finite sample behaviour of the test statistic proposed above by means of Montecarlo simulation techniques (the Fortran codes are available from the authors upon request). As Data Generating Processes, we use the GASDEV and RAN3 routines from Press et al. (1986) to obtain Gaussian series for different sample sizes T = 100, 500 and 1000 and carry out 10,000 replications in each case; the reported results are for a nominal size of 5%.

Table 1 displays the rejection frequencies of the test statistic $\hat{R}$ in (5) for three different samples sizes, T = 100, 500 and 1000 and a nominal size of 5%. It can be seen that the nominal sizes are too large in all cases, and they approach 0.05 as the sample size increases. There is also a bias in the size as higher values are obtained in all cases against alternatives of form $d < d_o$. Finally, the frequencies against departures from the null increase as the sample size increases, which is consistent with the asymptotic behaviour of the test.

**TABLES 1 AND 2 ABOUT HERE**

Table 2 is similar to Table 1 but reports the results based on the $t_3$-Student distribution for the error term. Once again the sizes are higher than the 5% level and higher values are observed against departures of the form $d < d_o$. The rejection frequencies are also higher for this type of departures, and even for small ones the rejection frequencies are relatively high.

## 4. Three Empirical Applications

For illustration purposes, we use the proposed framework to model three US time series. The first is the US real GNP per capita series analysed in Omay et al. (2016); it is quarterly and spans the period from 1947 Q1 to 2018 Q1, for a total of 285 observations (see Figure 5); the source is the FRED database of the Federal Reserve Bank of St Louis (https://www.stlouisfed.org/). The second one is the S&P500 weekly series from January 1, 1970 up to October 23, 2023, obtained from Yahoo Finance (see Figure 6). The third one is the US Consumer Price Index for All Urban Consumers, monthly, from January 1913 until October 2023 (see Figure 7). The issue of interest is whether the effects of exogenous shocks are transitory of permanent, and thus whether the series can be characterised as trend stationary or difference stationary (Omay et al., 2016).

Table 3 reports the results for US real GNP, more precisely the estimates of α, β, γ and d in the model given by equations (1) and (2) under the assumption that u(t) is a white noise process with zero mean and constant variance. It can be seen that when choosing values of γ from 0 to 1.50 with 0.10 increments, the estimates of d are very similar and range from 1.28 to 1.30. The estimated model exhibits an exponential trend with $\gamma = 0.80$, $d = 1.28$ and the 95% confidence interval being given by (1.17, 1.42), with the remaining two parameters, $\alpha$ and $\beta$, both being statistically significant. Thus, the unit root null hypothesis is rejected in favour of $d > 1$ and $\gamma < 1$, which indicates the presence of a concave time trend in the data.

Table 4 has the same layout as the previous one but concerns the S&P500 stock market index. The estimates of d now range between 0.91 and 1.24 and the lowest statistic is obtained with $\gamma = 1.00$ and $d = 0.97$ (0.92, 1.24). Thus, a linear time trend with a unit root seems to be a plausible hypothesis; this is consistent, for $t > 2$, with a random walk model with an intercept, and thus with the Efficiency Market Hypothesis (EMH) in its weak form (Fama, 1970).

Finally, Table 5 reports the corresponding results for the US Consumer Price Index. In this case d is much higher than 1 (specifically, 1.44), with a confidence interval given by (1.38, 1.52), and thus the unit root null hypothesis is rejected in favour of $d > 1$; also, the estimate of $\gamma = 1.10$ implies a convex time trend.

## 5. Conclusions

This paper puts forward a modelling and testing framework that allows for exponential deterministic trends in a fractional integration context. The Montecarlo simulations carried out to examine the properties of the proposed test indicate that it performs well in finite samples. As an illustration, the proposed framework is then applied to model the behaviour of US real GDP, the S&P500 stock market index, and US Consumer Prices. The empirical exercise shows that the suggested model captures well the behaviour of the series under examination and is data congruent.

The proposed modelling approach is widely applicable to time series exhibiting an exponential trend. However, it should be noted that, although unlimited exponential growth might characterise some economic and financial series, this is not likely to occur whenever real resources are involved. In this case there will necessarily be an upper bound which should also be introduced into the model, for instance through a logistic curve. This issue is left for future research.

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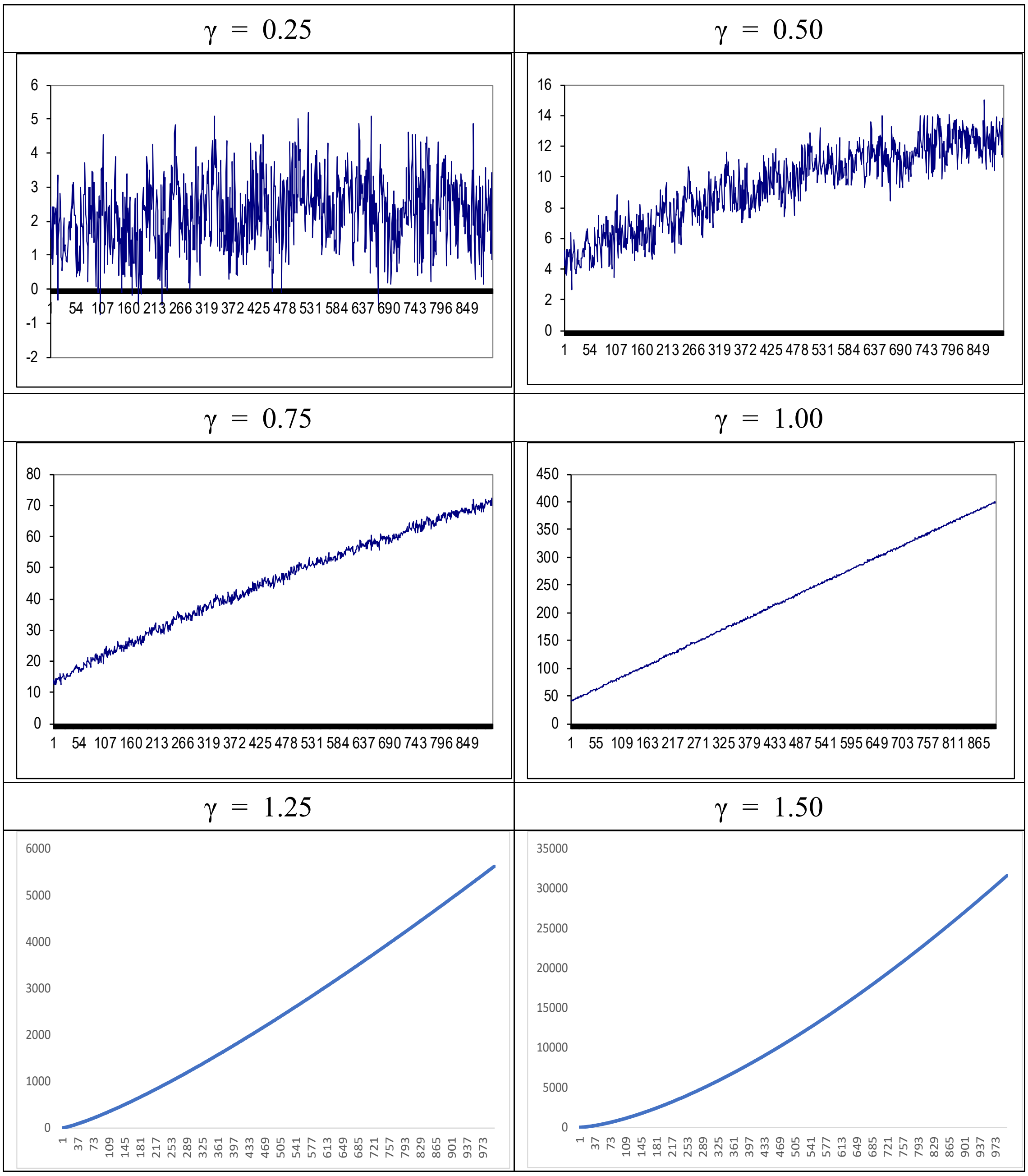
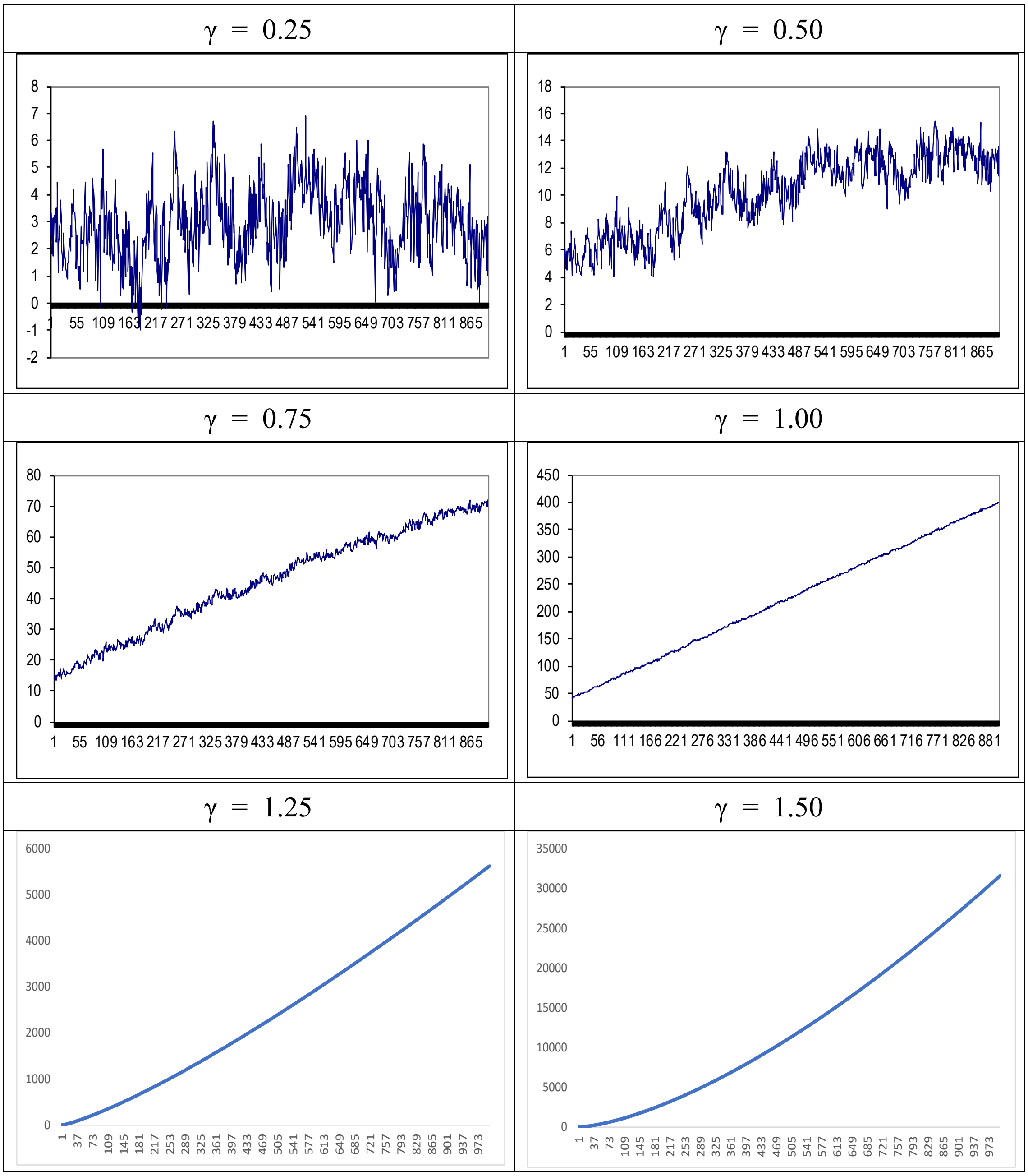
**Figure 1: Realisations from Equations (1) and (2) with d = 0.25**



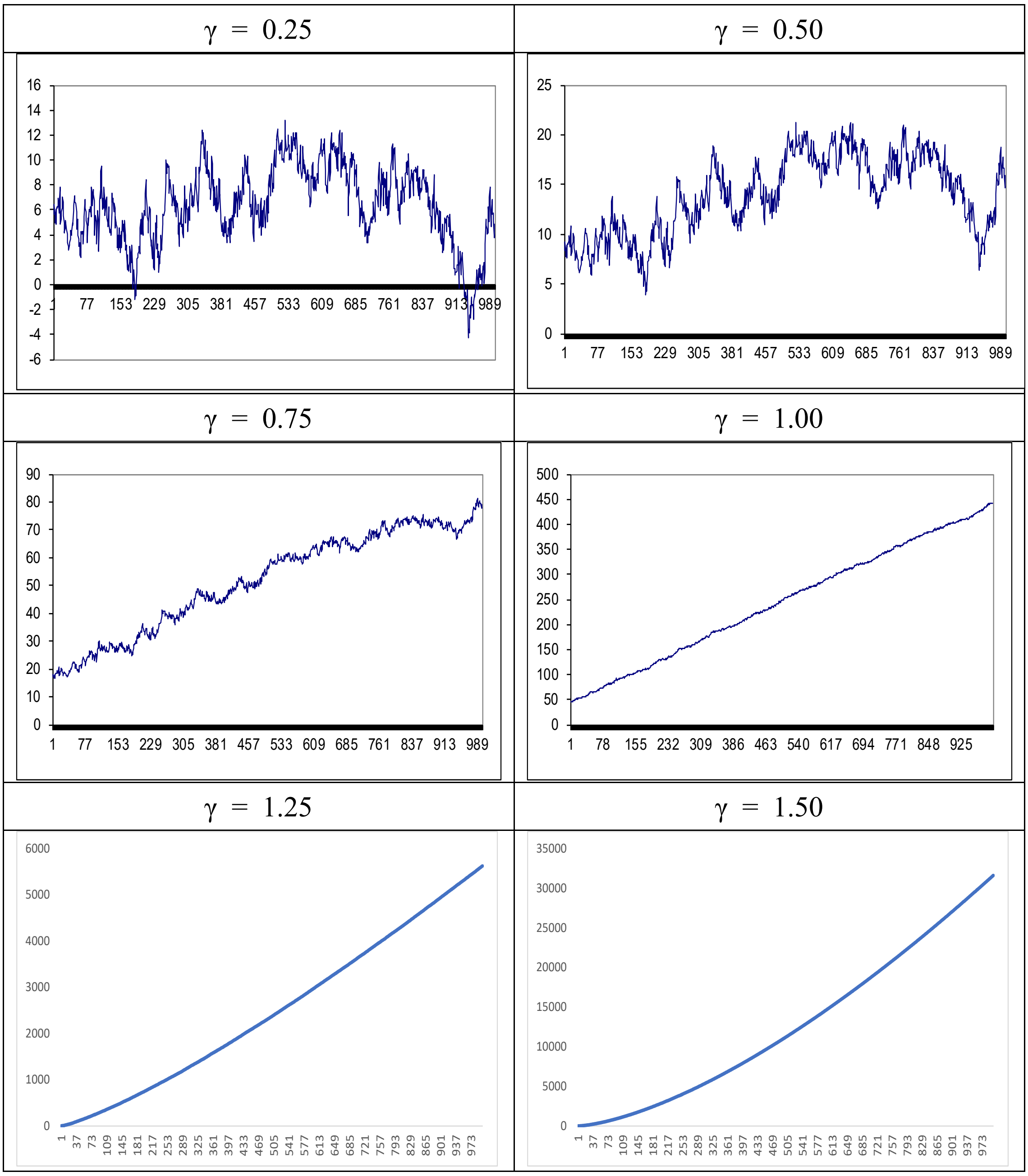

**Note:** We generate Gaussian series with T = 1000, and then produce the realisations of $y_t$ in (1) and (2) with d = 0.25.

**Figure 2: Realisations from Equations (1) and (2) with d = 0.50**



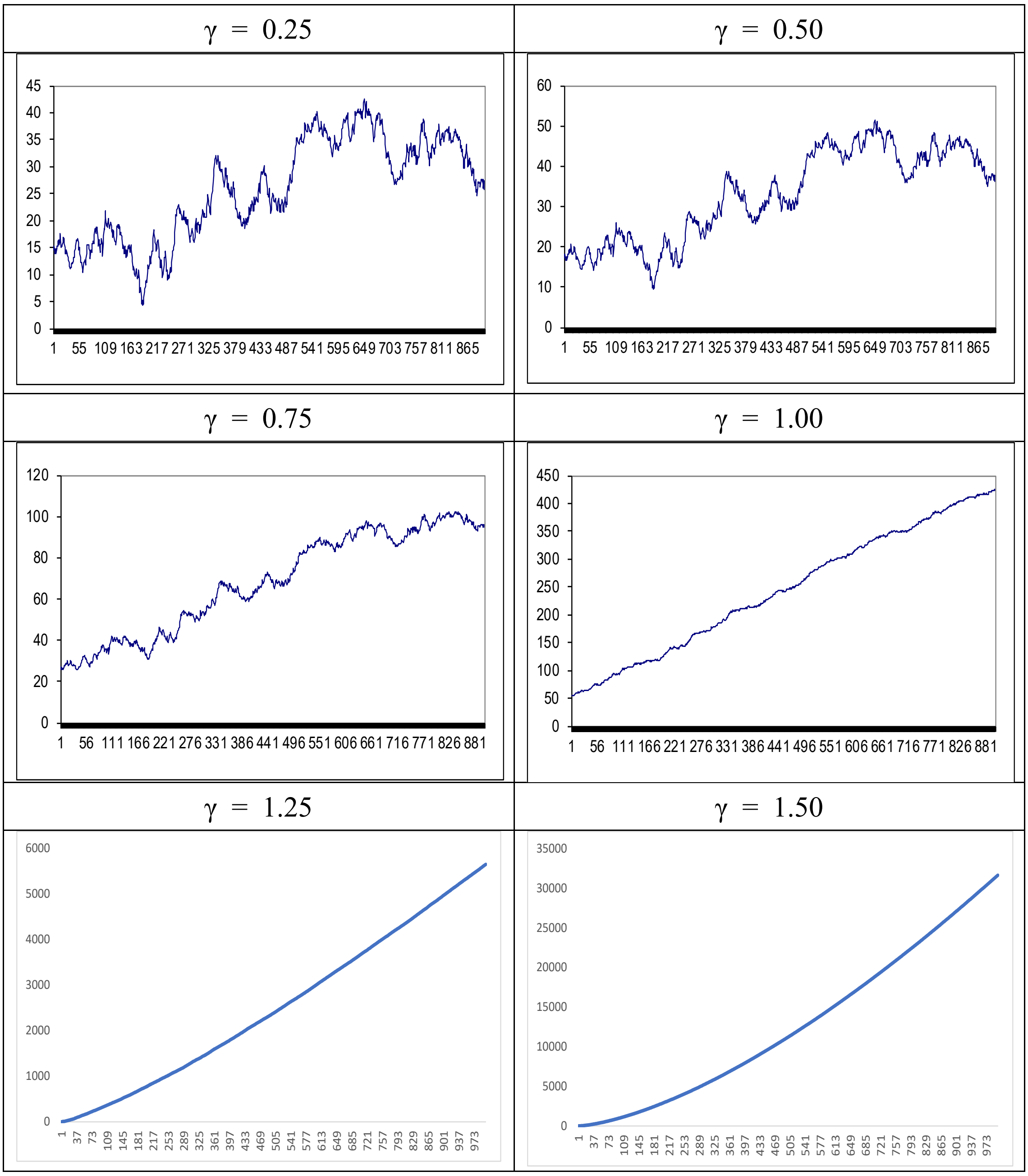

**Note:** We generate Gaussian series with T = 1000, and then produce the realisations of $y_t$ in (1) and (2) with d = 0.50.

**Figure 3: Realisations from Equations (1) and (2) with d = 0.75**



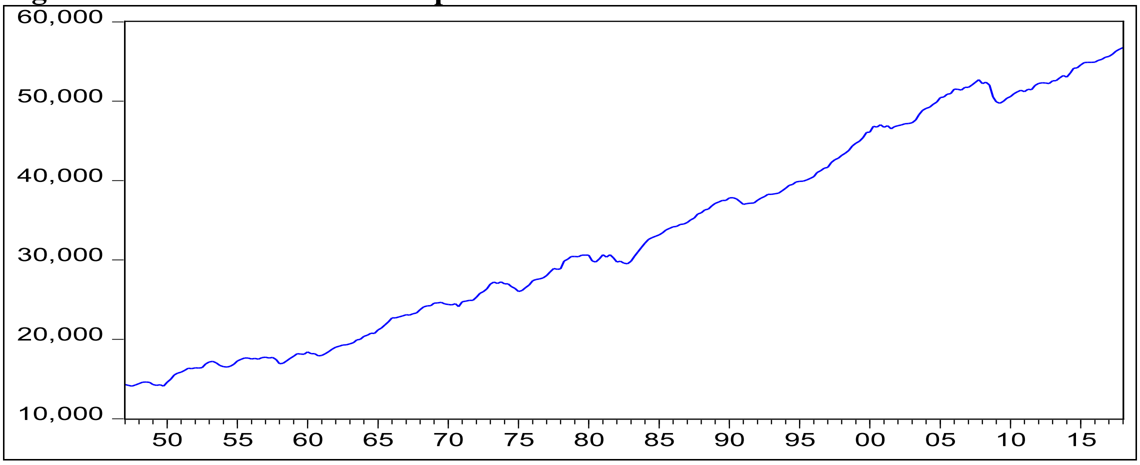

**Note:** We generate Gaussian series with T = 1000, and then produce the realisations of $y_t$ in (1) and (2) with d = 0.75.

**Figure 4: Realisations from Equations (1) and (2) with d = 1.00**



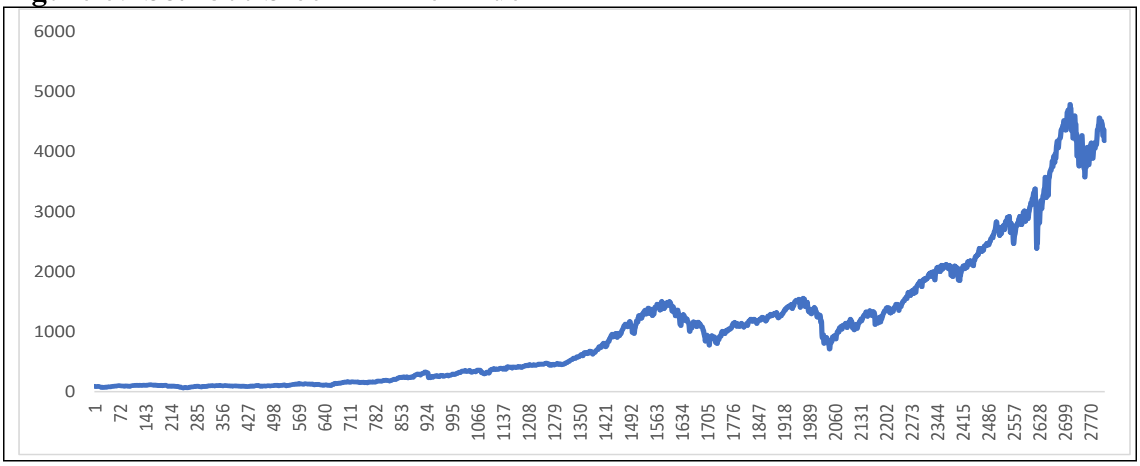

**Note:** We generate Gaussian series with T = 1000, and then produce the realisations of $y_t$ in (1) and (2) with d = 1.00.

**Figure 5: US Real GNP Per Capita**



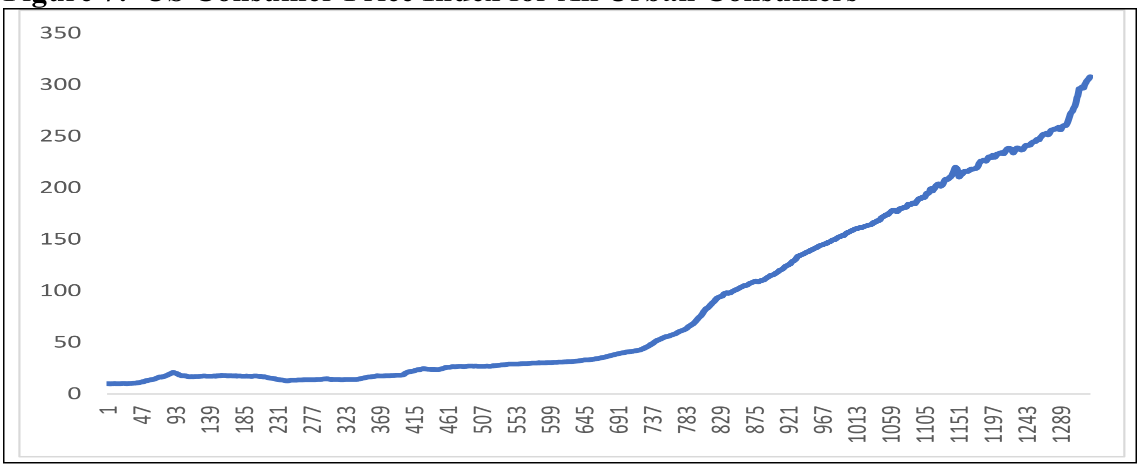

**Note:** the data source is the FRED database of the Federal Reserve Bank of St Louis (https://www.stlouisfed.org/); the series is quarterly and the sample period goes from 1947 Q1 to 2018 Q1.

**Figure 6: S&P500 Stock Market Index**




**Note:** the data source is the Yahoo finance (https://es.finance.yahoo.com/); the series is weekly and the sample period goes from January 1, 1970 to October 23, 2023.

**Figure 7: US Consumer Price Index for All Urban Consumers**




**Note:** the data source is the U.S. Department of Labor Bureau of Labor Statistic (https://www.bls.gov); the series is monthly and the sample period goes from 1913m1 to 2023m10.

**Table 1: Rejection frequencies against one-sided alternatives with Gaussian errors**

| | $d_o$ | T = 100 | T = 500 | T = 1000 |
|---|---|---|---|---|
| $H_o$: $d > d_o$ | 0.20 | 0.688 | 0.983 | 1.000 |
| | 0.30 | 0.478 | 0.688 | 0.808 |
| | 0.40 | 0.296 | 0.354 | 0.499 |
| | 0.50 | **0.099** | **0.069** | **0.057** |
| $H_o$: $d < d_o$ | 0.50 | **0.104** | **0.099** | **0.068** |
| | 0.60 | 0.319 | 0.449 | 0.676 |
| | 0.70 | 0.665 | 0.883 | 0.997 |
| | 0.80 | 0.997 | 1.000 | 1.000 |

**Note:** The values reported in this table are the rejection frequencies of the test against fractional alternatives. In bold the size of the test.

**Table 2: Rejection frequencies against one-sided alternatives with $t_3$-distributed errors**

| | $d_o$ | T = 100 | T = 500 | T = 1000 |
|---|---|---|---|---|
| $H_o$: $d > d_o$ | 0.20 | 0.709 | 0.878 | 0.998 |
| | 0.30 | 0.526 | 0.735 | 0.910 |
| | 0.40 | 0.314 | 0.359 | 0.651 |
| | 0.50 | **0.112** | **0.089** | **0.067** |
| $H_o$: $d < d_o$ | 0.50 | **0.127** | **0.109** | **0.083** |
| | 0.60 | 0.414 | 0.565 | 0.712 |
| | 0.70 | 0.727 | 0.899 | 1.000 |
| | 0.80 | 1.000 | 1.000 | 1.000 |

**Note:** The values reported in the table are the rejection frequencies of the test against fractional alternatives. In bold the size of the test.

**Table 3: Estimated coefficients for the log of US real GNP per capita**

| γ | d | 95% band | α (t-value) | β (t-value) | Stastistic |
|---|---|---|---|---|---|
| 0 | 1.29 | (1.18, 1.42) | 9.568 (1120.32) | --- | 0.02218 |
| 0.10 | 1.29 | (1.18, 1.43) | 9.625 (80.19) | -0.00584 (-0.47) | 0.04994 |
| 0.20 | 1.29 | (1.18, 1.43) | 9.585 (172.36) | -0.01811 (-0.31) | 0.04412 |
| 0.30 | 1.29 | (1.18, 1.43) | 9.572 (299.66) | -0.00473 (-0.13) | 0.03298 |
| 0.40 | 1.29 | (1.18, 1.43) | 9.565 (404.95) | 0.00347 (0.14) | 0.00959 |
| 0.50 | 1.29 | (1.18, 1.43) | 9.561 (550.26) | 0.00823 (0.46) | 0.02218 |
| 0.60 | 1.29 | (1.17, 1.42) | 9.559 (712.56) | 0.01068 (0.83) | -0.05273 |
| 0.70 | 1.28 | (1.17, 1.42) | 9.559 (882.03) | 0.01165 (1.33) | 0.05811 |
| **0.80** | **1.28** | **(1.17, 1.42)** | **9.561 (1008.38)** | **0.00981 (1.68)** | **0.00337** |
| 0.90 | 1.28 | (1.17, 1.42) | 9.583 (1079.09) | 0.00709 (1.92) | 0.01557 |
| 1.00 | 1.28 | (1.17, 1.42) | 9.565 (1107.19) | 0.00453 (2.04) | 0.01978 |
| 1.10 | 1.28 | (1.17, 1.42) | 9.567 (1115.28) | 0.02653 (2.05) | 0.05558 |
| 1.20 | 1.29 | (1.18, 1.42) | 9.567 (1119.87) | 0.00147 (1.91) | -0.03381 |
| 1.30 | 1.29 | (1.19, 1.42) | 9.568 (1119.53) | 0.00079 (1.83) | 0.02787 |
| 1.40 | 1.30 | (1.19, 1.42) | 9.568 (1122.12) | 0.00041 (1.63) | -0.06320 |
| 1.50 | 1.30 | (1.20, 1.43) | 9.568 (1121.49) | 0.02166 (1.54) | -0.01687 |

**Note:** The first column reports the values of the exponent for the trend. The second and third columns refers respectively to the estimated differecing parameter and the associated 95% confidence intervals. The following columns display the intercept and the slope of the exponential trend along with their associated t-values. The final column reports the test statistics.

**Table 4: Estimated coefficients for the log of US real GNP per capita**

| γ | d | 95% band | α (t-value) | β (t-value) | Stastistic |
|---|---|---|---|---|---|
| 0 | 0.96 | (0.93, 1.02) | 41.119 (0.171) | --- | 0.227 |
| 0.10 | 0.96 | (0.93, 1.02) | 42.633 (0.143) | 49.836 (0,16) | 0.219 |
| 0.20 | 0.97 | (0.92, 1.23) | 52.366 (0.399 | 39.924 (0.31) | 0.203 |
| 0.30 | 0.97 | (0.92, 1.23) | 54.121 (0.70) | 37.914 (0.56) | 0.239 |
| 0.40 | 0.95 | (0.91, 1.22) | 57.445 (1.09) | 34.327 80.90) | 0.202 |
| 0.50 | 0.95 | (0.91, 1.22) | 59.009 (1.18) | 34.327 (0.90) | 0.200 |
| 0.60 | 0.96 | (0.92, 1.22) | 73.080 (1.99) | 18.523 (1.71) | 0.188 |
| 0.70 | 0.97 | (0.92, 1.23) | 80.733 (2.28) | 10.997 (2.08) | 0.161 |
| 0.80 | 0.97 | (0.92, 1.23) | 85.950 (2.46) | 5.941 (2.38) | 0.091 |
| 0.90 | 0.97 | (0.92, 1.23) | 89.053 (2.55) | 3.004 (2.62) | 0.006 |
| **1.00** | 0.97 | (0.92, 1.22) | **90.718 (2.60)** | **1.461 (2.81)** | **-0.001** |
| 1.10 | 0.97 | (0.92, 1.23) | 91.577 (2.63) | 0.693 (2.96) | -0.154 |
| 1.20 | 0.98 | (0.92, 1.22) | 92.010 (2.64) | 0.324 (3.09) | -0.225 |
| 1.30 | 0.97 | (0.92, 1.23) | 92.228 (2.65) | 0.149 (3.20) | -0.289 |
| 1.40 | 0.98 | (0.93, 1.23) | 92.341 (2.65) | 0.068 (3.30) | -0.346 |
| 1.50 | 0.97 | (0.92, 1.24) | 92.042 (2.61) | 0.031 (3.39) | -0.398 |

**Note:** The first column reports the values of the exponent for the trend. The second and third columns refers respectively to the estimated differecing parameter and the associated 95% confidence intervals. The following columns display the intercept and the slope of the exponential trend along with their associated t-values. The final column reports the test statistics.

**Table 5: Estimated coefficients for the US Consumer Price Index**

| γ | d | 95% band | α (t-value) | β (t-value) | Stastistic |
|---|---|---|---|---|---|
| 0 | 1.43 | (1.36, 1.51) | 9.921 (3.13) | --- | 0.144 |
| 0.10 | 1.43 | (1.36, 1.51) | 9.938 (1.71) | -0.144 (-0.02) | 0.147 |
| 0.20 | 1.42 | (1.36, 1.50) | 9.851 (3.60) | -0.056 (-0.01) | 0.129 |
| 0.30 | 1.43 | (1.36, 1.52) | 9.814 (5.81) | -0.018 (-0.03) | 0.122 |
| 0.40 | 1.43 | (1.35, 1.52) | 9.784 (8.36) | -0.015 (-1.21) | 0.124 |
| 0.50 | 1.42 | (1.37, 1.50) | 9.871 (4.44) | -0.017 (-1.22) | 0.108 |
| 0.60 | 1.43 | (1.36, 1.51) | 9.742 (4.26) | 0.123 (-0.02) | 0.119 |
| 0.70 | 1.42 | (1.36, 1.50) | 9.796 (7.88) | 0.125 (0.59) | 0.114 |
| 0.80 | 1.43 | (1.37, 1.52) | 9.475 (5.76) | 0.166 (0.60) | 0.094 |
| 0.90 | 1.44 | (1.38, 1.52) | 9.697 (24.36) | 0.171 (0.87) | 0.088 |
| 1.00 | 1.44 | (1.37, 1.50) | 9.722 (26.15) | 0.151 (1.11) | 0.079 |
| **1.10** | **1.44** | **(1.38, 1.52)** | **9.755 (26.85)** | **0.106 (1.32)** | **-0.055** |
| 1.20 | 1.44 | (1.38, 1.50) | 9.780 (27.05) | 0.064 (1.91) | -0.079 |
| 1.30 | 1.44 | (1.37, 1.51) | 9.741 (27.11) | 0.034 (1.93) | -0.087 |
| 1.40 | 1.44 | (1.38, 1.51) | 9.799 (26.14) | 0.018 (1.98) | -0.119 |
| 1.50 | 1.44 | (1.37, 1.51) | 9.801 (27.13) | 0.009 (1.65) | -0.145 |

**Note:** The first column reports the values of the exponent for the trend. The second and third columns refers respectively to the estimated differecing parameter and the associated 95% confidence intervals. The following columns display the intercept and the slope of the exponential trend along with their associated t-values. The final column reports the test statistics.