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Retake Opportunities, Pass Probabilities, and Preparation for Exams

Abstract

Additional retake opportunities generally increase the probability of eventually passing a given threshold at given competence, and decrease preparation for exams. Preparation work performed before the first attempt may increase only for very weak students, and may decline so much as to decrease the total pass probability only for very strong students. If additional preparation is possible before retaking a failed exam, it is optimal for students to make early attempts without much preparation. Some will be lucky enough to pass, and others will make many attempts after gradually improving preparation. Hence, plentiful retake opportunities reduce the reliability of exam results as an indicator of competence, and increase the cost of taking and administering exams for students and teachers.

JEL-Codes: I210, C610.

Keywords: sequential options, information content of test results, graduate wage premia.

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1 Introduction

Exams are meant to certify a level of competence that depends on endogenous study effort and is measured imprecisely. Because tests cannot be perfectly accurate and competent students may test poorly when their performance is hampered by temporary shocks, allowing those who fail an exam to retake it can sensibly smooth out its result's random components. Everyone may deserve a second chance, but it is not obvious that a third and further chances are warranted, because the benefits of insurance have to be traded off its moral hazard implications. Incentives to study and prepare for exams are generally lessened in models with only one retake opportunity (Kooreman 2016, Michaelis and Schwanebeck 2016) or potentially infinite retakes in steady state (Krishna et al., 2018). Retake possibilities may however strengthen study incentives, e.g. if errors are uniform and effort is interior as assumed by Lewer et al. (2021), for the same reason that in that special case lets smaller measurement errors increase student effort (Bertola, 2021).

How many times must an exam be retaken before it's too many times? The answer requires a theoretical model of a single exam that students may retake a finite number of times, and is blowing in Italian universities where students can retake failed exams a large and variable number of times. Because only students who fail retake the exam, random factors are not smoothed out in the same way as if repeating it were mandatory: the good luck that may let poorly prepared students pass an early attempt cannot be evened out by bad luck in later attempts they will not make. Hence, plentiful retake opportunities increase the passing rate at given competence, and give students incentives to try and pass exams randomly at low competence.

The resulting pattern of outcomes and effort is applicable to Italy, where retake opportunities in higher education were strictly limited to one per year by a 1938 law until the government devolved regulation to universities in 1989, and have since grown to be remarkably numerous and heterogeneous. The lowest number of opportunities to take or retake exams currently appears to be 4 per year (for all courses at e.g. Politecnico di Torino and Università Bocconi). Exam sessions must be at least 6 per year at Università di Padova and 8 at Università di Bari and Università di Torino, and can be more numerous in specific degree programs (for the Law school, 11 in Bari and 9 in Turin). Each student may not take advantage of all opportunities (in Turin, Law degree programs allow only 3 or 5 attempts per year), but these limits are enforced loosely and were removed during the COVID pandemic in Turin, and possibly elsewhere. From 2023, even medical school admission tests may be taken twice a year by Italian students in the last two years of high school, and by any older individual with any secondary school degree.

Section 2 shows that a larger number of retake opportunities not only increases the probability that one will result in a pass at given per-attempt pass probability, but also makes the total pass probability a more concave function of its per-attempt counterpart. Section 3 allows students to increase their competence and characterizes their incentives to bear the cost of doing do so in terms of the optimal per-attempt pass probability. This reformulation of costly effort choices is similar to that of agency problems in Mirrlees (1999) and Holmström (1979), and it is equally convenient when exams are meant to elicit as well as to certify competence and examiners play the role of principals, students that of agents choosing unobservable effort. The derivations suppose that preparation either is all chosen before the first attempt, or can be flexibly distributed also after failed attempts, so that students solve a simple asset pricing problem with independent random and discrete payoffs and optimal exercise of sequential options to prepare. The two extreme cases have similar implications for the total pass probability and average preparation implied by additional retake opportunities. Section 4 shows that when preparation takes place before the first attempt only the best students, with nearly unitary pass probabilities, may pass with slightly lower probabilities because they find it optimal to work less. Weaker students also prepare less carefully, but not so much as to offset the higher total pass probability implied by a larger number of attempts. Only students whose total pass probability is very low, and therefore a very steep function of the per-attempt pass probability, find it optimal to prepare better when more numerous retake opportunities are available. If preparation is possible between attempts, additional early attempts at low but positive pass probabilities unambiguously increase the total pass probability. Section 5 shows that more numerous retake opportunities reduce preparation, except possibly that chosen before the first attempt by students with very low pass probabilities.

Not all these general results are intuitively obvious and some entail somewhat convoluted derivations, detailed in the Appendix. Sections 6 and 7 flesh them out discussing how interpretable parameters influence student choices and exam outcomes in a structural specification, and illustrating with calibrated examples the effects of a larger number of retake opportunities. One unpleasant implications is that the assessment technology becomes more costly and less effective, as students who attempt the exam many times on average devote work to taking exams rather than to improve preparation (and teachers to drafting and administering exams rather than to teaching). Another is that if preparation may be improved after failed attempts then it will be low on average and more widely dispersed among students who pass the exam, as pointed out in a single-retake-opportunity setting by Kooreman (2016) and confirmed in laboratory experiments by Nijenkamp et al. (2016). Opportunities to retake exams reduce the information conveyed by degrees, and may dangerously allow lucky early attempts to qualify poorly prepared

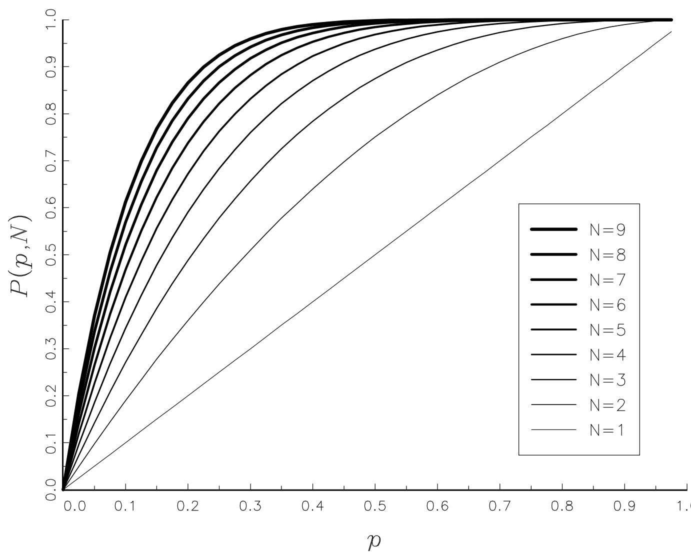

Figure 1: Total probability of passing an exam if N attempts are possible, as a function of the per-attempt pass probability.

individuals for tasks that require critical competencies. Section 8 discusses how the problematic implications of numerous retake opportunities may be remedied by structural features of exams in a reality that lies between the two extreme cases.

2 Total and per-attempt pass probabilities

If passing an exam is an independent event across attempts that has the same probability p at each attempt, the probability of passing at least one of N attempts is the complement to unity of the probability of failing all attempts,

$$P(p,N) = 1 - (1-p)^N . \tag{1}$$

This function has several properties, illustrated in Figure 1, that will be useful in what follows. An additional retake opportunity increase the total pass probability (1) at given p,

$$\begin{aligned} P(p,N+1) - P(p;N) &= \left(1-(1-p)^{N+1}\right) - \left(1-(1-p)^N\right) \\ &= p\,(1-p)^N > 0 \text{ if } 0<p<1: \end{aligned} \tag{2}$$

when a larger number of attempts is possible it is more likely that one will result in a pass. This is the reason why students like many retake opportunities. The pass probability effect (2) of an additional retake opportunity is zero for all N when $p = 0$ or $p = 1$, decreases in N at given $0 < p < 1$, and is most positive for each N at $p = 1/(N+1)$, where $d\left(p(1-p)^N\right)/dp = (1-p)^N - Np(1-p)^{N-1}$ equals zero.

At given N, the total pass probability (1) is increasing and concave in p:

$$\frac{d}{dp}\left(1-(1-p)^N\right) = N(1-p)^{N-1} > 0 \text{ if } p < 1, \tag{3}$$

$$\frac{d^2}{dp^2}\left(1-(1-p)^N\right) = -N(N-1)(1-p)^{N-2} < 0 \text{ if } N > 1. \tag{4}$$

A larger N increases the total pass probability at given p,

$$P(p, N+1) - P(p, N) = 1 - (1-p)^{N+1} - (1-(1-p)^N) = p(1-p)^N,$$

and makes it a steeper function of p when a low per-attempt probability p makes the maximum retakes constraint so likely to bind that relaxing it with a larger N makes expression (3) more positive:

$$(N+1)(1-p)^N - N(1-p)^{N-1} = ((N+1)(1-p) - N)(1-p)^{N-1} > 0$$

if $(N+1)(1-p) > N$, or

$$p < \frac{1}{N+1}. \tag{5}$$

A larger N strengthens the marginal effect of p on P only when p is below 50% for $N = 1$, and even smaller for larger N.

It will also be useful to compute the expected number $\eta(p, N)$ of attempts until the student passes or fails all N possible attempts. An additional attempt is made only if the N previous attempts fail, so

$$\eta(p, N+1) = \eta(p, N) + (1-p)^N. \tag{6}$$

The solution of this difference equation with boundary condition $\eta(p, 1) = 1$,

$$\eta(p, N) = \left(1-(1-p)^N\right)/p, \tag{7}$$

increases towards $1/p$ as N grows towards infinity at given $p < 1$, and depends on p with slope

$$\frac{d}{dp}\eta(p, N) = \left((1-p)^{N-1}\left(1-p+Np-(1-p)^{1-N}\right)\right)/p^2, \tag{8}$$

which the Appendix and Figure 2 show is negative for all $p \in [0, 1]$ if $N > 1$, and more negative for larger N: a larger probability of success reduces the average number of failures, hence attempts, more strongly if more numerous opportunities allow attempts to grow higher.

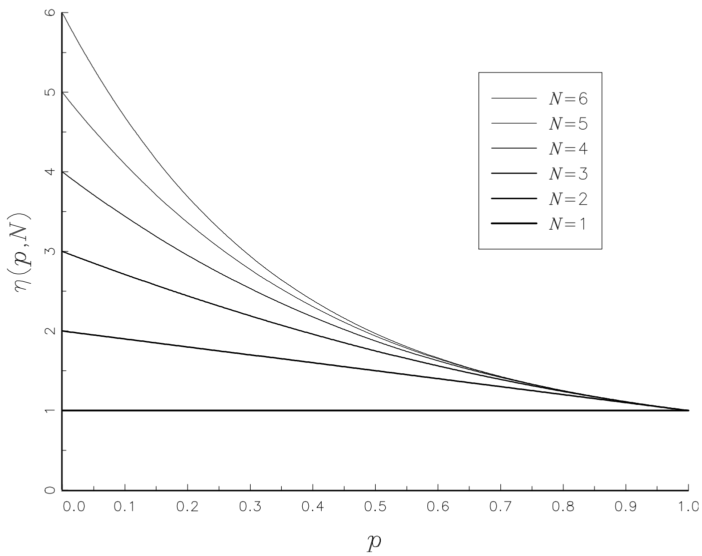

Figure 2: Expected number of attempts as a function of the per-attempt pass probability.

3 Endogenous preparation

Even when p is an exogenously given constant, the results of exams that can be repeated when failed can be as misleading as those of COVID tests by somebody who refuses to accept a possibly false positive result and continues to be tested until a possibly false negative result arrives. When testing rational students rather than viruses there is an additional problem: retake opportunities matter not only for the total pass probability at given exam structure and student competence, but also for incentives to increase competence by studying and learning.

Let x be a scalar index of increasingly difficult tasks that the student should be likely to perform correctly at the exam and in later life, and let achieving competence x cost $c(x)$ for a student who enjoys utility $v(x) > 0$ if the exam is passed. These functions are continuous and differentiable with $c'(x) \geq 0$ and $v'(x) \geq 0$ for all x. Some competence may be exogenously given and free to use, so $c'(x)$ can be zero at low x, but $c'(x)$ is strictly positive at levels of competence that require work. If higher competence is beneficial given passing, then $v'(x) > 0$.

Consider first the case where $N = 1$, and no retakes are allowed. Denoting with $x(\cdot)$ the inverse of $p(\cdot)$ and defining $g(p) \equiv c(x(p))/v(x(p))$, the problem of choosing x to maximize $p(x)v(x) - c(x)$ is the same as that of choosing p to maximize $p(x) - g(p)$. The monotonic function $p(x)$ maps competence x to the probability of passing each attempt. It depends on how costly it is to prepare better and on how rewarding it is to pass the exam, as well as on the structure of the exam, which will be modeled and discussed below but is assumed to be the same at each attempt. If in real life the tasks that students who pass the exam should perform correctly with high probability are similar to those examined, the pass probability p is the appropriate gauge of the competencies that are tested by the exam, and may be improved by students if they have incentives to do so. It will be dubbed "preparation" in what follows.

The level and the slope of $g(p)$ depend on the format of the exam and student characteristics. The value $\hat{p}$ such that $g(\hat{p}) = 0$ is higher for students with better exogenous competence, and the slope of $g(p)$ is flatter for individuals who find it easier to study productively. The marginal cost/reward $g'(p)$ of increasing p need not be positive for all p, but must be at the optimal choice

$$p^*_{1|1} = \arg\max_p \{p - g(p)\}, \tag{9}$$

which exists as long as $V(p, N)$ is continuous in p on the closed and bounded $[0, 1]$ interval.

For a single attempt, there is a $p^*_{1|1} = 0$ corner solution if $g(p) > p$ for all $p \in [0, 1]$ or $g'(0) > 1$, a $p^*_{1|1} = 1$ corner solution if $g'(1) < 1$, and any $p \in [0, 1]$ is optimal if $g(p) = p$ for all $p \in [0, 1]$. Otherwise, $p^*_{1|1}$ satisfies the first-order condition

$$1 - g'(p^*_{1|1}) = 0, \tag{10}$$

which is sufficient if $g''(p) \geq 0$. The student takes the exam if its maximized expected value exceeds the monetary and opportunity costs of doing so. One or more retake opportunities are available if $N > 1$.

3.1 All preparation before first attempt

Consider the extreme case where all preparation must take place before the first attempt, so a constant p is endogenous to maximization of the student's objective function. The student bears costs $g(p)$ before the first attempt and κ for each attempt, so the total expected cost of choosing p is $\bar{g}(p) \equiv g(p) + \kappa\eta(p, N)$. The undiscounted expected value of the exam sequence is maximized by

$$\bar{p}^*_N = \arg\max_p \bar{V}(p) \text{ for } \bar{V}(p) \equiv P(p, N) - \bar{g}(p). \tag{11}$$

For $N > 1$ the optimal internal choice $\bar{p}^*_N$ equates its pass probability implications (3) to its marginal cost/reward implications. That first-order condition, $N\left(1 - \bar{p}^*_N\right)^{N-1} - \bar{g}'(\bar{p}^*_N) = 0$, is sufficient if $\bar{g}''(p) \geq 0$, which with (4) ensures concavity of the objective function (11), and can be rearranged to

$$\bar{p}^*_N = 1 - \left(\frac{\bar{g}'(\bar{p}^*_N)}{N}\right)^{\frac{1}{N-1}}. \tag{12}$$

This expression is smaller when $g'(\bar{p}_N^*)$ is larger, as students with more strongly increasing preparation costs (or a lower reward for passing) optimally choose less preparation. It is positive when $\bar{g}'(\bar{p}_N^*) < N$, which for a convex $\bar{g}(\cdot)$ is implied by $\bar{g}'(0) < N$.

3.2 Preparation can increase between attempts

Consider next the opposite extreme case where the student may distribute preparation before and between attempts, moving up an invariant $g(p)$ cost/reward function.

Denote $p_{n|N}$ the pass probability of the n^{th} attempt out of the possible N. If the student ends up making N attempts, the choice of $p_{N|N}$ is the same as the choice of $p_{1|1}$ in (9), except for the fact that the cost of preparation $p_{N-1|N}$ has already been paid. If $g'(0) > 1$ or $g(p) > p$ at all $p \in [0, 1]$ then $p^*_{N|N} = 0$ and, because all previous attempts look exactly the same as the last, $p^*_{n|N} = 0$ for all n. If $g(p) < p$ at all $p \in [0, 1]$ and $g'(1) < 1$ then $p^*_{N|N} = 1$, and because $p^*_{n|N} = 1$ for all n the student passes at the first attempt. When the solution is not at those corners the value of the last attempt, net of all preparation costs to date, is

$$V_N = \max_p \{p - (g(p) + \kappa)\}. \tag{13}$$

The student, of course, does not need to prepare only when retake opportunities are about to run out. At attempts $1 \leq n \leq N - 1$, past preparation $p_{n-1|N}$ is sunk, and the optimal choice of $p_{n|N}$ takes into account that it removes a portion of future preparation costs. The next attempt will be needed with probability $(1 - p_{n|N})$, has value V_{n+1} including all cost, and value $V_{n+1} + g(p_{n|N})$ excluding previously incurred preparation costs. Hence, the preparation sequence maximizes the recursively defined undiscounted value function

$$\begin{aligned} V_n &= \max_p \{p - (g(p) + \kappa) + (1 - p)(V_{n+1} + g(p))\} \\ &= \max_p \{p - (pg(p) + \kappa) + (1 - p)(V_{n+1})\}. \end{aligned} \tag{14}$$

In this stylized timeless representation of sequential choices the student maximizes a current-period flow payoff where the increasing cost function $pg(p)$ effectively accounts for only a portion $-1 + (1 - p) = p$ of preparation costs and looks forward to the value of the next one, which is realized with probability $(1 - p)$ as long a further attempt is possible. A discount factor could appear alongside $1 - p$, and p could be allowed to depreciate before the next attempt, at the cost of some notational complication, and with obvious implications. The student cannot recover the cost of study by unlearning some material, so p cannot decrease endogenously between attempts. The $p_{n+1} \geq p_n$ irreversibility constraint is not binding, however, because in the absence of trends or shocks fewer remaining retakes can only increase the optimal p when it is not a corner solution.

4 Total pass probability

Because school administrator frown on high dropout rates, and failing exams is *ex post* problematic for students even when it follows from *ex ante* optimal enrolment choices under uncertainty (Bertola, 2023), it is interesting to see how exam failures depend on the number N of possible retakes.

Should all preparation occur before the first attempt, as in Section 3.1, the Appendix shows that when possible retakes grow from N to $N+1$ the eventual pass probability implied by the student's optimal preparation choice increases if

$$g'(p_N^*) > \left(\frac{g'(\bar{p}_{N+1}^*)}{g'(\bar{p}_N^*)}\right)^{N^2-1} N^{N^2} (N+1)^{1-N^2}. \tag{15}$$

Because the total pass probability increases both in p and in N by (2), this inequality is certainly satisfied when a larger N increases or leaves unchanged the optimal preparation (12), a condition that can be rearranged to read

$$g'(\bar{p}_N^*) \geq \left(\frac{g'(\bar{p}_{N+1}^*)}{g'(\bar{p}_N^*)}\right)^{\frac{N-1}{N}} N^N (N+1)^{1-N} \tag{16}$$

and, as the Appendix shows formally, is less restrictive than (15).

To see why additional retake opportunities may increase preparation only when the marginal cost of preparation is sufficiently large, note that a high marginal cost makes it optimal for a student to choose low preparation, which by (4) implies that additional retake opportunities strongly increase preparation's marginal impact on the total pass probability. To see that this can make it optimal to choose better preparation, it is useful to consider some special cases. Should $\bar{g}'(0) > 1$ make the $\bar{p}_1^* = 0$ corner solution optimal for a single attempt, an additional retake opportunity increases $\bar{p}_N^*$ to a positive value if $N > 1$ and $\bar{g}'(0) < N$. And if the cost/reward function $g(\cdot)$ is only weakly concave and has the same slope $g'(\bar{p}_{N+1}^*) = g'(\bar{p}_N^*) = \bar{g}'$ at both $\bar{p}_N^*$ and $\bar{p}_{N+1}^*$, then $\bar{p}_N^* = \bar{p}_{N+1}^* = 1 - (\bar{g}'/N)^{\frac{1}{N-1}}$ in (12), which in (5) implies a stronger total probability effect of better preparation if

$$1 - \left(\frac{\bar{g}'}{N}\right)^{\frac{1}{N-1}} < \frac{1}{N+1} \Longleftrightarrow \bar{g}' < \left(\frac{N}{N+1}\right)^{(N-1)} N,$$

which is the same as (16) with $g'(\bar{p}_{N+1}^*)/g'(\bar{p}_N^*) = 1$: better preparation is worthwhile for the

	(1) $P^*_{N+1} < P^*_N$ if	(2) minimal at	(3) if	P^*_N	P^*_{N+1}
$N=1$	$\bar{g}'$ <1.0000	-0.2500	$\breve{g}' = 1.0000$	1.0000	0.7500
$N=2$	$\bar{g}'$ <0.5926	-0.0093	$\breve{g}' = 0.3333$	0.9722	0.9630
$N=3$	$\bar{g}'$ <0.3003	-0.0014	$\breve{g}' = 0.1483$	0.9890	0.9876
$N=4$	$\bar{g}'$ <0.1407	-0.0003	$\breve{g}' = 0.0649$	0.9959	0.9956
$N=5$	$\bar{g}'$ <0.0629	-0.0001	$\breve{g}' = 0.0278$	0.9985	0.9984
$N=6$	$\bar{g}'$ <0.0272	0.0000	$\breve{g}' = 0.0117$	0.9994	0.9994

Table 1: Constant cost/reward slopes such that preparation can decline so strongly as to decrease the total pass probability, and such that it declines the most. The endogenous total pass probability P^*_N declines when N increases if the slope $\bar{g}'$ is constant and less than the critical value tabulated in column (1), expression (A.3) in the Appendix. The change is most negative at the values tabulated in column (2) when the slope is that listed in column (3), expression (A.4) in the Appendix.

weak or poorly motivated students whose constant marginal cost/reward is so high as to make it optimal to set p so low as to be below $1/(N+1)$.

If $g(\cdot)$ is strictly convex, then $g'(p^*_{N+1})/g'(p^*_N) > 1$ when $p^*_{N+1} > p^*_N$, and the inequality

$$p^*_N < 1 - \left(\frac{g'(\bar{p}^*_N)}{N}\right)^{\frac{1}{N-1}} = \frac{1 - \left(1 - \left(\frac{g'(\bar{p}^*_{N+1})}{g'(\bar{p}^*_N)}\right)^{\frac{1}{N}}\right) N}{N+1}$$

is less stringent than that in (5): hence, additional retake opportunities increase preparation for a broader range of high marginal cost/reward and low pass probabilities.

Outside that range, $p^*_{N+1} < p^*_N$ may but need not offset the positive implications of a larger N for the eventual pass probability of students with high per-attempt and overall pass probabilities. To see this, consider again the special case where $\bar{g}'(1) < 1$ makes $\bar{p}^*_N = 1$ the optimal one-shot choice. Because (3) is zero at $p = 1$, if $\bar{g}'(p) > 0$ for all p it cannot be optimal to choose $p = P(1, N) = 1$ when $N > 1$. Increasing N above 1 implies a switch from a corner to an interior solution, and adding just one retake possibility induces a student who would be sure of passing a single attempt to choose a smaller probability for both attempts, and fail them both with positive probability.

In the other special case where $g'(\bar{p}^*_{N+1}) = g'(\bar{p}^*_N) = \bar{g}'$, Appendix A.3 shows that (15) does not hold if $\bar{g}'$ is lower than the critical values tabulated for each N in column (1) of Table 1, and that the decline is largest at the values tabulated in column (2) of that table when $\bar{g}'$ takes the values tabulated in column (3). The implications of changing N from 1 to 2 are quite significant because, as discussed above, the optimal choice of $\bar{p}$ switches from a corner at unity to an interior solution, trading large cost/reward savings off a lower total pass probability (of course, if preparation may be improved after a failed attempt a student with low and constant marginal cost/reward will choose the corner solution at the second attempt, and restore the unitary probability). When N increases from larger values the total pass probability declines by less than a percentage point, because the small $\bar{g}'$ that makes it possible for it to decline when N increases implies that the optimal choice of $\bar{p}$ is high, hence in the region where $P(p, N)$ is flat as a function of p. When $g''(\bar{p}) > 0$, condition (15) implies a less stringent lower bound for the cost/reward slope, and a small degree of convexity in fact rules out a significant decline of P when N increases further from a moderately high value: as p declines only a little if it does, it is very difficult for additional retake opportunities to cause even the small decrease of P that is possible when $g'(\bar{p})$ is constant.

When as in Section 3.2 preparation can be improved after the first attempt and between the others then the total pass probability is obviously higher than the probability of passing a single attempt with no retakes, which is the same as the probability of passing the last retake opportunity. The last n choices are the same for any $N \geq n$, so an increase of possible attempts from N to $N+1$ simply makes it optimal to choose a lower value for the first attempt and, if it is unsuccessful, prepare for the second as when it was the first of N possible attempts. Because a larger N adds a poorly prepared attempt at the start of the optimal preparation sequence, it must increase the probability $\tilde{P}_N^* = 1 - \prod_{n=1}^{N}\left(1 - p_{n|N}^*\right)$ of not failing all N attempts. The Appendix shows that a unit increase of N increases that total pass probability by

$$\tilde{P}_{N+1}^* - \tilde{P}_N^* = p_{1|N+1}^*\left(1 - \tilde{P}_N^*\right), \tag{17}$$

an expression that is always positive, but small when a large N implies that the total pass probability $\tilde{P}_N^*$ is close to unity, and makes it optimal to choose a low preparation $p_{1|N+1}^*$ with a chance of passing that initial attempt and saving all further preparation work.

5 Preparation

Administrators and students may be thrilled by the higher pass rates implied by additional retake opportunities. The downside, however, is less preparation, and lower and variable competence among passing students.

When as in Section 3.1 endogenous preparation is chosen before the first attempt, it is the same regardless of which attempts results in a pass, and generally lower at larger N: only if the marginal cost of increasing preparation is high, and the optimal preparation is low, it may be higher at larger N.

Almost as obviously, and more interestingly, when as in Section 3.2 preparation can improve after the first attempt a larger number of retake opportunities lowers average preparation and increases its dispersion. Formally, denote $f(n)$ the probability that the student makes the n^{th} attempt. Because one attempt suffices to pass if it is successful, $f(1)=p^*_{1|N}$. The probability of passing at the n^{th} attempt is that of failing all previous ones and finally passing,

$$f(n) = \prod_{i=1}^{n-1} \left(1 - p^*_{i|N}\right) p^*_{n|N} \text{ for } 2 \leq n < N,$$

and all N attempts are needed with probability

$$f(N) = \prod_{i=1}^{N-1} \left(1 - p^*_{i|N}\right).$$

These are all the possible cases, so $\sum_{n=1}^{N} f(n) = 1$, as can be laboriously verified writing out the summation, collecting terms in $p^*_{i|N}$, and finding that their coefficient is zero.

Average preparation is an appropriate indicator of competent performance in later life of the tasks tested by the exam. Across all students it is $\sum_{n=1}^{N} f(n)p^*_{n|N}$, which may be of interest if education is beneficial also when the exam is failed. Dividing that expression by the total pass probability $\tilde{P}^*_N = 1 - \prod_{n=1}^{N} \left(1 - p^*_{n|N}\right)$ yields average preparation of students who do pass the exam. Because students may pass at low preparation earlier than at the last attempt, when the preparation is the same as in a one-shot exam, retake opportunities reduce average preparation,more strongly when additional retake opportunities make it possible to pass at earlier attempts. And because the preparation of those who pass depends on which attempt was successful it is dispersed around its average, and low when for students who pass at early attempts.

Less preparation work is additional good news for students who each take the reward v of passing as given, and may not realize that in equilibrium it should depend on how their individually optimal behavior determines the market value of the competence of all students who pass the exam. It would be cumbersome to model explicitly in this paper, but is interesting to note, that the negative effect of a larger N on endogenous preparation is reinforced if a lower value of passing the exam feeds back negatively on incentives to prepare.

6 A structural example

The magnitude of the theoretical effects derived so far depends on the structure of the student's problem. This section illustrates those and other implications of many retake opportunities

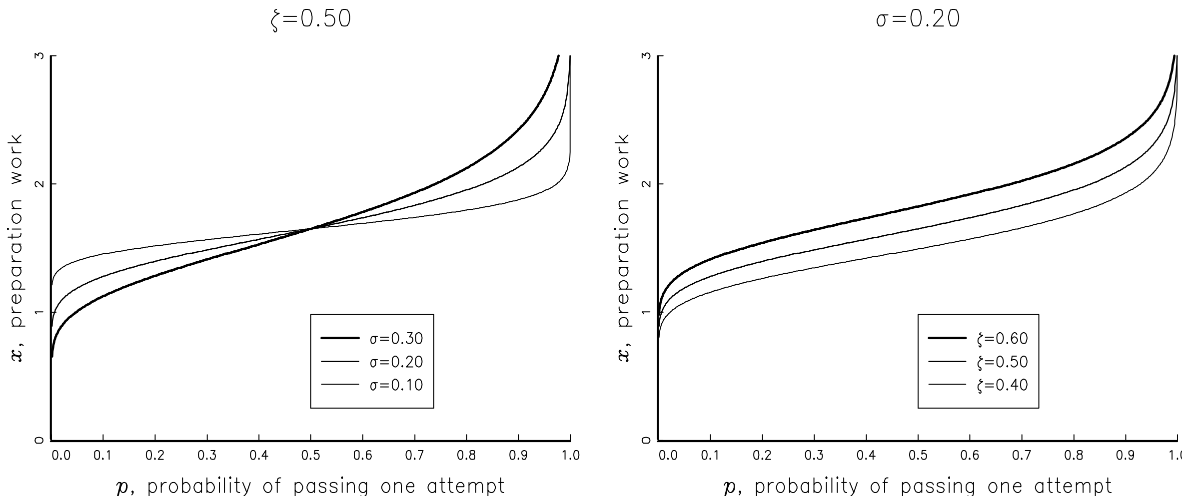

Figure 3: How exam imprecision and the passing threshold influence the relationship between preparation work and pass probability.

with tractable and fairly realistic explicit functions. To let the student's cost function depend on structural parameters that may differ across students and across exams, suppose the exam is passed if the competence index x multiplied by e^{ε} exceeds a threshold e^{ζ}, where $\varepsilon \sim N(0, \sigma^2)$. The pass probability is that of $\varepsilon > \zeta - \ln x$,

$$p(x) = 1 - \Phi\left(\left(\zeta - \ln x\right)/\sigma\right) = \Phi(\left(\ln x - \zeta\right)/\sigma)$$

for $\Phi(\cdot)$ the normal distribution function. The competence that implies pass probability p is

$$x(p) = e^{\zeta + \sigma\Phi^{-1}(p)}, \tag{18}$$

which Figure 3 plots for various values of the exam's structural parameters σ and ζ. Its slope with respect to p,

$$x'(p) = \frac{\sigma}{\phi(\Phi^{-1}(p))} e^{\zeta + \sigma\Phi^{-1}(p)}, \tag{19}$$

is more strongly positive if ζ or σ are larger, and close to zero if σ is small and the student can pinpoint very exactly the competence that ensures passing the exam. It tends to positive infinity as p tends to zero or one and the normal density $\phi(\cdot)$ in the denominator tends to zero.

This parameterization of the exam technology is very tractable, in particular because it conveniently rules out corner solutions. Zero competence is associated with zero probability of passing and, as the normal density is strictly positive everywhere if σ is finite, it is not possible for finite competence to ensure a pass: random errors and accidents are possible regardless of preparation.

Because $\Phi^{-1}(0.5) = 0$, the $x = \zeta$ competence that implies a 50% probability of passing does not depend on the result's noisiness σ, and is $x = \exp(0.5) \approx 1.65$. The additional work required by higher pass probabilities depends on σ. Competence x results in a fail with probability $\Phi((\zeta - \ln x)/\sigma)$, so to reduce to 5% or 1% the chance of failing because of bad luck a student should aim to competence about $\ln x - \zeta = \Phi^{-1}(0.05)\sigma \approx 1.64\sigma$ or $\Phi^{-1}(0.01)\sigma \approx 2.33\sigma$ percentage points above the threshold. To elicit such statistically insignificant failure probabilities, a fairly precise exam with a reasonable pass threshold might be parameterized by $\sigma = 0.2$ and a pass threshold about 30 or 40% below the competence level that examiners would like to elicit and certify.

Of course, heterogeneous students prepare at different levels and pass with different probabilities, and different exams may have different competence and reliability objectives. Bad luck cannot be eliminated either at exams or when similar tasks are performed in real life, but is tolerable if the probability of unacceptable performance is acceptably low, and very low in exams that assess critical competencies. In this stylized parametric model, exam failure may be triggered by exogenous shocks represented by the normal density's long tail, such as mishaps and accident on the way to the exam or at work in later life. Studying need not reduce the chances of such bad luck, but it might: a well-prepared student is more relaxed and confident, hence less likely to oversleep and miss the bus or look the wrong way when crossing the street.

Let the unit cost γ of the work that can endogenously improve x and the reward $v(x) = v$ of passing, measured in the same units, be constant for a given student (increasing functions would complicate the following characterization without changing its substantive implications). If $\gamma > 0$, it would be infinitely costly to be completely certain of passing the exam. The choice of preparation trades its cost off the failure probability. Because the cost/reward function $g(p) = x(p)\gamma/v$ is not everywhere convex in p, multiple local optima might exist, as they do in Bertola's (2024) model of pass-or-fail assessments where an increasing marginal cost schedule crosses the density of additive errors more than twice. Here, multiplicative uncertainty and constant marginal cost ensure that the optimal preparation is uniquely identified by the first- and second-order conditions, and is shown in the Appendix to be

$$p^*_{1|1} = \Phi\left(\sqrt{\sigma^2 - 2\left(\ln(\gamma/v) + \ln\left(\sigma\sqrt{2\pi}\right) + \zeta\right)} - \sigma\right) \tag{20}$$

for a single attempt without retake opportunities. When N attempts are possible, and each costs κ in the same units as γ and v, both studying and taking exams are costly activities.

If as in Section 3.1 all preparation takes place before the first attempt, the total expected cost/reward function starting from $x = 0$

$$\bar{g}(\bar{p}) \equiv \frac{\gamma x(\bar{p}) + \kappa \eta(\bar{p}, N)}{\upsilon}$$

accounts for the expected number of attempts (7). The function $\gamma x(p)$ is upward sloping, more strongly if γ is large. If κ is positive, at N the total cost/reward is larger at given $\bar{p}$ and, because preparation reduces the expected number of attempts, it is less a steeply increasing (and possibly declining) function of $\bar{p}$. The exam-taking cost $\kappa\eta(\bar{p}, N)$ is convex in $\bar{p}$, so $g(\bar{p})$ is convex if $x(\bar{p})$ is. At given N, the student chooses $\bar{p}$ to maximize

$$V(\bar{p}, N) = P(\bar{p}, N) - \bar{g}(\bar{p}).$$

Because $P(\bar{p}; N) \leq 1$ for all $\bar{p}$ and N, any $\bar{p}$ such that $g(\bar{p}) > 1$ implies $V(\bar{p}, N) < 0$, so participation cannot be optimal if it is optional and costly (if participation in the exam sequence entails an additional cost ω, such as a course fee, it cannot be optimal at any $\bar{p}$ such that $\gamma x(\bar{p}) + \kappa\eta(\bar{p}, N) > 1 - \omega$). At an interior optimal choice of $\bar{p}$ the $\bar{g}'(\bar{p})$ marginal cost/reward implied by (19) and (8) equals the total pass probability slope (3), which depends on $\bar{p}$ and N in the ways characterized in Section 2. That first-order condition has no closed-form solution if $N > 1$, but can help interpret numerical solutions.

In Section 3.2's opposite extreme case of sequential preparation, the cost/reward function is

$$g(p) \equiv \frac{\gamma x(p) + \kappa}{\upsilon}$$

at each attempt. At the last, $\Phi^{-1}(p^*_{N|N}) = \Phi^{-1}(p^*_{1|1})$ from (20). Previous optimal choices of $p^*_{n|N}$ can be computed recursively using numerical versions of (14). The expected number of attempts until the student passes or exhausts the N possible retakes is

$$\tilde{\eta}(p^*_{1|N}, p^*_{2|N}, ..., p^*_{N|N}, N) = p^*_{1|N} + \sum_{n=2}^{N-1} n \left(\prod_{i=1}^{n-1} \left(1 - p^*_{i|N}\right) p^*_{n|N} \right) + N \prod_{i=1}^{N-1} \left(1 - p^*_{i|N}\right).$$

This expression would coincide with (7) if $p^*_{i|N} = \bar{p}$ for all i. It may be larger or smaller than (7) evaluated at the $\bar{p}^*_N$ preparation chosen only before the first attempt, because the incremental preparation sequence starts below $\bar{p}^*_N$ and ends above $\bar{p}^*_N$, following a nonlinear path that depends on functional forms.

7 Calibration

The parameters that determine optimal preparation can be chosen so as to represent realistic students and exams. A student who intends to be so prepared as to have probability p of passing the threshold ζ with a single attempt needs to devote the time and effort cost needed to set expression (18) to p. When starting from $x = 0$, if a unit increase of x corresponds to γ days of study then $\exp(0.5) \approx 1.65\gamma$ days are needed to pass with 50% probability. The work needed to attain higher pass probabilities depends on σ, is proportionately more or less for the different γ values of students with different ability, and is additively different for students who start from positive x values when they begin to prepare for the exam.

Some examples help gauge the magnitude of the effects and see how they depend on student characteristics. If $\gamma = 20$ for student A and $\gamma = 30$ for student B, they would respectively take about 33 and about 50 days to prepare at $p = 0.5$ starting from $x = 0$. If $\sigma = 0.2$, student A needs about $20\exp(0.5+0.2\Phi^{-1}(0.9)) = 42.6$ days to prepare at $p = 90\%$, $20\exp(0.5+0.2\Phi^{-1}(0.95)) = 45.8$ days to prepare at $p = 95\%$, and $20\exp(0.5 + 0.2\Phi^{-1}(0.99)) = 52.6$ days to prepare at $p = 99\%$. For weaker student B, twice as many days are needed to attain each preparation level. The total cost is lower for students who already know some of the material before preparing this exam. Should these students start from the competence that implies a 50% probability of failure, A would need about $46 - 33 = 13$ days and B about $92 - 50 = 42$ days to prepare at $p = 95\%$.

Of course, students will study so much only if they have incentives to face a suitably low risk of mistakes that can be avoided by preparing well. Their preparation choice depend on the reward υ of passing the exam. If passing is rewarded by a fraction of a year's completed education, υ can be measured in "day" units as a portion of the student's labor market graduation premium over the work life. When a year of completed education increases income throughout 30 or 40 years of work by 5% or 10%, in discounted present value terms it benefits students like a couple of years of the income that could be earned (or of the time enjoyed as leisure) before graduating. In Italy, students have to pass six or so exams per year, so the examples illustrated and discussed below suppose that $\upsilon = 100$ days for both A and B, the right order of magnitude if passing the exam allows them to expect completing about a sixth of an additional year of education.

Of course not only the cost of competent performance but also the value υ of passing differs across students. The net-of-opportunity-cost expected value of taking the exam should be zero for a marginal student who barely finds it optimal to participate. Students A and B are not marginal: as we shall see, the net expected value of the exam is positive for both, larger for the stronger student A, and increases when more retake opportunities are available. Other students may find it optimal to take the exam only when many retakes are possible.

These parameters in (20) imply that if only one attempt were available A would prepare at $p^*_{1|1} = 95.6\%$ and B would take twice as long to attain $p^*_{1|1} = 93.1\%$. Both would approximate a reasonable 95% reliability objective, and both would somewhat unfairly fail with about 5% probability. Because $(1 - 0.95)^2 = 0.25\%$, they would hardly ever fail two attempts at their single-attempt optimal preparation if retakes simply ensured that bad luck plays a lesser role in exams. As shown above, however, retake opportunities influence their preparation, in ways that depend on the cost κ of each attempt. Suppose $\kappa = 1$ day for both A and B. It will be instructive to inspect also the behavior of students C and D, who have the same γ as A and B respectively (and would choose the same preparation in a single attempt) but need $\kappa = 3$ days for each attempt. The exam-taking costs of C and D may be higher because they live farther from the examining room, or perhaps, being more forgetful and emotional, need more time for a last-minute review of the material before each exam and to recover from the stress of examinations.

Figures 4 and 5 and Tables 2 and 3 illustrate and inspect the preparation chosen by the four students and its implications for total pass probabilities and other statistics, discussed below, when $N = 2$ and when $N = 5$. The large variation of γ and κ helps see clearly the qualitatively general implications of those parameters. Considering the implications of three additional attempts offers a sharper contrast than the unit-increment analytical expressions, and a comparison the single retake opportunity available in most university systems to the more numerous opportunities granted in Italy. The cost functions are plotted supposing that all students start from zero competence. Better starting points would shift them downwards, and imply that no preparation is needed when more than the optimal competence is already available.

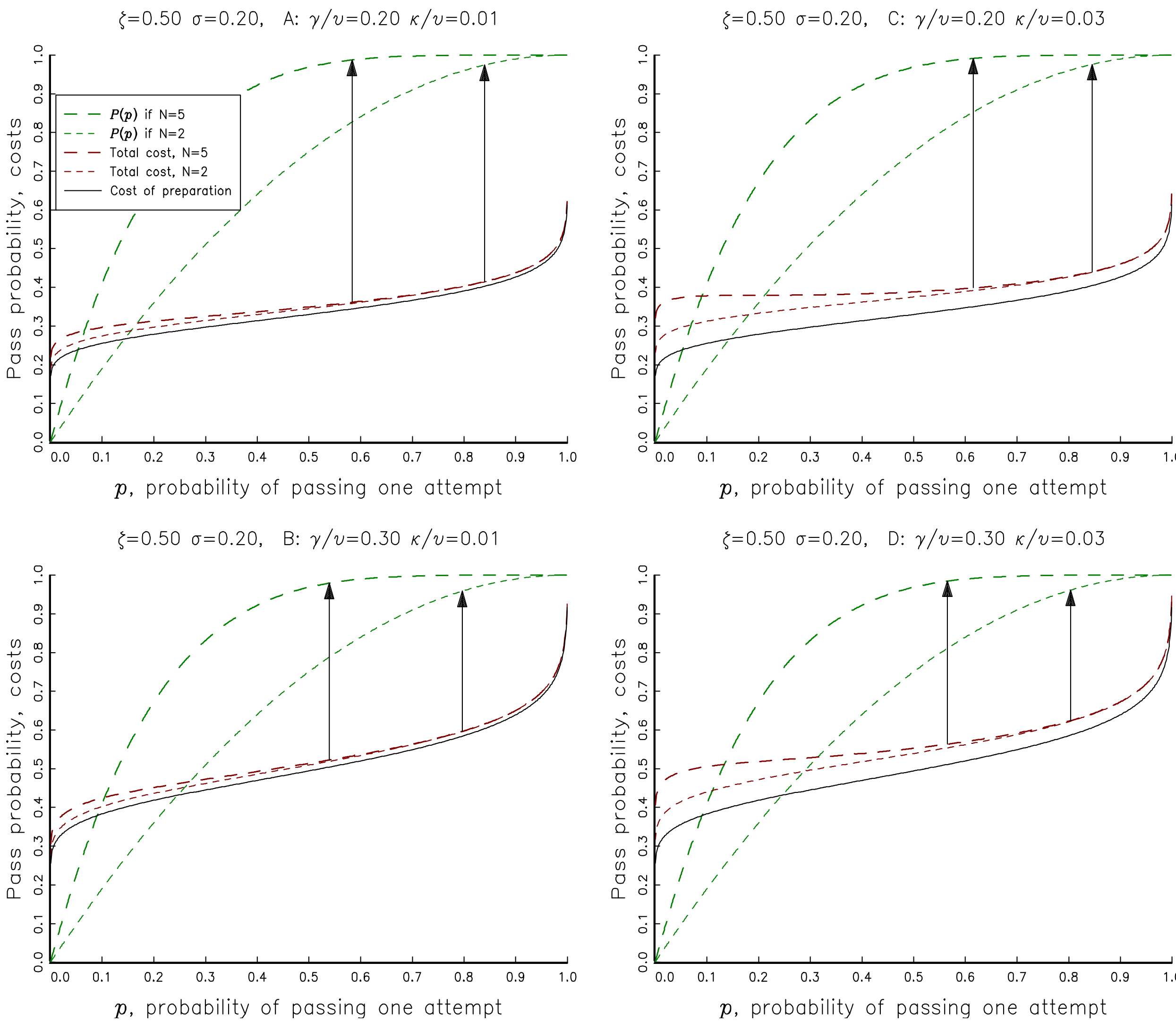

Figure 4: Four examples of students choosing optimal preparation only before the first attempt.

Student parameters	N	$\bar{p}_N^*$	P_N^*	$E[n]$	$\Pr(n = N)$	$\bar{V}(P_N^*)$
γ/υ =0.2, κ/υ =0.01	2	0.840	0.974	1.160	0.160	0.560
	5	0.584	0.987	1.692	0.03	0.627
γ/υ =0.3, κ/υ =0.01	2	0.797	0.959	1.203	0.203	0.363
	5	0.540	0.979	1.815	0.045	0.457
γ/υ =0.2, κ/υ =0.03	2	0.845	0.976	1.155	0.155	0.537
	5	0.616	0.992	1.611	0.022	0.594
γ/υ =0.3, κ/υ =0.03	2	0.803	0.961	1.197	0.197	0.339
	5	0.566	0.985	1.741	0.036	0.421

Table 2: Preparation only before the first attempt

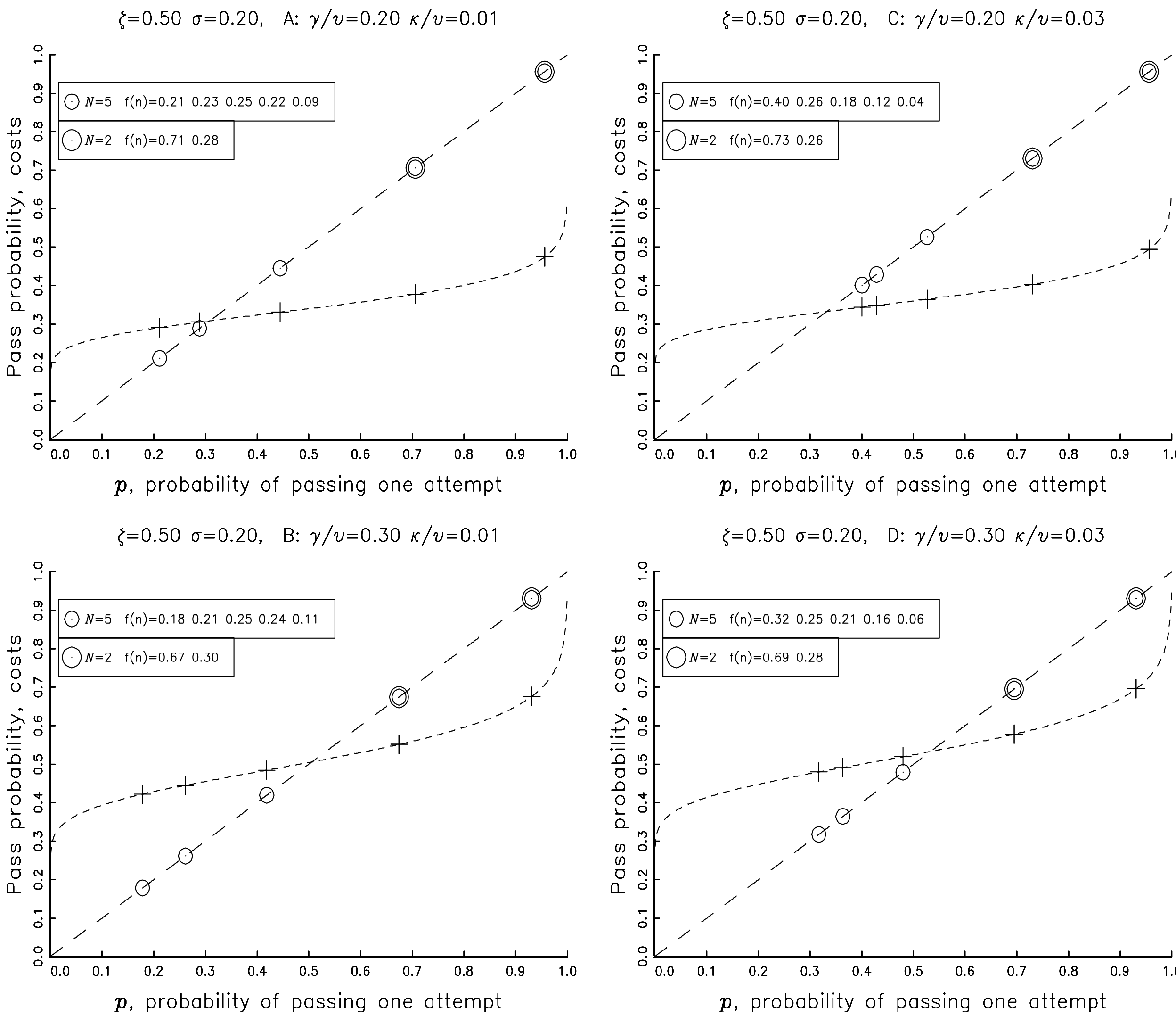

Figure 5: Four examples of students distributing optimal preparation across all attempts.

Student parameters	N	$E[p\|\text{pass}]$	$\tilde{P}^*_N$	$E[n]$	$\Pr(n = N)$	$V_1(p^*_{1\|N}, N)$
γ/υ =0.2, κ/υ =0.01	2	0.779	0.987	1.294	0.294	0.607
	5	0.464	0.996	2.752	0.092	0.688
γ/υ =0.3, κ/υ =0.01	2	0.758	0.978	1.326	0.326	0.423
	5	0.462	0.992	2.897	0.115	0.552
γ/υ =0.2, κ/υ =0.03	2	0.791	0.988	1.270	0.270	0.578
	5	0.494	0.998	2.148	0.044	0.620
γ/υ =0.3, κ/υ =0.03	2	0.767	0.979	1.305	0.305	0.392
	5	0.464	0.995	2.412	0.069	0.474

Table 3: Preparation also between attempts

7.1 Preparation

When as in Section 3.1 all preparation takes place before the first attempt, increasing $\bar{p}$ entails more preparation work, less exam taking work. Which of these effects is more relevant depends on the parameters of the students' problems. Figure 4 shows that for students who find it easier to study the cost/reward function is lower and flatter, and when κ is large has negative slope in the region where fewer average attempts more than offset the cost of better preparation. A larger N changes both the level of the total pass probability and its slope with respect to preparation, which as discussed above becomes steeper at low pass probabilities, but flatter at the optimal choices of the calibrated students, which maximize its distance from the cost/reward function of preparation as shown by arrows in the figure. The cost/reward function is very convex at high p and not so convex at low p so, as shown above, it is hardly possible for a larger N to decrease the total pass probability even when it is very high, or increase p even when it is very small. Its slope must be positive at the optimum, but it is rather small for the students considered: hence, the optimal p declines very significantly when a larger N flattens the total pass probability P as a function of p.

In Figure 5, as in Section 3.2, preparation may be freely distributed before and between attempts, hence starts lower and ends higher than the $\bar{p}^*$ chosen by the same students when preparation is only possible before the first attempt. The optimal preparation sequence for $N = 2$ coincides with the last two attempts of that for $N = 5$, which at earlier attempts is well below 50% in all cases, as low as 18% for weak students with small exam-taking cost κ, and works its way up the preparation cost function and the linear pass probability to eventually reach the $\bar{p}^*_{5|5} = \bar{p}^*_{1|1}$ level that is optimal when further retakes are impossible. Preparation does not depend on κ at the last attempt but a larger κ improves preparation at earlier attempts, especially for strong students with small marginal cost/reward: if taking exams is more costly, the optimal preparation strategy shifts towards fewer, better prepared expected attempts. The legends report the $f(n)$ probabilities of passing at each attempt, which when $N = 5$ are fairly flat before the last attempt, small but significantly larger than zero at the first low-preparation attempt, and slightly hump-shaped with mode $n = 2$ when κ is low.

7.2 Outcomes

The effect of additional retake opportunities on the pass probability and other statistics shown in Tables 2 and 3 is qualitatively sensible and quantitatively remarkable. The total pass probability is already very close to unity for $N = 2$, slightly higher if preparation is sequential as in Table

3 rather than pre-set as in Table 2, and becomes even higher for $N = 5$, especially for weaker students. In Table 2, when $N = 2$ the stronger student passes with total probability $P_2^* = 97.4\%$ with a preparation $\bar{p}_2^*$ that at 84% is much lower than what would be chosen for a single attempt; when $N = 5$ the same student chooses an even lower $\bar{p}_5^* = 58.4\%$, and still passes with higher $P_5^* = 98.7\%$ probability. The weaker student's preparation declines similarly from a lower level, and the resulting total pass probability remains very high.

Increasing N from 2 to 5 implies a dramatic decline of the preparation $\bar{p}^*$ chosen before the first attempt in Table 2. In Table 3, its average counterpart $E[p|\text{pass}]$ when preparation is distributed between attempts is significantly lower for $N = 2$, and declines even more for $N = 5$ and, as Figure 5 shows, all students choose very low preparation at early attempts.

The expected number of attempts $E[n]$ is lower when preparation is chosen before the first attempt in Table 2 than when it is chosen incrementally in Table 3. In both cases it grows larger as N increases from 2 to 5, and is almost 3 when $N = 5$, preparation is sequential, and taking exams is not very costly for students. Taking and administering many unsuccessful exams entails additional work by students and teachers which, unlike that devoted to studying and teaching, does not increase competence.

The tables also report the $\Pr(n = N)$ frequency of last attempts, which for $N = 5$ is less than 5% for all students when all preparation is chosen before the first attempt, and somewhat higher for weaker students with high γ or with low exam-taking cost κ. It is about twice as large, but still small, when preparation can proceed between attempts. Even though the retake limit is rarely binding *ex post*, knowing that many retakes are possible strongly reduces *ex ante* incentives to prepare well.

The expected value of the exam for a student at the first attempt, $\bar{V}(P_N^*)$ in Table 2 and $V_1(p_{1|N}^*, N)$ in Table 3, is computed supposing that all of x requires work (even though some may be costless), and normalized by a given value v of passing the exam (even though it should be smaller when the average competence certified by successful exams is lower). It is of course higher at larger N, partly because the overall pass probability is somewhat higher at given preparation and largely because of less preparation work, especially when sequential preparation makes it quite possible to pass with very little work and additional retake opportunities induce all students to prepare poorly the early attempts.

The intensity of the effect of additional retake opportunities depends in interesting ways on the exam-taking cost κ. Even though students only want to pass and do not care about learning, in the bottom half of the tables a larger κ implies stronger incentives to prepare the exam better and expect to take fewer costly attempts at given N. Average preparation increases slightly, but a larger κ reduces not only the expected number of attempts and the value of the exam but also the incidence of pass outcomes at poor sequential preparation which, even at similar average, confounds the information conveyed by exam results. When $N = 5$, both strong and weak students with $\kappa = 1$ make the first attempt with preparation so low as to pass with only about 20% probability. Similarly strong students with $\kappa = 3$ prepare for the first attempt at about 40%, and similarly weak students with $\kappa = 3$ also find it optimal to increase preparation, albeit only to about 30% because their work is more costly. Early preparation is higher for stronger students, who are more likely to pass at early attempts and preparation levels that for them are very low, though not as low as those of weaker students. In the top-left diagram of Figures 5 a strong student with $\kappa = 1$ who is allowed to retake up to 5 times passes with preparation below 40% with frequency $0.21 + 0.23 = 0.44$. In the top-right diagram, the same student's preparation sequence is bunched together by $\kappa = 3$ and is always at least 40% when the exam is passed, but still below 50% with more than 80% frequency.

8 Discussion

Neither Section 3.1's assumption that preparation is only possible before the first attempt, nor Section 3.2's timeless perspective on sequential preparation represents accurately a reality where students distribute effort over time but discount the future, and opportunities to take exams might become more frequent in a given period or extend to a longer period when N increases. Such extensions of the basic framework have ambiguous implications for preparation incentives. It might be less costly to improve preparation to a given level during longer periods between attempts, but if late success is discounted students should prepare better before the first attempt and after early failed attempts. It would also be possible to allow preparation to depreciate between attempts, even though skills that depreciate fast are not the most useful among those tested by exams.

Reality differs in these and other ways across exams and among students with heterogeneous ability, different ambitions, and more or less accurate information about the exam's syllabus and structure. Because the two stylized frameworks oversimplify reality in opposite ways, however, they bracket the possible implications of retakes in more complex intermediate situations, and their predictions must be realistic when they are similar. Both, and any mixture that allocates some preparation before and between attempts, predict that additional retake opportunities increase the total pass probability of all students (except that of the strongest before the first attempt), reduce the average preparation of all students (except that of the weakest before the first attempt), and increase the expected value of taking the exam for all students (including weak ones, who may prefer not to make any attempt if retake opportunities are few).

The derivation and illustration of these results above take as given the reward for passing, and so do the students and administrators who like many retake opportunities taking as given the market value of degrees. That value, however, must decrease as a larger number of retake opportunities implies lower and more variable preparation for students who pass exams. Preparation and the reward for passing might differ across attempts if their number is observable by third parties who infer preparation from passed exams. While the number of failed attempts is not reported by the transcripts issued by Italian universities, it may to some extent be inferred from the length of time taken to graduate. A further complication is that Italian students are often allowed to decline a passing grade and retake the exam voluntarily, aiming to improve it. If it is possible to decline a passing grade, and retake hoping to obtain a better one, the same logic that shapes incentives to increase the pass probability is applicable to the probability of eventually obtaining a desired grade that is equaled or exceeded with at each attempt with a probability that depends on preparation. The incentives implied by such optional retakes deserve to be characterized in further work, and are relevant not only to Italian university exams but also to standardized tests that providers allow students to retake many times and often do not report all scores on transcripts (perhaps imagining their tests to be so precise that only preparation and concentration may improve the score, and certainly appreciating the test fees paid by weak and rich students who make many attempts).

A smaller reward for passing decreases incentives for students to prepare well, amplifies the implications of retake opportunities at given reward for passing, and may trigger a doom loop towards equilibria where both student competence and degree values are low. The quality of education and exams will go down faster and deeper if, as a referee points out, instructors overwhelmed by large numbers of exams grade them more leniently and less precisely. This mechanism might partly explain why tertiary education in Italy is empirically associated with relative earnings premia that are much smaller than in most industrialized countries (e.g. OECD, 2022, Figure A4.1) and on average only about half as large as in the United States, where failed exams may be remedied only by retaking the whole course the next year. Fewer retake opportunities should lower the total passing rate as most students work harder (but not enough to offset the lower probability of succeeding at least once over a shorter sequence of attempts) and only some of the weakest self-select out of enrolment. However the function that maps competence into each attempt's pass probability, parameterized by the pass threshold and imprecision in the examples of Section 6, is unlikely to remain the same as the number of retake opportunities varies. Changes of the timing and format of exams introduced at the same time as a reduction of retake opportunities by a reform aimed at reducing dropout can explain why Bratti et al. (2024) find that it was associated with a higher frequency of successful exams, hopefully without reducing the competence elicited and certified by exams.

To the extent that retake opportunities have undesirable implications they should be limited in number, or made a less attractive option. Universities around the world enforce a variety of rules, some of which are mentioned and modeled in a less structural framework by Michaelis and Schwanebeck (2016). For exams that students need to pass before proceeding to the next year, British universities typically allow a single "resit capped at basic pass mark" in a late Summer session. In Germany and Switzerland a failed exam may often be retaken, but only once, and accumulating too many "malus" points from failed exams prevents graduation. In Spain, failed exams can be retaken a couple of times a year, and must have been passed by the end of the degree program. In France it is not usually possible to retake exams but very weak ("passable") or even dismal performance on some exams is condoned ("rattrapé") as long the grade average is sufficiently high that year. All such provisions are meant to avoid punishing students' bad luck but make it possible to obtain degrees while sparing the effort needed to be truly competent in a few subjects, and may induce some students to spend more time and effort to studying intricate rules and devising exam-taking strategies rather than to studying textbooks and paying attention to lectures.

When the number and type of retake opportunities must be taken as given, increasing the cost of taking exams should induce students to prepare better before the first attempt, whether or not preparation can be improved afterwards, and reduce the dispersion of preparation across students who pass the exam when preparation is possible between attempts. A student who might randomly and quickly attempt many multiple-choice online tests will find it optimal to prepare better if required to spend hours in the classroom before handing in the answers, or to pay a fee: Italian for-profit universities that offer distance degrees require a substantial payment for appointments to take exams, which are administered in person by proctors. Delayed publication of exam results is also useful because, by reducing the time available to improve preparation before the next attempt, it strengthens incentives to prepare well for early attempts. The model's characterization of student behavior is in fact applicable to researchers who "retake" editorial assessments resubmitting a revised version to the same journal when allowed to do so, or sending a rejected paper to another journal. To discourage frivolous attempts and encourage careful preparation, editors should and usually do refrain from providing editorial decisions quickly.

Appendix

A.1 Characterization of (8)

The derivative (8) is negative for $p \in (0,1)$ if

$$1 - p + Np - (1-p)^{1-N} < 0,$$

which is true for $N > 1$ because the left-hand side is zero at $p = 0$ and becomes increasingly negative as p grows:

$$\frac{d}{dp}\left(1 - p + Np - (1-p)^{1-N}\right) = \left(\frac{(1-p)^N - 1}{(1-p)^N}\right)(N-1) < 0.$$

To show that a larger N makes (8) more negative at each p, differentiate (6)

$$\frac{d}{dp}\eta(p, N+1) = \frac{d}{dp}\eta(p, N) - N(1-p)^{N-1},$$

so

$$\frac{d}{dp}\eta(p, N+1) - \frac{d}{dp}\eta(p, N) = -N(1-p)^{N-1} < 0.$$

A.2 Derivation of (15)

Additional retake opportunities increase the total pass probability if

$$P^*_{N+1} - P^*_N = -\left(\frac{g'(p^*_{N+1})}{N+1}\right)^{\frac{N+1}{N}} + \left(\frac{g'(p^*_N)}{N}\right)^{\frac{N}{N-1}} \tag{A.1}$$

is positive, or

$$\left(\frac{g'(p^*_N)}{N}\right)^{\frac{N}{N-1}} > \left(\frac{g'(p^*_{N+1})}{N+1}\right)^{\frac{N+1}{N}}$$

which can be rearranged and simplified to

$$g'(p^*_N)^{\frac{N}{N-1}} > g'(p^*_{N+1})^{\frac{N+1}{N}} N^{\frac{N}{1+N}} (N+1)^{\frac{N+1}{N}}$$

or, denoting $\frac{g'(p^*_{N+1})}{g'(p^*_N)} \equiv \xi$,

$$(g'(p^*_N))^{\frac{N}{N-1} - \frac{N+1}{N}} \xi^{-\frac{N+1}{N}} > N^{\frac{N}{N-1}} (N+1)^{-\frac{N+1}{N}}.$$

Using $\frac{N}{N-1} - \frac{N+1}{N} = \frac{1}{N(N-1)}$,

$$(g'(p^*_N))^{\frac{1}{N(N-1)}} \xi^{-\frac{N+1}{N}} > N^{\frac{N}{N-1}} (N+1)^{-\frac{N+1}{N}}$$

raising both sides to the power $N(N-1)$,

$$g'(p^*_N)\xi^{-\frac{N+1}{N}N(N-1)} > N^{\frac{N}{N-1}N(N-1)} (N+1)^{-\frac{N+1}{N}N(N-1)}$$

simplifying the exponents using $-\frac{N+1}{N}N(N-1) = 1 - N^2$,

$$g'(p_N^*)\xi^{1-N^2} > N^{N^2}(N+1)^{1-N^2}$$

and multiplying both sides by

$$\xi^{N^2-1} = \left(\frac{g'(\bar{p}_{N+1}^*)}{g'(\bar{p}_N^*)}\right)^{N^2-1}$$

yields (15). That condition can be shown to be less restrictive than (16), which with $\frac{g'(\bar{p}_{N+1}^*)}{g'(\bar{p}_N^*)} \equiv \xi$ reads

$$g'(\bar{p}_N^*) \geq (\xi)^{\frac{N-1}{N}} N^N (N+1)^{1-N},$$

comparing their right-hand sides:

$$\begin{aligned}
&\xi^{N^2-1} N^{N^2}(N+1)^{1-N^2} \div \xi^{\frac{N-1}{N}} N^N (N+1)^{1-N} \\
&\xi^{N^2-1-\frac{N-1}{N}} N^{N^2-N}(N+1)^{1-N^2+N-1} \div 1 \\
&\xi^{N^2-1-\frac{N-1}{N}} N^{N^2-N}(N+1)^{N-N^2} \div 1 \\
&\xi^{N^2-1-\frac{N-1}{N}} \left(\frac{N}{N+1}\right)^{N^2-N} \div 1
\end{aligned}$$

If $\xi = 1$ the left-hand expression is less than unity for $N > 1$, so the right-hand side of (A.3) is smaller than that of (16), and is a less restrictive lower bound on marginal cost/reward. If convexity of $g(\cdot)$ implies $\xi < 1$ and $\bar{p}_{N+1}^* < \bar{p}_N^*$ might imply a lower total pass probability when an additional retake becomes possible, then ξ raised to the positive power

$$N^2 - 1 - \frac{N-1}{N} = (N-1)\left(N + 1 - \frac{1}{N}\right) > 0 \text{ for } N > 1$$

moves that bound further below unity.

A.3 Total pass probability at constant cost/reward slope

When the marginal cost/reward is constant at $\bar{g}'$, more retake opportunities decrease the total pass probability (A.1) if

$$P_{N+1}^* - P_N^* = -\left(\frac{\bar{g}'}{N+1}\right)^{\frac{N+1}{N}} + \left(\frac{\bar{g}'}{N}\right)^{\frac{N}{N-1}} < 0 \tag{A.2}$$

or equivalently

$$\bar{g}' < N^{N^2}(N+1)^{1-N^2}, \tag{A.3}$$

tabulated in column (1) of Table 1. The right-hand side is unitary at $N = 1$, so the inequality is satisfied when $\bar{g}' < 1$ would make $P = 1$ optimal for a single attempt and increasing $N = 2$ reduces the pass probability below unity.

The negative change of P in (A.2) is minimal as a function of $\bar{g}'$ when

$$\frac{d}{d\bar{g}'}\left(-\left(\frac{\bar{g}'}{N+1}\right)^{\frac{N+1}{N}}+\left(\frac{\bar{g}'}{N}\right)^{\frac{N}{N-1}}\right)=\frac{1}{N\bar{g}'\left(N-1\right)}\left(N^2\left(\frac{\bar{g}'}{N}\right)^{\frac{N}{N-1}}+\bar{g}'\left(\frac{\bar{g}'}{N+1}\right)^{\frac{1}{N}}-N\bar{g}'\left(\frac{\bar{g}'}{N+1}\right)^{\frac{1}{N}}\right)$$

equals zero. For $N>1$ the minimum is at $\check{g}'$ such that

$$\begin{aligned} N^2\left(\frac{\check{g}'}{N}\right)^{\frac{N}{N-1}} &= \bar{g}'\left(N-1\right)\left(\frac{\check{g}'}{N+1}\right)^{\frac{1}{N}} \\ \left(\check{g}'\right)^{\frac{N}{N-1}-1-\frac{1}{N}} &= \left(N-1\right)\left(\frac{1}{N+1}\right)^{\frac{1}{N}}N^{-2+\frac{N}{N-1}} \end{aligned}$$

using $\frac{N}{N-1}-1-\frac{1}{N}=\frac{1}{N(N-1)}$

$$\check{g}'=\left(N-1\right)^{N(N-1)}\left(\frac{1}{N+1}\right)^{\frac{N(N-1)}{N}}N^{\left(-2+\frac{N}{N-1}\right)N(N-1)}$$

using $\left(-2+\frac{N}{N-1}\right)N\left(N-1\right)=-N\left(N-2\right)$

$$\check{g}'=\left(N-1\right)^{N(N-1)}\left(\frac{1}{N+1}\right)^{\frac{N(N-1)}{N}}N^{-N(N-2)}, \tag{A.4}$$

tabulated in column (3) of Table 1. Thus, the minimum value of (A.2) is

$$P^*_{N+1}-P^*_N=-\left(\frac{\left(N-1\right)^{N(N-1)}\left(\frac{1}{N+1}\right)^{\frac{N(N-1)}{N}}N^{-N(N-2)}}{N+1}\right)^{\frac{N+1}{N}}+\left(\frac{\left(N-1\right)^{N(N-1)}\left(\frac{1}{N+1}\right)^{\frac{N(N-1)}{N}}N^{-N(N-2)}}{N}\right)^{\frac{N}{N-1}},$$

a rather formidable function of N that returns the values tabulated in column (2) of Table 1.

A.4 Derivation of (17)

Because $p^*_{n+1|N+1}=p^*_{n|N}$ for the last N attempts, $\prod_{n=2}^{N+1}\left(1-p^*_{n|N+1}\right)=\left(1-p^*_{1|N+1}\right)\prod_{n=1}^{N}\left(1-p^*_{n|N}\right)$. Hence,

$$\begin{aligned} \tilde{P}^*_{N+1}-\tilde{P}^*_N &= 1-\prod_{n=1}^{N+1}\left(1-p^*_{n|N+1}\right)-1+\prod_{n=1}^{N}\left(1-p^*_{n|N}\right) \\ &= -\left(1-p^*_{1|N+1}\right)\prod_{n=1}^{N}\left(1-p^*_{n|N}\right)+\prod_{n=1}^{N}\left(1-p^*_{n|N}\right) \\ &= \left(-\left(1-p^*_{1|N+1}\right)+1\right)\prod_{n=1}^{N}\left(1-p^*_{n|N}\right) \\ &= p^*_{1|N+1}\prod_{n=1}^{N}\left(1-p^*_{n|N}\right) \\ &= p^*_{1|N+1}\left(1-\tilde{P}^*_N\right) \end{aligned}$$

as was to be shown.

A.5 Derivation of (20)

With $g'(p) = x(p)\gamma/v$ and $x(p)$ as in (18), the first-order condition (10) for a single attempt without retake opportunities is

$$1 - \frac{\gamma}{v}\frac{\sigma}{\Phi'(\Phi^{-1}(p^*_{1|1}))}e^{\zeta+\sigma\Phi^{-1}(p^*_{1|1})} = 0$$

and with $\Phi'(z) = \exp\left(-z^2/2\right)/\sqrt{2\pi}$, the normal density function, can be rearranged to

$$1 = \frac{\gamma}{v}\frac{\sigma}{\exp\left(-\left(\Phi^{-1}(p^*_{1|1})\right)^2/2\right)/\sqrt{2\pi}}e^{\zeta+\sigma\Phi^{-1}(p^*_{1|1})}.$$

Taking logs yields a quadratic equation in $\Phi^{-1}(p^*_{N|N})$

$$0 = \ln(\gamma/v) + \ln\left(\sigma\sqrt{2\pi}\right) + \zeta + \sigma\Phi^{-1}(p^*_{1|1}) + \left(\Phi^{-1}(p^*_{1|1})\right)^2/2. \tag{A.5}$$

If $\sigma^2 - 2\left(\ln(\gamma/v) + \ln\left(\sigma\sqrt{2\pi}\right) + \zeta\right) < 0$ there is no real solution, and the exam has negative expected value. Otherwise, the solutions of (A.5) are

$$\Phi^{-1}(p^*_{1|1}) = \pm\sqrt{\sigma^2 - 2\left(\ln(\gamma/v) + \ln\left(\sigma\sqrt{2\pi}\right) + \zeta\right)} - \sigma.$$

The derivative with respect to $p^*_{1|1}$ of the right-hand side of the transformed first-order condition (A.5) should be positive to satisfy the second-order condition which, because $\Phi^{-1}(p^*_{1|1})$ is a monotonically increasing function of $p^*_{1|1}$, requires $\sigma + \Phi^{-1}(p^*_{1|1}) > 0$ to be positive. The negative-root solution violates that condition, and identifies a local minimum of the student's objective function. The optimal choice of preparation takes the positive root, and rearranges to (20).

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