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Tax Treatment of Commuter Cost

Abstract

The paper discusses the tax treatment of commuting where wages and housing cost vary across locations. An income tax distorts the locational choices of agents, who dislike commuting and have preferences for place of residence Wages, housing cost and commuting cost determine how subsidising or taxing commuting affects behaviour and social efficiency. A subsidy encourages commuting and induces agents to choose a more favourable living place. The analysis clarifies the circumstances in which the subsidy alleviates or exacerbates the tax distortions, also where housing is tax favoured, as is often the case. The distributional impact depends on the effects of wages on commuting. An empirical illustration based on Norwegian data shows how one can infer efficiency effects of responses to subsidies on commuting.

JEL-Codes: H210, H240.

Keywords: income tax, commuting, commuter cost, subsidies on commuting, place of residence.

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1. Introduction[1]

Many workers incur sizeable travel expenses commuting between their home and their workplace. In many countries commuting to work is subsidised, for instance by making travel expenses deductible against gross income when computing taxable income[2]. There may be both social efficiency arguments and equity arguments for this arrangement. It may be a measure to mitigate the distortionary effect of income taxes by lowering the cost of working to stimulate labour supply. This is an argument in the spirit of a strand of literature on how commodity taxes or subsidies can alleviate labour supply distortions (see, for example, Corlett and Hague (1951), Christiansen (1984)). This is however a contentious issue. It has been argued that, rather than encouraging people to travel to well-paid jobs, a travel subsidy prompts people to choose attractive places to live even if it involves commuting to work.

The purpose of the current paper is to discuss subsidies on commuting primarily from a social efficiency perspective in a context where people choose place of residence and choose among workplaces offering different wages. The government taxes wages by an income tax, which discourages workers from pursuing higher earnings through choice of workplace. This is a standard tax distortion. In order to develop an analysis that is applicable to empirical cases, we also include subsidisation of housing due to various tax favours, which is another pre-existing distortion in many countries.

We analyse the tax treatment of commuting in a setting where the attainable wage and the housing cost are increasing along a line from a point on the periphery to some central point. Based on pure preferences, there is a preferred living place, and a living place is less valued the further away it is located from the favourite place of residence. Agents can have different preferences for place of residence. The geographical wage dispersion reflects that labour is not equally productive in all places. Besides the geographical variation, people have different innate abilities and are not equally productive even in the same place, as is a standard assumption in optimal income tax models following Mirrlees (1971). How steeply the wage of an agent increases depends positively on his/her productive skill. Each agent chooses a place of residence and a workplace along the wage line. The commuting distance is the distance between the two points. Beyond the concern with living place, the individuals derive utility from consumption and shortness of commuting distance. The circumstances determining the choices of the agent are the after-tax wage, the commuting cost, the cost of housing, and the preferences.

The location of the workplace, and hence the wage earned by an agent, is determined by the place of residence and the commuting distance from there. Where people have a preferred place of residence a subsidy has two effects. It encourages commuting and it induces an agent to move closer to the preferred place of residence, which may be a movement towards a higher-paying or a lower-paying workplace. In the former case the overall effect on earnings is unambiguously positive and a subsidy mitigates the tax distortion. In the latter case the opposite effect is a conceivable outcome. One should note that it is the change in *earnings* that determines the impact on social efficiency. Several empirical studies have estimated the effect on the length of commutes. This is an interesting positive issue, but from our normative

[1] This paper is part of the research at Norwegian Fiscal Studies at the Department of Economics, University of Oslo. Comments by Thor Olav Thoresen are gratefully acknowledged.

[2] Two methods are common: The deduction is based on declared actual cost or commuting distance.

perspective the crucial question is whether a longer commute leads to a larger wage and not only to longer journeys to better living place.

Combining our theory of the marginal trade-offs of commuters with data on the prices, wages and taxes determining behaviour, we show how one can infer empirically the efficiency effects of marginally changing a subsidy on commuting. Where a marginal relocation of place of residence or workplace is economically harmful (beneficial) the commuter must incur a non-pecuniary benefit (disbenefit) from the move to be initially at a private optimum. It follows whether the move is desirable or not according to the pure preferences of the commuter. We can then infer the direction of the responses to a marginal subsidy. Also computing empirically, the tax/subsidy wedges we can then assess whether the induced responses are socially beneficial or harmful. We illustrate this approach by using data for the region around Oslo in Norway. To capture differences in productive skill, we distinguish between more and less educated agents. We conclude that a larger subsidy on commuting enhances efficiency for those with lower education while it is uncertain but plausible that it also enhances efficiency for those with higher education, and accordingly aggregate efficiency. One should note that this is a case-specific result in our illustration. The general contribution is the approach developed to enable empirical inference about efficiency effects based on a limited set of data.

From a distributional perspective, subsidising commuting will benefit those with large commuter cost. How commuting varies with the wage is determined by income and substitution effects. Assuming that shortness of commuting distance is a non-inferior good, those with higher income tend to commute a shorter distance and benefit less from the subsidy. On the other hand, those with a higher wage have more to gain from commuting and various cross-effects come into play. It is an empirical question which are the dominant effects and who are the main beneficiaries.

Our approach should be interpreted as taking a long-term perspective. In the short term there are presumably considerable inertia, adjustment costs and constraints when it comes to change of residence and workplace. We shall briefly mention the polar cases where place of residence is fixed, but otherwise we consider the outcome of long-term optimisation that is unconstrained by short-term concerns.[3]

In practice, workers choose labour supply at many different margins (working hours, work intensity, occupation, type of job, retirement age, etc.). To focus on commuting, we abstract from other margins in our analysis. We should however bear in mind that where commuting is a cost of working a change in the commuter cost is likely to affect an agent's willingness to participate in the labour market.

The current paper has several features in common with Wrede (2009), while deviating in other respects. In both models, people reside along a straight line, earn wage income, pay a distortive income tax, and face a, possibly subsidised, commuting cost. Both papers have geographical wage dispersion which is modelled as discrete with two workplaces (central business districts) in Wrede's model and continuous in this paper. In the former model everybody earns the same income in each of the two workplaces, while in our model workers earn different incomes in each of a continuum of workplaces. In Wrede's model individuals have identical utility functions defined over consumption and land (housing lot size). In contrast to our model,

[3] However, there are findings in the literature indicating that our caveat may be of less relevance. Boehm (2013) claims that individuals react swiftly to changes in the tax treatment of commuting expenses.

all individuals achieve the same utility level in equilibrium. In our model, agents have utility functions defined over consumption, commuting distance, and, more importantly, heterogeneous preferences for place of residence. In Wrede's model, land/housing prices will vary with distances from the two workplaces. In the current paper the housing price is assumed to vary monotonically with location between the periphery and the centre. Wrede's paper is explicit about the market for land, which is suppressed in our paper. We simplify by assuming directly geographical variation in housing cost. On the other hand, the current paper addresses more explicitly the locational choice of residence and the choice of commuting distance, and how the two decisions may reinforce or oppose each other. It describes more explicitly how responses to a commuting subsidy is conditional on prices, agent-specific wages and heterogeneous preferences. It also captures conceivable pre-existing distortions of housing prices due to favourable tax treatment, often observed in practice. This is important when applying our analysis to assess policy in many countries, but absent in previous literature on the tax treatment of commuting expenses. A key contribution, facilitated by our set-up, is to present a framework for assessing empirically the efficiency effects of changing the subsidisation of commuting, based on a limited set of data on prices, wages, and tax rates.

In the next section we extend the presentation of related literature. Section 3 presents the model. In Section 4 we describe the agents' behaviour and responses to tax policy, while Section 5 addresses the social efficiency and welfare arguments. The subsequent section presents empirical illustrations based on Norwegian data. Section 7 concludes.

2. Related literature.

Previous literature on tax treatment of commuting expenses takes different approaches and address various aspects both akin to and beyond the scope of our analysis.

In an early contribution, Wrede (2001) discusses tax deductibility of commuting expenses in the presence of wage taxes and some degree of attachment to home in a model with two regions. In each region land is scarce, and residents demand land for housing and firms use land and labour as inputs in production. Commuting is costly in terms of goods (commuting expenses) and time (leisure foregone). The government has a revenue requirement. The key question is whether (first best) efficient allocations can be supported as market equilibria where the government imposes a wage tax but allow commuting expenses to be deductible at a certain rate, which may optimally exceed 100%. Wrede (2009) takes a second-best approach, as outlined above.

Agrawal et al. (2024) presents a positive and empirical analysis of how commuting subsidies affect commuting distances and the match quality of workers and firms. The basis is a model where workers are assumed to have a fixed residential location and choose work location trading off a higher wage against a longer commute. The empirical analysis exploits actual changes to the tax treatment of commuting expenses in German tax law. It documents that more generous commuting deductions increase commuting distance. A further result is that commuting subsidies allow for high-ability individuals to better match with higher-paying firms and thus disproportionally improving the earnings of already high-income workers. There is assortative matching.

Paetzold (2019) exploits Austrian data and tax rules to study the effect on commuting distances. of subsidising commuting expenses. The conclusion is that commuting subsidies do indeed enhance the length of the commute.

Boehm (2013) exploits data from Germany to estimate the effect of commuting costs on the decision to switch job and move houses. He concludes that individuals are more likely to move and switch jobs when tax breaks change, and they are more likely to do so for the resulting commuting distance to be shorter. A finding is that the average individual is more likely to be willing to switch jobs rather than to move houses.

Richter (2006) analyses tax treatment of commuting costs subject to the assumption that commuting does not generate additional income. Willingness to pay for commuting may rather be due to savings of housing cost. The analysis concludes that under these assumptions commuting should be taxed as such a tax would diminish the income tax distortion of labour supply defined as hours worked.

Borck and Wrede (2009) discuss the use of commuting subsidies to internalise agglomeration externalities. The paper also introduces a distinction between intracity and intercity commuting.

Wrede (2000) addresses a particular aspect of commuting, namely that commuters can cut their travel time by spending more on faster modes of travel. Time saving increases the time endowment available for work and leisure. Where the labour-leisure trade-off is distorted by an income tax, there is a case for letting commuting expenses be partially deductible when determining taxable income. The concern of that paper is neither the choice of locations of work and home nor commuting distance. The current paper, like most other papers in the field, leaves aside the choice of modes of travel.

We confine our attention to the case where there is no geographical differentiation of the tax rate across local communities within the commuting region. Wrede (2001) briefly discusses deductibility of commuting expenses where a wage tax is levied at different rates according to the residence principle. Agrawal and Hoyt (2018) address the spatial distortions created by inter-jurisdictional tax differentials.

Another aspect of transport subsidies, also beyond commuting subsidies, is concern with the spatial expansion of cities, which is addressed in Brueckner (2005). While most papers, like ours, take a normative approach focusing on social efficiency, Borck and Wrede (2005) present a political economy approach.

3. Model with preference for place of residence.

Suppose that, from some starting point, y is the distance to the workplace, and r is the distance to the place of residence, where the direction is from the periphery towards the centre. Obviously, the distance between the two places is the commuting distance. We denote by k the cost of travelling one unit of distance, say one kilometre or mile[4]. We assume that the output generated by an agent varies with location. We express the output as wy, where $w > 0$ is a

[4] Travelling can cause environmental and other external costs. This is not our concern in this paper. We assume that any external costs are internalised and included in k.

constant parameter, which determines the wage of an agent at various locations assuming that the agent's wage is equal to the output he produces [5]. The assumption is that the wage increases with centrality. As an analogy to a conventional labour supply model, we might conceive of w as a wage rate and y as a measure of labour supply. Consumption, c, is output (income) net of the commuting cost and cost of residence, p. We assume that the housing cost varies according to place of residence so that p is a function of r, $p(r)$. We shall assume that $p'(r) > 0$, i.e. $p(r)$ increases with proximity to the centre and is positively correlated with income. We assume that an agent derives utility from consumption, c, but dislikes commuting. We note that $y \geq r$. If the converse were true ($y < r$) the agent could both diminish commuting (the distance between r and y) and obtain a larger income by increasing y for a fixed r. Denoting the commuting distance by z, we have $z = y - r$. We can conceive of z as a bad or $-z$, i.e., short commuting distance, as a good.

We shall assume that, neglecting economic costs and resources, there is a preferred place to live, r^*, and what matters for utility is the departure of the actual place of residence, r, from r^*. The utility function is then

$$u(c, z, r - r*) \qquad (1)$$

The partial derivatives of the utility function are

$u_c > 0, u_z < 0, u_{r-r*}$, where $u_{r-r*} > 0$ for $r < r*$, $u_{r-r*} = 0$ for $r = r*$ and $u_{r-r*} < 0$ for $r > r*$.

We may note that this would be the case if we let the utility function capture the impact of the deviation of r from r^* through a term

$q = -(r * -r)^2$, and $dq/dr = 2(r * -r)$.

We let $m = \frac{-u_z}{u_c} = \frac{-u_y}{u_c}$ denote the marginal valuation of reduced commuting distance, which is equivalent to the marginal disbenefit incurred by travelling an extra unit of distance.

$n = \frac{u_{r-r*}}{u_c}$ denotes the marginal valuation of distance from the preferred place of residence, i.e. the positive marginal benefit of moving closer to the preferred place or the negative marginal benefit of moving further beyond it.

We assume that the marginal valuation of reduced commuting distance is higher when the initial distance is larger and when consumption is larger: $m_z > 0$ and $m_c > 0$, where, as above, subscripts denote partial derivatives.

Assumption 1. $m_z > 0$ *and* $m_c > 0$

How m is affected by living closer to or further away from the preferred place of residence, i.e., the sign of m_{r-r^*} , is an open question. Likewise, it is hard to tell how the valuation of place of residence is affected, if at all, by the commuting distance. The sign of n_z is indeterminate. Absent any clear notion of these signs, we shall at various stages invoke the following assumption.

[5] We may assume that nobody works at locations given by y-values close to zero.

Assumption 2. $m_{r-r*} = n_z = 0$.

We assume that where the marginal benefit of increasing r is positive ($r < r*$) it is larger the larger the consumption level, c: $n_c > 0$. If the marginal benefit is negative ($r > r*$) it is smaller (larger in absolute value) the larger c is: $n_c < 0$. This is tantamount to saying that a marginal move towards the preferred place is more highly valued at a larger consumption level.

We assume that if increasing r is beneficial ($r < r*$), the marginal benefit is smaller the closer one comes to the preferred place of residence, i.e., when $r - r*$ rises. When the marginal benefit of increasing r is negative ($r > r*$), it diminishes when moving further away from $r*$. The upshot is that $n_{r-r*} < 0$.

Assumption 3. *Where* $n > 0$, $n_c > 0$. *Where* $n<0$, $n_c < 0$. $n_{r-r*} < 0$.

In the absence of taxes, the budget constraint is

$$c = wy - k(y - r) - p(r) + a = wr + (w - k)z - p(r) + a$$

where a is an exogenous income. The budget available for consumption is income minus commuting and housing cost.

Alternately, we can express the budget constraint in terms of three goods. $a = c + [p(r) - wr] + [w - k](-z)$

where the exogenous income is available for acquiring general consumption, place of living and a short commuting distance. We note that we can perceive an augmentation of –z as a shortening of the commuting distance.

The unit cost of shortening the commuting distance for a fixed place of residence is the wage foregone net of the saving of commuting cost. The net cost of moving towards the centre, i.e., increasing r for a fixed commuting distance, is the additional housing cost minus the attained pay rise.

Let us now introduce taxes and subsidies into the model. We assume there is an income tax with tax rate t. Moreover, commuting is subsidised at a rate s and the housing cost may be subsidised at a rate x. The reason for considering the latter possibility is that an important part of housing cost is mortgage interest which can be tax favoured for instance by being tax deductible. If the alternative to investing in housing is to deposit the wealth into a bank account, the interest foregone is interest net of tax where the interest is taxed, while the imputed housing rent is untaxed. The budget constraint is then:

$$\begin{aligned} c &= (1 - t)wy - (1 - s)k(y - r) - (1 - x)p(r) + a \\ &= (1 - t)wr + ((1 - t)w - (1 - s)k)z - (1 - x)p(r) + a \end{aligned} \quad (2)$$

$$a = c + [(1 - t)w - (1 - s)k](-z) + [(1 - x)p(r) - (1 - t)wr] \quad (3)$$

We may now interpret a as a lump sum transfer or, if negative, a lump-sum tax.

The effect of a tax on earnings is to lower the private cost of shortening the commuting distance by depressing the earnings foregone, and to lower the return to a move towards a workplace offering a higher wage. We let $w \in [\underline{w}, \overline{w}]$ and $r^* \in [\underline{r}, \overline{r}]$. We shall assume that it is possible for all agents to have earnings exceeding the housing cost. To simplify, we shall assume that $p'(r) = p'$ is a positive constant and so is $(1-x)p'$.

4. Agent behaviour

In general, the agent is assumed to choose z and r (and hence $y = r + z$) to maximise utility (1) subject to the constraint (2). We may note that if one changes z for a fixed r or changes r for a fixed z there is in either case a change of workplace. The first order conditions are given by equations (4) and (5) below.

$$(1-t)w - (1-s)k - m = 0 \tag{4}$$

The pay rise acquired by extending the commuting distance, $(1-t)w$, is equated to the marginal pecuniary cost, $(1-s)k$, plus the marginal disbenefit in monetary terms, *m*, of extending the commuting distance.

We shall confine attention to the case where $w - k > 0$ and $w(1-t) - k(1-s) > 0$.

Assumption 4. $w - k > 0$ *and* $w(1-t) - k(1-s) > 0$.

We can note that in the special case where r is fixed the optimum is characterised by (4) alone. Where r is chosen, we also get the condition

$$n + (1-t)w - (1-x)p' = 0 \tag{5}$$

The marginal benefit, *n*, of increasing r, which is positive where $r < r*$ and negative where $r > r*$, plus the pay rise acquired by changing workplaces, $(1-t)w$, is equated to the increase in housing cost, $(1-x)p'$.

We should note that condition (5) enables us to infer the individuals' preferences for place of residence from information about housing prices and wages. We shall make use of this revealed preference approach in various places in the subsequent analysis.

To explore compensated effects, to be essential in the analysis below, we consider a fixed utility, denoted u_0. Then

$$u(c, z, r - r*) = u_0, \tag{6}$$

which implicitly defines c as a function of z and r: $c(z, r - r*)$. We shall refer to a compensated subsidy where utility is preserved. Before we discuss the general case, let us consider for a moment the special case where r is fixed. This is the case considered in the positive analysis of subsidy responses in Agrawal et al. (2024). We let subscript s denote the compensated derivative with respect to s. The compensated effects are derived in the appendix. In the special case where r is fixed (see eq. a2),

$$y_s = z_s = \frac{k}{c_{zz}} > 0 \tag{7}$$

The double subscript indicates a second order derivative. Where the place of residence is fixed, the compensated effect of a subsidy on commuting is to induce the agent to choose a better-paying workplace and a longer commuting distance.

If, on the other hand, we have the special case where workplace, y, is fixed a commuting subsidy can have various effects depending on prices, wages, and preferences. Neglecting conceivable housing subsidies, we would have a private optimum characterised by

$$n + m = p' - (1 - s)k$$

We can interpret $n+m$ as the non-pecuniary benefit from moving towards the centre and shortening the commuting distance by retaining the initial workplace. At the optimum, the marginal benefit, $n+m$, is equated to the net marginal cost of such a move, which is equal to the increment in housing cost minus the saving of commuter cost, $p' - (1 - s)k$.

Increasing s raises the marginal cost and provides an incentive to lower r, yielding a superior or inferior place of residence depending on whether n is positive or negative.

In the general case, where the agent chooses both z and r, we find, as shown in the appendix (see eqs. a3 and a4):

$$z_s = \frac{k}{D}(n_c n - n_{r-r*}) > 0 \qquad (8)$$

Workers will respond to a (larger) subsidy by commuting a longer distance.

$$c_s = -\frac{k}{D} m n_{r-r*} > 0 \qquad (9)$$

$$r_s = \frac{k}{D}(m_c n - m_{r-r*}) = \frac{k}{D}(n_c m + n_z)$$

Absent any clear notion of the respective signs of m_{r-r*} and n_z, we set $m_{r-r*} = n_z = 0$.

Then, under Assumption 2:

$$r_s = k\frac{m_c n}{D} \qquad (10)$$

$r_s > 0$ or $r_s < 0$ according as $n > 0$ $(r < r*)$ or $n < 0$ $(r > r*)$.

The upshot is that a compensated subsidy on commuting provides an incentive to move closer to the preferred place of residence. The crucial question is then in which direction workers prefer to move. Where $r < r^*$ a more favourable place of residence is reached by moving towards higher-paying workplaces. This happens where $n = (1 - x)p' - (1 - t)w > 0$. The housing cost increases more steeply than the wage. According to pure preferences, the agent would like to move closer to the centre, but the benefit from such a move is just offset by a financial cost. Where $r > r*$ a more favourable place of residence is reached by moving towards lower-paying workplaces. This happens where $n = (1 - x)p' - (1 - t)w < 0$.

The effects are pure substitution effects. The subsidy on commuting makes short commuting distance a more expensive good as it diminishes the private cost saving from reducing the distance. The cost increase induces substitution towards general consumption and a more attractive place of residence.

We can summarise effects in the following:

Proposition 1. *Where workers have preference for place of residence and choose place of residence and commuting distance, a compensated subsidy on commuting leads to longer commuting distance and induces workers to increase general consumption and to move closer to the preferred place of residence. The move will be towards the centre if the housing cost rises more steeply than does the disposable income* $(n = (1-x)p' - w(1-t) > 0)$*, and the move will be away from the centre if the disposable income rises more steeply than does the housing cost* $(n = (1-x)p' - w(1-t) < 0)$.

It is obvious that in the special case where place of residence is fixed, the subsidy induces workers to choose workplaces offering higher wages. In the general case, the effect on *y* is

$$y_s = z_s + r_s = \frac{k}{D}(n_c(n+m) - n_{r-r*} + n_z) = \frac{k}{D}(n(n_c + m_c) - n_{r-r*} - m_{r-r*})$$

Where $m_{r-r*} = n_z = 0$,

$$y_s = r_s + z_s = \frac{k}{D}(n(n_c + m_c) - n_{r-r*})$$
$$= \frac{k}{D}(n_c(n+m) - n_{r-r*}) = \frac{k}{D}n_c\big((1-x)p' - k(1-s)\big) - \frac{k}{D}n_{r-r*} \quad (11)$$

after invoking the first order condition.

When agents move closer to higher-wage workplaces ($r_s > 0$) and commute a longer distance on top of that ($z_s > 0$), it is trivial that y increases. Where $r > r*$ $(n < 0, n_c < 0)$ more favourable places of residence are attained by lowering *r*, i.e., moving to a less expensive living place and towards lower-wage locations. Then y will increase where $n + m = \big((1-x)p' - k(1-s)\big) < 0$, i.e., the travel cost per unit of distance exceeds the increase in housing cost[6]. We may observe that $n + m$ is the nonpecuniary valuation of increasing *r* and reducing *z* while keeping y unchanged, i.e., reducing the commuting distance by moving closer to a given workplace. We note that $n + m < 0$ implies that $m < -n$. The marginal disbenefit incurred by extending the commuting distance is smaller than the marginal benefit obtained by lowering *r*. Then the increase in *z* dominates the lowering of *r* and *y* increases. However, where the marginal disbenefit incurred by extending the commuting distance exceeds the marginal benefit obtained by lowering *r*, the net effect may be a lower *y*. A necessary condition for *y* to decline is that $n < 0$ $(n_c < 0)$ and $m > -n$, i.e., the marginal disbenefit incurred by extending the commuting distance is larger than the marginal benefit obtained by lowering *r*. We note that this is not a sufficient condition since in eq. (11) $-n_{r-r*} > 0$. To summarise these findings, we state

Proposition 2. *Where workers have preference for place of residence and choose commuting distance and place of residence, a compensated subsidy on commuting will induce workers to increase earnings in the case where the rise in housing cost exceeds the wage increase, and in the case where the rise in housing cost is below the commuting cost per unit of distance. Otherwise, the impact on earnings is indeterminate.*

The sign of $n = (1-x)p' - w(1-t)$ obviously depends on *w*. We note that if p' is independent of *w* there are three possibilities. It is possible that $n = (1-x)p' - w(1-t)$ is positive or negative for all *w* in the population, or it may be that it is positive for small values

[6] Since $m > 0$, $n + m < 0$ implies that $n < 0$ and $n_c < 0$.

of w and negative for large values of w[7]. It may, however, be that p', which measures how housing cost varies with distance, is not the same for all w. Due to income effects, it is plausible that that richer people have larger housing expenses so that p' increases with w. It is still possible that $(1-x)p'$ may remain above or below all values of w or above only lower values of w. However, a conceivable novel case is that $(1-x)p'$ is below w for smaller values of w and above for larger values of w. Which case that prevails is obviously an empirical question, to which we shall get back in Section 6.

Income effects are given by the effects of a on the various goods. We shall assume that all goods are non-inferior. In formal terms $c_a > 0$, $(-z)_a > 0$, i.e. $z_a < 0$. Where $r - r* < 0$, r is a good at the margin and $r_a > 0$. Where $r - r* > 0$, r is a bad at the margin and $r_a < 0$.

Invoking the envelope theorem, the effect of a subsidy increment ds in terms of income is equal to $kzds$, and the Slutsky equations are

$$\frac{\partial r}{\partial s} ds = r_s ds + kzr_a ds$$

$$\frac{\partial z}{\partial s} ds = z_s ds + kzz_a ds$$

where we recall that subscript s denotes compensated effects of a subsidy.

5. Social efficiency and welfare effects

The social gain from increasing z at the margin (and hence increasing y for a fixed r) is equal to the extra output, given by w, minus the additional pecuniary and non-pecuniary commuting cost, $k+m$, which yields the net effect $w-k-m$. Substituting for m from the first order conditions of the agent's optimisation, we get $(tw - sk)dz$. We can interpret $tw - sk$ as a net-of-subsidy tax wedge, i.e., the discrepancy between the social and the private gain. Where the tax wedge is positive there is a downward distortion of commuting. Commuters refrain from commuting extra kilometres to earn a larger income because part of the additional income is taxed away. We note that a subsidy on commuting diminishes the tax wedge. The social gain from increasing r at the margin (and hence increasing y for a fixed z) is equal to the extra output, given by w, plus the extra benefit from changing place of residence minus the additional housing cost, i.e. $w+n-p'$. Substituting for n from the first order condition of the agent's optimisation, we get $(tw - xp')dr$. Also $tw - xp'$ can be interpreted as a net-of-subsidy tax wedge. Where $x=0$, there is a pure income tax wedge, which discourages people from taking up residence in higher-wage, but more expensive, locations. The latter effect is however diminished where housing is subsidised through tax favours curtailing the additional housing cost. Where the subsidy is large it may override the income tax ($xp' > tw$) such that there would in fact be a negative tax wedge (a net subsidy) encouraging moves towards the centre. On the other hand, a larger x makes it more likely that $n = (1-x)p' - w(1-t)$ becomes negative so that r diminishes.

[7] We should however note that a rise in w is softened by an accompanying rise in t where there is marginal tax progressivity.

An agent's contribution to tax revenue is $\rho = twy - sk(y-r) - xp(r) = twy - skz - xp(r)$. Let us start by considering a single (type of) agent[8] and let us introduce a subsidy on commuting financed by a lump sum tax, i.e., a change in a. Note that in case there is a homogenous population it could be treated as a single individual by normalising the population to unity. Then ρ would be equal to total tax revenue, denoted by R. Welfare could then be enhanced for a fixed t if there is a pair of increments ds, da that would keep utility unchanged while raising more revenue, because the additional revenue could in a next step be transferred to the agent to enhance utility while maintaining the initial tax revenue. To have an optimal combination of s and a, the pair of utility-preserving increments would have to be revenue neutral. To keep utility unchanged, we must have $\frac{\partial u}{\partial a}da + \frac{\partial u}{\partial s}ds = u_c\left(da + kzds\right) = 0$ implying that $da + kzds = 0$. The effect on tax revenue is then

$$d\rho = twy_a(da + kzds) - skz_a(da + kzds) - xp'(r)r_a(da + kzds) - skz_s ds + twy_s ds - xp'(r)r_s ds$$

$$= twy_s ds - skz_s ds - xp'(r)r_s ds \qquad (12)$$

where y_s and z_s denote the respective compensated effects as the pair of increments da, ds keep utility unchanged.

Eq. (12) shows how induced changes in z and r affect social efficiency. We note that in case there is a positive tax wedge associated with z, $tw\text{-}sk>0$, as is obviously the case where $s=0$, $z_s > 0$ yields a partial beneficial social effect. Where there is a positive net tax wedge associated with r, $tw - xp' > 0$, there is a further beneficial social impact in case r increases whereas the converse effect occurs when r diminishes. As we observed above, r increases where the rise in housing cost exceeds the wage increase $(n = (1-x)p' - w(1-t) > 0)$, whereas r diminishes in the opposite case. We shall address empirical cases in Section 5.

Alternately, we can write

$$d\rho = (tw - sk)z_s + (tw - xp')r_s ds\,, \qquad (13)$$

which in turn is equivalent to

$$d\rho = (tw - sk)y_s + (sk - xp')r_s ds\,. \qquad (14)$$

The behavioural-induced change in tax revenue is a measure of the social efficiency enhancement generated by a subsidy on commuting. Increased earnings achieved by opting for a higher-wage workplace $(y_s > 0)$ will alleviate the net-of-subsidy income tax distortion, as reflected by the former term. Moving closer to the workplace (r increases for a fixed y) shortens the commuting distance entailing a saving of subsidy payments to commuting, while increasing subsidy payments to housing. We may note that in case there are initially no subsidies ($s=x=0$) the effect tends to zero where the change of earnings, y_s, vanishes. This confirms the finding

[8] This case is only helpful in order to highlight efficiency effects in a second best setting. If agents were indeed homogenous the first best optimum could be implemented by means of a poll tax.

of Richter (2006) that there is no case for a commuting subsidy if it has no effect on earnings ("does not earn taxable wage income", see *op cit*. p.689)[9].

Considering a departure from the situation without subsidies (x=$s=0$), we have

$$d\rho = twy_s ds = tw(z_s + r_s)ds$$

There is an efficiency enhancement where workers are induced to increase earnings. The result is similar to that of Wrede (2009, Proposition 1). In his model there are only two workplaces (central business districts) and earnings will increase where workers move from the low-productivity to high-productivity location. In the current model workers increase earnings by opting for workplaces with higher productivity along a line with a continuum of workplaces. As set out above, it studies more in depth the interaction between residence choice and choice of commuting distance. A reason for this concern is that it has a bearing on the choice of workplace. It throws light on the question whether the effect of a commuting subsidy is simply to induce agents to choose more attractive places of residence, as sometimes claimed in public debate. A second reason is that the residence choice has efficiency ramifications where the housing cost is subsidised, which is often an important concern, especially when considering empirical effects.

The circumstances in which a subsidy stimulates earnings and enhances utility are given in Proposition 2 above. The upshot is that a compensated subsidy on commuting induces substitution from shortness of commuting distance to general consumption and a more attractive living place. Earnings and efficiency increase in case the agent is encouraged to move house closer to the centre or there is only a weak inducement to move further away from the centre.

In general, we can consider the effect of slightly increasing a subsidy on commuting for any value of x and assuming that $tw > sk$. Then the former term in (13) is positive, whereas the latter term can have either sign.

We can state

Proposition 3.

Assuming that $tw > sk$, there is an unambiguous social efficiency case for increasing a subsidy on commuting

i. *where moving closer to the centre is desirable according to pure preferences, $n = (1-x)p' - w(1-t) > 0$, and there is a positive net tax wedge associated with r, $tw - xp'>0$, which is always the case where* x=0.

ii. *where moving further away from the centre is desirable according to pure preferences, $n = (1-x)p' - w(1-t) < 0$, and there is a negative net tax wedge associated with r, $tw - xp'<0$.*

In other cases, the net effect is ambiguous because the beneficial effect of increasing the commuting distance is to a smaller or larger extent offset by a change of living place with a negative impact on social efficiency.

[9] Unlike the assumption of this paper, Richter assumes that there is an endogenous labour supply in terms of hours worked, being distorted by an income tax. Then one could enhance efficiency by taxing commuting to raise revenue enabling the government to lower the income tax and ease the distortion.

Considering the special case where $s=0$, Proposition 3 states conditions under which it is desirable from a social efficiency perspective to introduce a subsidy on commuting.

Where there is a negative social efficiency effect of a (positive) subsidy, there would of course be an argument for a tax on commuting.

Based on eq. (14) rather than eq. (13) we can state conditions for an efficiency enhancement as follows.

Proposition 4.

Assuming that $tw > sk$, there is an unambiguous social efficiency case for increasing a subsidy on commuting

1. *where earnings increase (cf. Propostion 2), the distance to the living place increases, and $tw - xp'>0$ (which is always the case where* x=0),
2. *where earnings increase, the distance to the living place decreases, and $tw - xp'<0$.*

To study heterogeneity with respect to w, we now turn to the general case where there is a continuum of agent types, each characterised by a preference parameter, r^*, and a value of w, which reflects the skill or productivity of the agent[10]. We assume that the distribution of agents is given by the density function $h(r^*, w) = g(r^*)f(w)$, where $g()$ and $f()$ are the marginal density functions and $h()$ is the simultaneous density function. We normalise the size of the population to unity.

The aggregate tax revenue net of subsidies and transfer is

$$R = \int\int [twy - skz - xp(r) - a] f(w)dw\, g(r*)dr*.$$

Social welfare is given by

$$\Omega = \int\int u(c, z, r - r*)\, f(w)dw\, g(r*)dr* = \int\int u\big((1-t)wy - k(1-s)z - (1-x)p(r) + a, z, r - r*\big) f(w)dw\, g(r*)dr*$$

To characterise the optimal policy, we formulate the Lagrange function

$$\begin{aligned}\Lambda &= \Omega + \mu R \\ &= \int\int u\big((1-t)wy - k(1-s)z - (1-x)p(r) + a, z, r - r*\big) f(w)dwg(r*)dr* + \\ &\mu \int\int (twy - skz - xp(r) - a)\, f(w)dwg(r*)dr* \qquad (15)\end{aligned}$$

where μ is the shadow value of government revenue. A useful and widely used term in the optimal tax literature is Diamond's (1975) social marginal valuation of income defined as $\lambda = u_c + \mu twy_a$. Considering the no-subsidy case ($s=x=0$), and assuming that the transfer is set optimally, we have the special case

$$\frac{\partial\Lambda}{\partial s} = \int [cov(\lambda, kz) + \mu \int (twy_s)\, f(w)dw]\, g(r*)dr*$$

as derived in the appendix (see eq. a19). The covariance is computed across all values of w.

[10] This is similar to the assumption in the Mirrlees model (Mirrlees, 1971) where the wage per unit of labour supply is determined by the agent's skill level.

Where x and s may deviate from zero, we can define a modified marginal valuation of income as $\lambda^* = u_c + \mu\left(twy_a - xr_a p' - skz_a\right)$, where the terms in parentheses capture the marginal net propensity to pay taxes when the exogenous income increases, including effects on both positive and negative taxes (subsidies). Making use of λ^*, (see eq. a17), we can write

$$\frac{\partial \Lambda}{\partial s} = \int [cov(\lambda^*, kz) + \mu \int (twy_s - skz_s - p'xr_s) f(w) dw] g(r*) dr* \tag{16}$$

Departing from the initial situation with no commuting subsidy ($s=0$), we have

$$\frac{\partial \Lambda}{\partial s} = \int [cov(\lambda^*, kz) + \mu \int (twy_s - p'xr_s) f(w) dw] g(r*) dr* \tag{17}$$

A positive term is a case for a subsidy. A negative term is a case for a tax. In line with what is done in the optimum tax literature (see Dixit and Sandmo, 1977), we can interpret the former term as a distributional effect while the latter is an efficiency effect as discussed above.

Where w varies across the population, also the efficiency effect may differ. To illustrate this, consider the case where $x=s=0$. From Proposition 2 we know that y increases where $n + m = (p' - k) < 0$ or $n = p' - (1-t)w > 0$. The latter condition may hold only for a subset of w-values that are of moderate size, but where this is the dominant situation, we get $\int (twy_s) f(w) dw > 0$, and subsidising commuting enhances social efficiency. However, where the opposite sign dominates, a tax on commuting may be required to enhance social efficiency.

We recall that *kzds* is the income gain an agent receives when a small subsidy *ds* is introduced. Where $cov(\lambda^*, kz) > 0$ the benefit from a subsidy tends to be smaller at wage levels where the marginal valuation of income is smaller. Where $cov(\lambda^*, kz) < 0$ the benefit tends to be smaller at wage levels where the marginal valuation of income is larger.

Where increasing the income tax aggravates the inefficiency generated by distortionary taxes and the government is inequality averse, the optimal policy is characterised by a trade-off between efficiency and distribution, we have $cov(\lambda^*, wy) < 0$, i.e., λ^* is negatively correlated with w and earnings. It is an innocuous assumption that wy is an increasing function of w. A lower welfare weight is assigned to more productive agents (high-income agents). If agents with higher income incur a higher commuting cost and hence benefit more from the subsidy to commuting the covariance $cov(\lambda^*, kz)$ is negative, and there is a negative distributional effect. The partial effect is to weaken the case for a subsidy. We get the opposite conclusion where those with lower skills (income) are those who benefit more from a subsidy.

It is not obvious how commuting varies across people with different wages. There is an income effect, which is an argument for lower commuting expenses among those with higher wages when shortening of the commuting distance is a non-inferior good. But a higher wage rate also prompts substitution affecting both commuting and the location of residence. Substitution effects may reinforce or counteract the income effect. Let ω denote the after-tax wage rate, $(1-t)w$. From the appendix we have the compensated effects

$$\begin{aligned} z_\omega &= \frac{1}{D}(n_c n - n_{r-r*} + m_c n - m_{r-r*}) \\ &= \frac{1}{D}(n(n_c + m_c) - n_{r-r*}) = \frac{1}{D}(n_c(n+m) - n_{r-r*}) \end{aligned} \tag{18}$$

where $m_{r-r*} = 0$, and

$$r_\omega = \frac{1}{D}(m_c(m+n) + m_z). \quad (19)$$

The marginal cost of lowering z is $\omega - (1-s)k$ (income foregone minus saving of commuter cost). Where ω increases the marginal cost goes up. The marginal cost of increasing r is $(1-x)p' - \omega > 0$ (additional housing cost minus additional income). Where ω increases the marginal cost declines. First suppose that $n>0$, and r is a good at the margin. These marginal cost changes induce the agents to increase both z and r, which is the conclusion conveyed by eq. (18) and eq. (19).

The result that there is a positive substitution effect on z obtains also where $n+m<0$ (implying that $n<0$, $n_c<0$, and lowering r is a good at the margin). Where ω increases in the case where $n+m<0$ the marginal cost of shortening the commuting distance increases relative to the marginal cost of taking up residence further away from the centre. The relative cost change is an incentive to substitute a shorter commuting distance by a better living place. This substitution effect strengthens the case for extending z, while it is conceivable, but not necessarily the case, that r declines. Where ω increases in the case where $n<0$ and $n+m>0$ the marginal cost of moving further away from the centre increases relative to the marginal cost of shortening the commuting distance, and there is an incentive to increase r, and lower z. A good living place is to some extent substituted by a shorter commuting distance. There are ambiguous effects on z, and it is conceivable that it may decline. We elaborate on these substitution effects in the appendix.

We can summarise our results.

Proposition 5. *The sign of the distributional effect, $cov(\lambda^*, kz)$, is in general indeterminate due to conflicting income and substitution effects. Where shortening of the commuting distance is a non-inferior good the income effect is conducive to a positive covariance,* while *substitution effects work in favour of a negative covariance with one conceivable caveat: Where earnings increase more steeply than the housing cost* $(n<0)$ *and the housing cost exceeds commuting cost* $(n+m>0)$ *a conceivable outcome is that substitution effects strengthen the case for a positive covariance.*

6. Empirical illustrations

There are three key factors determining behaviour in our model: pecuniary travel expenses, geographical variation in earnings and geographical variation in housing cost. We want to show how empirical data on these factors can be exploited to assess the efficiency effects of changing actual subsidies on commuting. Combining theory and data, we can both compute the initial tax wedges and infer the direction of responses to a marginal reform, which in conjunction determine the sign and magnitude of the resulting efficiency effect.

To provide an empirical illustration of this approach, we have considered the region around Oslo in Norway. That is, we take Oslo as the centre in our model.

Each year, the tax authority in Norway announces a travel cost per kilometre that can be deducted against gross income by those eligible for a deduction. We use this figure for the pecuniary travel cost corrected for tax deduction.

We use register data on earnings in 2021, reported by the employers, to establish average earnings in each municipality in the region.[11] We distinguish between highly educated workers (university or college education) and workers with lower education. The reason is that we expect centrality of a workplace to be more important for some categories of workers than for others. Close to 60 per cent of the employees in the region have higher education.

For housing costs in 2021 ($p'(1-x)$), we use data commissioned by The National Federation of House Owners in Norway.[12] For each municipality several types of housing costs are assessed for an average 120 square metres sized dwelling. The items we include in the housing costs are interest payments (net of tax benefits due to deductions), maintenance, insurance, energy consumption, and municipal charges for water, wastewater treatment, garbage collection, etc. Moreover, we adjust the housing costs according to high or low education. Highly educated individuals will, due to higher income, enjoy higher housing standards (defined by size and quality). This is taken care of by estimating the difference in housing values between individuals with high and low education based on register data from the Income and Wealth Statistics.[13][14]

In addition to variation in housing standards across income groups, housing consumption in terms of standards will normally vary as part of behavioural responses to commuting subsidies. Larger consumption of most goods is obtained where a higher wage is attained at a more productive workplace. However, we do not single out housing standards as a separate good when considering the commuters' responses. Where the consumption bundle is optimally composed there is indifference between marginal consumption of various goods, and we can consider a change in housing standards as part of the change in general consumption, *c*.

By using Google maps, we measure the distance between the Oslo City Hall and the administrative centre of each of the surrounding municipalities. In Figures 1 and 2 we have plotted the average private housing costs, $p'(1-x)$, and the average earnings for the two educational groups in each municipality, depending on the distance to Oslo. It appears that this substantiates our assumption that both variables decline as we move away from Oslo.[15] To estimate the effects on earnings and housing costs of distance to Oslo, we run OLS regressions for each educational group. We report the results in Table 1 and, as expected, we get a significant negative effect on earnings and housing costs of moving away from Oslo. For instance, moving 10 kilometres closer to Oslo will generate more than NOK 11,400 in gross earnings for a highly educated individual. The housing cost rises by more than NOK 4,600. Furthermore, we need to estimate the marginal net subsidies on housing ($p'x$). Hence, we run

[11] For more information regarding the data on earnings, see https://www.ssb.no/en/arbeid-og-lonn/lonn-og-arbeidskraftkostnader/statistikk/lonn.

[12] The data are retrieved from several sources and compiled by the consulting company Economics Norway (see https://www.huseierne.no/huseiernes-bokostnadsindeks/).

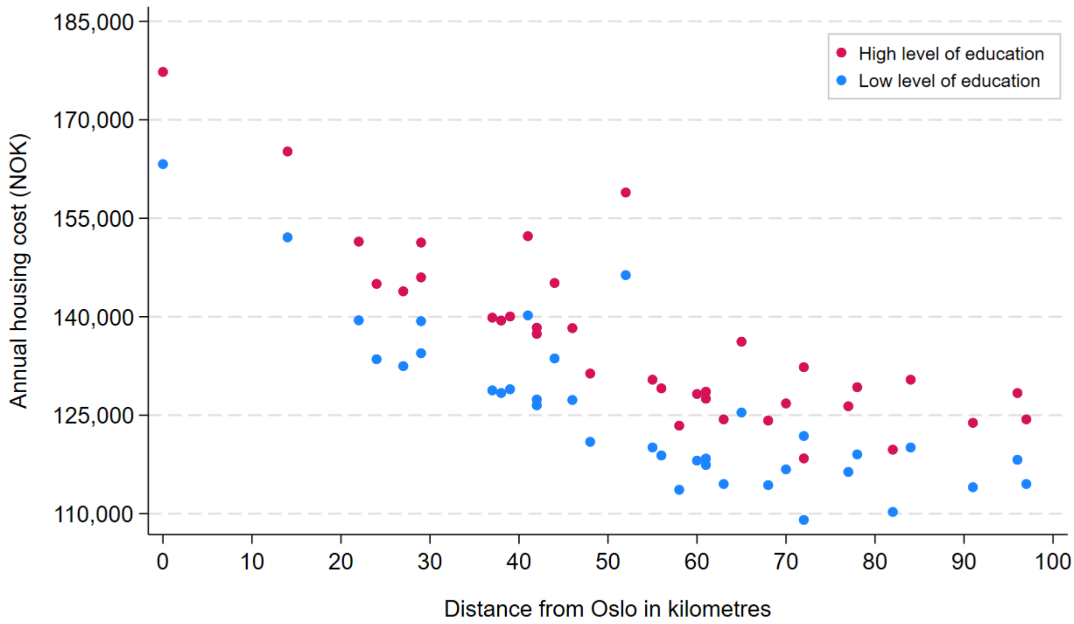
[13] We regress the log of housing value on a dummy for education, controlling for municipality and age. By knowing the average housing costs and the share of highly and less educated individuals, respectively, we can derive housing costs for each educational group. We estimate the costs to be 9 percentage higher for individuals with high education.

[14] For more information regarding the Income and Wealth Statistics, see https://www.ssb.no/en/inntekt-og-forbruk/inntekt-og-formue.

[15] We have confined our attention to the geographical areas located less than 100 km from the centre, since the decline tends to vanish beyond this distance.

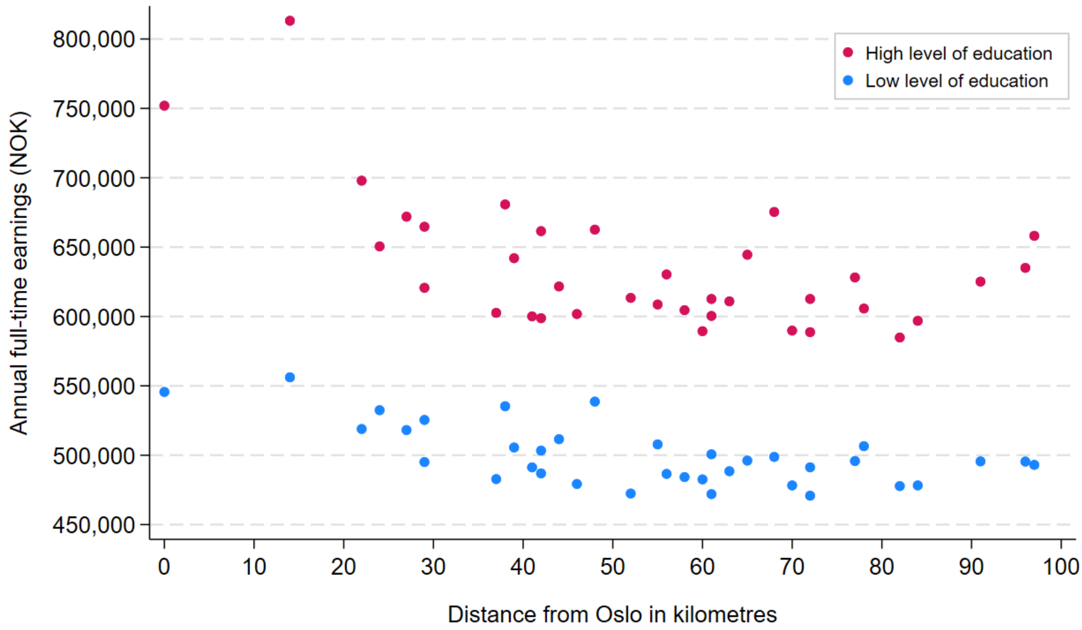
regressions with the tax benefits from interest deductions. The results reported in Table 1 show that housing subsidies rise with proximity to Oslo.

Figure 1. Average private housing costs by distance to Oslo for individuals with high and low level of education, respectively.



Note. Average private housing costs in each municipality in 2021. Distance by road from the administrative centre of the municipality to Oslo City Hall, estimated by using google maps.

Figure 2. Average earnings by distance to Oslo for individuals with high and low level of education, respectively.



Note. Average full-time gross earnings in each municipality in 2021, as reported by the employers. Distance by road from the administrative centre of each municipality to Oslo City Hall, estimated by using google maps.

Table 1. Estimates from OLS regressions based on data from 2021

	Low education			High education		
	Private housing costs	Tax benefit from interest deductions	Earnings	Private housing costs	Tax benefit from interest deductions	Earnings
Distance (10km)	- 4,282.1*** (507.8)	-1,260.4*** (160.6)	-5,758.3*** (1255.1)	-4,650.4*** (551.4)	-1,368.8*** (174.38)	-11,455.9*** (2,882.2)
Const.	148,739.1*** (2975.6)	17,368.0*** (940.9)	531,005.0*** (7,355.96)	161,530.7*** (3,231.4)	18,861.6*** (1,021.9)	696,685.9*** (16,889.42)
R-sq.	0.68	0.64	0.38	0.68	0.64	0.32
Obs.	36	36	36	36	36	36

Note: Earnings and housing costs in Norwegian kroner (NOK) and distance from Oslo in 10 kilometres. We use register data on earnings, reported by employers, to establish average earnings in municipalities. Data on housing costs are taken from The National Federation of House Owners in Norway, and data on distance to Oslo computed by using Google maps. Standard errors in parenthesis and *** $p < 0.001$.

Since we are interested in the marginal increase in disposable income due to higher earnings, we tax the marginal increase in earnings by using an average marginal tax rate based on the number of individuals in each tax bracket.[16] Based on this, the travel costs, and our estimation results, we compute values corresponding to different components and report them in Table 2.

For those with higher education, the implication is that one would lose financially by moving further away from the centre since the loss of income exceeds the decline of housing cost. The financial loss must then be balanced by a more attractive place of residence (r>r*). If it were not for the financial loss one would like to live further away from the centre. There is a non-pecuniary loss from moving closer to the centre (*n<0*) and a gain from moving further away from the centre. The commuting cost is NOK 2,799 and NOK 3,588 in the respective cases where the commuter is eligible or not for a deduction. In either case $n+m>0$.[17]

The efficiency effect of increasing s at the margin is $(tw - sk)y_s + (sk - xp')r_s = (tw - sk)(r_s + z_s) + (sk - xp')r_s = (tw - sk)z_s + (tw - xp')r_s = (tw - sk)z_s + (xp' - tw)(-r_s)$, where $-r_s$ is the decline of r. For those with higher education, we have *tw-sk*=3,450, $xp' - sk = 580$, and $xp' - tw = -2{,}870$. With the figures in our empirical illustration, the marginal efficiency effect is $3{,}450y_s - 580r_s = 3{,}450z_s - 2{,}870(-r_s)$. We know that z will increase, and r will decrease because *n<0*. Then we have opposing effects in the expression for the efficiency effect. The incentive to increase the commuting distance is beneficial because there is a pre-existing downward distortion of commuting when $tw - sk > 0$. However, there is an efficiency loss from lowering r $(-r_s > 0)$ because there is a pre-existing downward distortion since a lowering of r diminishes the income tax revenue by more than it diminishes the loss of revenue due to the subsidisation of housing. There will be a positive impact on social efficiency if $3{,}450z_s - 2{,}870(-r_s) > 0$, which is equivalent to $z_s > -r_s 2{,}870/3{,}450 = -0.83r_s$ or $-r_s < 1{,}20z_s$. The commuting distance must increase by more than 83 per cent of the decline of the distance to the place of residence, and latter must not exceed the former by more than 20 per cent. Otherwise, there is an efficiency loss at the margin. It does not seem plausible that a subsidy on commuting should generate a change of distance to the living place that significantly exceeds the increase in commuting distance. One might even argue that the commuting subsidy has a more direct and hence a larger impact on the commuting distance than the more indirect effect on the distance to the place of residence. If this argument is accepted the subsidy obviously enhances efficiency. It is also interesting to note that for the well-educated *m*=4,418, which is a fairly large disbenefit incurred by commuters.

For those with lower education, there is a moderate private financial loss from moving closer to the centre since the housing cost rises by more than the increase in disposable income. The implication is that to be at an equilibrium there must a non-pecuniary gain from moving closer to the centre ($r<r^*$ and *n>0*). It follows that the agents will respond to a larger commuting subsidy by extending the commuting distance ($z_s > 0$) and moving towards the centre ($r_s > 0$), and earnings will increase.

The efficiency effect is $(tw - sk)z_s + (tw - xp')r_s$.

[16] The marginal tax rate for individuals with high level of education is 37 percent, while for those with low level of education we use 36 percent.

[17] Commuting costs exceeding the amount of NOK 23,900 in 2021 were eligible for deductions. This implied that commuters had to live more than 33.5 kilometres away from the workplace to be eligible.

Using the numerical values from our empirical illustration, we get

$1{,}284 z_s + 813 r_s$

According to our analysis, the behavioural responses to a larger subsidy alleviate pre-existing distortions and enhances efficiency for those with lower education. For the less educated, we also find that m=886, which is a moderate disbenefit incurred by commuting.

Table 2. Derived values based on Norwegian data, 2021

	Low education	High education
$w(1-t)$	3,685	7,217
w	5,758	11,456
tw	2,073	4,239
$(1-x)p'$	4,282	4,650
p'	5,542	6,019
xp'	1,260	1,369
$(1-s)k$	2,799	2,799
k	3,588	3,588
sk	789	789
$xp' - tw$	-813	-2,870
$n = (1-x)p' - (1-t)w$	597	-2,567
$n + m = (1-x)p' - k$ *(s=0)*	694	1,062
m *(s=0)*	97	3,629
$n + m = (1-x)p' - (1-s)k$	1,483	1,851
m	886	4,418
$tw - sk$	1,284	3,450
$xp' - sk$	471	580

Note: The variables are defined in the main text. We use a marginal tax rate of 36 percent for less educated and a rate of 37 percent for highly educated. Commuting costs derived by using a cost of NOK 1.56 per kilometre and assuming 230 days of commuting.

Our empirical illustration shows the significance of variation in wage rates. On aggregate, 60 per cent of wage earners have higher education. We can conclude that a larger subsidy on commuting enhances efficiency for those with lower education while it is uncertain but plausible that it also enhances efficiency for those with higher education, and accordingly aggregate efficiency.

Another empirical finding is that there is a weak but positive relationship between commuting expenses and income. The correlation coefficient is approximately 0.05. This means that from a distributional perspective there is an argument against further subsidisation of commuting, but the case is indeed a very weak one. If one accepts that the more plausible outcome is that a larger subsidy on commuting enhances efficiency, we have opposing efficiency and distributional effects, which would be offsetting at a social optimum. However, as the distributional effect is indeed minor, it seems plausible that, based on our data, a further increase of the subsidy on commuting would be welfare enhancing. Our main contribution is however to illustrate how one can analyse the effects of a subsidy on commuting both theoretically and empirically.

7. Conclusion

Our paper makes several contributions. First, it analyses the circumstances in which subsidising commuting can alleviate the income tax distortions of locational choices of residence and workplace in a setting where agents have a favourite place of residence according to pure preferences. It elaborates on the interaction between the two decisions. Then we extend the concern with efficiency to distortionary tax favours to housing often encountered in practice. We establish a framework for assessing the efficiency effects of changing a subsidy on commuting based on a limited set of empirical data on prices, wages, and tax rates. A key element of the approach is to infer information about residential preferences from the market and tax data. Finally, the approach is illustrated by considering empirically commuting in the region around Oslo in Norway.

Our starting point is the recognition that an income tax causes a downward distortion of earnings. Earnings depend on workplace, which is determined by commuting distance and the place of residence from which the commuter departs. Therefore, the tax distortion of earnings is made up of the distortion of commuting and the distortion of choice of living place. A subsidy on commuting stimulates commuting to higher-wage locations, which mitigates the pre-existing downward distortion of earnings. The impact via the residential choice is less transparent. A subsidy on commuting induces workers to move closer to their preferred location of residence. The subsidy makes it more costly to shorten the commuting distance since the cost saving diminishes. When shortening the commuting distance becomes a more expensive good it is to some extent substituted by a more attractive living place. Whether the favourite place of residence is closer to or further away from the centre depends on the respective geographical variations in earnings and housing cost. Where the agents incur a financial loss due to unequal changes in housing cost and earnings when moving towards or away from the centre, the loss must be offset by a more attractive place of residence for the agent to be indifferent at the optimum. Where there is a preference for living closer to the centre, the subsidy will induce a move to better-paying locations and a further mitigation of the distortion of earnings. Where there is a preference for living further away from the centre the effect is reversed. Then the effects via the commuting decision and the residential choice are opposing each other and the net effect determines whether the earnings distortion is weakened or not.

However, there is a further complication. Housing is often subsidised through tax favours, the partial effect of which is to shrink, and conceivably reverse the sign of, the tax wedge created by the income tax. Then there are four possible cases when allowing for both the income tax and the subsidy on housing: i. There is a *net* downward distortion of the incentive to move to higher-wage locations and a subsidy on commuting encourages such a move. ii. There is a *net* upward distortion of the incentive to move to higher-wage locations and a subsidy on commuting encourages a move in the opposite direction. iii. There is a *net* downward distortion of the incentive to move to higher-wage locations and a subsidy on commuting encourages a move in the opposite direction. iv. There is a *net* upward distortion of the incentive to move to higher-wage locations and a subsidy on commuting encourages such a move. In cases i. and ii. the distortionary effect is mitigated and there is a beneficial social effect through the impact on the residential choice, which adds to the positive effect through the impact on commuting. In cases iii. and iv. there is a harmful social effect through the impact on the residential choice, which opposes the positive effect through the impact on commuting. The overriding effect determines whether there is an efficiency enhancement or deterioration.

Diversity of geographical wage changes across agents implies that there may be individual responses with opposing effects on social efficiency, and the overall impact depends on which response that dominates in the population.

The distributional effect of subsidising commuting depends on whether the main beneficiaries are low-income people or high-income people. How much a person benefits from subsidised commuting depends on his/her commuting distance. Both income and substitution effects are at work. High-income people may commute less where shortness of commuting distance is a non-inferior good. On the other hand, higher-wage persons have more to gain by commuting, and there are conceivable cross-effects between choice of living place and commuting. Distributional effects are unsettled on theoretical grounds. This is due both to the existence of conflicting income and substitution effects at the individual level and diverging responses across individuals.

To provide an empirical illustration of our theoretical analysis we have considered commuting in the region around Oslo in Norway, with Oslo as the centre in our model. Applying data on travel cost, geographical wage dispersion, and variation in housing cost, we can infer the responses to a subsidy on commuting for different wage groups, illustrated by highly educated and less educated people, respectively.

We have adopted the assumption of fixed pretax wages and prices, which is common in much of the optimum tax literature. Some caveats may however be in order. Where policy changes stimulate the employment density in certain locations, one may argue that there is a labour supply shift that tends to diminish local wages, and the converse is true in locations losing labour. On the other hand, one may argue that there can be agglomeration effects, especially in urban areas, implying that increasing employment raises the productivity of existing workers. Such effects are not taken into account by individual workers and can be perceived as agglomeration externalities that policy makers should allow for. Venables (2007) argues for inclusion of such effects in evaluation of urban transport projects, but may be equally relevant for tax and subsidy policy. Changes of residence may affect local scarcity of land and housing prices, as assumed in Wrede (2001). We should note that each agent is a price taker treating wages and housing prices as parameters. The (marginal) non-pecuniary benefits of changing location of residence or work are therefore given by individual first order optimality conditions even where general equilibrium effects occur. Policy-induced price changes may soften the impact of policy reforms. For instance, the tendency to move towards a preferred place of residence may be weaker without reversing the effect.

The recent Corona pandemic has provided an impetus for working from home, a phenomenon that may endure, at least to some extent, beyond the pandemic. Where a fixed part of the working week, say θ , can be spent working at home, less commuting is required to earn a certain income while living away from the workplace. We can then replace z by θz in the utility function, and a certain commuting distance becomes less costly. On the other hand, working from home may cause inconvenience that must be traded off against the cost of commuting where working conditions are flexible. Subsidising commuting will distort this trade-off, being another source of excessive commuting. Elaborating on these aspects of commuting may be a topic for future research.

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Appendix

We take as departure the fixed utility requirement:

$$u(c, z, r - r*) = u_0, \tag{a1}$$

which implicitly defines *c* as a function of *z* and *r*: $c(z, r - r*) = c(y - r, r - r*)$.

Differentiating (a1), we get

$$c_z = -\frac{u_z}{u_c} = m(c, z, r - r*)$$

$$c_{r-r*} = -\frac{u_{r-r*}}{u_c} = -n(c, z, r - r*)$$

Minimising the expenditure

$$e = c + [(1 - t)w - k + sk](-z) + (1 - x)p(r) - (1 - t)wr$$

with respect to *z* and *r*, allowing for (a1), we get the first order conditions

$$e_z = c_z - [(1 - t)w - k + sk] = 0$$

$$e_{r-r*} = c_{r-r*} + [(1 - x)p' - (1 - t)w] = 0$$

We find the following second order derivatives

$$c_{z,r-r*} = m_c c_{r-r*} + m_{r-r*} = -m_c n + m_{r-r*}$$

$$c_{r-r*,z} = -n_c c_z - n_z = -n_c m - n_z$$

$$c_{zz} = m_c c_z + m_z = m_c m + m_z$$

$$c_{r-r*,r-r*} = -n_c c_{r-r*} - n_{r-r*} = n_c n - n_{r-r*}$$

The second order conditions are

$$e_{zz} = c_{zz} > 0$$

$$e_{r-r*,r-r*} = c_{r-r*,r-r*} > 0$$

$$D = e_{zz} e_{r-r*,r-r*} - e^2_{z,r-r_*} = c_{zz} c_{r-r*,r-r*} - c^2_{z,r-r_*} > 0$$

To find the compensated effects of a subsidy by means of comparative statics, we differentiate the first order conditions and get

$$e_{zz}z_s + e_{z,r-r*}r_s - k = c_{zz}z_s + c_{z,r-r*}r_s - k = (m_c m + m_z)z_s + (-m_c n + m_{r-r*})r_s - k = 0$$

$$e_{r-r*,z}z_s + e_{r-r*,r-r*}r_s = c_{r-r*,z}z_s + c_{r-r*,r-r*}r_s = (-m_c n + m_{r-r*})z_s + (n_c n - n_{r-r*})r_s = 0$$

where (invoking Young's theorem), we have used $c_{r-r*,z} = c_{z,r-r*,z} = -m_c n + m_{r-r*} = -n_c m - n_z$.

In the special case where *r* is fixed

$$y_s = z_s = \frac{k}{c_{zz}} = \frac{k}{m_c m + m_z} \tag{a2}$$

Solving the general equation system

$$(m_c m + m_z)z_s + (-m_c n + m_{r-r*})r_s = k$$
$$(-m_c n + m_{r-r*})z_s + (n_c n - n_{r-r*})r_s = 0$$

for z_s and r_s, we obtain

$$z_s = \frac{k}{D}(n_c n - n_{r-r*}) \tag{a3}$$

$$r_s = \frac{k}{D}(m_c n - m_{r-r*}) = \frac{k}{D}(n_c m + n_z) \tag{a4}$$

The fixed utility (a1) implies that

$$u_c\, c_s + u_z z_s + u_{r-r*}r_s = 0$$

$$c_s - m\frac{k}{D}(n_c n - n_{r-r*}) + n\frac{k}{D}(n_c m + n_z) = 0$$

Where $n_z = 0$,

$$c_s = -\frac{k}{D}mn_{r-r*} > 0$$

$$y_s = \frac{k}{D}(n_c n - n_{r-r*} + m_c n - m_{r-r*}) = \frac{k}{D}(n(m_c + n_c) - n_{r-r*} - m_{r-r*})$$
$$= \frac{k}{D}(n_c n - n_{r-r*} + n_c m + n_z) = \frac{k}{D}(n_c(n+m) - n_{r-r*} + n_z) \tag{a5}$$

Absent any clear notion of the respective signs of m_{r-r*} and n_z, we set $m_{r-r*} = n_z = 0$.

Then

$$y_s = \frac{k}{D}(n(m_c + n_c) - n_{r-r*}) = \frac{k}{D}(n_c(n+m) - n_{r-r*}) \tag{a6}$$

Then we differentiate the first order conditions with respect to $\omega = (1-t)w$ and get

$$c_{zz}z_\omega + c_{z,r-r*}r_\omega = 1$$

$$c_{r-r*,z}z_\omega + c_{r-r*,r-r*}r_\omega = 1$$

$$z_\omega = \frac{1}{D}\left(c_{r-r*,r-r*} - c_{z,r-r*}\right)$$

$z_\omega = \frac{1}{D}(n_c n - n_{r-r*} + m_c n - m_{r-r*}) = \frac{1}{D}(n_c n - n_{r-r*} + n_c m + n_z)$

$$= \frac{1}{D}(n(n_c + m_c) - n_{r-r*}) = \frac{1}{D}(n_c(n+m) - n_{r-r*}) \quad \text{(a7)}$$

where $m_{r-r*} = n_z = 0$.

$r_\omega = \frac{1}{D}(c_{zz} - c_{z,r-r*})$

$r_\omega = \frac{1}{D}(m_c(m+n) + m_z)$

$r_\omega = \frac{1}{D}(m_c m + m_z + n_c m) = \frac{1}{D}((m_c + n_c)m + m_z)$

where $m_{r-r*} = n_z = 0$.

$y_\omega = z_\omega + r_\omega = \frac{1}{D}[(n_c(n+m) - n_{r-r*}) + (m_c(m+n) + m_z)]$

$y_\omega = \frac{1}{D}[(n_c + m_c)(n+m) - n_{r-r*} + m_z)]$

When utility is kept constant

$c_\omega = m z_\omega - n r_\omega$

$= m\frac{1}{D}(n(n_c + m_c) - n_{r-r*}) - n\frac{1}{D}((m_c + n_c)m + m_z)$

$= \frac{1}{D}(-m n_{r-r*}) - n\frac{1}{D}m_z$

Where $n \geq 0$ ($r \leq r*, n + m > 0$[18]), we see that $z_\omega > 0$ and $r_\omega > 0$. It obviously follows that $y_\omega > 0$. Increasing ω makes it more costly to shorten the commuting distance (the income foregone increases), the partial effect of which is to increase *z*. It also makes it cheaper to increase *r* when that is a good according to pure preferences, and the partial effect is to increase *r*.

Also, where $n + m = 0, r_\omega > 0, z_\omega > 0$ and $y_\omega > 0$.

Where $n < 0$ (and $n_c < 0$) there may be ambiguous effects. As $m_z > 0$ and $-n_{r-r*}>0$ there are partial effects in favour of increasing *z* and *r*. This is not surprising since a larger wage increase than before is achieved by moving or commuting to higher-wage locations. However, there is also a relative cost effect, which induces substitution. Let us first consider the case where $n < 0$ $(r > r*)$, but $n + m > 0$. This means that $n + m = (1-x)p' - \omega + \omega - (1-s)k > 0$. As we see, this is equivalent to $(1-x)p' > (1-s)k$ and $\omega - (1-s)k > \omega - (1-x)p'$. The left-hand side is the marginal cost of reducing the commuting distance, which is a good. The right-hand side is the marginal cost of moving further away from the centre, which is a good when $r > r*$. Where ω increases the former cost decreases relative to the latter, and there is a case for letting an attractive place of residence be substituted by a shorter commuting distance, i.e., a lower *z*. The case for increasing *r* is strengthened, while it is conceivable that *z* declines.

Then consider the case where $n < 0$ $(r > r*)$, and $n + m = (1-x)p' - \omega + \omega - (1-s)k < 0$. This is equivalent to $(1-x)p' < (1-s)k$ and $\omega - (1-s)k < \omega - (1-x)p'$. Where ω increases the marginal cost of shortening the commuting distance (the left-hand side). increases relatively more than the marginal cost of moving further away from the centre (the

[18] We may note that *m+n* is the marginal valuation of increasing *r* and lowering *z* to keep *y* unchanged. At the optimum, this is equal to the net cost of this substitution, $(1-x)p' - (1-s)k$.

right-hand side). The relative cost change is an incentive to substitute a shorter commuting distance by a more favourable living place, i.e., a case for lowering *r* and extending *z*. This strengthens the case for increasing *z*, but weakens the case for increasing *r*. It is a conceivable outcome that *r* diminishes.

We note that $z_\omega > 0$ where $n + m < 0$ and $r_\omega > 0$ where $n + m > 0$. It follows that we can never have both $z_\omega < 0$ and $r_\omega < 0$. We note that when $n<0$, $c_\omega = \frac{1}{D}(-mn_{r-r*}) - n\frac{1}{D}m_z > 0$.

Optimal policy

The aggregate tax revenue net of subsidies and transfer is

$R = \int\int[twy - skz - xp(r) - a]f(w)dw\, g(r*)dr*.$

The marginal effects of the policy instruments are

$$\frac{\partial R}{\partial a} = \int\int (twy_a - skz_a - xp'r_a - 1)f(w)dw\, g(r*)dr* \quad \text{(a8)}$$

$$\frac{\partial R}{\partial t} = \int\int \left[tw\frac{\partial y}{\partial t} - sk\frac{\partial z}{\partial t} - xp'\frac{\partial r}{\partial t} + wy\right] f(w)dw\, g(r*)dr*$$

$$\frac{\partial R}{\partial s} = \int\int \left[tw\frac{\partial y}{\partial s} - sk\frac{\partial z}{\partial s} - xp'\frac{\partial r}{\partial s} - kz\right] f(w)dwg(r*)dr*$$

Making use of the Slutsky decomposition,

$$\frac{\partial R}{\partial t} = \int\int[twy_t - skz_t - xp'r_t - wytwy_a + wyskz_a + wyxp'r_a + wy]f(w)dwg(r*)dr* \quad \text{(a9)}$$

$$\frac{\partial R}{\partial s} = \int\int[twy_s - skz_s - xp'r_s + twy_a kz - skz_a kz - xp'r_a kz - kz]f(w)dwg(r*)dr* \quad \text{(a10)}$$

Social welfare is given by

$$\Omega = \int\int u(c, z, r - r*)\, f(w)dw\, g(r*)dr*$$

$$= \int\int u((1-t)wy - k(1-s)z - (1-x)p(r) + a, z, r - r*)f(w)dw\, g(r*)dr*$$

A bar is used as notation for an average across values of *w*. Invoking the Envelope Theorem, we get

$$\frac{\partial \Omega}{\partial a} = \int\int u_c\, f(w)dw\, g(r*)dr* = \int \bar{u}_c g(r*)dr* \quad \text{(a11)}$$

$$\frac{\partial \Omega}{\partial t} = -\int\int u_c\, wyf(w)dwg(r*)dr* = \int[-\bar{u}_c \int wyf(w)dw - cov(u_c, wy)]g(r*)dr* \quad \text{(a12)}$$

$$\frac{\partial \Omega}{\partial s} = \int\int u_c\, kzf(w)dw\, g(r*)dr* = \int[\bar{u}_c k\bar{z} + cov(u_c, kz)]g(r*)dr* \quad \text{(a13)}$$

To characterise the optimal policy, we formulate the Lagrange function

$$\Lambda = \Omega + \mu R$$
$$= \int \int u\big((1-t)wy - k(1-s)z - (1-x)p(r) + a, z, r - r*\big) f(w)dw\, g(r*)dr* +$$
$$\mu \int \int (twy - skz - xp(r) - a)\, f(w)dwg(r*)dr*$$

Where the transfer is set optimally,

$$\frac{\partial \Lambda}{\partial a} = \frac{\partial \Omega}{\partial a} + \mu \frac{\partial R}{\partial a} = \int [\bar{u}_c + \mu \int (twy_a - skz_a - xp'r_a - 1) f(w)dw] g(r*)dr* \quad \text{(a14)}$$
$$= \int [\bar{u}_c - \mu + \mu t\overline{wy_a} - \mu sk\overline{z_a} - \mu xp'\overline{r_a}]\, g(r*)dr* = 0$$

The optimal income tax rate is characterised by

$$\frac{\partial \Lambda}{\partial t} = -\int [\int u_c wyf(w)dw + \mu \int (wy - wytwy_a + twy_t + wykskz_a + wyxp'r_a - ksz_t -$$
$$xp'r_t)\, f(w)dw] g(r*)dr* = 0 \quad \text{(a15)}$$

As it is beside the main issue we address, we shall not delve further into the optimal choice of *t*.

$$\frac{\partial \Lambda}{\partial s} = \frac{\partial \Omega}{\partial s} + \mu \frac{\partial R}{\partial s}$$
$$= \int \left[\int u_c kzf(w)dw + \mu \int \left(-kz + tw\frac{\partial y}{\partial s} - ks\frac{\partial z}{\partial s} - p'x\frac{\partial r}{\partial s}\right) f(w)dw\right] g(r*)dr*$$

Deploying the Slutsky equation, we can rewrite the expression for $\frac{\partial \Lambda}{\partial s}$ as

$$\frac{\partial \Lambda}{\partial s} = \int [\bar{u}_c k\bar{z} + cov(u_c, kz) + \mu \int \big(-kz + twkzy_a - kskzz_a - xp'kzr_a - p'xr_s + twy_s -$$
$$ksz_s\big)\, f(w)dw\,] g(r*)dr*$$

where we recall that subscripts *s* and *t* denote partial compensated effects. Moreover, we get

$$\frac{\partial \Lambda}{\partial s} = \int \bar{u}_c k\bar{z} - \mu k\bar{z} + \mu k\bar{z}t\overline{wy_a} - \mu k\bar{z}sk\overline{z_a} - x\bar{z}p'k\overline{r_a}$$
$$+ cov(u_c, kz) + \mu\, cov(kz, twy_a) - \mu\, cov(kz, skz_a) - \mu\, cov(kz, xp'r_a) + \mu \int (twy_s - ksz_s -$$
$$p'xr_s)\, f(w)dw] g(r*)dr*$$

And where *a* is set optimally according to (a14)

$$\frac{\partial \Lambda}{\partial s} = \int [cov(u_c, kz) + \mu\, cov(kz, twy_a) - \mu\, cov(kz, skz_a) - \mu\, cov(kz, xp'r_a) + \mu \int (twy_s - ksz_s -$$
$$p'xr_s)\, f(w)dw] g(r*)dr* \quad \text{(a16)}$$

Making use of λ^*, we can write

$$\frac{\partial \Lambda}{\partial s} = \int [cov(\lambda^*, kz) + \mu \int (twy_s - skz_s - p'xr_s)\, f(w)dw] g(r*)dr* \quad \text{(a17)}$$

Taking as point of departure the no-subsidy case (*s*=*x*=*0*), and assuming that the transfer is set optimally, we have the special case

$$\frac{\partial \Lambda}{\partial s} = \int [cov(\lambda, kz) + \mu \int (twy_s)\, f(w)dw]\, g(y*)dy* \quad \text{(a18)}$$