ON THE LONG-RUN EVOLUTION OF TECHNOLOGICAL KNOWLEDGE

HENDRIK HAKENES
ANDREAS IRMEN

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Abstract

This paper revisits the debate about the appropriate differential equation that governs the evolution of knowledge in models of endogenous growth. We argue that the assessment of the appropriateness of an equation of motion should not only be based on its implications for the future, but that it should also include its implications for the past. We maintain that the evolution of knowledge is plausible if it satisfies two asymptotic conditions: Looking forwards, infinite knowledge in finite time should be excluded, and looking backwards, knowledge should vanish towards the beginning of time (but not before). Our key results show that, generically, the behavior of the processes under scrutiny is either plausible in the future and implausible in the past or vice versa, or implausible at both ends of the time line.

JEL Code: O11, O31, O40.

Keywords: endogenous technological change, Malthus, long-run growth.

Hendrik Hakenes
Max Planck Institute for Research on Collective Goods
Kurt-Schumacher-Str. 10
53113 Bonn
Germany
hakenes@coll.mpg.de

Andreas Irmen
Free University of Bozen – Bolzano
School of Management and Economics
Sernesistr. 1
39100 Bozen
Italy
airmen@unibz.it

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1 Introduction

Most models of endogenous technological change posit a relationship that links the change in aggregate technological knowledge to the level of existing knowledge and the amount of human capital employed in research activity. The focus on steady states has led many authors to depict this link by means of two particular differential equations. The first variant has constant returns to the existing stock of knowledge in the creation of new knowledge (see, e. g., Romer, 1990, Grossman and Helpman, 1991, and Aghion and Howitt, 1992). In these models the level of research activity is endogenous, and it reflects the extent to which the economy allocates its time-invariant labor endowment to manufacturing and research. A second variant was developed by Jones (1995). He assumes decreasing returns to the existing stock of knowledge in conjunction with exponential population growth.

These variants are by no means universally accepted as an appropriate description of the production of knowledge. For instance, critics like Solow (2000) have pointed to the knife-edge character of the first variant. Indeed, with increasing returns, knowledge becomes infinite in finite time; with decreasing returns, the growth of knowledge peters out over time, precluding the possibility of steady-state growth. In a sense, Jones (1995) responds to the latter criticism by showing that steady-state growth is possible if exponential population growth acts as a countervailing force to decreasing returns.

This paper revisits the debate about the appropriate differential equation that governs the evolution of knowledge in models of endogenous growth. We argue that the assessment of the appropriateness should not only be based on the forward-looking properties of such an equation. Rather, we show that the analysis of the backward-looking properties generates criteria that should be included in the overall assessment. By including the past, this approach extends and tightens Solow's critique, imposing a further constraint on the set of plausible processes.

Our analysis is based on the insight that once we stipulate an "initial" value for the level of knowledge, the solution to the chosen differential equation for knowledge determines its evolution for the time after *and* before the initial period. We may therefore look forwards and backwards at the implied evolution of knowledge. Essentially, by including the past, this approach deviates from the often encountered interpretation of the initial condition, which is seen as a "historically given constant." This interpretation tacitly assumes a structural break that must have occurred right before the initial period, such that the specified differential equation cannot account for how the economy arrived at the stipulated initial value.

We take the view that a plausible description of the evolution of knowledge should satisfy two asymptotic conditions. Looking *forwards*, we follow Solow in maintaining that infinite knowledge in finite time is impossible. Looking *backwards*, we require knowledge to vanish in the infinite past, but not in finite time. We call an evolution *plausible* if it satisfies these criteria. Intuitively, these conditions may be seen as minimum requirements to exclude absurd evolutions of key economic magnitudes. For instance, if we think of knowledge as an essential input in an aggregate Cobb-Douglas production function, then the idea behind Solow's criterion is that infinite per-capita income for a strictly positive population must be unattainable in the finite future. For the past, we add the requirement that per-capita income should vanish in the infinite past, yet not before. Since the level of knowledge remains bounded away from zero, this stipulation excludes an evolution where a small and vanishing population becomes tremendously rich.[2]

The key result of our analysis is that standard equations for the evolution of knowledge used in the modern literature on endogenous growth satisfy these requirements only under non-generic circumstances.[3] Roughly speaking, a plausible evolution is a probability-zero event. Moreover, we stress that, generically, the behavior of the processes under scrutiny is either implausible in the past, or in the future, or both.

To build intuition, Section 2 starts off with the analysis of the first variant used in the literature. We fully characterize and interpret the implied asymptotic evolution of knowledge. For the case of decreasing returns, we show that knowledge must have been zero in the finite past. Next, we show that this result remains valid for *two* more elaborate processes of knowledge growth. *First*, in Section 3, we extend the analysis to a scenario similar to Jones (1995) and allow for exogenous exponential population

[2]Our stipulation that knowledge must approach zero in the infinite past limit is consistent with the view that any species is born with some innate knowledge comprising basic rules of survival. If the population of the species in question vanishes towards the beginning of time, so does knowledge.

[3]*Non-generic* means that the set of parameters under which a phenomenon, behavior, or event occurs has measure zero. In our case, given all other parameters of the model, there is only one initial level of knowledge such that the evolution of knowledge is plausible. In other words, there is one bogus degree of freedom: the set of *plausible* exogenous parameters has one dimension less than the set of parameters itself.

growth. We prove for generic sets of initial values that two scenarios can arise. Either zero-knowledge occurs in the finite past, or the level of knowledge remains strictly positive throughout history. A plausible evolution of knowledge obtains only for a non-generic set of initial values. Section 4 deals with the *second* extension and introduces endogenous population growth in a Malthusian manner. More precisely, we reconsider Kremer's (1993) setup and study the implications of his dynamical system when time approaches the infinite past. We find that, generically, two scenarios occur that share the key properties of the two scenarios we identified under exogenous population growth. The first exhibits zero knowledge in the finite past; the second exhibits a positive level of knowledge even in the infinite past. Thus, for both scenarios, the evolution of knowledge is implausible. Section 5 concludes. Proofs are in the Appendix.

2 Knowledge Growth

Consider the evolution of knowledge governed by the differential equation

$$\dot{A}(t) = g\, A(t)^{\phi}, \tag{1}$$

where $g \in \mathbb{R}_{++}$, $A(t) \in \mathbb{R}_{+}$, $t \in \mathbb{R}$, and $\phi \in \mathbb{R}_{++}$.[4] The *algebraic* solution to the initial value problem with $A(0) = A_0$ is

$$A(t) = \begin{cases} \left[{A_0}^{1-\phi} + (1-\phi)\, g\, t\right]^{\frac{1}{1-\phi}} & : \quad \phi \neq 1 \\ A_0 \exp(g\, t) & : \quad \phi = 1. \end{cases} \tag{2}$$

In addition, for $A_0 = 0$ there is the trivial solution with $A(t) = 0$ for all t. Accordingly, for this initial value the algebraic solution may not be unique. Consider the case $\phi < 1$. A look backwards leads to the insight that

$$A(t) = 0 \quad \text{for} \quad t = t_c := -\frac{{A_0}^{1-\phi}}{(1-\phi)\, g}, \tag{3}$$

where t_c is negative and finite. Observe that $A(t) = 0$ for all $t \leq t_c$. This follows since $A(t) > 0$ for some $t < t_c$ would imply $\dot{A}(t) < 0$ for some $t < t_c$, which is

[4]Throughout the analysis, we focus on $\phi > 0$ and neglect the possibility of independence ($\phi = 0$) and of fishing-out ($\phi < 0$). The latter turn out to generate straightforward dynamics that violate what we call a plausible evolution. Detailed results for $\phi \leq 0$ are available from the authors upon request.

inconsistent with (1). Furthermore, negative knowledge is impossible. Thus, we have zero knowledge at some finite $t_c < 0$. Moreover, since knowledge cannot shrink below zero, it must have been zero before date t_c. As a consequence, an economy starting before t_c must have taken-off without knowledge.[5]

Proposition 1 *Let technological knowledge evolve according to the differential equation (1) with $A_0 > 0$.*

1. *If $\phi < 1$, then the economy takes off from zero-knowledge in the finite past. In the future, knowledge converges to infinity; its growth rate converges to zero.*
2. *If $\phi > 1$, knowledge is always strictly positive; it vanishes only in the limit $t \to -\infty$ and reaches infinity in the finite future.*
3. *If $\phi = 1$, knowledge is always strictly positive; it vanishes only in the limit $t \to -\infty$ and does not become infinite in the finite future.*

Hence, for $\phi \neq 1$, either knowledge is zero in the finite past and its path is not unique, or Solow's critique applies. A plausible evolution in the remote future includes an implausible evolution in the remote past, and vice versa. A plausible evolution of knowledge obtains only for the non-generic case $\phi = 1$, for which equation (1) exhibits exponential growth.

3 Knowledge and Exogenous Population Growth

This section follows Jones (1995) and replaces g in (1) by $g(t) = \gamma\, N(t)^{\lambda}$, where γ, $\lambda > 0$, and $N(t)$ denotes population at time t. If $n > 0$ is the constant population growth rate and $N_0 > 0$ the initial population size, then the evolution of knowledge is $\dot{A}(t) = \gamma\, {N_0}^{\lambda}\, e^{n\,\lambda\,t}\, A(t)^{\phi}$. Define $g' := \gamma\, {N_0}^{\lambda}$ and $n' := n\,\lambda$, to simplify the latter to

$$\dot{A}(t) = g'\, e^{n'\,t}\, A(t)^{\phi}. \tag{4}$$

[5]The phenomenon that the process of knowledge growth may spontaneously take off from a zero-knowledge state implies an indeterminacy of its evolution, which is due to the missing Lipschitz continuity of (1) at $A = 0$. Lipschitz continuity requires $\partial\dot{A}/\partial A$ to be finite. However, from (1) we see that this expression becomes infinite as $A \to 0$.

The key question is whether this extension admits a plausible evolution of knowledge for the generic case $\phi \neq 1$. The differential equation (4) has the algebraic solution[6]

$$A(t) = \begin{cases} \Big[{A_0}^{1-\phi} + \dfrac{g'(1-\phi)}{n'}(e^{n'\,t}-1)\Big]^{\frac{1}{1-\phi}} & : \ \phi \neq 1 \\ A_0 \, \exp\Big(\dfrac{g'}{n'}\,(e^{n'\,t}-1)\Big) & : \ \phi = 1. \end{cases} \tag{5}$$

Again, we must add the trivial solution with $A(t) = 0$ at all times if $A_0 = 0$. Consider the generic case $\phi \neq 1$. Then, the term in squared brackets of (5) is zero if and only if

$$0 = {A_0}^{1-\phi} + (1-\phi)\frac{g'}{n'}(e^{n'\,t}-1), \qquad \text{or equivalently}$$
$$t = t_c := \frac{1}{n'} \ln\Big(1 - \frac{{A_0}^{1-\phi}}{1-\phi}\,\frac{n'}{g'}\Big). \tag{6}$$

The latter extends (3) to the case $n' > 0$. At $t = t_c$, knowledge vanishes if $\phi < 1$, and it becomes infinite for $\phi > 1$. If $\phi < 1$, then $t_c < 0$, and zero-knowledge may have occurred in the past. If $\phi > 1$, then $t_c > 0$, and an infinite level of knowledge may be reached in the future. However, t_c need not be finite.

Let $\phi < 1$. We deduce from (6) that t_c is finite if and only if

$$0 < 1 - \frac{{A_0}^{1-\phi}}{1-\phi} \cdot \frac{n'}{g'}, \quad \text{i. e.} \quad \frac{\lambda\, A_0^{1-\phi}}{\gamma\, N_0^{\lambda}}\, n + \phi < 1. \tag{7}$$

Hence, $\phi < 1$ is no longer sufficient for zero-knowledge to occur in the finite past. What is needed in addition is moderate population growth. The intuition is as follows. Looking backwards from period $t = 0$ the stock of knowledge must decline. According to (4), the decline grinds to a halt if either the level of knowledge or population vanishes. Condition (7) assures us that population does not shrink too fast relative to the decline of knowledge for given initial conditions and technological parameters.[7]

If n' violates condition (7), then knowledge is not zero in the finite past. While this case does not exhibit the indeterminacy problem, it gives rise to another remarkable property. Generically, if knowledge is not zero in the finite past, then it must have always existed, even in the limit (thus at a vanishing population), i. e.

$$\lim_{t \to -\infty} A(t) = \Big[{A_0}^{1-\phi} - \frac{(1-\phi)\, g'}{n'}\Big]^{\frac{1}{1-\phi}} > 0. \tag{8}$$

[6]The solution follows from the fact that (4) is a Bernoulli equation that can be solved by appropriate substitution (see, e. g., Gandolfo, 1997, p. 436).

[7]As before, at $A(t) = 0$ the evolution of knowledge is ambiguous, which can be traced back to a violation of Lipschitz continuity.

In other words, knowledge was around before man. As a consequence, we obtain a plausible evolution of knowledge only in the non-generic case, where (7) holds as an equality. Here, the path of knowledge is exponential with $\lim_{t\to-\infty} A(t) = 0$; knowledge and man disappear "contemporaneously" in the infinite past. Equipped with these intuitions, the following proposition states and proves all possible asymptotic cases.

Proposition 2 *Let technological knowledge evolve according to the differential equation (4) with $A_0 > 0$.*

1. *If $\phi < 1$, then in the past either*
   - *(a) the economy takes off from zero-knowledge in finite time,*
   - *(b) knowledge converges to a strictly positive level, or*
   - *(c) knowledge converges to zero, but does not reach zero in finite time. This evolution is non-generic.*

   Furthermore, in the future, knowledge converges to infinity and its growth rate to a constant level.

2. *If $\phi > 1$, knowledge converges to a strictly positive level in the infinite past; it becomes infinite in the finite future.*

3. *If $\phi = 1$, knowledge converges to a strictly positive level in the infinite past; it does not become infinite in the finite future.*

Interestingly, a plausible evolution of knowledge no longer obtains for $\phi = 1$ since the presence of population growth prevents knowledge from disappearing in the limit $t \to -\infty$. Essentially, a plausible evolution requires $\phi < 1$ and a non-generic set of initial conditions.[8]

[8] The key findings of Proposition 2 hold true if we allow for multiple research inputs. Moreover, they extend to a setting with depreciation of knowledge. Details are available from the authors.

4 Knowledge and Endogenous Population Growth

Next, we add more flexibility to the evolution of population and revisit Kremer's (1993) setup, where knowledge and population are endogenous complementary state variables. Does this setup generically allow for a plausible evolution of knowledge?

Following Kremer (1993), we consider the system

$$\dot{A} = N^{\lambda}\, A^{\phi} \tag{9}$$

$$\dot{N} = N\, f(y) \quad \text{with} \quad y = A\, N^{\alpha-1}, \tag{10}$$

where $\lambda > 1$, $\phi \in \mathbb{R}_{++}$, and $\alpha \in (0, 1)$. Here, y denotes per-capita income. Let $f(y)$ be continuous and monotonously increasing on $y \in \mathbb{R}_{++}$ with the Malthusian feature $f(y) > 0$ iff $y > \bar{y}$ for some $\bar{y} > 0$. The following proposition characterizes the asymptotic behavior of knowledge for all admissible parameter constellations.

Proposition 3 *Let technological knowledge evolve according to the system of differential equations (9) and (10) with $A(0) = A_0 > 0$ and $N(0) = N_0 > 0$.*

1. *If $\phi < 1$, then in the past either*

   (a) the economy takes off from zero-knowledge in finite time,

   (b) knowledge converges to a strictly positive level, or

   (c) knowledge converges to zero, but does not reach zero in finite time. This evolution is non-generic.

   Furthermore, in the future, knowledge converges to infinity. However, its growth rate does not converge.

2. *If $\phi > 1$, the economy always maintains a strictly positive level of knowledge and reaches infinite knowledge in the finite future.*

3. *If $\phi = 1$, knowledge is always strictly positive and does not become infinite in the finite future. This evolution is non-generic.*

According to Proposition 3, a plausible evolution of knowledge under endogenous, Malthusian population growth can only obtain in cases 1c and 3. Both cases are non-generic. To gain intuition for this finding, consider the phase-diagram of Figure 1, where

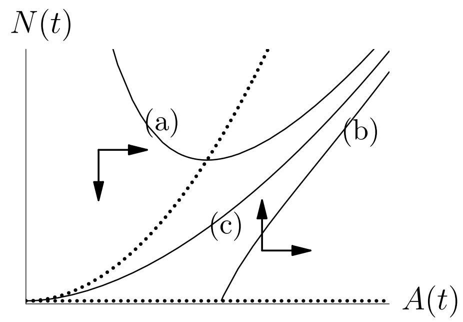

Figure 1: The Trajectories of A and N.

the dotted curves indicate the locus where $\dot{N} = 0$. Depending on the initial conditions (A_0, N_0) three types of trajectories may occur. For $\phi < 1$, all trajectories of type (a) and (b) imply an implausible asymptotic behavior of knowledge in the past. Along type-(a) trajectories knowledge becomes zero in finite time. This is not obvious from the phase-diagram, as N converges to ∞ at the same time. Along type-(b) trajectories knowledge remains positive even in the limit $t \to -\infty$. Only the type-(c) trajectory has the property that knowledge vanishes in this limit. Interestingly, these qualitative properties are similar to those of case 1 in Proposition 2. A plausible evolution may also obtain for $\phi = 1$, if the economy is on a trajectory of type (a) or (c). Along these paths, knowledge does not vanish in finite time. For $\phi > 1$, trajectories are similar to those of Figure 1, all three cases (a), (b) and (c) can occur. However, now knowledge reaches infinity in the finite future – an implausible evolution.

5 Concluding Remarks

The equation of motion that governs the evolution of technological knowledge is an essential part of the theory of endogenous growth. It is therefore important to understand the properties of the implied evolution. We argue that both the forward-looking and the backward-looking properties are important in assessing the appropriateness of a specific functional form. In particular, we claim that a plausible evolution should never yield infinite or zero knowledge in finite time, but that zero knowledge should be reached in the infinite past limit.

As Solow suggests, the usual equation of motion generates an implausible evolution of knowledge for $\phi > 1$. When we look forwards, knowledge becomes unbounded in finite time. Our results emphasize that, generically, implausible results also obtain for $\phi < 1$. Then, looking backwards, knowledge either vanishes already in the finite past, or it does not even vanish in the limit.

Most dynamic models focus the discussion about stipulated differential equations on their implications for the *future*. This paper stresses that dynamic models should also apply to the *past*. Recognizing this fact helps to judge the plausibility of the assumed equations from a theoretical point of view. While we remain at the macroeconomic level, our results impose constraints on attempts to derive an equation of motion for technological knowledge from microeconomic principles (see the pioneering contributions of Weizman (1998) and Olsson (2000, 2005)).

Our paper gives rise to further research questions. Since we point to a weakness of most specifications of knowledge growth, one question concerns how to more satisfactorily specify an equation of motion. Several directions of research seem possible. For instance, one may want to replace continuous-time models altogether and switch to discrete-time models. Alternatively, one may introduce stochastic factors. Within our analytical framework, one could try to modify the key differential equation itself. However, our results suggest that, generically, peculiarities in the past and in the future are substitutes – processes that behave well in the future do badly in the past, and vice versa. Finding the "right" equation may not be an easy task.

A Appendix

Proof of Proposition 1 For $\phi < 1$, the asymptotic behavior for $t \to -\infty$ is explained in the main text. As to the limit $t \to -\infty$, we have from (2) that $A(t) \to \infty$. Moreover, since $\dot{A}(t)/A(t) = g/A(t)^{1-\phi}$, this growth rate tends to zero. If $\phi > 1$, then we have $\lim_{t\to-\infty} A(t) = 0$, which is readily verified from (2). To see, that the level of knowledge becomes infinite in finite time, consider (2) and find that

$$A(t) \to \infty \quad \text{iff} \quad t \to t_c := \frac{{A_0}^{1-\phi}}{(\phi-1)\,g} < \infty.$$

For $\phi = 1$, equation (1) exhibits exponential growth. ■

Proof of Proposition 2 For $\phi < 1$, the asymptotic behavior for $t \to -\infty$ is fully characterize in the main text. As to the limit $t \to \infty$, we find from (5) that $\lim_{t\to\infty} A(t) = \infty$. Moreover, $A(t) \approx (1-\phi)\,\frac{g'}{n'}\,e^{n'\,t/(1-\phi)}$ for $t \to \infty$ such that the growth rate converges to $\dot{A}(t)/A(t) \to n'/(1-\phi)$. This is, of course, Jones' (1995) steady-state growth rate. For $\phi > 1$, one readily verifies from (5) that the asymptotic behavior of knowledge for $t \to -\infty$ is as in (8). Looking forwards, infinite knowledge is reached at date $t_c < \infty$. This follows since $0 < 1 + {A_0}^{1-\phi}\,n'/((\phi-1)\,g')$. Hence, form (6) the term in squared brackets in (5) is zero for a finite t_c.

For $\phi = 1$, the asymptotic behavior of knowledge is similar to the case $\phi > 1$. From (5) we obtain $\lim_{t\to-\infty} A(t) = A_0\,\exp(-g'/n')$ and $\lim_{t\to\infty} A(t) = \infty$. ■

Proof of Proposition 3 We start with the simpler cases 2 and 3. Recall that the statements of Proposition 2 for $\phi > 1$ and $\phi = 1$ hold true for any population size and population growth rate. Hence, these statements hold also true under the assumptions of Proposition 3.

Next, we turn to the proof of case 1. Since A is monotonous the path of knowledge exhibits either **(i)** $\lim_{t\to-\infty} A = 0$ and $A(t_c) = 0$ for a finite t_c, or **(ii)** $\lim_{t\to-\infty} A = \bar{A} > 0$, or **(iii)** $\lim_{t\to-\infty} A = 0$ and $A(t) > 0$ for all t. The remainder of the proof proceeds in proving the following three claims.

Claim 1 consists of three parts. *First*, if A evolves as in **(i)**, then $\lim_{t\to-\infty} N = \infty$. *Second*, if A evolves as in **(ii)**, then $\lim_{t\to-\infty} N = 0$. *Third*, if A evolves as in **(iii)**, $\lim_{t\to-\infty} N = 0$.

Claim 2. Case 3 is non-generic. Therefore, case 1c not generic either.

Claim 3. The growth rate of A does not converge generically.

Proof of Claim 1: Since the limit of A and N for $t \to -\infty$ is either zero, positive and finite, or infinite, we have to consider nine cases. ¿From (9), we deduce that $\dot{A} \geq 0$ for all t. Hence, $\lim_{t\to-\infty} A = \infty$ is impossible. This reduces the number of possible cases to six: The limit of A may be zero, or positive and finite, and the limit of N may be zero, positive and finite, or infinite. Each of the following three cases **(iv)**, **(v)** and **(vi)** leads to a contradiction and can therefore be discarded.

Case **(iv)** with properties $\lim_{t\to-\infty} A = \bar{A} \in \mathbb{R}_{++}$ and $\lim_{t\to-\infty} N = \infty$: From (9), A is monotonous. Hence $\lim_{t\to-\infty} A = \bar{A}$ implies $\lim_{t\to-\infty} \dot{A} = 0$. However,

$$\lim_{t\to-\infty} \dot{A} = \lim_{t\to-\infty} N^{\lambda} A^{\phi} = \bar{A}^{\phi} \lim_{t\to-\infty} N^{\lambda} = \bar{A}^{\phi} \cdot \infty > 0,$$

which is a contradiction. Case **(v)** with $\lim_{t\to-\infty} A = \bar{A} \in \mathbb{R}_{++}$ and $\lim_{t\to-\infty} N = \bar{N} \in \mathbb{R}_{++}$: The proof is analogous to case (iv). Case **(vi)** $\lim_{t\to-\infty} A = 0$ and $\lim_{t\to-\infty} N = \bar{N} \in \mathbb{R}_{++}$: From (10), $\lim_{t\to-\infty} y = \lim_{t\to-\infty} A\, N^{\alpha-1} = \bar{N}^{\alpha-1} \lim_{t\to-\infty} A = 0$. This implies $\lim_{t\to-\infty} f(y) < 0$. Thus, in the limit $\dot{N} = N\, f(y) < 0$, N is monotonous, and $\lim_{t\to-\infty} N = \bar{N}$ implies $\lim_{t\to-\infty} \dot{N} = 0$. Hence, $\lim_{t\to-\infty} n = 0$ is necessary, but contradictory to $\lim_{t\to-\infty} n = \lim_{t\to-\infty} f(y) < 0$. The remaining three cases are **(i)**, **(ii)** and **(iii)** from above. □

Proof of Claim 2: Since we are only interested in the shape of the (A, N)-trajectories and not in the speed of the evolution, we eliminate time from equations (9) and (10) and obtain

$$\frac{\partial N}{\partial A} = \frac{\partial N}{\partial t}\frac{\partial t}{\partial A} = \frac{\partial N}{\partial t}\left(\frac{\partial A}{\partial t}\right)^{-1} = \frac{N\, f(A\, N^{\alpha-1})}{N^{\lambda}\, A^{\phi}} = N^{1-\lambda}\, A^{-\phi}\, f(A\, N^{\alpha-1}) =: g(A, N).$$

The differential equation $\partial N/\partial A = g(A, N)$ characterizes the shape of a trajectory as a function $N(A)$. If we can show that only one trajectory starts in the origin, then we know that for any A_0, there is only one N_0 such that the limit $t \to -\infty$ for both variables is zero, hence the case is non-generic. Unfortunately, $\partial N/\partial A$ is not well-defined in the origin. Therefore, we cannot examine Lipschitz continuity (otherwise we would know that the initial value problem starting in the origin has a unique solution; as a consequence, the trajectory would have dimension one, and the proof would be complete). Instead, we show that $g_N(A, N)$ is negative. As a result, any two solutions to an initial value problem should converge for rising A. However, if several trajectories started in the origin, they would have to diverge at least at some point, thus leading to a contradiction. We have

$$\begin{aligned} g_N(A,N) &= (1-\lambda)\,N^{-\lambda}\,A^{-\phi}\,f(A\,N^{\alpha-1}) + (\alpha-1)\,N^{\alpha-\lambda-1}\,A^{1-\phi}\,f'(A\,N^{\alpha-1}) \\ &= N^{-(1+\lambda)}\,A^{-\phi}\left[(1-\lambda)\,N\,f(A\,N^{\alpha-1}) - (1-\alpha)\,A\,N^{\alpha}\,f'(A\,N^{\alpha-1})\right]. \end{aligned}$$

Because $\lambda > 1$ and $\alpha < 1$, the proof is complete if we can show that $f(A\,N^{\alpha-1}) > 0$. Hence, population growth must be strictly positive. Assume for the moment that $f(A\,N^{\alpha-1}) < 0$, then as a consequence $\dot{N} < 0$, but $\dot{A} > 0$ (this is always true). Therefore, a logical second before, population must have been even larger, knowledge even smaller, and per-capita income even lower. As a result, $\dot{N}$ is negative for every point in time before. Hence, the trajectory cannot pass through the origin. Consequently, we have $f(A\,N^{\alpha-1}) > 0$ on every trajectory that starts in the origin. Hence, $g_N(A,N) < 0$ on every such trajectory, and, therefore, there is only one such trajectory.

Note that $\lambda > 1$ is a sufficient criterium, albeit not a necessary criterium. A tighter (but still not necessary) condition is $g_N(A,N) < 0$ if $(1-\lambda)\,f(A\,N^{\alpha-1}) - (1-\alpha)\,A\,N^{\alpha-1} \cdot f'(A\,N^{\alpha-1}) < 0$, i. e. if $(1-\lambda)\,f(y) - (1-\alpha)\,y\,f'(y) < 0$, i. e. if $\partial f(y)/\partial y \cdot y/f(y) > (1-\lambda)/(1-\alpha)$. Hence if $\lambda < 1$, the elasticity of population growth with respect to per capita income must be small enough. □

Proof of Claim 3: Assume that knowledge reaches a steady state, i. e. $A \sim \exp(a\,t)$. Then, from (9), $a = \dot{A}/A = N^{\lambda}\,A^{\phi-1}$, and N grows at rate $n = a\,(\phi-1)/\lambda$. However, from (10), N grows at rate $n = \dot{N}/N = f(y)$. As a consequence, $y = f^{-1}(a\,(\phi-1)/\lambda)$ is a constant. This leads to a contradiction since, $y = A\,N^{\alpha-1}$ and grows at rate $a + n\,(\alpha-1)$. Thus, unless we have the non-generic constellation $a = n\,(1-\alpha)$, there is no steady-state path for A. ■

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