PROBABILISTIC AGING

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Abstract

The paper develops an overlapping generations model with probabilistic aging of households. We define age as a set of personal attributes such as earnings potential, health and tastes that are characteristic of a person's position in the life-cycle. In assuming a limited number of different states of age, we separate the concepts of age and time since birth. Agents may retain their age characteristics for several periods before they move with a given probability to another state of age. Different generations that share the same age characteristics are aggregated analytically to a low number of age groups. The probabilistic aging model thus allows for a very parsimonious yet rather close approximation of demographic structure and life-cycle differences in earnings, wealth and consumption. Existing classes of overlapping generations models follow as special cases.

JEL Code: D58, D91, H55, J21.

Keywords: overlapping generations, aging, demographics, life-cycle earnings.

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1 Introduction

Many issues of fiscal policy and intertemporal macroeconomics are concerned with the behavior of overlapping generations (OLG) of households. Depending on the specific issues to be analysed, economists have a range of alternative models at hand. At one extreme end, one finds the representative agent model of infinitely lived consumers due to Ramsey (1928), Cass (1965) and Koopmans (1965). Weil (1987) has shown that this model follows from an OLG model if current generations are perfectly linked with future generations by an operative, altruistic bequest motive. Under this and a number of further conditions, the Ramsey model gives rise to the famous Ricardo-Barro (1974) debt neutrality theorem. If this perfect intergenerational link is cut either by the entry of new, unconnected generations at the extensive margin of population growth (Weil, 1989) or by a non-altruistic or non-operative bequest motive, then the debt neutrality theorem is violated and fiscal policy tends to redistribute across generations.[1]

At the other extreme is the two period OLG model without bequests, as pioneered by Samuelson (1958) and Diamond (1965), which gives rise to the standard crowding out hypothesis. The model is analytically very tractable but cuts down life-cycle detail to the bare bones. It is thus less useful for quantitative empirical analysis since the length of a period featuring constant interest rates and constant rates of consumption and investment etc. covers about thirty years in real time. At the expense of analytical tractability, empirical applications thus rely on numerically solved models with a large number of generations and detailed life-cycle patterns of earnings, consumption and savings. These models were pioneered by Auerbach and Kotlikoff (1987) and Hubbard and Judd (1987) for the analysis of tax and fiscal policy including demographic change and social security reform. More recent and refined applications include Altig and Carlstrom (1999), Altig, Auerbach, Kotlikoff, Smetters and Walliser (2001), Imrohoroglu, Imrohoroglu and Joines (1999), Heckman, Lochner and Taber (1998) and Blundell, Costa Dias and Meghir (2003)

[1] Smetters (1999) shows, however, that at least the long-run capital intensity is robust to most assumptions of the standard Ricardian model although neutrality tends to fail in the short-run.

for tax and social security reform as well as education and labor market policies.

Life-cycle models with many generations can take account of detailed differences in wealth, marginal propensity to consume and in labor supply and earnings of different agents. But precisely for this reason, these models tend to be analytically intractable and must be solved numerically. They operate in a state space of as many as 108 dimensions as Laitner (1990) has shown.[2] Therefore, they are very expensive to implement and the results are difficult to replicate. By way of contrast, the perpetual youth model due to Yaari (1965), Blanchard (1985) and Buiter (1988) is an analytically very tractable OLG model with a realistic period length. Because of its simple empirical application, it has become a widely used tool of quantitative analysis as well. The main drawback, however, is the rigid demographic assumption of an age independent mortality rate and the absence of life-cycle detail of earnings, consumption and savings.[3] For this reason, the model is not well suited for the analysis of aging or old age insurance. The recent extension by Gertler (1999) reconciles the perpetual youth model with an important aspect of life-cycle behavior by allowing for a stochastic transition from work into retirement and from retirement into death. It thereby opens up useful applications with an easily implemented model that includes no more than two stock variables to model the household sector. Although somewhat more complex, it is analytically tractable and easily implemented empirically. According to Cooley (1999), however, it is not really an improvement in quantitative terms over existing life-cycle models with many generations.

This paper develops an alternative approach that retains the simplicity of analytically aggregated OLG models and yet succeeds to replicate the rich life-cycle details of the high dimensional finite horizon models. The model is easily empirically implemented with a few state variables only and thereby makes it easier to replicate empirical studies on life-

[2] The state space would double if one considered two skill groups. The high dimensionality is a difficult constraint to real business cycle analysis which has lately turned to life-cycle models for exploring labor market fluctuations, see Gomme et al. (2005), for a recent example.

[3] Heijdra and Romp (2005), however, have included some life-cycle features in a continuous time version of the model that still allows for analytical solutions.

cycle effects of public policy and other shocks. The key idea is to separate the concept of "age" from "time since birth". We define *age* as a set of physical or mental attributes such as earnings potential, health, tastes, and other characteristics that may not change every period. Aging thus refers to the change in life-cycle characteristics rather than the passing of calender time. This is the key difference to existing models, where aging and the passing of time are perfectly synchronized and age is understood as time since birth. In these models, there are as many states of age as there are life-cycle periods, and an age period is identical to a time period. This is most obvious in the two period OLG model which distinguishes between youth and old age only. A period length corresponds to about thirty years in real life which imposes severe limits in empirical applications. An OLG model with 55 life-cycle periods, in contrast, distinguishes between 55 states of age to replicate as close as possible real world life-cycles.

Our alternative approach builds on the insight that, for empirical purposes, it is enough to distinguish only a few different states of age in order to capture empirically realistic life-cycle differences. Consequently, age and time since birth become very different. Aging occurs much less frequently than the passing of calender time and tends to occur stochastically. It seems, in fact, a realistic real world feature that some people age faster than others. Some people retain their health, earnings potential and youthfulness for many periods and thereby invite comments such as "that person still looks the same as ten years ago". Obviously, such a statement means that a person has not visibly aged. Other people age much faster, possibly as a result of accidents or simply due to personal health characteristics. Since aging is assumed to occur stochastically, we label the proposed framework the "probabilistic aging" (PA) model. It will usually be sufficient to distinguish only a few different states of age. This gives rise to the concept of an age group which consists of the collection of generations or cohorts that find themselves in the same state of age. Choosing the number of age states and, correspondingly, of age groups is a matter of how closely the empirical life-cycle features should be replicated.

In the PA model, people move stochastically from one age state to another with an age group specific transition rate. Our PA approach is thus a natural generalization of the model by Gertler (1999). He distinguishes only two age groups that refer to workers and retirees. He also assumes, somewhat restrictively, that mortality risk sets in only after retirement while workers face no such risk. We allow for more age groups and find that about eight age states already yield a close approximation of empirical life-cycle properties. Another generalization and extension is that the PA model allows for mortality already in younger age groups and thereby yields a closer approximation of demographic characteristics. Furthermore, it is not only important to distinguish several worker groups to capture life-cycle earnings detail but also to distinguish several groups of retirees to take account of the substantial heterogeneity among old and very old generations. Health deteriorates and mortality increase rapidly among the very old generations.

In limiting the number of age groups, the state space of the PA model is drastically reduced as compared to standard life-cycle models with many generations. The PA model allows for analytical aggregation of generations into age groups and is, thus, easily implemented empirically. For example, the youngest age group could be chosen to correspond to the 20 to 30 year olds. With agents normally staying a number of periods in the same group, age and time since birth become entirely different concepts. Despite of distinguishing only a few age states, the PA model nevertheless supports a period length of one year, or a quarter if desired, for realistic dynamic properties. Another insightful feature of the PA model is that it replicates a wide range of intertemporal household models as special cases by appropriately choosing the parameters governing the aging process.

Section 2 now explains how an agent's life-cycle is modeled by defining age states with different characteristics such as earnings potential, mortality rate and possibly other attributes as well. It is then shown how the population is analytically aggregated into age groups. Section 3 solves the intertemporal optimization problem of households in the presence of mortality and aging risks. We delineate existing classes of intertemporal models as special cases of the PA model by appropriately parameterizing the aging process. Section 4 presents an illustrative empirical application and section 5 concludes.

2 The Probabilistic Aging Model

In line with popular observations, we view aging as discrete events that occur stochastically. Some people remain young for many periods, with their labor productivity, health and other attributes being unchanged before an aging event moves them to the next state. The less fortunate ones grow old much more rapidly, maybe due to sickness, accidents and other key events. They might age every single year and arrive within a few periods at the final state of life before extinction. To capture these statements, one needs two different clocks to measure the passing of real time and the speed of the aging process. We measure real time in regular annual periods, with interest, prices and quantities appropriately defined per year. The aging clock runs slower and stochastically.

2.1 Life-Cycle Histories

We define a discrete number of A states of increasing age, and accordingly collect all agents with identical characteristics in the same age group $a \in \{1, \ldots, A\}$. People start life in state $a = 1$ with a given set of attributes. The life-cycle characteristics could include a person's earnings potential or her mortality risk but possibly other attributes as well. Aging means that an individual's life-cycle characteristics change when she grows older by switching to state $a+1$. If a person does not age from one period to the next, she remains in the same group and keeps her characteristics unchanged. Although aging shocks arrive stochastically, the average outcome of heterogeneous aging patterns across individuals leads to smooth profiles of expected life-cycle earnings and other characteristics.

Households differ not only by their date of birth, but also by their diverse life-cycle histories. An agent's life-cycle history is her biography of aging events that have happened since birth. It is represented by a vector α that records the past dates of aging events. At date t, the set of possible histories of a household that belongs to age group a is

$$\mathcal{N}_t^a \equiv \{(\alpha_1, \ldots, \alpha_a) : \alpha_1 < \ldots < \alpha_a \leq t\}. \tag{1}$$

A particular life-cycle history is represented by a vector $\alpha \in \mathcal{N}_t^a$. The element α_i, $i \in \{1, \ldots, a\}$, denotes the date at which the household who was formerly in age group $i-1$, became a member of group i. In denoting the unborn by a virtual age group zero, the element α_1 lists the date of birth when an agent switches from the group of unborn to the first age group. We say that a member of group one aged only once with no further aging since birth. Nevertheless, different persons of the first age group are heterogeneous since they were born at different moments in the past. The set of possible biographies is $\mathcal{N}_t^1 = \{(\alpha_1) : \alpha_1 \leq t\}$. By the same logic, α_2 is the date when an agent moved from group 1 into 2. People in age group 2 have aged twice. The set of life-cycle histories in this case is $\mathcal{N}_t^2 = \{(\alpha_1, \alpha_2) : \alpha_1 < \alpha_2 \leq t\}$. With this notation, the vectors α describing the biography of people in group a have a elements since such persons have aged a times.

Individual biographies are updated when a person experiences an aging event. Suppose a person is in age group $a-1$ and is identified by a biography $\alpha = (\alpha_1, \ldots, \alpha_{a-1})$. When the next aging shock occurs at the end of period t, she arrives in group a next period. Her biography is appended by the entry $t+1$ and reads $(\alpha_1, \ldots, \alpha_{a-1}, t+1)$. Accordingly, the set $\mathcal{N}_t^a$ of biographies of age group a will be augmented next period by all the biographies $\alpha' \in \mathcal{N}_t^{a-1} \times (t+1)$ that have $t+1$ as their last entry and refer to people who currently switch from group $a-1$ to a. The set of biographies $\mathcal{N}_{t+1}^a$ is thus divided into two disjoint sets where the first refers to all those who were already in group a in period t while the second refers to the newly aged people who were in group $a-1$,

$$\mathcal{N}_{t+1}^a = \mathcal{N}_t^a \cup \mathcal{N}_t^{a-1} \times (t+1), \quad a \in \{1, \ldots, A\}. \tag{2}$$

To model demographics, we allow for mortality among younger age groups. When an individual with an arbitrarily given life-cycle history plans for next period, she faces the risk of aging and dying. She must thus reckon with three possible events: (i) she dies with probability $1-\gamma^a$; (ii) she survives without aging and remains in the same age group with probability $\gamma^a \omega^a$, and (iii) she survives and ages and belongs to age group $a+1$ next period with probability $\gamma^a(1-\omega^a)$. Individuals in the last age group have exhausted the aging process. They may either survive with probability γ^A within group A or die with probability $1-\gamma^A$. Observe that only the last age group behaves according to the mortality and demographic assumptions of Blanchard's (1985) perpetual youth model.

Since the characteristics of people such as their earnings potential differ across age groups, an agent's consumption, assets and other economic variables will generally depend on her particular life-cycle history. For example, assets depend on the agent's past earnings history which, in turn, is linked to her aging trajectory. To keep track of the population's heterogeneity, one must thus very carefully identify each agent by her age group as well as her aging biography α. The number of agents at date t, in state of life a and with aging history α is given by $N_{\alpha,t}^a$. Within this group, agents are identical and face the same independent probability of moving to one of the alternative states. With stochastically independent risks, the law of large numbers implies that the individual probabilities for a certain event correspond to the fraction of people that are subject to this event. Consequently, age group a is divided into three subgroups next period: (i) those who die and whose biography is updated to $\alpha^\dagger$; (ii) those who survive within the same age group a; and (iii) those who are hit by an aging shock and switch to the next group. The biography α remains unchanged in case (ii) while switching to the next group $a+1$ in case (iii) adds another event in a person's life-cycle history α and thereby results in a new biography α':

$$\begin{array}{llcll} (i) & N_{\alpha^\dagger,t+1}^{\dagger} & = & N_{\alpha,t}^a \cdot (1-\gamma^a)\,, & \text{death,} \\ (ii) & N_{\alpha,t+1}^a & = & N_{\alpha,t}^a \cdot \gamma^a\omega^a, & \text{no aging,} \\ (iii) & N_{\alpha',t+1}^{a+1} & = & N_{\alpha,t}^a \cdot \gamma^a(1-\omega^a)\,, & \text{aging.} \end{array} \tag{3}$$

2.2 Demographic Structure and Earnings Profiles

People with the same biography are identical. At date t, age group a includes a number $N_{\alpha,t}^a$ of agents with the same biography α. The total number of people in age group a is obtained by adding up over all possible histories α ending up in this group a,

$$N_t^a \equiv \sum_{\alpha\in\mathcal{N}_t^a} N_{\alpha,t}^a. \tag{4}$$

The aggregation formula (4) takes the sum over all possible biographies with varying dates of birth that could conceivably lead to age group a in period t. Given identical death and aging probabilities within a given group, one can now use the law of large numbers for analytical aggregation. The aggregate population groups evolve deterministically over time where $N^1_{(t+1),t+1}$ refers to the mass of newborns who arrive at the beginning of period $t+1$ in the first age group. Proofs are found in Appendix B.

Proposition 1 ***(Demographic Structure)*** *The laws of motion for age groups are*

$$\begin{aligned} (a) &: N^a_{t+1} = \gamma^a\omega^a \cdot N^a_t + \gamma^{a-1}\left(1-\omega^{a-1}\right) \cdot N^{a-1}_t, \quad \omega^A = 1, \\ (b) &: N^1_{t+1} = \gamma^1\omega^1 \cdot N^1_t + N^1_{(t+1),t+1}, \\ (c) &: N_{t+1} = N_t + N^1_{(t+1),t+1} - \sum_{a=1}^{A}\left(1-\gamma^a\right) N^a_t, \quad N_t \equiv \sum_{a=1}^{A} N^a_t. \end{aligned} \tag{5}$$

The key demographic parameters are the birth rate, the transition rates to successive age groups, and the mortality rates. Since all are exogenous, the demographic subsystem is independent of economic influences and evolves autonomously according to (5.a-c). The demographic steady state results from the requirement that inflows and outflows of any age group must balance to yield constant group size. Using (5.a-b),

$$N^1 = \frac{N^1_{t,t}}{1-\gamma^1\omega^1}, \quad N^a = \frac{\gamma^{a-1}\left(1-\omega^{a-1}\right)}{1-\gamma^a\omega^a} \cdot N^{a-1}. \tag{6}$$

The stationary size of group 1 is determined by the exogenous inflow of newborns. The long-run magnitude of other groups and of the total population results upon recursively applying the second equation. For any demographic transition, the exogenous driving force is the inflow $N^1_{t,t}$ of newborns. With this flow exogenously specified and constant, the system arrives at a stationary population N.

The empirical implementation of the PA model on real population data rests on the fact that it contains the annual "cohort" model as a special case. Setting $\omega^a = 0$ in equation (5) implies that an aging event occurs with probability one in each period, leading to $\tilde{N}^t = \tilde{\gamma}^{t-1}\tilde{N}^{t-1}$. The tilde indicates the decomposition in annual cohorts or population vintages $\tilde{N}^t$. The concept of an age group thus becomes identical with a cohort or vintage where age t is measured by time since birth. We now take the age dependent survival rates $\tilde{\gamma}^t$ from official mortality tables and construct the cohort composition of the population in a demographic steady state. Recursively applying $\tilde{N}^t = \tilde{\gamma}^{t-1}\tilde{N}^{t-1}$ yields the size of cohort t relative to the size of a new cohort. Summing up over all cohorts fixes the size of the new cohort compared to total population size N,

$$\tilde{N}^t = \tilde{N}^1 \prod_{s=1}^{t-1} \tilde{\gamma}^s, \quad \tilde{N}^1 = N \Big/ \sum_{t=1}^{T} \prod_{s=1}^{t-1} \tilde{\gamma}^s . \tag{7}$$

Taking a total length of life of T years, and based on actual survival rates, we have thus found the stationary decomposition of the population into a total of T cohorts or vintages. Alternatively, the total population may be decomposed into broader age groups N^a. Each of these contains all vintages that share the same age characteristics,

$$\sum_{t=1}^{T} \tilde{N}^t = N = \sum_{a=1}^{A} N^a. \tag{8}$$

For the sake of concreteness, assume that total life-time consists of $T = 70$ periods as in Table 1 which considers the active population starting at age 20 and living until age 90. The total population is divided into a cross section of eight cohort groups. The first six are equally spaced and contain 10 cohorts each, the very old are subdivided into two smaller groups with five cohorts each. The first group contains vintages $N^1 = \sum_{t=20}^{29} \tilde{N}^t$. Line 3 of Table 1 reflects actual, non-stationary population data. Line 4 reports the population shares based on (7) that reflect a demographic steady state with observed mortality rates.

How is the PA model matched with data obtained from the annual cohort model? Aggregation collects a subgroup of neighboring cohorts into an age group and takes the average wage and other average attributes to define the characteristics of the corresponding age group. Taking the first group as an example, we have three restrictions to be fulfilled: (i) the mass of agents in age group one must be identical with the mass of people in cohorts 20-29; (ii) the average wage of cohorts 20-29 must correspond to the uniform wage in age group one; and (iii) the expected duration that an agent spends in a given age group corresponds to the number of vintages that define this group. Since cohorts

20-29 define the first group, the expected duration in that group is ten years. For the last two age groups in Table 1, the expected duration is assumed 5 years to approximate more closely the increase in mortality rates in the last phase of life.

Table 1: Demographic and Life-Cycle Parameters

1. Age groups	1	2	3	4	5	6	7	8
2. Cohorts	20-29	30-39	40-49	50-59	60-69	70-79	80-84	85-89
3. Data N^a/N	0.168	0.222	0.192	0.168	0.120	0.089	0.025	0.016
4. Model N^a/N	0.179	0.177	0.175	0.168	0.148	0.107	0.031	0.016
5. Labor prod. θ^a	1.000	1.362	1.561	1.582	1.295	0.000	0.000	0.000
6. Prob. $1-\gamma^a$	0.001	0.001	0.004	0.012	0.028	0.042	0.096	0.200
7. Prob. $1-\omega^a$	0.099	0.099	0.096	0.089	0.074	0.061	0.115	0.000
8. Propens. $1/\Delta^a$	0.047	0.052	0.059	0.069	0.086	0.110	0.168	0.230

Notes: θ^a life-cycle labor productivity determines wage $w^a = w\theta^a$, $1-\gamma^a$ probability of dying, $1-\omega^a$ probability of aging, $1/\Delta^a$ marginal propensity to consume. Data sources: BFS (2004), and own calculations.

We now show how these restrictions identify the parameters of the PA model. Setting the date of birth at $\alpha_1 = 20$, the aggregation key chosen in Table 1 corresponds to a particular life-cycle biography $\alpha = (20, 30, 40, \ldots, 80, 85)$. An agent with this biography would spend exactly the average duration in the respective age groups and would thus always belong to the cohorts that defined this group. Having chosen the aggregation key, one may aggregate T cohorts into A age groups by

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_A), \quad \alpha_A < T, \quad N^a = \sum_{t=\alpha_a}^{\alpha_{a+1}-1} \tilde{N}^t. \tag{9}$$

Equation (9) satisfies the first restriction. To fulfill the second, one uses actual wage data to find life-cycle productivity θ^a. For any given wage w per efficiency unit of labor, one obtains life-cycle wages $w^a = w\theta^a$. The life-cycle profiles of annual wages $\tilde{w}^t$ are readily available from cohort data. Condition (9) imposes that the total mass of an age group must be identical to the mass of cohorts which define the characteristics of this group. In perfect analogy, we require that the aggregate wage bill of a given age group is the same as the accumulated wage bill from all cohorts that define this group:

$$w^a N^a = \sum_{t=\alpha_a}^{\alpha_{a+1}-1} \tilde{w}^t \tilde{N}^t. \tag{10}$$

This restriction determines the group specific wages w^a as an average of cohort specific wages $\tilde{w}^t$, taking the shares $\tilde{N}^t/N^a$ of the cohorts in that age group as weights. Figure 1 illustrates how they approximate the empirical wage profile. The solid, ragged line shows actual wage data for each cohort in Switzerland 2000. The solid step function with 10 year intervals corresponds to line 5 in Table 1. Note that the wage of age group 5 is an average of the relatively high wage prior to retirement and the zero wage after retirement when people are in their late sixties. The dashed line with 20 year intervals represents only 4 age groups with the first three comprising 20 cohorts and the last one 10 cohorts. Obviously, the approximation is rather crude for periods early and late in the life-cycle while it makes not much difference for agents in their mid-life. Finally, the dashed line with 5 year spacing comes considerably closer to the true wage profile by taking averages over fewer cohorts. The gains in empirical approximation are highest for the youngest cohorts facing a steep wage profile, and for the cohorts close to retirement.

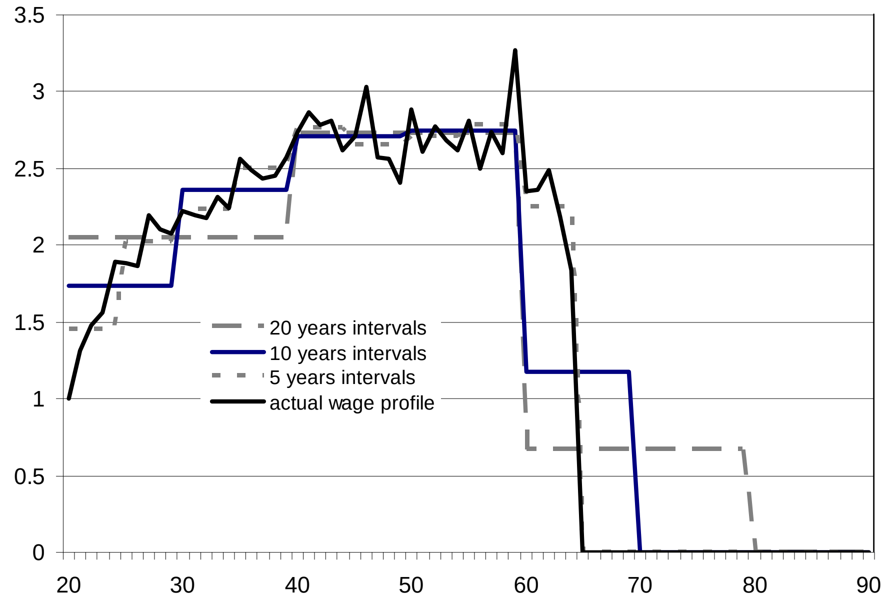

Fig. 1: Life-Cycle Wages. Data source: BFS (2004)

The last restriction must yield the demographic parameters that replicate the population shares based on (6) which are listed in line 4 of Table 1. To identify the two parameters ω^a and γ^a, we have in fact two restrictions, the expected duration in group a and the size of group a. Linking the aggregation of cohorts to a life-cycle history as in (9) implies that an agent expects to remain $\alpha_{a+1} - \alpha_a$ periods in age group a. In the above example, the average time spent in the first age group would be 10 years while expected duration in the last group is five years.[4] Given that the instantaneous probability of

[4] To allow convenient aggregation, the perputual youth model of Blanchard (1985) postulates a single mortality rate $1 - \gamma$ which is the same for all vintages and thereby implies very rigid demographic structure. This literally implies that the expected *remaining* life-time is the same for a new born and a person 100 years old. The PA model shares the same property within each age group implying that a newborn expects the same duration in group one than a person who already spent 20 years in this group and still has not aged. In the PA model, however, mortality rates increase substantially upon aging as Table 1 illustrates. A person in the last group, with characteristics corresponding to people aged 85-90, expects a remaining life-time of only five years which essentially cuts off the population at very high ages!

staying is $\omega^a\gamma^a$ according to (3.ii), the expected duration in group a is

$$\alpha_{a+1} - \alpha_a = 1/\left(1 - \omega^a\gamma^a\right). \tag{11}$$

One can now recover the demographic parameters of the PA model from our knowledge of aggregated population data N^a and duration in group a. In a demographic steady state, each age group must fulfill the restriction (5.a) which is

$$\left(1 - \gamma^a\omega^a\right) \cdot N^a = \left(\gamma^{a-1} - \gamma^{a-1}\omega^{a-1}\right) \cdot N^{a-1}. \tag{12}$$

At this stage, one knows N^a from aggregated population data and $\gamma^a\omega^a$ from the age group duration as implied by the chosen aggregation. One can now easily solve (12) for all γ^{a-1} except for the last one. Having computed γ^{A-1}, the survival rate γ^A of the last group follows directly from (12) on account of the restriction $\omega^A = 1$. Lines 6 and 7 of Table 1 list the resulting values of the exit probabilities.[5] Figure 3 illustrates how these values approximate the true mortality rates from demographic data. The step function reflects the fact that mortality rates are identical for all agents within an age class.

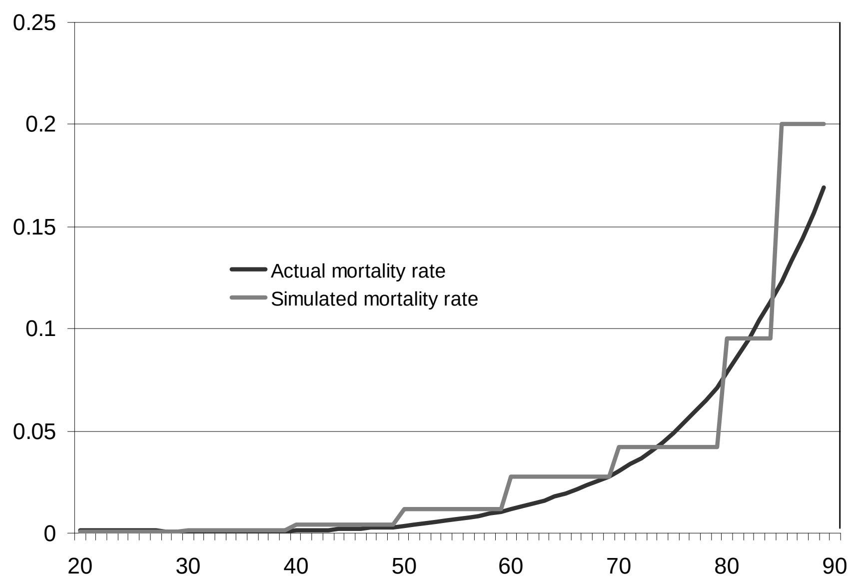

Fig. 2: Mortality Rates

[5]The marginal propensity to consume in line 8 will be discussed in section 4.

3 Life-Cycle Economies

3.1 Consumptions and Savings

The theory of probabilistic aging is now combined with an intertemporal general equilibrium model with life-cycle decisions and perfect foresight. In economic applications, a key distinction between age groups is the definition of wage related income,

$$y_t^a = \begin{cases} (1-\tau)\, w_t \theta_t^a & : a \in \left\{1, \ldots, a^R - 1\right\}, \\ p_t & : a \in \left\{a^R, \ldots, A\right\}. \end{cases} \tag{13}$$

There are several states of work and retirement life where a^R denotes the retirement date in a person's biography. Wage related income per capita, y_t^a, differs across age groups but is identical for all agents within the same group. Importantly, it does not depend on life-cycle history α which is generally different for people in the same age group. Each worker in group a is thus endowed with θ^a efficiency units of labor which earn a wage of w_t per unit. Consequently, a worker's gross wage per unit of labor is $w_t^a = w_t \theta_t^a$. The government levies a proportional wage tax τ to finance pensions and other public spending. Retirees receive a pension of p_t per capita from pay-as-you-go (PAYG) social security.

Given current financial wealth and labor income, agents accumulate assets

$$\gamma^a A_{\alpha,t+1}^a = R_{t+1} \left[A_{\alpha,t}^a + y_t^a - C_{\alpha,t}^a\right], \quad A_{\alpha,t+1}^a = A_{\alpha',t+1}^{a+1}, \tag{14}$$

where C stands for consumption, A for assets, and $R = 1+r$ is the interest factor equal to one plus the annual interest rate. The value of assets is measured at the end of the period and earns, together with new savings $y - C$, an interest r_{t+1} until the end of next period. Note that a person inherits the same asset wealth in period $t+1$ from savings in period t irrespective of whether he ages or not. The term γ^a on the left-hand side arises due to the assumption of reverse life-insurance. In the absence of a bequest motive, the agent wants to get compensated during her life-time for any accidental bequests that she leaves upon death. Such compensation is assumed to be available from a competitive insurance sector. It is assumed that an actuarially fair, group specific insurance scheme is available to all age groups. Suppose the agent has assets of $S^a_{\alpha,t}$ at the end of period t, equal to the square bracket in (14). Aggregate end of period assets of this group are S^a. Since a fraction $1-\gamma^a$ dies, the insurance sector collects assets with a total value of $(1-\gamma^a) S^a$. On the other hand, premiums must be paid to those who survive, adding up to $\pi^a\gamma^a S^a$. The insurance sector breaks even with a premium of $\pi^a = (1-\gamma^a)/\gamma^a$ or $1+\pi^a = 1/\gamma^a$. If such insurance is available, the agent's assets next period are $A^a_{\alpha,t+1} = R_{t+1}(1+\pi^a_t) S^a_{\alpha,t}$ if she survives, or $\gamma^a A^a_{\alpha,t+1} = R_{t+1} S^a_{\alpha,t}$ as is stated in (14).

Assumed preferences reflect CES non-expected utility theory as proposed by Farmer (1990) and Weil (1990) and recently reviewed by Backus, Routledge and Zin (2005). These preferences restrict individuals to be risk neutral with respect to variations in income but allow for an arbitrary intertemporal elasticity of substitution. Let β be a subjective discount factor reflecting the pure rate of time preference. Agents maximize expected welfare over the remaining life-time. Except for the last age group, all agents are potentially subject to an aging shock. Hence, a person's expected utility next period, conditional on surviving, is

$$\bar{V}^a_{\alpha,t+1} \equiv \omega^a V^a_{\alpha,t+1} + (1-\omega^a) V^{a+1}_{\alpha',t+1}. \tag{15}$$

With probability ω^a, the agent is not aging and expects welfare $V^a_{\alpha,t+1}$. With probability $1-\omega^a$, she ages, switches to the next higher age state and expects welfare $V^{a+1}_{\alpha',t+1}$. The recursive form of intertemporally separable preferences yields the Bellmann equation

$$V\left(A^a_{\alpha,t}\right) = \max_{C^a_{\alpha,t}} \left[\left(C^a_{\alpha,t}\right)^\rho + \gamma^a\beta\left(\bar{V}^a_{\alpha,t+1}\right)^\rho\right]^{1/\rho}. \tag{16}$$

The CES parameter $\rho = 1-1/\sigma$ reflects the constant elasticity of intertemporal substitution σ. To solve for the optimal consumption policy, it is useful to define

$$\eta^a_{\alpha,t} \equiv \frac{dV^a_{\alpha,t}}{dA^a_{\alpha,t}}\left(V^a_{\alpha,t}\right)^{\rho-1}, \quad \bar{\eta}^a_{\alpha,t+1} \equiv \left[\omega^a \frac{dV^a_{\alpha,t+1}}{dA^a_{\alpha,t+1}} + (1-\omega^a)\frac{dV^{a+1}_{\alpha',t+1}}{dA^{a+1}_{\alpha',t+1}}\right]\left(\bar{V}^a_{\alpha,t+1}\right)^{\rho-1}, \tag{17}$$

where the square bracket is a weighted shadow price of next period's assets that emerges due to the possibility of aging.

Solving the dynamic programming problem (16) subject to (14) yields the optimality and envelope conditions for consumption and assets,

$$\left(C^a_{\alpha,t}\right)^{\rho-1} = \beta R_{t+1}\cdot\bar{\eta}^a_{\alpha,t+1}, \quad \eta^a_{\alpha,t} = \beta R_{t+1}\cdot\bar{\eta}^a_{\alpha,t+1}. \tag{18}$$

When the consumer postpones consumption and accumulates more assets, she compares the current utility loss with the *expected* utility gain next period as reflected in $\bar{\eta}$. The optimality condition reflects the usual tangency condition in intertemporal optimization that requires equality of the marginal rate of transformation and the agent's marginal rate of intertemporal substitution between current consumption and next period assets, $MRT = MRIS$. Suppressing indices other than t, the differential of the budget constraint yields $MRT^a \equiv -dA^a_{t+1}/dC^a_t = R_{t+1}/\gamma^a$. The rate of substitution follows from the differential of the Bellmann equation,[6] $MRIS^a \equiv -dA^a_{t+1}/dC^a_t\big|_{dV^a_t=0} = \left(C^a_t\right)^{\rho-1}/\left[\gamma^a\beta\bar{\eta}^a_{t+1}\right]$. Equating them gives the optimality condition in (18).

The necessary conditions in (18) lead to a modified Euler equation of consumption growth. To interpret, it will be useful to state the marginal rate of substitution across age states which reflects the consumer's trade-off between an extra Euro in the two alternative states that she may end up next period. Since life-cycle characteristics change when an agent moves from age state a to $a+1$, an extra Euro will be valued differently in the two states. The differential of expected utility next period, $d\bar{V}=0$, yields

$$\begin{aligned} MRS^a &\equiv dA^a/dA^{a+1}\big|_{d\bar{V}=0} = \tfrac{1-\omega^a}{\omega^a}\,\tfrac{dV^{a+1}/dA^{a+1}}{dV^a/dA^a} = \tfrac{1-\omega^a}{\omega^a}\left(\Lambda^a\right)^{1-\rho}, \\ \Omega^a &\equiv \omega^a\cdot\left[1+MRS^a\right] = \omega^a + \left(1-\omega^a\right)\left(\Lambda^a\right)^{1-\rho}, \quad \Lambda^a \equiv \tfrac{V^{a+1}/C^{a+1}}{V^a/C^a}. \end{aligned} \tag{19}$$

The last equality combines $\eta^a = \left(C^a\right)^{\rho-1}$ from (18) with $dV^a/dA^a = \eta^a\left(V^a\right)^{1-\rho} = \left(V^a/C^a\right)^{1-\rho}$ from (17). The consumer would be willing to give up MRS^a Euros in state a if she could thus obtain an extra Euro in state $a+1$. The fact that an Euro saved yields an extra gain MRS^a in state $a+1$ acts like a magnification of the interest factor in the

[6] Suppressing α, we calculate the current utility loss from postponed consumption, $dV^a_t/dC^a_t = \left(V^a_t/C^a_t\right)^{1-\rho}$, and the expected utility gain per additional Euro tomorrow, $dV^a_t/dA^a_{t+1} = \left(V^a_t\right)^{1-\rho}\gamma^a\beta\bar{\eta}^a_{t+1}$. Since an Euro saved yields R_{t+1}/γ^a Euros tomorrow, the total gain from postponed consumption is $dV^a_t/dC^a_t = \left(R_{t+1}/\gamma^a\right)\cdot dV^a_t/dA^a_{t+1}$. Dividing through by the last term yields $MRIS = MRT$.

Euler equation. To see this, take out ω^a and $dV^a/dA^a = (V^a/C^a)^{1-\rho}$ from the square bracket in (17). Therefore, $\bar{\eta}^a = \Omega^a \left[\bar{V}^a/ (V^a/C^a)\right]^{\rho-1}$ with Ω^a given in (19). Writing expected utility as $\bar{V}^a = [\omega^a C^a + (1-\omega^a)\Lambda^a C^{a+1}] (V^a/C^a)$ results, upon substitution, in $\bar{\eta}^a = \Omega^a [\omega^a C^a + (1-\omega^a)\Lambda^a C^{a+1}]^{\rho-1}$. Using this in (18) yields the modified Euler equation where $\sigma = 1/(1-\rho)$,

$$\omega^a C^a_{\alpha,t+1} + (1-\omega^a)\Lambda^a_{\alpha,t+1} C^{a+1}_{\alpha',t+1} = \left(\beta R_{t+1}\Omega^a_{\alpha,t+1}\right)^\sigma \cdot C^a_{\alpha,t}. \tag{20}$$

Observe that probabilistic aging results in an increased mortality rate which is reflected in the magnification $\Omega^a_{\alpha,t+1}$ of the interest factor as will be further discussed below. The Euler equation states that desired consumption growth is a function of the expected interest rate relative to the subjective discount rate implicit in the factor β. A higher interest tilts the consumption profile towards the future, relative to a given life-cycle income profile, and thus implies higher savings. The sensitivity with respect to interest depends on the elasticity of intertemporal substitution. One can now obtain a closed form solution for consumption and indirect utility in terms of the agent's financial assets and the present value of her expected future pension benefits, see Appendix B for a proof.

Proposition 2 ***(Consumption)*** *Optimal consumption $C^a_{\alpha,t}$ and indirect utility $V^a_{\alpha,t}$ are*

$$\begin{array}{rlcl}
(i) & C^a_{\alpha,t} & = & (1/\Delta^a_t)\left(A^a_{\alpha,t} + H^a_{\alpha,t}\right), \quad \sigma = 1/(1-\rho), \\
(ii) & V^a_{\alpha,t} & = & (\Delta^a_t)^{1/\rho} C^a_{\alpha,t}, \\
(iii) & \Delta^a_t & = & 1 + \gamma^a\beta^\sigma\left(\Omega^a_{t+1}R_{t+1}\right)^{\sigma-1}\Delta^a_{t+1}, \\
(iv) & \Omega^a_{t+1} & = & \omega^a + (1-\omega^a)\left(\Lambda^a_{t+1}\right)^{1-\rho}, \quad \Lambda^a_{t+1} = \left(\Delta^{a+1}_{t+1}/\Delta^a_{t+1}\right)^{1/\rho}, \\
(v) & H^a_{\alpha,t} & = & y^a_t + \gamma^a \bar{H}^a_{\alpha,t+1}/\left(\Omega^a_{t+1}R_{t+1}\right), \\
(vi) & \bar{H}^a_{\alpha,t+1} & = & \omega^a H^a_{\alpha,t+1} + (1-\omega^a)\left(\Lambda^a_{t+1}\right)^{1-\rho} H^{a+1}_{\alpha',t+1}.
\end{array} \tag{21}$$

Δ^a_t is the inverse of the marginal propensity to consume and H denotes human capital equal to the present value of future wages and pension benefits.

The last group is conceptually different from earlier ones because no further aging is possible ($\omega^A = 1$). A person may either survive to the next period within the same age group, or die. In that case, $\Omega^A = 1$ and $MRS^A = 0$. Imposing these restrictions yields the optimal consumption policy of the last age group from (21). In particular, human wealth and the inverse of the marginal propensity to consume follow $H^A_{\alpha,t} = y^A_t + \gamma^A H^A_{\alpha,t+1}/R_{t+1}$ and $\Delta^A_t = 1 + \gamma^A \beta^\sigma (R_{t+1})^{\sigma-1} \Delta^A_{t+1}$.[7] The behavior of the last group is fully described by the perpetual youth model of Blanchard (1985) albeit with one important difference. Since the last is only one of several age groups, average time spent in this group is rather limited. The mortality rate must thus be chosen much larger which implies a much higher marginal propensity to consume, compared to younger age groups.

If all age classes face the same survival probability $\gamma^a = \gamma$ and thereby differ only in their earnings, then the marginal propensity to consume is invariant across age classes. This is most easily seen in case of a unit elasticity $\sigma = 1$ which yields $1/\Delta = 1 - \beta\gamma$ by (21.iii) but is also true more generally. Suppose $\Delta^{a+1} = \Delta^a$, then Λ^a as well as Ω^a are both equal to one, implying $\Delta^a_t = 1 + \gamma\beta^\sigma (R_{t+1})^{\sigma-1} \Delta^a_{t+1}$. Hence, Δ^a must indeed be the same for all groups which, by (21.iii), is possible only with a uniform survival rate γ. Age classes then differ only in terms of human wealth and consumption levels but all choose the same intertemporal consumption structure. The arguments show that a change in the marginal propensity to consume from one age class to the next exclusively reflects a change in the mortality rate.

Consider the most realistic case that the mortality rate increases between any two consecutive groups as in Table 1. Again, the Cobb Douglas case shows most transparently that the marginal propensity to consume $1/\Delta^a = 1 - \gamma^a \beta$ rises with a lower survival rate γ^a. Given that the ratio Δ^{a+1}/Δ^a falls below unity, and that $\rho < 0$ for an intertemporal substitution elasticity $\sigma < 1$, the factor Λ^a exceeds one and thereby raises Ω^a above one as well. Agents start to discount the future more heavily, at an effective rate $\Omega^a R$, as the end of life becomes a more probable event on account of an increased mortality rate.

[7] The reader may verify the solution in (21) for the oldest life-cycle group by solving the problem $V\left(A^A_{\alpha,t}\right) = \max_{C^A_{\alpha,t}} \left[\left(C^A_{\alpha,t}\right)^\rho + \gamma^A \beta \left(V^A_{\alpha,t+1}\right)^\rho\right]^{1/\rho}$ subject to $\gamma^A A^A_{\alpha,t+1} = R_{t+1}\left[A^A_{\alpha,t} + y^A_t - C^A_{\alpha,t}\right]$. One may also verify that the solution satisfies the intertemporal budget constraint that is obtained by solving forward the periodic budget in (14).

Consequently, the marginal rate of substitution MRS^a across age states as defined in (19) becomes larger when mortality is more common. With a higher probability of extinction, agents are less inclined to postpone consumption and thus consume a larger fraction of their resources immediately. Quite apparently, saving for the future makes less sense. Agents value consumption in state $a+1$ relatively more than in state a which explains the increase in the marginal rate of substitution across age classes.

3.2 Aggregation

The simplicity and tractability of the simple perpetual youth model with a constant mortality rate rests on the fact that it allows for analytical aggregation. The same applies to the PA model. The advantage of approximating actual demographic and life-cycle properties with a low dimensional system is possible only if age groups can be aggregated analytically. In section 2, we have argued that agents are completely identified by their biography α which also includes the date of birth as a first entry. In age group a, one records at date t a mass of agents $N^a_{\alpha,t}$ with the same life-cycle history. These agents are economically identical and all consume the same quantity $C^a_{\alpha,t}$. Aggregate consumption of age group a is obtained by summing over all possible biographies. Further adding over all age groups yields total, economy wide consumption,

$$C^a_t \equiv \sum_{\alpha \in \mathcal{N}^a_t} C^a_{\alpha,t} N^a_{\alpha,t}, \quad C_t \equiv \sum_{a=1}^{A} C^a_t. \tag{22}$$

The same principle applies to other static variables. When aggregating expressions that multiply with variables identical within an age group, or identical over the entire population, these constant terms drop out of the summation. For example, aggregating wage related income in (13) yields $\sum_\alpha y^a_t N^a_{\alpha,t} = y^a_t N^a_t$. Denoting labor supply in efficiency units by L^S and the retirees by N^R, one obtains

$$Y^a_t = y^a_t N^a_t, \quad Y_t = (1-\tau)\, w_t L^S_t + p_t N^R_t, \quad L^S_t \equiv \sum_{a=1}^{a^R-1} \theta^a_t N^a_t, \quad N^R_t \equiv \sum_{a=a^R}^{A} N^a_t. \tag{23}$$

Similarly, when aggregating consumption of age group a as stated in (21.i), we observe that Δ_t^a, Λ_t^a and Ω_t^a are identical within each group. They implicitly depend only on the common interest rate and demographic parameters that differ across but are constant within each group. Hence, aggregate consumption is simply $C_t^a = (A_t^a + H_t^a)/\Delta_t^a$.

Consider now the aggregation of dynamic variables such as human capital and asset wealth. Human capital is particularly simple since wage related income and, thus, human capital per capita is the same for all persons within each group. This is most evident for human capital of the last age group. Solving forward $H_{\alpha,t}^A = y_t^A + \gamma^A H_{\alpha,t+1}^A / R_{t+1}$ yields the same present value of future income, discounted at a common rate, for all retirees. Writing the per capita value as $H_{\alpha,t}^A = h_t^A$, aggregate human capital is simply $H_t^A = h_t^A N_t^A$. The same holds for all earlier age groups as is obvious from (21). Hence,

$$H_t^a = h_t^a N_t^a, \quad h_t^a \equiv H_{\alpha,t}^a, \quad h_t^a = y_t^a + \gamma^a \cdot \frac{\omega^a h_{t+1}^a + (1-\omega^a)\left(\Lambda_{t+1}^a\right)^{1-\rho} h_{t+1}^{a+1}}{\Omega_{t+1}^a R_{t+1}}. \tag{24}$$

The evolution of human capital per capita is repeated from (21.v-vi). The equation for the oldest group follows on account of the restriction $\omega^A = 1$, implying $\Omega^A = 1$.

Proposition 3 ***(Asset Accumulation)*** *Using the aggregator in (22), $A_t \equiv \sum_a A_t^a$ and $A_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t}^a N_{\alpha,t}^a$, aggregate assets evolve as*

$$\begin{aligned} (a) &: \quad A_{t+1}^1 = R_{t+1} \omega^1 S_t^1, \quad S_t^a \equiv A_t^a + Y_t^a - C_t^a, \\ (b) &: \quad A_{t+1}^a = R_{t+1}\left[\omega^a S_t^a + \left(1-\omega^{a-1}\right) S_t^{a-1}\right], \\ (c) &: \quad A_{t+1} = R_{t+1}\left[A_t + Y_t - C_t\right], \end{aligned} \tag{25}$$

where Y is aggregate wage related income as given in (23).

The proof is in Appendix B. Upon aggregation, the survival factor γ^a cancels out. The insurance mechanism redistributes, without any net loss in the aggregate, the assets unintentionally left behind by the deceased to the surviving fraction in each age group.

3.3 A Synthesis of Models

The PA framework encompasses a wide range of intertemporal models as special cases and thereby shows how they are related by specific assumptions with respect to aging and mortality. The simplest case is the Ramsey model with an infinitely lived representative agent. Weil (1987) showed that it follows from an OLG model of households linked by an operative, altruistic bequest motive. It emerges by excluding both aging and mortality, $\omega^a = \gamma^a = 1$ for all a which implies $\Omega^a = 1$ as well. With a constant population, savings and consumption result from the problem $V(A_t) = \max\left[C_t^\rho + \beta V(A_{t+1})^\rho\right]^{1/\rho}$ subject to $A_{t+1} = R_t\left[A_t + y_t - C_t\right]$. The standard decision rules are easily recovered from (21) under the given parameter restriction. Indices a and α become irrelevant. The marginal propensity to consume Δ_t and human wealth obviously remain independent of any age characteristics. Human wealth is the simple present value of future earnings derived from $H_t = y_t + H_{t+1}/R_{t+1}$. The Euler equation (20) reduces to $C_{t+1} = (\beta R_{t+1})^\sigma C_t$ and requires the interest rate to satisfy $\beta R = 1$ in a stationary long-run equilibrium. The PA model can also represent Weil's (1989) model of overlapping families of infinitely lived agents by introducing a positive birth rate while setting the mortality rate to zero.[8]

The perpetual youth model of Blanchard (1985) is a model with an infinity of cohorts but only one age group. It follows by setting $\gamma^a = \gamma < 1$ and $\omega^a = 1$. In any period, the event space is reduced to the first two events in (3) where the instantaneous mortality rate is invariant with respect to time and age. In the absence of aging, the Bellmann equation in (16) becomes independent of age characteristics a. An agent's life-cycle history reduces to a single event which is birth at date $\alpha = \alpha_1$. Note that time since birth, $t - \alpha_1$, does not translate into aging which we defined as a change in life-cycle characteristics. Hence, this model is quite appropriately termed a "perpetual youth" model. The constant mortality rate simply increases the effective discount factor to $\gamma\beta$. With $\omega^a = \Omega^a = 1$, the marginal propensity to consume, the inverse of (21.iii), is independent of age and the same for young and old generations since they all expect the same remaining life-time. However, it

[8] Buiter (1988) was the first to separate birth and death rates in the perpetual youth model.

is higher compared to the Ramsey model and thereby reflects the finiteness of life. For the same reason, future wage income is less valuable since it might not be available in case of premature death, and is thus discounted at a higher effective rate. Human capital follows from $H_t = y_t + \gamma H_{t+1}/R_{t+1}$ and is independent of age, see (21.v-vi), since all generations receive the same wage. Agents differ only in the amount of financial assets that were accumulated since birth, and the level of consumption.

Extensions of the basic model have also made labor income dependent on time since birth to approximate the empirical life-cycle pattern of wage income. While such extensions introduce some heterogeneity with respect to human wealth, they do not improve on the life-cycle pattern of the marginal propensity to consume which remains age independent as long as the survival rate remains constant. The PA model can reproduce this case by assuming annual aging, $\omega^a = 0$, where aging occurs each single period. An age group becomes identical to a cohort so that the index for the age group measures time since birth. One need not keep track of life-cycle history separately. The Bellmann equation simply becomes $V(A_t^a) = \left[(C_t^a)^\rho + \gamma\beta\left(V_{t+1}^{a+1}\right)^\rho\right]^{1/\rho}$. The decision rules are again recovered from (21). Since people necessarily age until next period, $\Delta_{t+1}^a = \Delta_{t+1}^{a+1}$ holds by definition, implying $\Lambda = 1$ and $\Omega = 1$. In fact, the marginal propensity to consume remains age independent as long as the survival factor is constant, $\Delta_t = 1 + \gamma\beta^\sigma (R_{t+1})^{\sigma-1}\Delta_{t+1}$. Human wealth follows $H_t^a = y_t^a + \gamma H_{t+1}^{a+1}/R_{t+1}$ with y_t^a now depending on time since birth.

Gertler's (1999) extension of the perpetual youth model, recently applied by Keuschnigg and Keuschnigg (2004), is easily reproduced by allowing for only two age groups corresponding to two aging events, birth and retirement. Agents are born as workers in age group one and then switch with constant probability into retirement. Mortality sets in only after retirement. The demographic assumptions are $\gamma^1 = 1$, $\gamma^2 < 1$, $\omega^1 < 1$ and $\omega^2 = 1$, leading to two age groups and an infinity of cohorts. This model not only introduces heterogeneity in human wealth. Retirees have a higher marginal propensity to consume than workers since they face a higher mortality rate. The PA model is a natural extension and generalization of this basic approach.

The PA model also encompasses the original Auerbach Kotlikoff (1987) model with an annual, although degenerate aging process. Each year since birth is taken as a separate age group a. In that sense, aging occurs every single period. The number of age groups thus coincides with the number of cohorts which is the number of life-cycle periods. The life-cycle characteristics, measured in terms of earnings potential θ^a, change annually to reproduce realistic age-earnings profiles until the fixed date of retirement approaches. Upon retirement, wage earnings drop to zero and are replaced by pension income. In the basic model, people are assumed to live with certainty until a fixed finite period and then die with certainty: $\gamma^a = 1$ for $a < A$ and $\gamma^A = 0$. The aging process is degenerate in the sense that people switch with certainty to the next higher age group every period, $\omega^a = 0$ all a, except $\omega^A = 1$. In particular, people switch to retirement at a fixed date with certainty when labor earnings drop to zero. This reproduces the *deterministic* segmentation between workers and retirees.

Recent variations of Auerbach Kotlikoff type OLG models also account for demographic change by deterministically shrinking the size of cohorts from period to period. The PA model would replicate this with $\gamma^a < 1$ for $a < A$ and $\gamma^A = 0$. These models still retain a fixed, deterministic time horizon and a fixed retirement date. Again, annual aging implies $\Lambda = 1$ and $\Omega = 1$. Time since birth fully describes an agent's biography which can thus be suppressed. Both human wealth and the marginal propensity to consume become age dependent. From (21), we recover $\Delta_t^a = 1 + \gamma^a \beta^\sigma \left(R_{t+1}\right)^{\sigma-1} \Delta_{t+1}^a$ and $H_t^a = y_t^a + \gamma^a H_{t+1}^{a+1}/R_{t+1}$. Now, however, life-time ends with certainty after A years which introduces the end point restrictions $\Delta_t^A = 1$ and $H_t^A = y_t^A$ on account of $\gamma^A = 0$. In particular, the marginal propensity to consume increases up to unity when the last period of life approaches. The mortality rates and consequently the marginal propensities to consume increase rapidly as the end of life approaches. As Table 1 demonstrates, this feature can be approximated by the PA model when choosing shorter durations of average age periods for the last age groups.

Finally, the PA model collapses to the two period OLG model of Diamond (1965) and Samuelson (1958) when $\gamma^1 = 1$, $\gamma^2 = 0$, $\omega^1 = 0$ and $\omega^2 = 1$ and $y^2_{t+1} = 0$. There is no life-time uncertainty ($\gamma^1 = 1$). People die with probability one after two periods, $\gamma^2 = 0$. Aging occurs after one period, $\omega^1 = 0$, when people retire with zero labor earnings, $y^2_{t+1} = 0$. Agents are necessarily in age group 1 in their first period of life and in group 2 in their retirement period. Cohorts and age groups are thus exactly identical.[9] In this simplest case with periodic aging, people are fully identified by their age group which makes the indexing of life-cycle history superfluous. Agents are not endowed with any assets in their first period. This leaves the familiar problem of maximizing $V^1_t = \left[(C^1_t)^\rho + \beta \left(C^2_{t+1}\right)^\rho\right]^{1/\rho}$ subject to $C^2_{t+1} = R_{t+1}\left[y^1_t - C^1_t\right]$. We summarize:

Proposition 4 ***(Synthesis of Models)*** *The PA model includes existing OLG models as special cases reflecting different assumptions on aging frequency, mortality and period length: (a) Infinitely lived agents (Weil, 1987, 1989): $\omega^a = \gamma^a = 1$; (b) Perpetual youth (Blanchard, 1985): $\gamma^a = \gamma < 1$ and $\omega^a = 1$ without and $\omega^a = 0$ with wage profiles; (c) Perpetual youth with two age groups (Gertler, 1999): $\gamma^1 = 1$ and $\omega^1 < 1$ for workers and $\gamma^2 < 1$ and $\omega^2 = 1$ for retirees; (d) Life-cycle models with many periods (e.g. Auerbach and Kotlikoff, 1987): $\omega^a = 0$ and $\gamma^a = 1$ and $\gamma^A = 0$; (e) Two period OLG model (e.g. Diamond, 1965): $\gamma^1 = 1$, $\omega^1 = 0$ for workers and $\gamma^2 = 0$, $\omega^2 = 1$ for retirees.*

3.4 General Equilibrium

Firms combine capital K_t and efficiency units of labor L_t to produce $Q_t = F(K_t, L_t)$. Technology is linear homogeneous and quasiconcave. For realistic dynamics of investment in a small open economy, investment costs J are assumed to increase progressively with the rate of gross investment, reflecting installation costs of capital. The installation technology is linear homogeneous and satisfies $\phi_I > 0$, $\phi_{II} > 0$, and $\phi_K < 0$, where I is the amount

[9] The case $\gamma^1 < 1$, $\gamma^2 = 0$, $\omega^1 < 1$ and $\omega^2 = 1$ would introduce a generalized two period OLG model with life-time uncertainty ($\gamma^1 < 1$) and second period labor income risk on account of probabilistic aging ($\omega^1 < 1$). Second period labor income would be y^1_{t+1} with no aging and $y^2_{t+1} < y^1_{t+1}$ with aging. Cohorts and age groups would not be identical anymore.

of gross investment. Capital depreciates at rate δ. As a normalization, we may specify $\delta = \phi(\delta, 1)$, implying that $J = I$ in a steady state. Dividends χ_t are

$$\chi_t = Q_t - w_t L_t - J_t, \quad J = \phi(I, K), \quad K_{t+1} = I_t + (1-\delta) K_t. \tag{26}$$

Defining the end of period, cum dividend value of capital by V^K, optimal investment and employment policies follow from $V^K(K_t) = \max_{I_t, L_t} \ \chi_t + V^K(K_{t+1}) / R_{t+1}$. Discounting with the households' interest factor R_{t+1} assures that the investors' no arbitrage-condition is satisfied, i.e. the return on dividend generating capital must equal the market rate of interest available on alternative assets. Denoting the shadow price of capital by $\lambda_t^K \equiv \partial V_t / \partial K_t$, the optimality and envelope conditions are $\phi_I = \lambda_{t+1}^K / R_{t+1}$, $w_t = F_L$ and $\lambda_t^K = F_K - \phi_K + (1-\delta) \lambda_{t+1}^K / R_{t+1}$. It is well known since Hayashi (1982) that the marginal and average shadow prices of capital are identical, $V_t^K \equiv \lambda_t^K K_t$. Finally, note that in a steady state, $I = \delta K$. Due to our normalization of adjustment costs ϕ, we have $\phi_I = 1$ and $\phi_K = 0$ in a steady state which yields the familiar condition $F_K = r + \delta$.

The model is closed by imposing the government budget constraint. Wage taxes must finance public consumption G and PAYG pensions where L^S and N^R are given in (23),

$$\tau w_t L_t^S = p_t N_t^R + G_t. \tag{27}$$

Finally, we state the capital and labor market clearing conditions. In equilibrium, demand for efficiency units of labor corresponds to aggregate household sector labor supply, scaled by worker productivity, $L_t = L_t^S$. Further, accumulated household sector financial wealth $A_{t+1} = V_{t+1}^K + D_{t+1}$ absorbs the value of domestically issued equity, V_{t+1}, and foreign bonds D_{t+1}. When assets are perfectly substitutable, they must earn an identical rate of return equal to the market interest r_{t+1}. Given optimal household and firm behavior, and with all budget constraints fulfilled, Walras' Law implies the current account

$$D_{t+1} = R_{t+1} \left[D_t + Q_t - J_t - G_t - C_t \right], \tag{28}$$

where the stock of foreign net assets D_t is measured at the end of the period.

4 An Illustrative Application

We highlight the usefulness of the PA model with a simple policy experiment that would be difficult to analyse with a standard perpetual youth model due to its lack of life-cycle structure. We consider the consequences of an increase in pay as you go pensions financed by a wage tax on the active population.[10] The quantitative application is based on an extended model with endogenous labor supply as explained in appendix A. The intensive margin of labor supply refers to hours worked of the active population while the extensive margin refers to the discrete retirement choice of people in their sixties.

The quantitative model is calibrated to stylized data of a representative economy. Table 2 states key parameters that characterize preferences and technology. These are commonly used values as in Altig et. al (2001) and Blundell et al. (2003), for example, or in the real business cycle literature as in Baxter and King (1993) or King, Plosser and Rebelo (1988). To keep the discussion short, we refer to these papers for a review of the econometric evidence. The replacement rate gives the size of the pension compared to the last net wage income. Tax revenue is used to finance pension payments and public spending as in (27) and requires a wage tax rate of roughly 30% to balance the budget. According to Gruber and Wise (2005) and Boersch-Supan (2000), the retirement decision is arguably the most important behavioral response to pension reform. The optimality condition in (A.2) means that people in age group a^R postpone retirement, and a larger share δ of the agents in this group remain actively working, if work becomes more attractive relative to retirement. The retirement elasticity is taken from empirical studies. Börsch-Supan (2000) estimates that a decrease of benefits by 12% would decrease the retirement probability of the 60 years old from 39.3% to 28.1%. This amounts to a semielasticity of the retirement decision of $\varepsilon^R = 1$, see Table 2. The parameters controlling the life-cycle properties of the PA model are in Table 1 of section 2.2.

[10] Since unfunded pension obligations are equivalent to government debt, the scenario is also informative about the effects of public debt.

Table 2: Taste and Technology Parameters

Real interest rate r	0.050
Depreciation rate δ^K	0.100
Output elasticity of capital α	0.350
Subjective discount factor β	0.978
Intertemporal elasticity of substitution σ	0.400
Wage elasticity of labor supply ε^L	0.300
Retirement semielasticity ε^R	1.000
Proportional wage tax rate τ	0.308
Pension replacement rate ζ	0.381

Notes: $Q = XK^{\alpha}(ZL)^{1-\alpha}$ is the production technology, $\zeta = p/[(1-\tau)w\theta^{a^R}]$ is the pension replacement rate and $\varepsilon^L \equiv \varphi'/(l\varphi'')$ is the labor supply elasticity, see Appendix A.

Table 3 reports the long-run consequences when pension spending of the pay as you go type is raised from 5 to 7 percent of GDP. Initially, net foreign assets of the model economy are zero. The first column refers to the small open economy case where real interest is exogenously given on world capital markets. Any imbalance of domestic savings and investment is then reflected in the current account. The immediate implication of the scenario is that higher pension spending necessitates an increase in the labor tax. The scenario thus tilts the life-cycle profile of disposable income towards the future and thereby undermines the income smoothing motive as a prime determinant of life-cycle savings. In consequence, accumulated savings and household asset wealth declines substantially. Furthermore, the scenario importantly discourages aggregate labor supply. First, the higher tax rate discourages hours worked and labor supply on the intensive margin. In equilibrium, per capita labor supply of active workers shrinks by 2.2%. Second, the scenario strongly induces early retirement since pension income upon retirement becomes much more generous compared to net of tax wages from continued employment. The replacement rate, defined by the pension as a fraction of the last net of tax wage, increases by 7 percentage points from 38 to 45%. The incentives for early retirement reduce the share of the active population in age group five, corresponding to people aged 60 to 70 in the cohort model, from 45 to 14%. Early retirement shrinks the workforce and inflates the number of retirees. The dependency ratio thus increases by almost 9 percentage points to 39%. The induced early retirement reinforces the necessary adjustments of taxes and benefits to keep the system sustainable.

The combined effect of early retirement and hours worked leads to a decline in aggregate employment by about 7.8%. Since the real interest is exogenously fixed in a small open economy, the capital stock falls in parallel with the reduction in aggregate employment to keep the capital labor ratio constant. Output thus declines by exactly the same amount. Since the scenario shifts the individual income profile from work to retirement, individual savings incentives to provide for old age income are much reduced. Aggregate assets fall by almost 15%. Since the decline in asset wealth exceeds the reduction in the value of capital, capital market equilibrium $A = V^K + D$ as noted in section 3.4 implies an increase in net foreign debt which eventually amounts to almost 19% of GDP in Table 3. The long-run adjustment of the savings investment balance is also demonstrated in Figure 3. However, the figure also points to a potential short-run non-monotonicity which results from the fact that the adjustment dynamics of aggregate savings in an OLG model is much slower than the adjustment of investment. Note that net foreign debt is zero initially and historically predetermined while the value of capital is a jump variable reflecting the future value of capital. Consequently, the revaluation of equity implies a revaluation of household sector assets by exactly the same percentage amount. However, since the subsequent decline in accumulated savings occurs with rather slow speed, the reduction in the value of capital exceeds the decline in aggregate savings in a first adjustment phase. Consequently, the economy first builds up a net asset position before it starts to incur net foreign debt.

Table 3: Long-Run Impact of Pension Increase

Key Macro Variables in %		Open	Open†	Closed
r	real interest* (0.050)	0.050	0.050	0.053
D^f/Y	debt GDP ratio* (0.000)	-0.189	-0.156	0.000
τ	contribution rate* (0.308)	0.358	0.354	0.363
ζ	replacement rate * (0.381)	0.454	0.452	0.452
δ	retirement* (0.450)	0.139	0.144	0.121
N^R/N^W	dependency ratio* (0.307)	0.391	0.390	0.396
$(1-\tau)w$	net of tax wage	-7.271	-6.673	-9.010
l	labor supply per capita	-2.239	-0.688	-2.793
L^D	employment, efficiency units	-7.816	-6.266	-8.658
K	capital stock	-7.816	-6.266	-11.477
Y	gross domestic product	-7.816	-6.266	-9.654
C	aggregate consumption	-11.060	-8.918	-11.313
A	aggregate assets	-14.917	-12.226	-11.216

Note: %) Percent changes relative to ISS. *) Absolute values, initial values in brackets. †) Simulation with $\varepsilon^L = 0.1$.

The last two columns of Table 3 are for sensitivity analysis. The second column simulates the effects of the same policy shock subject to a lower wage elasticity of hours worked. Labor supply per capita of the active population falls by only 0.7 instead of 2.2% while the tendency for early retirement is largely of the same magnitude. The macroeconomic response is scaled down on account of a smaller reduction in aggregate employment which predominantly stems from the less elastic labor supply response on the intensive margin. The last column reports the long-run effects in a closed economy where net foreign assets are restricted to be zero and the domestic real interest rate must adjust to balance savings and investment. Since the immediate impact of the pension reform is a reduction in life-cycle savings, the interest rate must increase to bring accumulated savings and investment into balance again. The higher cost of capital leads firms to choose a lower capital intensity which magnifies the reduction of the capital stock (-

11.5%) compared to aggregate employment (-8.6%). Consequently, the reduction of the net of tax wage becomes larger since it reflects not only the higher tax rate but also a reduced gross wage rate on account of a lower capital intensity. The quantitatively larger decline in net wages exacerbates not only the reduction of hours worked but also reinforces the trend to early retirement. Aggregate employment thus falls by 8.6 compared to 7.8% in the base case scenario. The loss in aggregate output turns out much larger on account of reduced factor inputs.

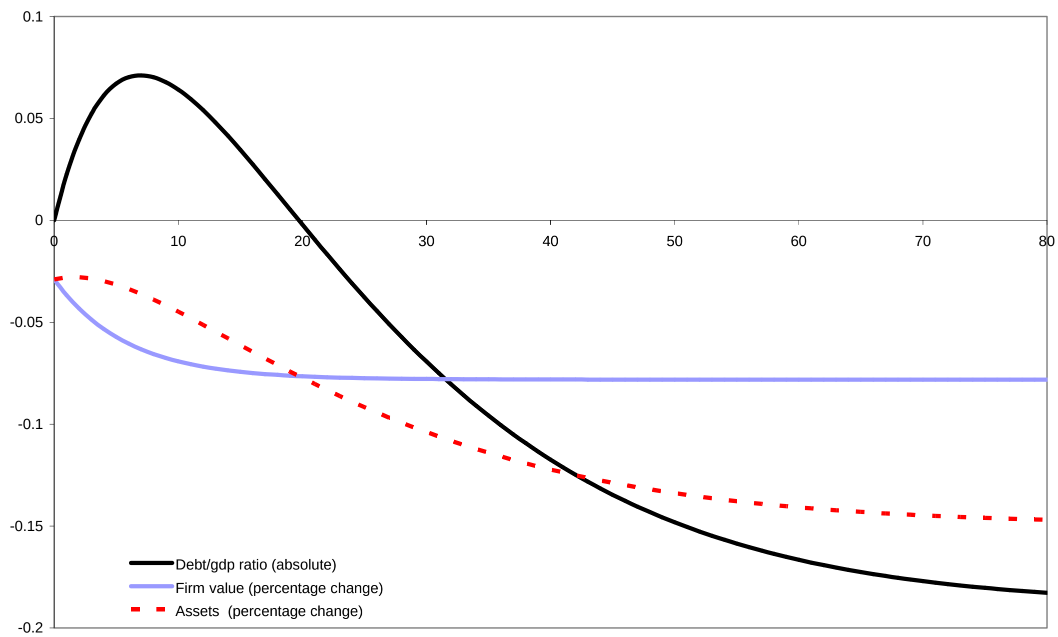

Fig. 3: Savings, Investment and Net Foreign Assets

5 Conclusions

The theory of probabilistic aging separates the notion of age from time since birth. Economic age is a set of personal characteristics such as labor productivity, tastes, health, mortality etc. that change less frequently over the life-cycle than periods since birth. People can retain their characteristics over several periods before they change as a result of aging. Aging occurs stochastically. Identifying a person with her biography which is the sequence of discrete aging events since birth, allows to analytically aggregate a multitude of different generations into a low number of age groups. The PA model is thus a generalized OLG model. Existing models follow as special cases which reflect specific assumptions with respect to aging frequency, mortality and period length.

In standard OLG models, aging occurs every period and the probability of dying is unity in the last period of life. The number of life-cycle periods corresponds to the number of age groups or cohorts. As Laitner (1990) has shown, typical models with 55 generations operate in a state space of 108 dimensions and are thus very expensive to implement. Results are difficult to replicate. If aging occurs less frequently, the number of age groups becomes smaller than the number of cohorts. We found that allowing for eight groups already leads to a quite accurate approximation of life-cycle properties. Counting as in Laitner (1990), the state space would be 14 instead of 108 dimensions. This low dimensionality greatly facilitates empirical implementation and numerical solution. It becomes much easier to replicate results from existing studies if necessary. An illustrative application relating to pension reform has demonstrated that the PA model is a much more powerful tool of policy analysis compared to the perpetual youth model due to Blanchard (1985) and Yaari (1965) as well as the recent extension by Gertler (1999) with two age groups. These models are popular because of analytical tractability and their simple empirical implementation. Yet they are quite limited when it comes to trace out the life-cycle implications of certain policy shocks. The PA model is in between the Auerbach-Kotlikoff and Blanchard-Yaari models, allowing for analytical aggregation to a low number of age groups combined with a realistic modeling of the life-cycle.

Appendix

A Endogenous Labor Supply and Retirement

Consider age group $a = a_R$ with characteristics of people in their sixties. Assume that a fraction δ of this group is still active while the rest is retired. To retain symmetry, we assume that each agent receives a share δ of her labor income as wages and a share $1-\delta$ as an old age pension. A higher value of δ means postponed retirement which reflects the extensive margin of labor supply. We refer to δ as the retirement date since postponed retirement raises the share of wages in average labor income in that period. If actively employed, agents may work a variable number of l^a hours, reflecting the intensive margin. They trade off income and consumption against disutility of work. A particularly simple and yet realistic approach is to exclude income effects.[11] We thus assume an additively separable subutility over consumption and foregone leisure, $\bar{C}^a \equiv C^a - \delta\varphi(l^a) - \phi(\delta)$, where $\varphi(l)$ and $\phi(\delta)$ stand for convex increasing disutility of work and postponed retirement.[12] Appropriately expanding (13) and (14) yields

$$\begin{array}{lrcl}
(i) & V\left(A^a_{\alpha,t}\right) & = & \max_{C^a_{\alpha,t},l^a_{\alpha,t},\delta^a_{\alpha,t}} \ \left[\left(\bar{C}^a_{\alpha,t}\right)^\rho + \gamma^a\beta\left(\bar{V}^a_{\alpha,t+1}\right)^\rho\right]^{1/\rho} \quad s.t. \\
(ii) & \gamma^a A^a_{\alpha,t+1} & = & R_{t+1}\left[A^a_{\alpha,t} + \bar{y}^a_{\alpha,t} - \bar{C}^a_{\alpha,t}\right], \\
(iii) & \bar{C}^a_{\alpha,t} & = & C^a_{\alpha,t} - \delta_{\alpha,t}\varphi\left(l^a_{\alpha,t}\right) - \phi\left(\delta_{\alpha,t}\right), \\
(iv) & \bar{y}^a_{\alpha,t} & = & \max_{l^a_{\alpha,t},\delta_{\alpha,t}} \ \delta_{\alpha,t}\left[w^a_t l^a_{\alpha,t} - \varphi\left(l^a_{\alpha,t}\right)\right] + \left(1-\delta_{\alpha,t}\right)p_t - \phi\left(\delta_{\alpha,t}\right), \\
(v) & y^a_{\alpha,t} & = & \delta_{\alpha,t} w^a_t l^a_{\alpha,t} + \left(1-\delta_{\alpha,t}\right)e_t, \quad w^a_t \equiv (1-\tau)\, w_t\theta^a_t.
\end{array} \tag{A.1}$$

A simple solution in three stages is possible. First, choose intensive and extensive labor supply to maximize effort adjusted wage income in (A.1.iv), yielding

$$w^a_t = \varphi'\left(l^a_t\right), \quad w^a_t l^a_t - \varphi\left(l^a_t\right) - p_t = \phi'\left(\delta_t\right). \tag{A.2}$$

[11] Excluding income effects is quite common in the literature on optimal income taxation or on real business cycles. See, for example, Heijdra (1998) and Greenwood et al. (1988) in intertemporal macroeconomics and Saez (2002) and Immervoll et al. (2004) on optimal income taxation.

[12] See Cremer and Pestieau (2003) for modeling postponed retirement this way.

Intensive labor supply exclusively depends on the current wage. On the extensive margin, the chosen retirement date reflects the difference between utility adjusted wages and alternative pension income. The more generous pensions are relative to net wages, the less people are inclined to incur the utility cost of postponed retirement, and the earlier they choose to retire. The share of wages in average labor income declines. Since wages and pensions are independent of history, labor supply and the retirement date are symmetric, $l_{\alpha,t}^a = l_t^a$ and $\delta_{\alpha,t} = \delta_t$, implying symmetry of y_t^a and $\bar{y}_t^a$ as well.

The second step solves the intertemporal problem in (A.1.i-ii) of allocating subutility $\bar{C}_{\alpha,t}^a$ over time exactly as in Proposition 2, except that C is replaced by $\bar{C}$ and y by $\bar{y}$. The solution of the first two steps gives optimal values for $\bar{C}$, l and δ. As a last step, it remains to compute optimal commodity consumption $C_{\alpha,t}^a$ by inverting (A.1.iii).

For age groups $a < a^R$, labor supply is reduced to the intensive dimension. Consequently, $\delta = 1$ in (A.1) and ϕ is a constant that is set to zero. Finally, groups $a > a^R$ are fully retired. Endogenous labor supply and retirement affects the definition of effective labor supply and the decomposition of population in workers and retirees,

$$L_t^S = \theta^{a^R} l_t^{a^R} \cdot \delta_t N_t^{a^R} + \sum_{a=1}^{a^R-1} \theta^a l_t^a N_t^a, \quad N_t^R = (1-\delta_t) N_t^{a^R} + \sum_{a=a^R+1}^{A} N_t^a. \tag{A.3}$$

B Proofs

Proposition 1: The stock of people in age group $a > 1$ in period $t+1$ is the current stock minus outflows plus inflows. Summing over people by their biography α as in (4),

$$\begin{aligned} \sum_{\mathcal{N}_{t+1}^a} N_{\alpha,t+1}^a &= \sum_{\mathcal{N}_t^a} N_{\alpha,t}^a - \sum_{\mathcal{N}_t^a \times (t+1)} N_{\alpha,t+1}^{a+1} - \sum_{\mathcal{N}_t^a \times (t+1)} N_{\alpha,t+1}^{\dagger} + \sum_{\mathcal{N}_t^{a-1} \times (t+1)} N_{\alpha,t+1}^a, \\ N_{t+1}^a &= N_t^a - \gamma^a (1-\omega^a) N_t^a - (1-\gamma^a) N_t^a + \gamma^{a-1} (1-\omega^{a-1}) N_t^{a-1}. \end{aligned} \tag{B.1}$$

The first term on the right is the current stock as defined in (4). The second term is the outflow from group a to $a+1$ which is defined by the subset $\mathcal{N}_t^a \times (t+1) \subset \mathcal{N}_{t+1}^{a+1}$. It counts all agents with the last entry $t+1$ in their biography, meaning that they arrived in group $a+1$ exactly at date $t+1$ and not earlier. Since all agents in a face the same independent probability of the joint event surviving and aging, irrespective of the past biography (see 3.iii), the mass of movers is given by the second term in the second line of (B.1). The third term represents the outflow due to death. It is given by the subset $\mathcal{N}_t^a \times (t+1)$ of the set $\mathcal{N}_{\alpha,t+1}^{\dagger}$ of all deceased. This subset consists of all biographies listing $t+1$ as a last entry and earlier entries $\alpha \in \mathcal{N}_t^a$. By the same arguments as before, a fraction $1-\gamma^a$ of the current age group is subject to the mortal event as in (3.i) which gives the third term in the second line. Finally, the last term is analogous to the second one, except that it applies to group $a-1$ and stands for an inflow to a. Adding up the first three terms yields $\gamma^a \omega^a N_t^a$ which corresponds to (3.ii). Since the last group cannot age further, the event in (3.iii) is precluded, implying the restriction $\omega^A = 1$ in (5.a).

The aggregation of group 1 in (5.b) is analogous. Replacing a by 1 in the last term of (B.1) gives $\sum_{\alpha \in \mathcal{N}_t^0 \times (t+1)} N_{\alpha,t+1}^1$. The set of biographies $\mathcal{N}_t^0$ referring to unborns consists of an empty vector () since no aging event has occurred. The Cartesian product yields a set with a single element $\alpha = (t+1)$. Hence, the last term gives the number of newborns $N_{(t+1),t+1}^1$ flowing into group 1. Adding up the terms in the second line in (B.1) yields (5.b). Equation (5.c) follows by adding up (5.a) and (5.b) over all age groups.

Proposition 2 We first show that the consumption function fulfills the Euler equation and is thus the optimal policy. Note that (21.ii) and (19) imply Λ^a as in (21.iv). Since (21.ii-iv) depend only on the interest rate and other factors independent of α, the terms Δ_t^a and by implication Λ_t^a and Ω_t^a are independent of α. Insert (21.i) into the left hand term of (20), use $A_{\alpha,t+1}^a = A_{\alpha',t+1}^{a+1}$, collect terms, note the definitions of Ω_{t+1}^a and $\bar{H}_{\alpha,t+1}^a$, and get $\left[\Omega_{t+1}^a A_{\alpha,t+1}^a + \bar{H}_{\alpha,t+1}^a\right] / \Delta_{t+1}^a = \left(\beta \Omega_{t+1}^a R_{t+1}\right)^{\sigma} \cdot C_{\alpha,t}^a$. Multiply by $\gamma^a \Delta_{t+1}^a / \left(\Omega_{t+1}^a R_{t+1}\right)$ and use (21.v) and (14) on the left side: $A_{\alpha,t}^a + H_{\alpha,t}^a - C_{\alpha,t}^a = \gamma^a \beta^{\sigma} \left(\Omega_{t+1}^a R_{t+1}\right)^{\sigma-1} \Delta_{t+1}^a \cdot C_{\alpha,t}^a$. Substitute again (21.i) on the left side and cancel $C_{\alpha,t}^a$. The result corresponds to (21.iii). Hence, the stated policy is optimal since it satisfies (20) which is merely a reformulation of the necessary conditions in (18).

The second step shows that (21.ii) gives indirect utility as it identically fulfills the

Bellmann equation. Again, start with the Euler equation and replace all consumption terms by the appropriate version of (21.ii). Multiply the result by $\left(\Delta_{t+1}^a\right)^{1/\rho}$, use the definitions of Λ_{t+1}^a and $\bar{V}_{\alpha,t+1}^a$ and get $\bar{V}_{\alpha,t+1}^a = \left(\beta\Omega_{t+1}^a R_{t+1}\right)^\sigma \cdot V_{\alpha,t}^a \left(\Delta_{t+1}^a/\Delta_t^a\right)^{1/\rho}$. Next, take the power of ρ, write $\rho\sigma = \sigma - 1$, multiply by $\gamma^a\beta$, and use (21.iii) to substitute $\gamma^a\beta^\sigma \left(\Omega_{t+1}^a R_{t+1}\right)^{\sigma-1} \Delta_{t+1}^a = \Delta_t^a - 1$. The result is $\gamma^a\beta \left(\bar{V}_{\alpha,t+1}^a\right)^\rho = (\Delta_t^a - 1)\cdot\left(V_{\alpha,t}^a\right)^\rho/\Delta_t^a$. By (21.ii), $\left(V_{\alpha,t}^a\right)^\rho/\Delta_t^a = \left(C_{\alpha,t}^a\right)^\rho$, which is used on the right hand side. A minor rearrangement shows that the Bellmann equation (16) is identically fulfilled.

Proposition 3 Multiply (14) by $N_{\alpha,t}^a$ and sum over all biographies α,

$$X_t \equiv \textstyle\sum_{\alpha\in\mathcal{N}_t^a} A_{\alpha,t+1}^a \gamma^a N_{\alpha,t}^a = R_{t+1}\cdot S_t^a, \quad S_t^a \equiv A_t^a + Y_t^a - C_t^a. \tag{B.2}$$

Denoting the left hand term by X and multiplying it by ω^a leads to

$$\begin{aligned} \omega^a X &= \textstyle\sum_{\alpha\in\mathcal{N}_t^a} A_{\alpha,t+1}^a N_{\alpha,t+1}^a = A_{t+1}^a - \sum_{\alpha\in\mathcal{N}_t^{a-1}\times(t+1)} A_{\alpha,t+1}^a N_{\alpha,t+1}^a \\ &= A_{t+1}^a - \gamma^{a-1}\left(1-\omega^{a-1}\right)\textstyle\sum_{\alpha\in\mathcal{N}_t^{a-1}} A_{\alpha,t+1}^{a-1} N_{\alpha,t}^{a-1}. \end{aligned} \tag{B.3}$$

The first equality uses (3.ii). From all $N_{\alpha,t}^a$ people in age group a at date t, only a fraction $\gamma^a\omega^a$ survives and remains in the same group which corresponds to the sum of the first three terms in (B.1). The other part $1-\gamma^a\omega^a$ represents outflows from the current age group due to aging or dying. The next equality in (B.3) expresses the fact that aggregate assets A_{t+1}^a of age a next period is composed of assets $\omega^a X$ of all those who are already in group a in period t, plus the inflowing assets of all those who were in age group $a-1$ in period t, were hit by an aging event and are now part of group a. The subset flowing into age group a is the set of biographies which figure $t+1$ in the last entry, $\mathcal{N}_t^{a-1}\times(t+1)\subset\mathcal{N}_{t+1}^a$. The last equality looks more precisely at the newcomers in group a. Corresponding to case (3.iii), the mass of newcomers is $N_{\alpha',t+1}^a = \gamma^{a-1}\left(1-\omega^{a-1}\right)N_{\alpha,t}^{a-1}$. Since all agents in group $a-1$ have the same chance of aging and surviving, the law of large numbers implies that an equal fraction $\gamma^{a-1}\left(1-\omega^{a-1}\right)$ out of each class of biographies is moving to group a. Hence, the second line sums over the entire set $\mathcal{N}_t^{a-1}$, but takes only the common fraction of each biography group, and thereby precisely identifies the number of movers. Each of the movers has assets $A^{a}_{\alpha',t+1} = A^{a-1}_{\alpha,t+1}$. Performing now the aggregation in the second line, one finds $\sum_{\alpha\in\mathcal{N}^{a-1}_{t}} A^{a-1}_{\alpha,t+1}N^{a-1}_{\alpha,t} = \left(R_{t+1}/\gamma^{a-1}\right) S^{a-1}_{t}$ by applying (B.2) to group $a-1$. Substituting the resulting expression for $\omega^{a}X$ into (B.2) yields aggregate assets in (25.b). The newborns entering the first group are bare of any assets since unborn agents in group 0 cannot save by definition, leading to (25.a). Finally, take the sum of (25.b) over all groups and use the restriction $\omega^{A} = 1$ to arrive at (25.c).

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