CAKE DIVISION BY MAJORITY DECISION

HANS GERSBACH
BERNHARD PACHL

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Abstract

We consider a collective choice process where three players make proposals sequentially on how to divide a given quantity of resources. Afterwards, one of the proposals is chosen by majority decision. If no proposal obtains a majority, a proposal is drawn by lot. We establish the existence of the set of subgame perfect equilibria, using a suitable refinement concept. In any equilibrium, the first agent offers the whole cake to the second proposal-maker, who in turn offers the whole cake back to the first agent. The third agent is then indifferent about dividing the cake between himself and the first or the second agent.

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Hans Gersbach
Center of Economic Research
at ETH Zurich
Zürichbergstr. 18
8092 Zurich
Switzerland
hgersbach@ethz.ch

Bernhard Pachl
Department of Economics
University of Heidelberg
Grabengasse 14
69117 Heidelberg
Germany
Bernhard.Pachl@urz.uni-heidelberg.de

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1 Introduction

In this paper, we study the division of a cake by majority decision with lotteries to break ties. Each of three players proposes a division of the cake. Afterwards, they choose one of the proposals by majority voting. If no proposal receives a majority, one of the proposals is chosen by lottery. The essential features of this model are that there is a definite end of the collective choice process and that no resources are discarded in case of disagreement.

There is a number of real-world examples in which proposals are made sequentially and voting takes place after all the proposals have been made. For instance, in legislative bargaining with simple open rules, agents are appointed sequentially for agenda-setting and they can make proposals, knowing the proposals made so far. If an agent brings a set of proposals to vote and one proposal passes, the process ends [see Krehbiel (1991) for a survey of such legislative organizations]. Moreover, many collective decisions by committees of public or private organizations are governed by open sequential proposals followed by a majority vote. The lottery rule for tie-breaking is somewhat less common, although there is a number of real-world substitutes more or less resembling a lottery. In some legislatures, for instance, the chairman of a committee does not participate in the voting itself, but he has a casting vote allowing him to break ties. If his preferences are not known to the committee members, to them, his tie-breaking move may be similar to a lottery. One can experience such decision schemes personally in committee meetings on budget allocations at German universities.

We show here that there is an infinite number of subgame perfect equilibria that all yield the same two outcomes. In both outcomes, the first player making a proposal offers the whole cake to the second proposal-maker, who in turn offers the whole cake back to the first player. The difference between the two outcomes is as follows: The third player offers half of the cake to himself in both outcomes, while he offers the other half to the first player in one outcome and to the second player in the other. The proposal will be chosen by a majority in the voting stage. Hence, in both outcomes, the cake will merely be divided among 2 players. Player 3 is always one of the two players receiving half of the cake.

Player 3 is in a dominant position because given any two former proposals, he can decide which proposal should get a majority by making an adequate proposal himself. Since Player 2 is aware of this, he will always try to design his proposal in such a way that Player 3 will choose him as a partner, thus getting more than Player 1. Ultimately, this is the reason why Player 1 tries to disadvantage Player 2 as much as possible by offering the whole cake to Player 2, because then, the only chance Player 2 has of countering this proposal is to offer the whole cake to Player 1 as well. This competition results in symmetric disadvantages for Players 1 and 2 and it allows Player 3 to choose a partner at random. Player 3 is indifferent about whom to cooperate with because he has to offer both players the same utility, compensating one of them for a tie and drawing him into a coalition. Therefore, any mixed strategy by Player 3, to offer half of the cake to Player 1 or 2, can be played in equilibrium with some probability.

2 Relation to the Literature

Majority rule and drawing lots are standard procedures for dividing resources in collective choice processes of the kind encountered in legislatures or committees [e.g. Baron and Ferejohn (1989), Bernholz and Breyer (1994)]. Accordingly, we are interested in the positive analysis of the division of resources we obtain in such cases. Our work is related to two strands of literature. First, the division of resources has been studied from the perspective of alternative collective choice processes. Mueller (1978) examined the veto rule, under which the resources are thrown away in case in disagreement. In such games, equilibria show a strong tendency towards equal shares for each individual. Our analysis deals with majority decisions and takes the view that resources are not thrown away in case of disagreement, but are subject to a tie-breaking procedure.

Second, Baron and Ferejohn (1989) have examined the division of resources by majority rule. If no agreement is reached, players can make new proposals. The proposer receives disproportionate benefits and the number of recipients of positive shares of the cake is at least a bare majority, but may also exceed this figure. In our case, there is a definite end to the collective choice process, which forces players to choose between agreement and tie-breaking procedure.

3 The Game

We consider a game involving three players denoted by i, j, and $k = 1, 2, 3$ who wish to divide a cake in the following way:

First, at the proposal stage, the players sequentially make open proposals about the division of the cake.

1. Player 1 proposes a division of the cake to Players 2 and 3, not knowing their proposals.
2. Player 2 makes a second proposal, knowing Player 1's proposal.
3. Player 3 suggests a division, knowing the proposals of the two former players.

Then, at the voting stage, the group selects one of these proposals[1] by majority voting. An offer receiving 2 or 3 votes will be implemented. If no majority can be reached, i.e. if each proposal gets one vote,[2] the winning proposal is selected by drawing lots and each proposal is selected with a probability of $\frac{1}{3}$. Hence, the cake is either divided by majority decision or by lot.

Note that without loss of generality, the labeling of the players and hence the sequence of proposals by the players is given exogenously.[3]

A proposal by Player i is denoted by $D_i = (a_{i1}, a_{i2}, a_{i3})$. Accordingly, a_{i1}, a_{i2}, and a_{i3} denote the shares of the cake offered by Player i to Players 1, 2, and 3 respectively. The resource constraint implies that $\sum_{j=1}^{3} a_{ij} = 1, \forall i$. Thus, after every player has made his proposal, we obtain

$$D_1 = (a_{11}, a_{12}, a_{13})$$
$$D_2 = (a_{21}, a_{22}, a_{23})$$
$$D_3 = (a_{31}, a_{32}, a_{33})$$

To simplify our presentation, we rewrite the complete set of three proposals as the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1}$$

Every player is risk-neutral and individual j derives utility[4]

$$U : [0,1] \to [0,1] \ , U(a_{ij}) = a_{ij} \tag{2}$$

[1]If two proposals coincide, they are treated as one proposal.

[2]The omission of abstentions will be explained in the characterization of equilibria.

[3]We might add a pre-stage at which the label of an agent is selected at random. Such an additional stage would equalize expected utilities across players in case of identical utility functions. Since we are focusing on actual resource divisions, we skip the pre-stage.

[4]Since all players are identical, we drop the index for the utility function.

if the proposal D_i is selected by majority voting. The expected utility u_j for Player j in the case of drawing lots is then given by

$$u_j = \frac{1}{3}\sum_{i=1}^{3} a_{ij} \ , j = 1, 2, 3 \tag{3}$$

In many parts of the analysis, we need the relative quality of the offers made to the specific players. Therefore, we introduce the rank matrix R. In this matrix, the best offer (the biggest share of the cake) a_{ij} for Player j is labeled 3, the second best 2, and the worst 1. If two or more offers are identical, they have the same rank. For example the proposal matrices

$$A_1 = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.7 & 0.2 \\ 0.6 & 0.3 & 0.1 \end{pmatrix} A_2 = \begin{pmatrix} 0.3 & 0.6 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{pmatrix} \tag{4}$$

then convert into the rank matrices

$$R_1 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} R_2 = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix} \tag{5}$$

Additionally, we denote the biggest share a_{ij} of the cake offered to Player j by X_j, the second biggest by Y_j, and the worst by Z_j.

If we label the entries of R as r_{ij} $(i, j = 1, 2, 3)$, where r_{ij} is the rank of Player i's proposal made to Player j, then the conversion function $\Phi[A] = R$ is technically given by

$$\Phi : r_{ij} = \sum_{k=1}^{3} \Theta(a_{ij} - a_{kj}) \tag{6}$$

where $\Theta(p - q)$ $(p, q \in \mathbb{R})$ is the Heavyside function given by

$$\Theta(p - q) = \begin{cases} 1 & p - q \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

4 Voting Equilibria and Proposal-Making

In this section, we first derive the voting equilibria based on a given set of proposals. The voting strategy of Player i is to select one proposal.[5] We note that the concept

[5]Allowing abstention does not change the voting equilibria. Abstention is weakly dominated by voting for the best proposal for Player i.

of a Nash equilibrium is insufficient for voting games with non-unanimity voting rules, as it can involve weakly dominated strategies. For that purpose, we use the following refinements and tie-breaking rules (for cases of indifference between payoffs). These hold throughout the paper.

4.1 Refinements

Refinement 1
A Nash equilibrium of the voting game has to be trembling hand perfect.[6]

Trembling hand perfection has two immediate consequences:

Lemma 1
Suppose that refinement 1 holds. Then

(i) A player never votes for the least favorable proposal made to him, i.e. a proposal labeled 1 in the rank matrix.

(ii) In any Nash equilibrium where agents i and j $(i \neq j)$ vote for the same proposal, agent k $(k \neq i,\ j)$ votes for his best proposal.

The elimination of weakly dominated strategies is standard. The second property also follows directly from the definition of trembling hand perfection, as any error by Players i or j leads either to a tie-break or to a majority win for the best proposal of voter i.

The next refinement eliminates voting equilibria that are payoff-dominated.

Refinement 2
If only one proposal includes 2 maxima, then it is the unique equilibrium.

This property follows from payoff dominance (see Fudenberg and Tirole 1992). A proposal with 2 maxima for, say, individuals i and j, is a Nash equilibrium supported by the votes of i and j. Any other possible Nash equilibrium is worse for i and j and hence payoff-dominated for the coalition $\{i, j\}$.

Even if both refinements are applied, we will still have multiple equilibria. For that purpose, we use the notion of correlated equilibria with public randomization introduced by Aumann (1974) (see e.g. Myerson 1991 for discussion). This concept assumes that voters engage in pre-play communication and use a coordination device to settle for

[6] See Selten (1975) for the original formalization and Fudenberg and Tirole (1992) for a survey.

a particular equilibrium. Such a device is a publicly observable random variable that agents use to determine which equilibrium should be played. For instance, agents may flip a coin, or a mediator may announce the outcome of the randomization process.

This is illustrated in the following example:

$$A = \begin{pmatrix} 0.6 & 0.0 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.2 & 0.8 & 0.0 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \tag{8}$$

In this case, D_1 and D_2 are equilibria and Player 1 and Player 3 prefer their second-best offer to a tie-break. Furthermore, these proposals contain the first- and second-best offer by Player 1 and Player 3. Therefore, we assume that Player 1 and Player 3 are playing a correlated strategy. With probability $p_1 = \frac{1}{2}$, both play D_1, and with $p_2 = \frac{1}{2}$ they play D_2.

Accordingly, we use the concept of correlated equilibria as follows:

Refinement 3
A correlated equilibrium arises if two proposals D_i, D_j $(i \neq j,\ i,j = 1,2,3)$ are equilibria[7] and they contain the best and second-best offer by the same two players. In the correlated equilibrium, both players under consideration will vote with probability $\frac{1}{2}$ for D_i or D_j. The expected payoffs for all players are then given by $C_{ij} = \frac{1}{2}(D_i + D_j)$.

Note that correlated equilibria cannot be used in particular cases, e.g. when two equilibria occur:

$$A = \begin{pmatrix} 0.0 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.1 & 0.3 & 0.6 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \tag{9}$$

Here, D_2 and D_3 are equilibria, but Player 2, who is needed for the majority of D_3, has no incentive to establish a coordination for D_2 and D_3 because D_3 contains his second-best and D_2 his worst offer. These cases are discussed in subsection 4.3 and ruled out by Refinement 4.

4.2 Tie-breaking Rules

In this subsection, we introduce some tie-breaking rules to simplify the exposition. Since agents try to receive a share as big as possible, there is only one possible motive

[7] We can neglect correlated equilibria with public randomization over all three proposals, as they are equivalent to selecting proposals by lot.

for an agent to deviate from voting for the best proposal and to vote for the second-best offer instead: If he cannot establish a majority of votes for his best proposal, the player will prefer his second-best offer to a tie-break. To illustrate this case, consider the following example involving the proposal matrix:

$$A = \begin{pmatrix} 0.7 & 0.0 & 0.3 \\ 0.5 & 0.5 & 0.0 \\ 0.0 & 0.1 & 0.9 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \tag{10}$$

Because of Refinement 1, Player 2 votes for D_2 and Player 3 for D_3. D_1, favored by Player 1, thus has no chance of becoming selected, but $Y_1 = \frac{1}{2}$ is greater than $u_1 = \frac{1}{3}(\frac{7}{10} + \frac{1}{2} + 0) = \frac{2}{5}$, his expected share in the case of drawing lots. Therefore, Player 1 votes for D_2 together with Player 2 to avoid a tie-break decision. The case where the payoffs of the second-best offer and of drawing lots coincide is resolved in the following tie-breaking rule:

Tie-Break Rule 1

If $Y_j = u_j$, all players will prefer their second-best bid to drawing lots.

Note that "prefer" in tie-breaking rule 1 means that Player i will avoid drawing lots if he is indifferent between his second-best proposal and drawing lots. The tie-breaking rule means that Player i will vote for his best proposal if it receives a majority. Otherwise, he will vote for the second-best proposal. The tie-breaking rule immediately implies

Lemma 2

Suppose that there exists at least one voting equilibrium where a majority is formed. Then drawing lots will not occur as an equilibrium.

To formulate the next tie-breaking rule, we denote an equilibrium as a single-proposal equilibrium if that single proposal is selected without the use of random selection devices. A single-proposal equilibrium necessarily requires that pure voting strategies are played and that a majority supports one proposal.

Tie-Break Rule 2

If the payoffs coincide, a player will prefer a single-proposal equilibrium to a correlated equilibrium and a correlated equilibrium to drawing lots.

4.3 Proposal-Making

Having characterized the structure of the voting equilibria, we turn now to proposal-making. In formulating the proposal-making stage, we face two problems.

First, multiple voting equilibria may still exist (see e.g. the proposal matrix in equation 9). Second, players may be indifferent between several proposals (e.g. Player 3 is always indifferent between at least two proposals if $D_1 = (a_{11}, 1 - a_{11}, 0)$ and $D_2 = (1 - a_{11}, a_{11}, 0)$).

Addressing the first point, we now introduce another refinement.

Refinement 4
Given the proposals D_1 and D_2, Player 3 makes his proposal in such a way that the proposal matrix A exhibits either a single-proposal equilibrium, a correlated equilibrium, or an equilibrium with drawing lots.

This refinement ensures that payoffs are well-defined at the voting stage. This can be justified by the aversion against strategic uncertainty. A priori, it is unclear whether Player 3 can always choose among the three options in the refinement. In the proof of the overall equilibrium, we will show that this is always possible for Player 3.

The second point is handled by an additional tie-breaking rule.

Tie-Break Rule 3
If Player i is indifferent between making several proposals, every proposal is submitted with the same probability.

4.4 The Structure of Voting Equilibria

After these preparations, we can now provide an overview of the structure of voting equilibria. We start by calculating equilibria in two examples. For this purpose, we extend the conversion of proposal matrix A to rank matrix R, since it is often necessary to distinguish whether it is possible for a player to deviate to his second-best proposal or not. This is done by introducing rank 2^* if any $Y_j \geq u_j$ $(j = 1, 2, 3)$. If we consider the example given in (10), we get

$$A = \begin{pmatrix} 0.7 & 0.0 & 0.3 \\ 0.5 & 0.5 & 0.0 \\ 0.0 & 0.1 & 0.9 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 3 & 1 & 2 \\ 2^* & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \tag{11}$$

To calculate the equilibria, we ask whether Player i deviates, given the votes of Players j and k ($i \neq j \neq k$ and the players are labeled P_1, P_2 and P_3). If Player i does deviate, his payoff is underlined.

$P_1 \longrightarrow D_1$

$P_2 \backslash P_3$	D_1	D_2	D_3
D_1	$\underline{0.7}$ 0.0 0.3	$\underline{0.7}$ 0.0 0.3	$\underline{0.7}$ 0.0 $\underline{0.3}$
D_2	$\underline{0.7}$ $\underline{0.0}$ 0.3	$\underline{0.5}$ $\underline{0.5}$ 0.0	0.4 $\underline{0.2}$ $\underline{0.4}$
D_3	$\underline{0.7}$ 0.0 0.3	0.4 0.2 0.4	$\underline{0.0}$ 0.1 $\underline{0.9}$

$P_1 \longrightarrow D_2$

$P_2 \backslash P_3$	D_1	D_2	D_3
D_1	0.7 0.0 0.3	0.5 0.5 0.0	0.4 0.2 $\underline{0.4}$
D_2	0.5 $\underline{0.5}$ 0.0	0.5 $\underline{0.5}$ 0.0	$\underline{0.5}$ $\underline{0.5}$ $\underline{0.0}$
D_3	0.4 0.2 0.4	$\underline{0.5}$ 0.5 0.0	0.0 0.1 $\underline{0.9}$

$P_1 \longrightarrow D_3$

$P_2 \backslash P_3$	D_1	D_2	D_3
D_1	0.7 0.0 0.3	0.4 0.2 0.4	0.0 0.1 $\underline{0.9}$
D_2	0.4 $\underline{0.2}$ 0.4	0.5 $\underline{0.5}$ 0.0	0.0 $\underline{0.1}$ $\underline{0.9}$
D_3	0.0 0.1 0.9	0.0 0.1 0.9	0.0 0.1 $\underline{0.9}$

In this case, the only equilibrium voting scheme is

$$\left.\begin{array}{l} P_1 \longrightarrow D_2 \\ P_2 \longrightarrow D_2 \\ P_3 \longrightarrow D_3 \end{array}\right\} \Longrightarrow \ D_2 \text{ is chosen} \tag{12}$$

We see that these calculations can be done simply by regarding the extended rank matrix. For this purpose, we look at the example given in (8). We obtain the extended rank matrix

$$A = \begin{pmatrix} 0.6 & 0.0 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.2 & 0.8 & 0.0 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 3 & 1 & 2^* \\ 2^* & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \tag{13}$$

and end up with

$P_1 \longrightarrow D_1$

$P_2 \backslash P_3$	D_1	D_2	D_3
D_1	$\underline{3}$ 1 2^*	$\underline{3}$ 1 $\underline{2}^*$	$\underline{3}$ 1 2^*
D_2	$\underline{3}$ 1 2^*	$\underline{2}^*$ 2 $\underline{3}$	T T T
D_3	$\underline{3}$ $\underline{1}$ $\underline{2}^*$	T $\underline{T}$ T	$\underline{1}$ $\underline{3}$ 1

$P_1 \longrightarrow D_2$

$P_2 \backslash P_3$	D_1	D_2	D_3
D_1	3 1 2^*	2^* 2 $\underline{3}$	T T T
D_2	2^* 2 3	2^* 2 $\underline{3}$	$\underline{2}^*$ 2 3
D_3	T $\underline{T}$ T	$\underline{2}^*$ $\underline{2}$ $\underline{3}$	1 $\underline{3}$ 1

$P_1 \longrightarrow D_3$

$P_2 \backslash P_3$	D_1	D_2	D_3
D_1	3 1 $\underline{2}^*$	T T T	1 3 1
D_2	T T T	2^* 2 $\underline{3}$	1 3 1
D_3	1 $\underline{3}$ 1	1 $\underline{3}$ $\underline{1}$	1 $\underline{3}$ 1

Now, two voting schemes are equilibria.

$$\left.\begin{array}{l} P_1 \longrightarrow D_1 \\ P_2 \longrightarrow D_3 \\ P_3 \longrightarrow D_1 \end{array}\right\} \Longrightarrow \ D_1 \text{ is chosen} \quad \left.\begin{array}{l} P_1 \longrightarrow D_2 \\ P_2 \longrightarrow D_3 \\ P_3 \longrightarrow D_2 \end{array}\right\} \Longrightarrow \ D_2 \text{ is chosen} \tag{14}$$

Since players 1 and 3 form the majority in both voting schemes, these equilibria are correlated.

In the following, we discuss the possibility of extending rank matrices and their equilibria. We denote a single-proposal equilibrium as D_i^*, an equilibrium with drawing lots as T^* and a correlated equilibrium as C_{ij}^* ($i \neq j$). In the discussion, the matrices described are representatives of a whole class of matrices that can be derived from the given matrix by interchanging columns and rows, or two entries.

For example

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

is, among other things, a representative of

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Where convenient, we omit cases with $X_i \neq Y_i = Z_i$, because they are qualitatively equal to cases with $X_i \neq Y_i \neq Z_i \wedge Y_i < u_i$, and we use $r_{ij} = 2^{(*)}$ if the conversion of a_{ij} in 2 or 2^* does not change the equilibrium outcome.

1. $D_1 = D_2 = D_3 \longrightarrow R$ shrinks to a (3×1) matrix and we have

$$\begin{pmatrix} 3 & 3 & 3 \end{pmatrix} \longrightarrow \; D_1^* \tag{15}$$

2. $D_1 \neq D_2 = D_3 \longrightarrow R$ shrinks to a (3×2) matrix and we have

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 2 & 3 \end{pmatrix} \longrightarrow \begin{cases} D_1^* \\ D_2^* \end{cases} \quad \begin{pmatrix} 3 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} \longrightarrow D_1^* \tag{16}$$

3. $D_1 \neq D_2 \neq D_3$

   (a) $X_1 = Y_1 = Z_1 \;\wedge\; X_2 \neq Y_2 \neq Z_2 \;\wedge\; X_3 \neq Y_3 \neq Z_3$

$$\begin{pmatrix} 3 & 3 & 1 \\ 3 & 2^{(*)} & 2^{(*)} \\ 3 & 1 & 3 \end{pmatrix} \longrightarrow \begin{cases} D_1^* \\ D_3^* \end{cases} \tag{17}$$

   (b) $X_1 = Y_1 \neq Z_1 \;\wedge\; X_2 = Y_2 \neq Z_2 \;\wedge\; X_3 = Y_3 \neq Z_3$

$$\begin{pmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix} \longrightarrow \begin{cases} D_1^* \\ D_2^* \\ D_3^* \end{cases} \tag{18}$$

   (c) $X_1 = Y_1 \neq Z_1 \;\wedge\; X_2 = Y_2 \neq Z_2 \;\wedge\; X_3 \neq Y_3 \neq Z_3$

$$\begin{pmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 2^{(*)} \end{pmatrix} \longrightarrow \begin{cases} D_1^* \\ D_2^* \end{cases} \tag{19}$$

(d) $X_1 = Y_1 \neq Z_1 \wedge X_2 \neq Y_2 \neq Z_2 \wedge X_3 \neq Y_3 \neq Z_3$

$$\begin{pmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 2^{(*)} & 2^{(*)} \end{pmatrix} \longrightarrow \begin{cases} D_1^* \\ D_2^* \end{cases} \quad \begin{pmatrix} 3 & 1 & 2^{(*)} \\ 3 & 3 & 1 \\ 1 & 2^{(*)} & 3 \end{pmatrix} \longrightarrow D_2^*$$
$$\begin{pmatrix} 3 & 1 & 2^{(*)} \\ 3 & 2^{(*)} & 1 \\ 1 & 3 & 3 \end{pmatrix} \longrightarrow D_3^* \tag{20}$$

(e) $X_1 \neq Y_1 \neq Z_1 \wedge X_2 \neq Y_2 \neq Z_2 \wedge X_3 \neq Y_3 \neq Z_3$

i. Doublemax:=$\big(\exists\, r_{ij} = r_{ik} = 3 \quad j \neq k \quad i, j, k \in \{1, 2, 3\}\big)$

$$\begin{pmatrix} 3 & 3 & 2 \\ 1 & 2^{(*)} & 3 \\ 2^{(*)} & 2 & 1 \end{pmatrix} \longrightarrow D_1^* \tag{21}$$

ii. $Y_1 < u_1 \wedge Y_2 < u_2 \wedge Y_3 < u_3 \wedge$ no Doublemax

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow L^* \tag{22}$$

iii. $Y_1 \geq u_1 \wedge Y_2 < u_2 \wedge Y_3 < u_3 \wedge$ no Doublemax

$$\begin{pmatrix} 1 & 3 & 2 \\ 2^* & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow D_2^* \tag{23}$$

iv. $Y_1 \geq u_1 \wedge Y_2 \geq u_2 \wedge Y_3 < u_3 \wedge$ no Doublemax

$$\begin{pmatrix} 1 & 3 & 2 \\ 2^* & 1 & 3 \\ 3 & 2^* & 1 \end{pmatrix} \longrightarrow \begin{cases} D_2^* \\ D_3^* \end{cases} \quad \begin{pmatrix} 1 & 1 & 3 \\ 2^* & 3 & 2 \\ 3 & 2^* & 1 \end{pmatrix} \longrightarrow C_{23}^*$$
$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 2 \\ 2^* & 2^* & 3 \end{pmatrix} \longrightarrow D_3^* \tag{24}$$

v. $Y_1 \geq u_1 \wedge Y_2 \geq u_2 \wedge Y_3 \geq u_3 \wedge$ no Doublemax

$$\begin{pmatrix} 1 & 3 & 2^* \\ 2^* & 1 & 3 \\ 3 & 2^* & 1 \end{pmatrix} \longrightarrow \begin{cases} D_1^* \\ D_2^* \\ D_3^* \end{cases} \quad \begin{pmatrix} 1 & 1 & 3 \\ 2^* & 3 & 2^* \\ 3 & 2^* & 1 \end{pmatrix} \longrightarrow C_{23}^* \tag{25}$$

5 Division of the Cake

In this section, we derive the overall equilibrium of the game. For that purpose, we have to compare the expected payoffs for the different players. Accordingly, we denote the expected payoff of Player i by π_i $(i = 1, 2, 3)$.

5.1 Overall Equilibrium

The solution of the game is given by the following theorem.[8]

Theorem 1

There exist two overall equilibria of the game

$$\begin{array}{rcl} D_1 & = & (0,1,0) \\ D_2 & = & (1,0,0) \\ D_3 & = & (\frac{1}{2},0,\frac{1}{2}) \end{array} \quad or \quad \begin{array}{rcl} D_1 & = & (0,1,0) \\ D_2 & = & (1,0,0) \\ D_3 & = & (0,\frac{1}{2},\frac{1}{2}) \end{array} \tag{26}$$

where $D_3 = (\frac{1}{2}, 0, \frac{1}{2})$ is a single-proposal equilibrium supported by Players 1 and 3, and $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ is a single-proposal equilibrium supported by Players 2 and 3. As Player 3 is indifferent between $D_3 = (\frac{1}{2}, 0, \frac{1}{2})$ and $D_3 = (0, \frac{1}{2}, \frac{1}{2})$, he will make either proposal with a probability of $\frac{1}{2}$ (Tie-Break Rule 3). For the payoffs, this implies

$$\pi_1 = \frac{1}{4} \quad \pi_2 = \frac{1}{4} \quad \pi_3 = \frac{1}{2}$$

The proof of the theorem follows directly from Corollary 1 and Propositions 5 and 6 in subsections 5.3, 5.4, and 5.5.

This theorem can be motivated as follows: If Player 1 makes any proposal $D_1 \neq (0, 1, 0)$, Player 2 will counter it with a proposal D_2 such that π_1 is less than $\frac{1}{4}$. But by proposing $D_1 = (0, 1, 0)$, Player 1 makes Player 2 so unattractive that any counter-proposal $D_2 \neq (1, 0, 0)$ will give Player 2 a payoff π_2 of less than $\frac{1}{4}$. This implies that ultimately, the cake is divided half-by-half between Players 1 and 3 or Players 2 and 3.

5.2 Strategy of the Proof

We prove Theorem 1 by backward induction. This is illustrated in the following steps:

Step 1: We determine the best response of Player 3 given any D_1 and D_2 and calculate the minimum payoff π_3, given specific relations between D_1 and D_2.

[8] The solution has been conjectured in Gersbach and Wehrspohn (2001).

Step 2: Given $D_1 \neq (0,1,0)$, we determine a proposal D_2 giving Player 2 a payoff of π_2 such that, together with the minimum payoff of π_3 of Player 3 from step 1, the resource constraint implies $\pi_1 < \frac{1}{4}$ for Player 1.

Step 3: Given $D_1 = (0,1,0)$, we calculate the payoff π_2 if Player 2 makes any proposal $D_2 \neq (1,0,0)$ and compare this with his payoff π_2 when proposing $D_2 = (1,0,0)$.

5.3 Strategy of Player 3

Given D_1, D_2, Player 3's ambition is to maximize his share of the cake by making an appropriate proposal D_3.

Given D_1 and D_2, Player 3 will generally consider four possible equilibrium outcomes when designing his proposal D_3.

I: D_3 is a single-proposal equilibrium.

II: D_1 or D_2 is a single-proposal equilibrium.

III: A correlated equilibrium arises.

IV: The cake is divided by drawing lots.

We now give examples of cases where the different designing principles (I-IV) are the best reactions of Player 3. The detailed rationalization of the best reaction of Player 3 will be given in the proof.

$$\text{I} \quad : \quad \left.\begin{array}{l} D_1 = (0.7, 0.3, 0.0) \\ D_2 = (0.3, 0.5, 0.2) \end{array}\right\} \Longrightarrow D_3 = (0.0, 0.4, 0.6) \qquad \Longrightarrow D_3^*$$

$$\text{II}^9 \quad : \quad \left.\begin{array}{l} D_1 = (0.1, 0.3, 0.6) \\ D_2 = (0.5, 0.5, 0.0) \end{array}\right\} \Longrightarrow D_3 = (1.0, 0.0, 0.0) \qquad \Longrightarrow D_1^*$$

$$\text{III} \quad : \quad \left.\begin{array}{l} D_1 = (0.0, 0.9, 0.1) \\ D_2 = (0.4, 0.1, 0.5) \end{array}\right\} \Longrightarrow D_3 = (0.2, 0.0, 0.8) \qquad \Longrightarrow C_{23}^*$$

$$\text{IV} \quad : \quad \left.\begin{array}{l} D_1 = (0.1, 0.1, 0.8) \\ D_2 = (0.6, 0.4, 0.0) \end{array}\right\} \Longrightarrow D_3 = (0.0, 0.7+\epsilon, 0.3-\epsilon) \quad \Longrightarrow L^*$$

with $\epsilon > 0$ and infinitesimally small.

[9] In this case, we have a continuum of best proposals $D_3 = (a_{31}, a_{32}, 1-a_{31}-a_{32})$ with $a_{31} \in (0.9, 1]$ and $a_{32} \in [0, 1-a_{31}]$.

The kind of equilibrium outcome (I–IV) Player 3 prefers depends strongly on the relations between the proposals D_1 and D_2, characterized by the submatrix a of the proposal matrix A, which is defined by

$$A = \begin{pmatrix} & \text{a} & a_{13} \\ & & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \Longrightarrow \quad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{27}$$

and the corresponding rank matrix ρ, which is calculated from a in a similar way as R from A:

$$\rho = \phi[a] = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

where ϕ is defined in a similar way to Φ (see (6)) as

$$\phi : \rho_{ij} = \sum_{k=1}^{2} \Theta(a_{ij} - a_{kj}) \tag{28}$$

The relevance of a and ρ will be shown in the proofs of the following propositions and corollaries.

Proposition 1
Given D_1 and D_2 and that ρ is symmetric $\Longrightarrow$ determining the best reaction of Player 3 implies $\pi_3 \geq \frac{3}{8}$.

Corollary 1
Given $D_1 = (0, 1, 0)$ and $D_2 = (1, 0, 0) \Longrightarrow$ Player 3's best proposal is $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ or $D_3 = (\frac{1}{2}, 0, \frac{1}{2})$, which implies $\pi_1 = \pi_2 = \frac{1}{4}$ and $\pi_3 = \frac{1}{2}$.

Proposition 2
Given D_1, D_2, $a \neq \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \vee \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ and that ρ is non-symmetric $\Longrightarrow$ determining the best reaction of Player 3 implies $\pi_3 > \frac{1}{4}$.

Corollary 2
Given $a = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \vee \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \Longrightarrow$ Player 3's best proposal is $D_3 = (\frac{1}{2} + \epsilon, \frac{1}{4}, \frac{1}{4} - \epsilon)$ or $D_3 = (\frac{1}{4}, \frac{1}{2} + \epsilon, \frac{1}{4} - \epsilon)$, which implies $\pi_1 = \pi_2 = \frac{7}{16} + \frac{\epsilon}{4}$ and $\pi_3 = \frac{1}{8} - \frac{\epsilon}{2}$.

(The proofs are given in the appendix).

5.4 Strategy of Player 2

In this section, we analyze possible reactions by Player 2, given D_1 and the reaction of Player 3 derived in section 5.3.

For that purpose, we divide the proposal set of D_1 given by

$$\mathcal{P} = \{(a_{11}, a_{12}) | (a_{11}, a_{12}) \in [0,1] \times [0, 1 - a_{11}]\} \tag{29}$$

into four subsets:

$$\begin{aligned} \mathcal{A} &= \left[\tfrac{1}{2}, 1\right] \times [0, 1 - a_{11}] \setminus \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \\ \mathcal{B} &= \left[0, \tfrac{1}{2}\right] \times [0, 1 - a_{11}] \setminus (0, 1) \\ \mathcal{C} &= \left[0, \tfrac{1}{2}\right) \times \left[0, \tfrac{1}{2}\right) \\ D^* &= (0, 1) \end{aligned} \tag{30}$$

Proposition 3
$\forall\ D_1 \in \mathcal{A} \cup \mathcal{B}\ \exists$ a proposal[10] *D_2^s of Player 2, such that ρ is symmetric and $\pi_2 > \frac{3}{8}$.*

Corollary 3
If for $D_1 \in \mathcal{A} \cup \mathcal{B}\ \exists$ a proposal[11] *D_2^{ns} of Player 2, such that ρ is non-symmetric and $\pi_2(D_2^{ns}) \geq \pi_2(D_2^s) \Longrightarrow \pi_2(D_2^{ns}) \geq \frac{1}{2}$ or $\pi_1(D_2^{ns}) < \frac{1}{4}$.*

Proposition 4
$\forall\ D_1 \in \mathcal{C}\ \exists$ a proposal D_2^{ns} of Player 2, such that ρ is non-symmetric and $\pi_2 \geq \frac{1}{2}$ or the best proposal D_2^{ns} with ρ non-symmetric for Player 2 implies $\pi_1 = 0$.

Corollary 4
If for $D_1 \in \mathcal{C}\ \exists$ a proposal D_2^s of Player 2, such that ρ is symmetric and $\pi_2(D_2^s) \geq \pi_2(D_2^{ns}) \Longrightarrow \pi_2(D_2^s) > \frac{3}{8}$.

Proposition 5
Given $D_1 = (0, 1, 0)$, the best reaction of Player 2 is $D_2 = (1, 0, 0)$.

(The proofs are given in the appendix).

[10] D_2^s is a proposal made by Player 2 such that a is symmetric.
[11] D_2^{ns} is a proposal made by Player 2 such that a is non-symmetric.

5.5 Strategy of Player 1

Since Player 1 anticipates the reactions of Players 2 and 3, we obtain the following proposition:

Proposition 6

The best proposal of Player 1 is $D_1 = (0, 1, 0)$.

Proof of Proposition 6

(i) Suppose $D_1 \neq (0, 1, 0)$ and D_2 with ρ symmetric is the best reaction of Player 2
$\Longrightarrow \pi_3 \geq \frac{3}{8}$ (Proposition 1) and $\pi_2 > \frac{3}{8}$ (Proposition 3 and Corollary 3)
$\Longrightarrow$ together with the resource constraint, we obtain $\pi_1 + \overset{>\frac{3}{8}}{\pi_2} + \overset{\geq\frac{3}{8}}{\pi_3} = 1 \Longrightarrow \pi_1 < \frac{1}{4}$.

(ii) Suppose $D_1 \neq (0, 1, 0)$ and D_2 with ρ non-symmetric is the best reaction of Player 2
$\Longrightarrow \pi_3 > \frac{1}{4}$ Proposition[12] 2) and $\pi_2 \geq \frac{1}{2} \vee \pi_1 < \frac{1}{4}$ (Corollary 3 and Proposition 4)
$\Longrightarrow$ together with the resource constraint, we obtain $\pi_1 + \overset{\geq\frac{1}{2}}{\pi_2} + \overset{>\frac{1}{4}}{\pi_3} = 1 \Longrightarrow \pi_1 < \frac{1}{4}$,
or we have directly $\pi_1 < \frac{1}{4}$.

$\Longrightarrow$ (i) and (ii), together with Corollary 1 and Proposition 5, imply that $D_1 = (0, 1, 0)$ is the best proposal for Player 1.

■

6 Discussion and Conclusion

We have examined a common collective choice process to study the allocation of resources among a group of people. The analysis reveals that the first two agents want to make each other as unattractive as possible with regard to the third agent's proposal. To do so, they offer each other the whole cake, and the third player can ensure that he obtains one half of the cake.

This outcome exhibits a powerful last-mover advantage, whereas the other players are forced to make strategic proposals involving zero resources for themselves. They expect $\frac{1}{4}$ of the cake. Additionally, it seems surprising that the first player can totally

[12] Note that neither $D_2 = (\frac{1}{2}, \frac{1}{2}, 0)$ is the best reaction of Player 2 given that $D_1 = (0, 0, 1)$, nor is $D_2 = (0, 0, 1)$ the best reaction of Player 2 given that $D_1 = (\frac{1}{2}, \frac{1}{2}, 0)$.

outweigh the second-mover advantage of the second player. The way in which these characteristics extend to group decisions with a larger number of individuals is an important avenue to future research. Of course, there is a variety of game-theoretic considerations and alternative refinement concepts that can be examined. How robust our main findings are with regard to such extensions remains to be explored. A further useful extension of our framework would be to consider the role of risk preferences. A large literature has generated the finding that it is disadvantageous to be relatively risk averse in bargaining settings.[13] Harrington (1989 and 1990) has, however, shown that in bargaining games in which acceptance of a proposed allocation only requires the approval of a majority, it is advantageous for a player to be relatively risk averse. How risk aversion in our model affects the utility of each member would be be an interesting research project.

[13] At least since Zeuthen (1930), risk preferences have been thought to be an important determinant of the outcome of bargaining. The role of risk preferences in a bargaining setting has been examined using both an axiomatic framework [see Roth (1979), Kihlstrom, Roth and Schmeidler (1981), Nielsen (1984)] and a non-cooperative game framework [Roth (1985), Binmore, Rubinstein and Wolinksy (1986) and Harrington (1986)]. These studies have confirmed that the more risk averse an agent is, the lower his share. Notable exceptions to this finding are described by Roth and Rothblum (1982) and Osborne (1985).

7 Appendix

We have already mentioned that the specific relations between D_1 and D_2 are relevant for the construction of the best reaction D_3 given by ρ and a. Now we need a more detailed specification of the relations between the entries of a. Therefore we define

$$\begin{aligned}
\mu_j &= \tfrac{1}{2}(a_{1j} + a_{2j}) \\
\overline{\mu} &= \max\{\mu_1 \,, \mu_2\} \\
\underline{\mu} &= \min\{\mu_1 \,, \mu_2\} \\
x_j &= \max\{a_{1j} \,, a_{2j}\} \\
y_j &= \min\{a_{1j} \,, a_{2j}\} \\
\underline{x} &= \min\{x_1 \,, x_2\}
\end{aligned}$$

7.1 Proof of Propositions 1 and 2 and Corollaries 1 and 2

Before we start, note that the best reaction D_3 is not always unique (i.e. $x_1 = x_2 \wedge y_1 = y_2$, or see footnote 9). But as we only want to determine the minimum share of the cake for Player 3, it is sufficient to give only one best reaction.

7.1.1 Proof of Proposition 1

There are three different possibilities for ρ to be symmetric:

$$\rho_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \rho_{1'} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \tag{31}$$

For ρ_2 we have $D_1 = D_2$, and the best proposal D_3 is given by

1. If $D_1 = (0, 0, 1) \Rightarrow D_3 = (0, 0, 1)$
2. If $D_1 \neq (0, 0, 1) \Rightarrow D_3 = (\underline{x}+\epsilon, 0, 1-\underline{x}-\epsilon)$ if $x_1 = \underline{x}$ and $D_3 = (0, \underline{x}+\epsilon, 1-\underline{x}-\epsilon)$ if $x_2 = \underline{x}$.

For ρ_1 or $\rho_{1'}$ it is sufficient to analyze ρ_1 only, as the same arguments follow for $\rho_{1'}$ by exchanging the columns in matrix a.

For ρ_1 A is given by

$$A = \begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \text{with} \qquad \begin{aligned} a_{13} &= 1 - x_1 - y_2 \\ a_{23} &= 1 - x_2 - y_1 \\ a_{33} &= 1 - a_{31} - a_{32} \end{aligned} \tag{32}$$

To calculate the best reaction of Player 3, the following properties of A are relevant:

(i) $x_i > a_{3i} \geq \mu_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 1, \Phi(a_{3i}) = 2^*$

(ii) $x_i > y_i > a_{3i} > 2y_i - x_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 2, \Phi(a_{3i}) = 1$

(iii) $x_i > y_i > a_{3i}$ and $a_{3i} \leq 2y_i - x_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 2^*, \Phi(a_{3i}) = 1$

(iv) $x_i = a_{3i} > y_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 1, \Phi(a_{3i}) = 3$

With these properties we distinguish six different kinds of proposals D_3 discussed in detail below:

$$D_3^{\underline{\mu}} \qquad D_3^{\overline{\mu}} \qquad D_3^{\mu_i} \qquad D_3^{\underline{x}} \qquad D_3^{Cor} \quad D_3^{L}$$

$D_3^{\underline{\mu}}$: Player 3 offers Player 1 or 2 $\underline{\mu}$ and keeps the rest for himself, such that D_3 is a unique equilibrium and he receives a payoff $\pi_3 = 1 - \underline{\mu}$. For example

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ \underline{\mu} & 0 & 1-\underline{\mu} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 2^* & 1 & 3 \end{pmatrix}; \qquad \begin{array}{l} D_1 = (0.7 \quad 0.1 \quad 0.2) \\ D_2 = (0.1 \quad 0.8 \quad 0.1) \\ D_3^{\underline{\mu}} = (0.4 \quad 0.0 \quad 0.6) \end{array} \tag{33}$$

$D_3^{\overline{\mu}}$: Player 3 offers Player 1 or 2 $\overline{\mu}$ and keeps the rest for himself, such that D_3 is a unique equilibrium and he receives a payoff $\pi_3 = 1 - \overline{\mu}$. For example

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ 0 & \overline{\mu} & 1-\overline{\mu} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 2^* & 3 \end{pmatrix}; \qquad \begin{array}{l} D_1 = (0.6 \quad 0.4 \quad 0.0) \\ D_2 = (0.2 \quad 0.6 \quad 0.2) \\ D_3^{\overline{\mu}} = (0.0 \quad 0.5 \quad 0.5) \end{array} \tag{34}$$

$D_3^{\mu_i}$: Player 3 offers Player i $a_{3i} = \mu_i$, Player j $a_{3j} = 2y_j - x_j + \epsilon$ and keeps the rest for himself, such that D_3 is a unique equilibrium and he receives a payoff $\pi_3 = 1 - (\mu_i + 2y_j - x_j + \epsilon)$ $(i, j = 1, 2\ i \neq j$ and $2y_j - x_j \geq 0)$. For example

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ \mu_1 & 2y_2 - x_2 + \epsilon & 1 - (\mu_1 + 2y_2 - x_2 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 2^* & 1 & 3 \end{pmatrix} \tag{35}$$

$$\begin{array}{lcl} D_1 & = & (0.58 \quad 0.38 \quad 0.04 \ ) \\ D_2 & = & (0.32 \quad 0.68 \quad 0.00 \ ) \\ D_3^{\mu_1} & = & (0.45 \quad 0.08 + \epsilon \quad 0.47 - \epsilon \ ) \end{array}$$

$D_3^{\underline{x}}$: Player 3 offers Player 1 or 2 $\underline{x}$ and keeps the rest for himself, such that D_3 is a unique equilibrium and he receives a payoff $\pi_3 = 1 - \underline{x}$. For example

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ \underline{x} & 0 & 1-\underline{x} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2^* & 1 \\ 2 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix}; \qquad \begin{array}{l} D_1 = (0.52 \quad 0.40 \quad 0.08) \\ D_2 = (0.30 \quad 0.54 \quad 0.16) \\ D_3^{\underline{x}} = (0.52 \quad 0.00 \quad 0.48) \end{array} \tag{36}$$

D_3^{Cor}: Player 3 offers Player i $a_{3i} = \mu_i$, Player j $a_{3j} = 0$

and keeps the rest for himself, such that D_3 is part of the correlated equilibrium and he receives a payoff $\pi_3 = \frac{1}{2}(1 - \mu_i + a_{i3})$. For example

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ \mu_1 & 0 & 1-\mu_1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 2^* \\ 1 & 3 & 1 \\ 2^* & 1 & 3 \end{pmatrix}; \quad \begin{array}{ll} D_1 & = (0.57 \;\; 0.07 \;\; 0.46) \\ D_2 & = (0.37 \;\; 0.66 \;\; 0.00) \\ D_3^{cor} & = (0.47 \;\; 0.00 \;\; 0.53) \end{array} \tag{37}$$

D_3^T: (a) Player 3 proposes $D_3 = (0, 0, 1)$ if $2y_i < x_i$ $(i = 1, 2)$, such that the proposal is selected by drawing lots and he receives a payoff $\pi_3 = 1 - \frac{1}{3}(x_1 + x_2 + y_1 + y_2)$. For example

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix}; \quad \begin{array}{lll} D_1 & = & (0.61 \;\; 0.09 \;\; 0.30) \\ D_2 & = & (0.29 \;\; 0.71 \;\; 0.00) \\ D_3^T & = & (0.00 \;\; 0.00 \;\; 1.00) \end{array} \tag{38}$$

(b) Player 3 offers Player i $a_{3i} = 2y_i - x_i + \epsilon$ if $2y_i \geq x_i$ and Player j $a_{3j} = 0$ if $2y_j < x_j$ $(i \neq j)$ and keeps the rest for himself, such that the proposal is selected by drawing lots and he receives a payoff $\pi_3 = 1 - \frac{1}{3}(3y_i + x_j + y_j + \epsilon)$. For example

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ 0 & 2y_2 - x_2 + \epsilon_2 & 1 - (2y_2 - x_2 + \epsilon_2) \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix} \tag{39}$$

$$\begin{array}{lll} D_1 & = & (0.58 \quad 0.30 \quad 0.12 \;) \\ D_2 & = & (0.05 \quad 0.55 \quad 0.40 \;) \\ D_3^T & = & (0.00 \quad 0.05 + \epsilon \quad 0.95 - \epsilon \;) \end{array}$$

Note that constructing D_3 in such a way that D_1 or D_2 are unique equilibria or D_1 and D_2 form a correlated equilibrium can never be better than $D_3^{\underline{x}}$, since $1 - \underline{x} \geq a_{i3}$ $(i = 1, 2)$ and offering $D_3^{\underline{x}}$ is always possible for Player 3 if ρ symmetric. This ensures that Refinement 4 can always be satisfied and Player 3's best reaction is an element of $\{D_3^{\underline{\mu}}, D_3^{\overline{\mu}}, D_3^{\mu_i}, D_3^{\underline{x}}, D_3^{Cor}, D_3^T\}$

In the following we calculate for every proposal $D_3^{\underline{\mu}}$, $D_3^{\overline{\mu}}$, $D_3^{\mu_i}$, $D_3^{\underline{x}}$, D_3^{Cor} and D_3^L the feasible set in which it is the best reaction of Player 3. Also, we minimize the payoff π_3 giving the best reaction and the feasible set. In the course of these calculations we will often draw upon the argument of contradiction. Accordingly, we introduce the sign $\lightning$ to indicate a conclusion that contradicts the assumptions. Since every feasible set is bounded by linear inequalities we can calculate $\min\{\pi_3\}$ given the best reaction of Player 3 by a simplex algorithm.[14]

[14]This minimization is done by Maple. Since the constraints are linear, there exists a finite algorithm for calculating the extrema. These are located in the corners of the 4-dimensional hyper-polyeder. The number of constraints does not exceed 30. Thus the problem has less than $\binom{30}{4} \approx 10^5$ corners, which can be checked in a few seconds using a 2GHz processor.

1. $D_3^{\underline{\mu}}$

   Constraints on $D_3^{\underline{\mu}}$ being the best proposal for Player 3 (w.l.o.g $\mu_1 = \underline{\mu}$):

1.	μ_1	$\leq$	μ_2	
2.	$2y_2$	$<$	x_2	
3.	a_{i3}	$\leq$	a_{j3}	$\}$ $i,j = 1,2$ $i \neq j$
4.	$1 - \mu_1 + a_{i3}$	$>$	$2a_{j3}$	

   - Drawing lots versus $D_3^{\underline{\mu}}$:

$$\begin{aligned} u_3^{max} &= \tfrac{1}{3}\big(\overbrace{1 - x_1 - y_2}^{a_{13}} + \overbrace{1 - x_2 - y_1}^{a_{23}} + \overbrace{1}^{a_{33}^{max}}\big) \\ &\overset{\mu 1 \leq \mu_2}{\leq} 1 - \tfrac{2}{3}(x_1 + y_1) < 1 - \tfrac{1}{2}(x_1 + y_1) = 1 - \underline{\mu} = a_{33}(D_3^{\underline{\mu}}) \end{aligned} \tag{40}$$

   - Correlated equilibrium versus $D_3^{\underline{\mu}}$:
     $1 - \underline{\mu} > \frac{1}{2}(1 - \underline{\mu} + a_{i3})$, $i = 1, 2$.
   - Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\underline{\mu}}) = \tfrac{1}{2} \tag{41}$$

   - General example: $\begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.8 & 0.0 \\ 0.4 & 0.0 & 0.6 \end{pmatrix}$

     Minium payoff example: $\begin{pmatrix} \frac{7}{10} & \frac{3}{10} & 0 \\ \frac{3}{10} & \frac{7}{10} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

2. $D_3^{\overline{\mu}}$

   Constraints on $D_3^{\overline{\mu}}$ being the best proposal for Player 3 (w.l.o.g $\mu_2 = \overline{\mu}$):

1.	$2y_2$	$\geq$	x_2	
2.	$2y_1$	$<$	x_1	
3.	μ_2	$\leq$	$\underline{x}$	
4.	a_{i3}	$\leq$	a_{j3}	$\}$ $i,j = 1,2$ $i \neq j$
5.	$1 - \mu_2 + a_{i3}$	$>$	$2a_{j3}$	
6.	$\mu_1 + 2y_2 - x_2$	$\geq$	μ_2	

   - Drawing lots versus $D_3^{\overline{\mu}}$:

$$\begin{aligned} u_3^{max} &= \tfrac{1}{3}\big(\overbrace{1 - x_1 - y_2}^{a_{13}} + \overbrace{1 - x_2 - y_1}^{a_{23}} + \overbrace{1 - (2y_2 - x_2)}^{a_{33}^{max}}\big) \\ &= 1 - \tfrac{1}{3}(x_1 + y_1) - y_2 \end{aligned} \tag{42}$$

We have

$$x_1 + y_1 \overset{\mu_2 \le \underline{x}}{\ge} \tfrac{1}{2}(x_2 + y_2) \overset{2y_2 \ge x_2}{\ge} \tfrac{3}{2}(x_2 - y_2)$$

$$\Longrightarrow \underbrace{1 - \frac{1}{3}(x_1 + y_1) - y_2}_{u_3^{max}} \le \underbrace{1 - \frac{1}{2}(x_2 + y_2)}_{a_{33}(D_3^{\overline{\mu}})} \tag{43}$$

- Correlated equilibrium versus $D_3^{\overline{\mu}}$:
  Suppose $\pi_3(\text{Correlated equilibrium}) > a_{33}(D_3^{\overline{\mu}}) \Longrightarrow$

$$\overbrace{\frac{1}{2}(1 - \mu_1 + 1 - x_1 - y_2)}^{\pi_3^{max}(\text{Correlated equilibrium})} > \overbrace{1 - \mu_2}^{a_{33}(D_3^{\overline{\mu}})} \Longrightarrow \underbrace{\frac{1}{2}(y_1 - x_1)}_{<0} > \underbrace{2 - x_2}_{>0} \quad \lightning \tag{44}$$

- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\overline{\mu}}) = \frac{3}{7} \tag{45}$$

- General example: $\begin{pmatrix} 0.6 & 0.34 & 0.06 \\ 0.2 & 0.50 & 0.30 \\ 0.0 & 0.42 & 0.58 \end{pmatrix}$

  Minium payoff example: $\begin{pmatrix} \frac{4}{7}-\epsilon & \frac{3}{7}+\epsilon & 0 \\ \frac{2}{7}+\epsilon & \frac{5}{7}-\epsilon & 0 \\ 0 & \frac{4}{7} & \frac{3}{7} \end{pmatrix}$

3. $D_3^{\mu_i}$

   Constraints on $D_3^{\mu_i}$ being the best proposal for Player 3 (w.l.o.g. we consider only the constraints for $D_3^{\mu_1}$):

(i)

1.	$2y_2$	$\ge$	x_2
2.	$2y_1$	$<$	x_1
3.	$\mu_1 + 2y_2 - x_2$	$<$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$<$	μ_2
5.	a_{i3}	$\le$	a_{j3}
6.	$1 - (\mu_1 + 2y_2 - x_2 + \epsilon) + a_{i3}$	$>$	$2a_{j3}$

$\}$ $i, j = 1, 2 \; i \ne j$ (rows 5–6)

- Drawing lots versus $D_3^{\mu_1}$:
  (a) $\mu_2 < \underline{x}$ see argumentation for $D_3^{\overline{\mu}}$.
  (b) $\mu_2 \ge \underline{x} \Longrightarrow \underline{x} = x_1$ and with $\mu_1 + 2y_2 - x_2 < x_1 \Longrightarrow$
  $x_1 + y_1 < 2x_1 + 2x_2 - 4y_2 < 4(x_2 - y_2) < 6(x_2 - y_2) \Longrightarrow$
  $u_3^{max} = 1 - \frac{1}{3}(x_1 + y_1) - y_2 < 1 - \mu_1 - (2y_2 - x_2) = a_{33}(D_3^{\mu_1})$

- Correlated equilibrium versus $D_3^{\mu_i}$
  For constraint set (i) we have $\mu_2 > \mu_1 + 2y_2 - x_2$
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\mu_i}(i)) > \frac{3}{7} \tag{46}$$

- General example: $\begin{pmatrix} 0.55 & 0.4 & 0.15 \\ 0.27 & 0.7 & 0.03 \\ 0.41 & 0.1+\epsilon & 0.49-\epsilon \end{pmatrix}$

  Minium payoff example: $\begin{pmatrix} \frac{4}{7} & \frac{3}{7} & 0 \\ \frac{2}{7}-2\epsilon & \frac{7}{10} & 2\epsilon \\ \frac{3}{7}-\epsilon & \frac{1}{7} & \frac{3}{7}+\epsilon \end{pmatrix}$

(ii)

1.	$2y_1$	$\geq$	x_1
2.	$2y_2$	$\geq$	x_2
3.	$\mu_1 + 2y_2 - x_2$	$<$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$\leq$	$\mu_2 + 2y_1 - x_1$

- Drawing lots versus $D_3^{\mu_1}$:

$$\begin{aligned} u_3^{max} &= \tfrac{1}{3}\big(\overbrace{1 - x_1 - y_2}^{a_{13}} + \overbrace{1 - x_2 - y_1}^{a_{23}} + \overbrace{1 - 2y_1 + x_1 - \epsilon - 2y_2 + x_2 - \epsilon'}^{a_{33}^{max}}\big) \\ &= 1 - (y_1 + y_2) - \tfrac{1}{3}(\epsilon + \epsilon') \end{aligned} \tag{47}$$

and

$$\begin{aligned} & y_1 + y_2 \overset{\substack{2y_1 \geq x_1 \\ 2y_2 \geq x_2}}{\geq} \underline{x} \qquad \Longrightarrow \\ & \underbrace{1 - (y_1 + y_2) - \frac{1}{3}(\epsilon + \epsilon`)}_{u_3^{max}} < 1 - \underline{x} \overset{\mu_1 + 2y_2 - x_2 < \underline{x}}{\leq} \underbrace{1 - (\mu_1 + 2y_2 - x_2 + \epsilon)}_{a_{33}(D_3^{\mu_1})} \\ & \qquad (\epsilon \leq \underline{x} - (\mu_1 + 2y_2 - x_2)) \end{aligned} \tag{48}$$

- Correlated equilibrium versus $D_3^{\mu_i}$
  $\mu_2 + 2y_1 - x_1 \geq \mu_1 + 2y_2 - x_2$
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\mu_i}(ii)) > \frac{3}{8} \tag{49}$$

- Examples

General example: $\begin{pmatrix} 0.55 & 0.4 & 0.05 \\ 0.27 & 0.7 & 0.03 \\ 0.41 & 0.1+\epsilon & 0.49-\epsilon \end{pmatrix}$

Minium payoff example: $\begin{pmatrix} \frac{5}{8}+2\epsilon & \frac{3}{8}-4\epsilon & 2\epsilon \\ \frac{3}{7} & \frac{5}{8} & 0 \\ \frac{4}{8}+\epsilon & \frac{1}{8}-8\epsilon+\epsilon' & \frac{3}{8}+7\epsilon-\epsilon' \end{pmatrix}$ $(\epsilon' < \epsilon)$

4. $D_3^{\underline{x}}$

Constraints on $D_3^{\underline{x}}$ being the best proposal for Player 3:

(i)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	μ_2	$\geq$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$>$	$\underline{x}$

- Drawing lots versus $D_3^{\underline{x}}$:

$$u_3^{max} = \frac{1}{3}(a_{13} + a_{23} + \underbrace{1 - (\mu_1 + 2y_2 - x_2 + \epsilon)}_{a_{33}(D_3^{\underline{x}})}) \overset{\substack{a_{13} \leq 1-\underline{x} \\ a_{23} \leq 1-\underline{x}}}{\leq} a_{33}(D_3^{\underline{x}})$$

- Correlated equilibrium versus $D_3^{\underline{x}}$:
  For a correlated equilibrium we need $\mu_1 = \underline{\mu}$. Otherwise it could not be better than $D_3^{\underline{x}}$, since $1 - \mu_1$ is part of the correlated payoff of Player 3 and $1 - \mu_1 \leq 1 - \underline{x}$.
  Additionally we need the following rank matrix:

$$\begin{pmatrix} x_1 & y_2 & 1 - x_1 - y_2 \\ y_1 & x_2 & 1 - y_1 - x_2 \\ \mu_1 & 0 & 1 - \mu_1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2^* & 2^* \\ 1 & 3 & 1 \\ 2^* & 1 & 3 \end{pmatrix} \tag{50}$$

  This implies the following three conditions for the existence of a correlated equilibrium:

  (a) $\mu_1 < \mu_2$
  (b) $a_{23} < a_{13}$
  (c) $1 - \mu_1 + a_{23} < 2a_{13}$

  Two cases are possible $x_1 = \underline{x} \vee x_2 = \underline{x}$

  (a) Suppose $x_1 = \underline{x} \Longrightarrow$

$$\begin{aligned} &1 - \mu_1 + a_{23} < 2a_{13} \Longleftrightarrow 3x_1 + 4y_2 < 3y_1 + 2x_2 \quad \text{and} \\ &2y_1 < x_1 \Longleftrightarrow 3x_1 + 4y_2 > 6y_1 + 4y_2 \overset{2y_2 \geq x_2}{\geq} 6y_1 + 2x_2 \geq 3y_1 + 2x_2 \quad ⚡ \end{aligned} \tag{51}$$

(b) Suppose $x_2 = \underline{x} \Longrightarrow \rho$ is no longer symmetric.

- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\frac{x}{3}}(i)) > \frac{3}{7} \tag{52}$$

- Examples

General example: $\begin{pmatrix} 0.55 & 0.45 & 0.00 \\ 0.27 & 0.72 & 0.01 \\ 0.55 & 0.00 & 0.45 \end{pmatrix}$

Minium payoff example: $\begin{pmatrix} \frac{4}{7}-\epsilon & \frac{3}{7}+\epsilon & 0 \\ \frac{2}{7}-11\epsilon & \frac{5}{7}-3\epsilon & 14\epsilon \\ \frac{4}{7}-\epsilon & 0 & \frac{3}{7}+\epsilon \end{pmatrix}$

(ii)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	μ_2	$\leq$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$\geq$	μ_2
5.[15]	a_{13}	$\leq$	a_{23}
6.	$1 - \mu_2 + a_{13}$	$\leq$	$2a_{23}$
7.	$\frac{1}{2}(1 - \mu_2 + a_{23})$	$\leq$	$1 - \underline{x}$

- Drawing lots versus $D_3^{\frac{x}{3}}$:
  If ($x_1 = \underline{x}$) constraint (4.) fails and if ($x_2 = \underline{x}$) the additionally required constraint $\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23}) > 1 - \underline{x}$ fails.
- Correlated equilibrium versus $D_3^{\frac{x}{3}}$:
  See constraint (7.)
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\frac{x}{3}}(ii)) = \frac{3}{7} \tag{53}$$

- Examples

General example: $\begin{pmatrix} 0.55 & 0.45 & 0.00 \\ 0.07 & 0.56 & 0.37 \\ 0.55 & 0.00 & 0.45 \end{pmatrix}$

Minium payoff example: $\begin{pmatrix} \frac{4}{7} & \frac{3}{7} & 0 \\ \frac{1}{14} & \frac{4}{7} & \frac{5}{14} \\ \frac{4}{7} & 0 & \frac{3}{7} \end{pmatrix}$

[15] Note that $a_{13} > a_{23}$ is not possible. Suppose $a_{13} > a_{23} \Longrightarrow$ condition (6.) converts to $1-\mu_2+a_{23} \leq 2a_{13} \Longrightarrow 4x_1 - 2y_1 + 3y_2 \leq 3x_2 \overset{\mu_2 \leq \mu_1 + 2y_2 - x_2}{\leq} x_1 + y_1 + 3y_2 \Longrightarrow 3x_1 \leq 2y_1$ $\lightning$

(iii)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	$\mu_1 + 2y_2 - x_2$	$\leq$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$\leq$	μ_2
5.[16]	a_{13}	$\leq$	a_{23}
6.	$1 - (\mu_1 + 2y_2 - x_2) + a_{13}$	$\leq$	$2a_{23}$
7.	$\frac{1}{2}(1 - \mu_2 + a_{23})$	$\leq$	$1 - \underline{x}$
8.	$1 - (2y_2 - x_2) + a_{13}$	$>$	$2a_{23}$
9.	$\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23})$	$\leq$	$1 - \underline{x}$

- Drawing lots versus $D_3^{\underline{x}}$:
  The same argumentation holds as for constraint set (i) of $D_3^{\mu_i}$.
- Correlated equilibrium versus $D_3^{\underline{x}}$:
  The same argumentation holds as for constraint sets (i) and (ii) of $D_3^{\underline{x}}$
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\underline{x}}(\text{iii})) = \frac{3}{8} \tag{54}$$

- Examples

General example: $\begin{pmatrix} 0.6 & 0.38 & 0.02 \\ 0.1 & 0.61 & 0.29 \\ 0.6 & 0.00 & 0.40 \end{pmatrix}$

Minium payoff example: $\begin{pmatrix} \frac{5}{8} & \frac{3}{8} & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \\ \frac{5}{8} & 0 & \frac{3}{8} \end{pmatrix}$

(iv)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$\geq$	x_1
3.	$\mu_1 + 2y_2 - x_2$	$\geq$	$\underline{x}$
4.	$\mu_2 + 2y_1 - x_1$	$\geq$	$\underline{x}$

- Drawing lots versus $D_3^{\underline{x}}$:
  $u_3^{max} \leq 1 - \underline{x}$ follows directly from constraints (3.) and (4.).
- Correlated equilibrium versus $D_3^{\underline{x}}$:
  $\pi_3^{max}(\text{Correlated equilibrium}) \leq 1 - \underline{x}$ follows directly from constraints (3.) and (4.).

[16] Note that $a_{23} < a_{13}$ is not possible. Suppose $a_{23} < a_{13} \Longrightarrow$ condition 6. converts to $1 - (\mu_1 + 2y_2 - x_2) + a_{23} \leq 2a_{13} \Longrightarrow x_1 < y_1$ ↯ .

- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\underline{x}}(\text{iv})) = \frac{3}{8} \tag{55}$$

- Examples

General example: $\begin{pmatrix} 0.60 & 0.40 & 0.0 \\ 0.35 & 0.65 & 0.0 \\ 0.60 & 0.00 & 0.4 \end{pmatrix}$

Minium payoff example: $\begin{pmatrix} \frac{5}{8} & \frac{3}{8} & 0 \\ \frac{3}{8} & \frac{5}{8} & 0 \\ \frac{5}{8} & 0 & \frac{3}{8} \end{pmatrix}$

5. Constraints on D_3^{Cor} being the best proposal for Player 3:

	(i)
1.	$2y_2 \geq x_2$
2.	$2y_1 < x_1$
3.	$\mu_2 \leq \underline{x}$
4.	$\mu_1 + 2y_2 - x_2 \geq \mu_2$
5.[17]	$a_{13} \leq a_{23}$
6.	$1 - \mu_2 + a_{13} \leq 2a_{23}$
7.	$\frac{1}{2}(1 - \mu_2 + a_{23}) < 1 - \underline{x}$
8.	$1 - (2y_2 - x_2) + a_{13} > 2a_{23}$
9.	$\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23}) \leq \frac{1}{2}(1 - \mu_2 + a_{23})$

- Drawing lots versus D_3^{cor}: See constraint (9.).
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{cor}(i)) > \frac{3}{8} \tag{56}$$

- Examples

General example: $\begin{pmatrix} 0.58 & 0.42 & 0.00 \\ 0.06 & 0.58 & 0.36 \\ 0.00 & 0.50 & 0.50 \end{pmatrix}$

Minium payoff example: $\begin{pmatrix} \frac{5}{8}-\epsilon & \frac{3}{8}+\epsilon & 0 \\ \frac{1}{8}-5\epsilon & \frac{5}{8}-\epsilon & \frac{1}{4}+6\epsilon \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

[17] See footnote 15.

(ii)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	$\mu_1 + 2y_2 - x_2$	$<$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$<$	μ_2
5.[18]	a_{13}	$\leq$	a_{23}
6.	$1 - (\mu_1 + 2y_2 - x_2) + a_{13}$	$\leq$	$2a_{23}$
7.	$\frac{1}{2}(1 - \mu_2 + a_{23})$	$>$	$1 - \underline{x}$
8.	$1 - (2y_2 - x_2) + a_{13}$	$>$	$2a_{23}$
9.	$\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23})$	$\leq$	$\frac{1}{2}(1 - \mu_2 + a_{23})$

- Drawing lots versus D_3^{cor}: See constraint (9.).
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{cor}(i)) > \frac{3}{8} \tag{57}$$

- Examples:

General example: $\begin{pmatrix} 0.61 & 0.39 & 0.00 \\ 0.04 & 0.65 & 0.31 \\ 0.00 & 0.52 & 0.48 \end{pmatrix}$

Minium payoff example: $\begin{pmatrix} \frac{5}{8}+\epsilon & \frac{3}{8}-\epsilon & 0 \\ \frac{1}{8}-7\epsilon & \frac{5}{8}+\epsilon & \frac{1}{4}+6\epsilon \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

6. D_3^T

Constraints on D_3^T being the best proposal for Player 3:

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	$\mu_1 + 2y_2 - x_2$	$\leq$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$\leq$	μ_2
5.[19]	a_{13}	$\leq$	a_{23}
6.	$1 - (\mu_1 + 2y_2 - x_2) + a_{13}$	$\leq$	$2a_{23}$
7.	$\frac{1}{2}(1 - \mu_2 + a_{23})$	$<$	$\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23})$
8.	$1 - (2y_2 - x_2) + a_{13}$	$>$	$2a_{23}$
9.	$\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23})$	$>$	$1 - \underline{x}$

- Drawing lots versus D_3^{cor}: See constraint (9.).

[18] See footnote 16.
[19] See footnote 16.

- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^L) > \frac{3}{8} \tag{58}$$

- Examples

General example: $\begin{pmatrix} 0.64 & 0.36 & 0.00 \\ 0.08 & 0.64 & 0.28 \\ 0.00 & 0.08+\epsilon & 0.92-\epsilon \end{pmatrix}$ $(\epsilon < 0.01)$

Minium payoff example: $\begin{pmatrix} \frac{5}{8}+\epsilon & \frac{3}{8}-\epsilon & 0 \\ \frac{1}{8}-3\epsilon & \frac{5}{8}+\epsilon & \frac{1}{4}+2\epsilon \\ 0 & \frac{1}{8}-3\epsilon+\epsilon' & \frac{7}{8}+3\epsilon-\epsilon' \end{pmatrix}$ $(\epsilon > 3\epsilon')$

Since the minimum payoff of $D_3^{\underline{x}}$ is given by $\pi_3^{min}(D_3^{\underline{x}}) = \frac{3}{8}$ and the constraints for D_3^{cor} and D_3^L directly imply that $\pi_3(D_3^{cor}) > 1 - \underline{x}$ and $\pi_3(D_3^L) > 1 - \underline{x}$, we obtain $\pi_3^{min}(D_3^{cor}) > \frac{3}{8}$ and $\pi_3^{min}(D_3^L) > \frac{3}{8}$.
Altogether we obtain

$$\min_{\alpha \in \{D_3^{\underline{\mu}}, D_3^{\overline{\mu}}, D_3^{\mu_i}, D_3^{\underline{x}}, D_3^{Cor}, D_3^L\}} \{\pi_3(\alpha)\} = \min_{\rho \text{ symmetric}} \{\pi_3\} = \frac{3}{8} \tag{59}$$

■

7.1.2 Proof of Corollary 1

Given $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the constraints on $D_3^{\underline{\mu}}$ hold when $\mu_1 = \mu_2 = \underline{\mu} = \frac{1}{2}$, which implies that $D_3 = (\frac{1}{2}, 0, \frac{1}{2})$ or $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ are the best reactions for Player 3 resulting in the payoffs $\pi_1 = \pi_2 = \frac{1}{4}$ and $\pi_3 = \frac{1}{2}$

■

By direct comparisons between the six possible proposals ($D_3^{\underline{\mu}}$, $D_3^{\overline{\mu}}$, $D_3^{\mu_i}$, $D_3^{\underline{x}}$, D_3^{Cor}, D_3^L) for Player 3, we also obtain the following corollary, which will be used later:

Corollary 5
Given symmetric $\rho \Longrightarrow \pi_2 > \underline{x}$ iff D_3^{cor} or D_3^L is the best reaction for Player 3.

7.1.3 Proof of Proposition 2

There are six possibilities for ρ to be non-symmetric:

$$\begin{aligned} \rho_1^{ns} &= \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad \rho_{1'}^{ns} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \\ \rho_3^{ns} &= \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \quad \rho_{3'}^{ns} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \quad \rho_{3''}^{ns} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \quad \rho_{3'''}^{ns} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \end{aligned} \tag{60}$$

In the following we only discuss ρ_1^{ns}, as the other cases arise by interchanging rows and observing that for ρ_3^{ns} nothing changes qualitatively. For ρ_1^{ns} A is given by

$$A = \begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{array}{l} a_{13} = 1 - y_1 - y_2 \\ a_{23} = 1 - x_1 - x_2 \\ a_{33} = 1 - a_{31} - a_{32} \end{array} \quad ; \tag{61}$$

First we observe that Player 3 has to offer at least Player i more than x_i to gain more than $\pi_3 = 1 - x_1 - x_2$, because otherwise D_2 contains two maximum shares and is a single-proposal equilibrium.[20]

As in the proof of Proposition 1, we look for the best offer D_3 and see that the following properties of a are relevant:

- $1 \geq a_{3i} > 2x_i - y_i \Longrightarrow \Phi(x_i) = 2, \Phi(y_i) = 1, \Phi(a_{3i}) = 3$
- $y_i > a_{3i} > 2y_i - x_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 2, \Phi(a_{3i}) = 1$

With these properties we can distinguish four different kinds of proposals D_3 discussed in detail below:

$$D_3^{\underline{x}^\epsilon} \qquad D_3^o \qquad D_3^3 \qquad D_3^{L^\epsilon}$$

1. $D_3^{\underline{x}^\epsilon}$

   Player 3 offers Player 1 or 2 $\underline{x} + \epsilon$ and keeps the rest for himself, such that D_3 is a single-proposal equilibrium and he receives a payoff $\pi_3 = 1 - \underline{x} - \epsilon$ (i.e):

$$\begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ \underline{x} + \epsilon & 0 & 1 - \underline{x} - \epsilon \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 2^* & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix} ; \tag{62}$$

$$\begin{array}{lcllll} D_1 & = & (0.2 & 0.4 & 0.4 &) \\ D_2 & = & (0.4 & 0.6 & 0.0 &) \\ D_3^{\underline{x}^\epsilon} & = & (0.4 + \epsilon & 0.0 & 0.6 - \epsilon &) \end{array}$$

2. D_3^o

   Player 3 offers Player i $a_{3i} = 2x_i - y_i + \epsilon$ and keeps the rest for himself, such that D_1 is a single-proposal equilibrium and he receives a payoff $\pi_3 = 1 - y_1 - y_2$, i.e.

$$\begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ 2x_1 - y_1 + \epsilon & 0 & 1 - (2x_1 - y_1 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2^* & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{pmatrix} \tag{63}$$

$$\begin{array}{lcllll} D_1 & = & (0.1 & 0.25 & 0.65 &) \\ D_2 & = & (0.5 & 0.40 & 0.10 &) \\ D_3^o & = & (0.9 + \epsilon & 0.00 & 0.1 - \epsilon &) \end{array}$$

[20] If Player 3 only offers Player 1 x_1, D_2 is still the only proposal with two maxima, or D_2 and D_3 are both voting equilibria but are not correlated.

3. D_3^3

   Player 3 offers Player i $2x_i - y_i + \epsilon$ ($i \in \{1,2\}$) and keeps the rest for himself, such that D_3 is a single-proposal equilibrium and he receives a payoff $\pi_3 = 1 - (2x_i - y_i + \epsilon)$, i.e.

$$\begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ 2x_1 - y_1 + \epsilon & 0 & 1 - (2x_1 - y_1 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2^* \end{pmatrix} \tag{64}$$

$$\begin{array}{lcl} D_1 & = & (0.1 \quad 0.25 \quad 0.65 \quad) \\ D_2 & = & (0.3 \quad 0.60 \quad 0.10 \quad) \\ D_3^3 & = & (0.5 + \epsilon \quad 0.00 \quad 0.5 - \epsilon \quad) \end{array}$$

4. $D_3^{T^\epsilon}$

   Player 3 offers Player i $a_{3i} = 2x_i - y_i + \epsilon$ and keeps the rest for himself, such that the proposal is chosen by drawing lots and he receives a payoff $\pi_3 = \frac{1}{3}(1 - (2x_1 - y_1 + \epsilon) + a_{13} + a_{23})$, i.e.:

$$\begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ 2x_1 - y_1 + \epsilon & 0 & 1 - (2x_1 - y_1 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \tag{65}$$

$$\begin{array}{lcl} D_1 & = & (0.1 \quad 0.1 \quad 0.8 \quad) \\ D_2 & = & (0.4 \quad 0.6 \quad 0.0 \quad) \\ D_3^{T^\epsilon} & = & (0.7 + \epsilon \quad 0.0 \quad 0.3 - \epsilon \quad) \end{array}$$

1. $D_3^{\underline{x}^\epsilon}$

   Constraints on $D_3^{\underline{x}^\epsilon}$ being the best proposal for Player 3:

1.	$y_1 + y_2 \quad > \quad \underline{x}$

   Since Player 3 needs to offer at least Player i $a_{3i} = \underline{x} + \epsilon$ to prevent D_2 from being a single-proposal equilibrium and $1 - \underline{x} + \epsilon > a_{i3}$ $i = 1, 2$, $D_3^{\underline{x}^\epsilon}$ is better than all other proposals.

   - Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{\underline{x}^\epsilon}) > \tfrac{1}{2} \tag{66}$$

   - General example: $\begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.0 \\ 0.4 + \epsilon & 0.0 & 0.6 - \epsilon \end{pmatrix}$

     Minium payoff example: $\begin{pmatrix} \frac{1}{4} - 3\epsilon & \frac{1}{4} + \epsilon & \frac{1}{2} + 2\epsilon \\ \frac{1}{2} - 4\epsilon & \frac{1}{2} + 4\epsilon & 0 \\ \frac{1}{2} - 4\epsilon + \epsilon' & 0 & \frac{1}{2} + 4\epsilon - \epsilon' \end{pmatrix} \epsilon' < \epsilon$

2. D_3^o
   Constraints on D_3^o being the best proposal for Player 3:

1.	$y_1 + y_2$	$\leq$	$\underline{x}$	
2.	$2y_i$	$\geq$	x_i	$\}\ i,j = 1,2\ i \neq j$
3.	$2x_j - y_j$	$<$	1	

   Note that $r_{23} = 2^*$ is not possible, as we have

$$2x_j > y_j + \epsilon \Longleftrightarrow \underbrace{1-(2x_j - y_j + \epsilon)}_{a_{33}(D_3^o)} + \underbrace{1-(y_1+y_2)}_{a_{13}} > \underbrace{2(1-(x_1+x_2))}_{23}$$

   and if $a_{33}(D_3^o) = 1-(2x_1-y_1+\epsilon) > a_{23}$, Player 3 can rise a_{j3} so that $1-a_{3j} = a_{23}$. Since $\pi_3 = 1 - y_1 - y_2$ is only ϵ worse than the payoff of $D_3^{\underline{x}^\epsilon}$, D_3^o is the best offer.

   - Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^o) = \tfrac{1}{2} \tag{67}$$

   - General example:
$$\begin{pmatrix} 0.1 & 0.25 & 0.65 \\ 0.5 & 0.40 & 0.10 \\ 0.9+\epsilon & 0.0 & 0.1-\epsilon \end{pmatrix}$$

     Minium payoff example:
$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{4}+\epsilon & \frac{1}{4}-\epsilon \end{pmatrix}$$

3. D_3^3
   Constraints for D_3^3 being the best proposal for Player 3:

   (i)

1.	$y_1 + y_2$	$\leq$	$\underline{x}$	
2.	$2y_1$	$<$	x_1	
3.	$2y_2$	$<$	x_2	
4.	$2x_i - y_i$	$<$	1	$\}\ (i = 1 \vee i = 2)$
5.	$1-(2x_i - y_i + \epsilon)$	$\geq$	$\frac{1}{2}(a_{13}+a_{23})$[21]	

   - Drawing lots versus D_3^3:
     With drawing lots, Player 3 cannot gain more than $\pi_3 = \frac{1}{2}(a_{13}+a_{23}) - \epsilon' \leq \pi_3(D_3^3(i))$ because of constraint (5.).

[21] If $2x_i - y_i < 1$ and $1-(2x_i - y_i + \epsilon) \geq \frac{1}{2}(a_{13}+a_{23})$ holds simultaneously for $i = 1,2$, Player 3 chooses $\min_{i=1,2}\{2x_i - y_i\}$.

- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^3(i)) > \tfrac{1}{4} \tag{68}$$

- General example:
$$\begin{pmatrix} 0.1 & 0.25 & 0.65 \\ 0.3 & 0.60 & 0.10 \\ 0.5+\epsilon & 0.0 & 0.5-\epsilon \end{pmatrix}$$

  Minium payoff example:
$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4}-\epsilon & \frac{1}{2}+\epsilon \\ \frac{1}{2}+\epsilon & \frac{1}{2}-\epsilon & 0 \\ 0 & \frac{3}{4}-\epsilon+\epsilon' & \frac{1}{4}+\epsilon-\epsilon' \end{pmatrix}$$

(ii)

1.	$y_1 + y_2$	$\leq$	$\underline{x}$	
2.	$2y_i$	$<$	x_i	$i, j = 1, 2 \; i \neq j$
3.	$2y_j$	$\geq$	x_j	
4.	$2x_j - y_j$	$<$	1	
5.	$2x_i - y_i$	$\geq$	1	
6.	$1 - (2x_j - y_j + \epsilon)$	$\geq$	$\frac{1}{2}(a_{13} + a_{23})$	

- Drawing lots versus D_3^3:
  With drawing lots, Player 3 cannot gain more than $\pi_3 = \frac{1}{2}(a_{13} + a_{23}) - \epsilon' \leq \pi_3(D_3^3(ii))$ because of constraint (6.).
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^3(ii)) > \tfrac{1}{3} \tag{69}$$

- General example:
$$\begin{pmatrix} 0.00 & 0.40 & 0.60 \\ 0.55 & 0.45 & 0.00 \\ 0.00 & 0.5+\epsilon & 0.5-\epsilon \end{pmatrix}$$

  Minium payoff example:
$$\begin{pmatrix} 0 & \frac{1}{3}+\epsilon & \frac{2}{3}-\epsilon \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{2}{3}-\epsilon+\epsilon' & \frac{1}{3}+\epsilon-\epsilon' \end{pmatrix}$$

4. $D_3^{T\epsilon}$

   Constraints on $D_3^{T\epsilon}$ being the best proposal for Player 3:

(i)

1.	$y_1 + y_2$	$\leq$	$\underline{x}$
2.	$2y_1$	$<$	x_1
3.	$2y_2$	$<$	x_2
4.	$2x_i - y_i$	$<$	1
5.	$1 - (2x_i - y_i + \epsilon)$	$<$	$\frac{1}{2}(a_{13} + a_{23})$
6.	$1 - (2x_j - y_j + \epsilon)$	$<$	$\frac{1}{2}(a_{13} + a_{23})$
7.	$1 - (2x_j - y_j)$	$\leq$	$1 - (2x_i - y_i)$

$\Big\} \; i, j = 1, 2 \; i \neq j$

- Since all other kinds of proposal are excluded, $D_3^{L^\epsilon}$ is the best proposal.
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{T^\epsilon}(i)) > \tfrac{1}{4} \tag{70}$$

- General example:
$$\begin{pmatrix} 0.1 & 0.10 & 0.80 \\ 0.4 & 0.6 & 0.00 \\ 0.7 + \epsilon & 0.0 & 0.3 - \epsilon \end{pmatrix}$$

  Minium payoff example:
$$\begin{pmatrix} \frac{1}{4} - \epsilon & \frac{1}{4} - \epsilon & \frac{1}{2} + 2\epsilon \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{4} - \epsilon + \epsilon' & \frac{1}{4} + \epsilon - \epsilon' \end{pmatrix}$$

(ii)

1.	$y_1 + y_2$	$\leq$	$\underline{x}$
2.	$2y_i$	$<$	x_i
3.	$2y_j$	$\geq$	x_j
4.	$2x_j - y_j$	$<$	1
5.	$2x_i - y_i$	$\geq$	1
6.	$1 - (2x_j - y_j + \epsilon)$	$<$	$\frac{1}{2}(a_{13} + a_{23})$

$\Bigg\} \; i, j = 1, 2 \; i \neq j$

- Since all other kinds of proposal are excluded, $D_3^{L^\epsilon}$ is the best proposal.
- Simplex minimization for Player 3:

$$\pi_3^{min}(D_3^{T^\epsilon}(ii)) \geq \tfrac{1}{3} - \epsilon \tag{71}$$

- General example:
$$\begin{pmatrix} 0.00 & 0.30 & 0.80 \\ 0.51 & 0.49 & 0.00 \\ 0.00 & 0.68 + \epsilon & 0.32 - \epsilon \end{pmatrix}$$

  Minium payoff example:
$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{2}{3} + \epsilon & \frac{1}{3} - \epsilon \end{pmatrix}$$

Calculating the feasible set of $(D_3^{\underline{x}^\epsilon} \cup D_3^o \cup D_3^3 \cup D_3^{T^\epsilon})$, we find that

1.	$y_1 + y_2$	$\leq$	$\underline{x}$
2.	$2y_1$	$<$	x_2
3.	$2y_2$	$<$	x_2
4.	$2x_1 - y_1$	$\geq$	1
5.	$2x_2 - y_2$	$\geq$	1

is missing. The only combination of $(D_1\,, D_2)$ to fulfill this condition is given by
$A = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
Note that this is also submatrix a, excluded in Proposition 2.

Altogether we obtain for $a \neq \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \wedge a \neq \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$

$$\min_{\alpha \in \{D_3^{\underline{x}^\epsilon}, D_3^o, D_3^3, D_3^{T^\epsilon}\}} \{\pi_3(\alpha)\} = \min_{\rho \text{ non-symmetric}} \{\pi_3\} > \frac{1}{4} \tag{72}$$

■

7.1.4 Proof of Corollary 2

W.l.o.g we assume $a = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \implies$ Player 3 must offer Player i at least $a_{3i} = \mu_i$ and Player j at least $a_{3j} = x_j + \epsilon$ $(i, j = 1, 2\ i \neq j)$ to prevent D_2 from being a single-proposal equilibrium with $\pi_3 = 0$. $\implies D_3 = (\frac{1}{4}, \frac{1}{2} + \epsilon, \frac{1}{4} - \epsilon)$ or $D_3 = (\frac{1}{2} + \epsilon, \frac{1}{4}, \frac{1}{4} - \epsilon)$ with the correlated equilibrium C_{23} is the best reaction for Player 3, resulting in the payoffs $\pi_1 = \pi_2 = \frac{7}{16} + \frac{\epsilon}{4}$ and $\pi_3 = \frac{1}{8} - \frac{\epsilon}{2}$ (given D_1 and D_2, this is also the worst situation for Player 3).

■

7.2 Proof of Propositions 3, 4, and 5, and of Corollaries 3 and 4

We prove these propositions and corollaries by constructing D_2 in such a way that we satisfy different constraint sets in the proofs of Propositions 1 and 2. But since we have only shown Propositions 1 and 2 for representative matrices a, we sometimes have to interchange indices in the constraint sets to adapt them for the following proofs.

7.2.1 Proof of Proposition 3

1. Suppose that $D_1 \in \mathcal{A} \Longrightarrow$ Player 2's payoff will be at least $\pi_2 > \frac{3}{8}$ and $\rho = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, if he proposes $D_2 = (\frac{1}{2}, \frac{1}{2}, 0)$, $D_2 = (a_{12}, a_{11} - \epsilon, a_{13} + \epsilon)$, or $D_2 = (\frac{1}{2} - \epsilon, \frac{1}{2} - \epsilon, 2\epsilon)$.

$\mathcal{A}_1$) Suppose $D_2 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ is the best reaction for Player 3 and a single-proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & 2 \\ 2^* & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix}; \begin{array}{ll} D_1 & = (0.6 \quad 0.1 \quad 0.3) \\ D_2^{\mathcal{A}_1} & = (0.5 \quad 0.5 \quad 0.0) \\ D_3 & = (0.0 \quad 0.5 \quad 0.5) \end{array} \tag{73}$$

This can be satisfied if constraint set (i) of $D_3^{\underline{x}}$ holds. The feasible set $\mathcal{S}_{\mathcal{A}_1}$ is then determined by[22]

$$\begin{array}{lrcccl} 1. & 2a_{21} & \geq & a_{11} & \Longrightarrow & a_{11} \leq 1 \\ 2. & 2a_{12} & < & a_{22} & \Longrightarrow & a_{12} < \frac{1}{4} \\ \mathbf{3}. & \mu_1 & > & \underline{x} & \Longrightarrow & a_{11} > \frac{1}{2} \\ \mathbf{4}. & \mu_2 + 2y_1 - x_1 & \geq & \underline{x} & \Longrightarrow & a_{12} \geq 2a_{11} - \frac{3}{2} \end{array}$$

After comparison of the inequalities, only 3. and 4. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{A}_1} \quad = \quad \left\{ \left(\tfrac{1}{2}, \tfrac{3}{4}\right] \times \left[0, \tfrac{1}{4}\right] \cup \left[\tfrac{3}{4}, \tfrac{5}{6}\right] \times \left[2a_{11} - \tfrac{3}{2}, 1 - a_{11}\right] \right\} \tag{74}$$

(See Figure 1)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{A}_1) = \frac{1}{2} \tag{75}$$

$\mathcal{A}_2$) See $\mathcal{A}_1$, but now with constraint set (iv) of $D_3^{\underline{x}}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & 2 \\ 2^* & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix}; \begin{array}{ll} D_1 & = (0.55 \quad 0.4 \quad 0.05) \\ D_2^{\mathcal{A}_2} & = (0.50 \quad 0.5 \quad 0.00) \\ D_3 & = (0.00 \quad 0.5 \quad 0.50) \end{array} \tag{76}$$

This requires

$$\begin{array}{lrcccl} 1. & 2a_{21} & \geq & a_{11} & \Longrightarrow & a_{11} \leq 1 \\ \mathbf{2}. & 2a_{12} & \geq & a_{22} & \Longrightarrow & a_{12} \geq \frac{1}{4} \\ 3. & \mu_2 + 2a_{21} - a_{11} & \geq & \underline{x} & \Longrightarrow & a_{12} \geq 2a_{11} - \frac{3}{2} \\ 4. & \mu_1 + 2a_{12} - a_{22} & \geq & \underline{x} & \Longrightarrow & a_{12} \geq \frac{3}{8} - \frac{1}{4}a_{11} \\ \mathbf{5}. & a_{11} & > & \underline{x} & \Longrightarrow & a_{11} \geq \frac{1}{2} \end{array}$$

[22] The bold numbers will denote those constraints which are binding.

After comparison of the inequalities, only 2. and 5. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{A}_2} \quad = \quad \left\{ \left(\tfrac{1}{2}, \tfrac{3}{4}\right] \times \left[\tfrac{1}{4}, 1 - a_{11}\right] \right\} \tag{77}$$

(See Figure 1)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{A}_2) = \frac{1}{2} \tag{78}$$

$\mathcal{A}_3$) Suppose $D_2 = (a_{12}, a_{11} - \epsilon, a_{13} + \epsilon)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is a single-proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{11} - \epsilon & a_{13} + \epsilon \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 2^* & 3 \end{pmatrix}; \tag{79}$$

$$\begin{array}{ll} D_1 & = (0.90 \quad 0.05 \quad 0.05) \\ D_2^{\mathcal{A}_3} & = (0.05 \quad 0.9 - \epsilon \quad 0.05 + \epsilon) \\ D_3 & = (0.00 \quad 0.45 - \frac{\epsilon}{2} \quad 0.55 + \frac{\epsilon}{2}) \end{array}$$

This can be satisfied if the constraint set of D_3^{μ} holds (note that $\mu_2 = \underline{\mu}$). The feasible set $\mathcal{S}_{\mathcal{A}_3}$ is then determined by

$$\begin{array}{lrcllcl} \mathbf{1}. & 2a_{21} & < & a_{11} & \Longrightarrow & a_{11} & > & \frac{3}{4} \\ 2. & a_{23} & \geq & a_{13} & \Longrightarrow & \epsilon & > & 0 \\ 3. & 1 - \mu_2 + a_{13} & > & 2a_{23} & \Longrightarrow & a_{12} & \geq & \epsilon' - a_{11} \\ \mathbf{4}. & \mu_2 & > & \frac{3}{8} & \Longrightarrow & a_{12} & > & \frac{3}{4} - a_{11} \end{array}$$

After comparison of the inequalities, constraints 1. and 4. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{A}_3} \quad = \quad \left\{ \left(\tfrac{3}{4}, 1\right] \times [0, 1 - a_{11}] \right\} \tag{80}$$

(See Figure 1)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{A}_3) > \frac{3}{8} \tag{81}$$

$\mathcal{A}_4$) To obtain complete cover for $\mathcal{A}$ there remains the line given by $a_{11} = \frac{1}{2}$. Suppose $D_2 = (\frac{1}{2} - \epsilon, \frac{1}{2} - \epsilon, 2\epsilon)$ ($\epsilon < \frac{1}{2} - a_{12}$) and $D_3 = (0, \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$ is the best reaction for Player 3 and a single-proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} \frac{1}{2} & a_{12} & a_{13} \\ \frac{1}{2} - \epsilon & \frac{1}{2} - \epsilon & 2\epsilon \\ 0 & \frac{1}{2} - \epsilon & \frac{1}{2} + \epsilon \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 3 & 1 & 2 \\ 2^* & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix} \tag{82}$$

$$\begin{array}{lcl} D_1 & = & (0.5 \quad\quad\quad 0.4 \quad\quad 0.1) \\ D_2^{\mathcal{A}_4} & = & (0.5-\epsilon \quad 0.5-\epsilon \quad\quad 2\epsilon) \\ D_3 & = & (0.00 \quad\quad 0.5-\epsilon \quad 0.5+\epsilon) \end{array}$$

Taking $a_{11} = \frac{1}{2}$, $a_{21} = a_{22} = \frac{1}{2} - \epsilon$, we see that either of the constraints on $\mathcal{A}_1$ or $\mathcal{A}_2$ hold and thus we obtain

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{A}_4} \quad = \quad \left\{\left[\tfrac{1}{2}, \tfrac{1}{2}\right] \times \left[0, \tfrac{1}{2}\right]\right\} \tag{83}$$

(See Figure 1)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{A}_4) = \frac{1}{2} - \epsilon \tag{84}$$

2. Suppose $D_1 \in \mathcal{B} \Longrightarrow$ Player 2's payoff will be at least $\pi_2 > \frac{3}{8}$ and $\rho = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, if he proposes $D_2 = (a_{12} + \epsilon, 1 - (a_{12} + \epsilon), 0)$, $D_2 = (\frac{3}{4} - a_{11} + 3\epsilon, \frac{3}{4} - a_{12} + 2\epsilon, a_{11} + a_{12} - \frac{1}{2} - 5\epsilon)$, $D_2 = (a_{12} + \epsilon, 1 - (a_{12} + \epsilon), 0)$, $D_2 = (a_{12} + \frac{1}{4} - 2\epsilon, \frac{3}{4} - a_{12} + 2\epsilon, 0)$, $D_2 = (\frac{5}{8} + 4\epsilon, \frac{3}{8} - 4\epsilon, 0)$, $D_2 = (a_{12} + 4\epsilon, 1 - (a_{12} + 4\epsilon), 0)$ or $D_2 = (1, 0, 0)$.

$\mathcal{B}_1$) Suppose $D_2 = (a_{12} + \epsilon, 1 - (a_{12} + \epsilon), 0)$ and $D_3 = (0, a_{12}, 1 - a_{12})$ is the best reaction for Player 3 and a single-proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} + \epsilon & 1 - (a_{12} + \epsilon) & 0 \\ 0 & a_{12} & 1 - a_{12} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 1 \\ 1 & 3 & 3 \end{pmatrix} \tag{85}$$

$$\begin{array}{lcl} D_1 & = & (0.1 \quad\quad\quad 0.55 \quad\quad 0.35) \\ D_2^{\mathcal{B}_1} & = & (0.55+\epsilon \quad 0.45-\epsilon \quad\quad 0) \\ D_3 & = & (0.00 \quad\quad\quad 0.55 \quad\quad 0.45) \end{array}$$

This can be satisfied if the constraint set (ii) of $D_3^{\underline{x}}$ holds (by construction we have $\underline{x} = a_{12}$). The feasible set $\mathcal{S}_{\mathcal{B}_1}$ is then determined by

$$\begin{array}{rrcl ccl} 1. & 2a_{22} & \geq & a_{12} & \Longrightarrow & a_{12} & < & \frac{2}{3} \\ 2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & \geq & 2a_{11} \\ \mathbf{3}. & \mu_2 & < & \underline{x} & \Longrightarrow & a_{12} & \geq & \frac{1}{2} \\ 4. & \mu_1 + 2a_{22} - a_{12} & \geq & \mu_2 & \Longrightarrow & a_{12} & < & \frac{3}{5} + \frac{a_{11}}{5} \\ 5. & a_{23} & < & a_{13} & \Longrightarrow & a_{12} & < & 1 - a_{11} \\ \mathbf{6}. & 1 - \mu_2 + a_{23} & \leq & 2a_{13} & \Longrightarrow & a_{12} & < & \frac{3}{4} - a_{11} \\ \mathbf{7}. & \frac{1}{2}(1 - \mu_2 + a_{13}) & \leq & 1 - \underline{x} & \Longrightarrow & a_{12} & < & \frac{1}{2} + a_{11} \end{array}$$

After comparison of the inequalities, only 3., 6., and 7. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_1} \quad = \quad \left\{\left[0, \tfrac{1}{8}\right] \times \left[\tfrac{1}{2}, \tfrac{1}{2} + a_{11}\right) \cup \left[\tfrac{1}{8}, \tfrac{1}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{4} - a_{11}\right)\right\} \tag{86}$$

(See Figure 2)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{B}_1) = \frac{1}{2} \tag{87}$$

$\mathcal{B}_2$) Suppose $D_2 = (\frac{3}{4} - a_{11} + 3\epsilon, \frac{3}{4} - a_{12} + 2\epsilon, a_{11} + a_{12} - \frac{1}{2} - 5\epsilon)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is the best reaction for Player 3 and a single-proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{3}{4} - a_{11} + 3\epsilon & \frac{3}{4} - a_{12} + 2\epsilon & 1 - a_{21} - a_{22} \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{88}$$

$$\begin{array}{lcllll} D_1 & = & (0.2 & 0.55 & 0.25) \\ D_2^{\mathcal{B}_2} & = & (0.55 + 3\epsilon & 0.2 + 2\epsilon & 0) \\ D_3 & = & (0.00 & 0.375 + \epsilon & 0.625 - \epsilon) \end{array}$$

This can be satisfied if the constraints of D_3^{μ} hold. The feasible set $\mathcal{S}_{\mathcal{B}_2}$ is then determined by

$$\begin{array}{lrclcrcl} 1. & \mu_2 & < & \mu_1 & \Longrightarrow & \epsilon & > & 0 \\ 2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{11} & \leq & \frac{1}{4} \\ \mathbf{3}. & a_{23} & \leq & a_{13} & \Longrightarrow & a_{12} & \leq & \frac{3}{4} - a_{11} \\ \mathbf{4}. & 1 - \mu_2 + a_{23} & > & 2a_{13} & \Longrightarrow & a_{12} & > & \frac{5}{8} - a_{11} \end{array}$$

After comparison of the inequalities, only 3. and 4. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_2} \quad = \quad \left\{ \left[0, \tfrac{1}{8}\right] \times \left(\tfrac{5}{8} - a_{11}, \tfrac{3}{4} - a_{11}\right] \cup \left(\tfrac{1}{8}, \tfrac{1}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{4} - a_{11}\right] \right\} \tag{89}$$

(See Figure 2)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{B}_2) > \frac{3}{8} \tag{90}$$

$\mathcal{B}_3$) Suppose $D_2 = (a_{12} + \epsilon, 1 - (a_{12} + \epsilon), 0)$ and $D_3 = (0, a_{12}, 1 - a_{12})$ is the best reaction for Player 3 and a single-proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} + \epsilon & 1 - (a_{12} + \epsilon) & 0 \\ 0 & a_{12} & 1 - a_{12} \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 1 \\ 1 & 3 & 3 \end{pmatrix} \tag{91}$$

$$\begin{array}{lcllll} D_1 & = & (0.35 & 0.55 & 0.10) \\ D_2^{\mathcal{B}_2} & = & (0.55 + \epsilon & 0.45 - \epsilon & 0.00) \\ D_3 & = & (0.00 & 0.55 & 0.45) \end{array}$$

This can be satisfied if the constraint set (iv) of D_3^x holds. The feasible set $\mathcal{S}_{\mathcal{B}_3}$ is then determined by

$$\begin{array}{llcllcl}
1. & 2a_{22} & \geq & a_{12} & \Longrightarrow & a_{12} & < & \frac{2}{3} \\
2. & 2a_{11} & \geq & a_{21} & \Longrightarrow & a_{12} & < & 2a_{11} \\
3. & \mu_1 + 2a_{22} - a_{12} & \geq & a_{12} & \Longrightarrow & a_{12} & < & \frac{1}{7}a_{11} + \frac{4}{7} \\
\mathbf{4}. & \mu_2 + 2a_{11} - a_{21} & \geq & a_{12} & \Longrightarrow & a_{12} & < & \frac{1}{4} + a_{11}
\end{array}$$

After comparison of the inequalities only 4. remains.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_3} \quad = \quad \left\{\left[\tfrac{1}{4}, \tfrac{3}{8}\right] \times \left[\tfrac{1}{2}, \tfrac{1}{4} + a_{11}\right) \cup \left(\tfrac{3}{8}, \tfrac{1}{2}\right] \times \left[\tfrac{1}{2}, 1 - a_{11}\right]\right\} \tag{92}$$

(See Figure 2)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{B}_3) = \frac{1}{2} \tag{93}$$

$\mathcal{B}_4$) Suppose $D_2 = (a_{12} - \frac{1}{4} - 2\epsilon, \frac{3}{4} - a_{12} + 2\epsilon, 0)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is the best reaction for Player 3 and a single-proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} + \frac{1}{4} - 2\epsilon & \frac{3}{4} - a_{12} + 2\epsilon & 0 \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{94}$$

$$\begin{array}{lcllll}
D_1 & = & (0.2 & 0.7 & 0.1) \\
D_2^{\mathcal{B}_2} & = & (0.95 - 2\epsilon & 0.05 - 2\epsilon & 0.0) \\
D_3 & = & (0.00 & 0.375 + \epsilon & 0.625 - \epsilon)
\end{array}$$

This can be satisfied if the constraints of D_3^μ hold (a_{22} has to be non-negative). The feasible set $\mathcal{S}_{\mathcal{B}_4}$ is then determined by

$$\begin{array}{llcllcl}
1. & \mu_2 & < & \mu_1 & \Longrightarrow & a_{12} & > & \frac{1}{2} - a_{11} \\
\mathbf{2}. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & > & 2a_{11} - \frac{1}{4} \\
3. & a_{23} & \leq & a_{13} & \Longrightarrow & a_{12} & \leq & 1 - a_{11} \\
\mathbf{4}. & 1 - \mu_2 + a_{23} & > & 2a_{13} & \Longrightarrow & a_{12} & > & \frac{11}{16} - a_{11} \\
\mathbf{5}. & a_{22} & \geq & 0 & \Longrightarrow & a_{12} & \leq & \frac{3}{4}
\end{array}$$

After comparison of the inequalities, only 2., 4. and 5. remain.

$$\begin{aligned}
\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_4} \quad = \quad & \left\{\left[0, \tfrac{3}{16}\right] \times \left(\tfrac{11}{16} - a_{11}, \tfrac{3}{4}\right] \cup \left(\tfrac{3}{16}, \tfrac{1}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{4}\right] \cup \right. \\
& \left. \left(\tfrac{1}{4}, \tfrac{3}{8}\right] \times \left[\tfrac{1}{2}, 1 - a_{11}\right] \cup \left[\tfrac{3}{8}, \tfrac{5}{12}\right] \times \left(2a_{11} - \tfrac{1}{4}, 1 - a_{11}\right]\right\}
\end{aligned} \tag{95}$$

(See Figure 2)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{B}_4) > \frac{3}{8} \tag{96}$$

$\mathcal{B}_5$) Suppose $D_2 = (\frac{5}{8} + 4\epsilon, \frac{3}{8} - 4\epsilon, 0)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is the best reaction for Player 3 and implies the correlated equilibrium $C_{13} \Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{5}{8} + 4\epsilon & \frac{3}{8} - 4\epsilon & 0 \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 3 & 2^* \\ 3 & 1 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{97}$$

$$\begin{array}{lcllll} D_1 & = & (0.05 & 0.61 & 0.34) \\ D_2^{\mathcal{B}_2} & = & (0.625 - 4\epsilon & 0.375 - 4\epsilon & 0.0) \\ D_3 & = & (0.00 & 0.4925 - 2\epsilon & 0.5075 + 2\epsilon) \end{array}$$

This can be satisfied if the constraint set (ii) of D_3^{cor} holds ($a_{12} = \underline{x}$). The feasible set $\mathcal{S}_{\mathcal{B}_5}$ is then determined by the following and the payoff is greater than $\frac{3}{8}$

$$\begin{array}{lrclcrcl}
1. & 2a_{22} & \geq & a_{12} & \Longrightarrow & a_{12} & < & \frac{3}{4} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{11} & \leq & \frac{5}{16} \\
3. & \mu_1 + 2a_{22} - a_{12} & \leq & a_{12} & \Longrightarrow & a_{12} & \geq & \frac{17}{32} + \frac{1}{4}a_{11} \\
\mathbf{4.} & \mu_1 + 2a_{22} - a_{12} & \leq & \mu_2 & \Longrightarrow & a_{12} & \geq & \frac{7}{12} + \frac{1}{3}a_{11} \\
5. & a_{23} & \leq & a_{13} & \Longrightarrow & a_{12} & \leq & 1 - a_{11} \\
6. & 1 - (\mu_1 + 2a_{22} - a_{12}) + a_{23} & \leq & 2a_{13} & \Longrightarrow & a_{12} & < & \frac{11}{16} - \frac{1}{2}a_{11} \\
7. & \frac{1}{2}(1 - \mu_2 + a_{13}) & \leq & 1 - a_{12} & \Longrightarrow & a_{12} & \geq & \frac{3}{8} + 2a_{11} \\
8. & 1 - (2a_{22} - a_{12}) + a_{23} & > & 2a_{13} & \Longrightarrow & a_{12} & \geq & \frac{7}{12} - \frac{2}{3}a_{11} \\
9. & \frac{1}{3}(1 - (2a_{22} - a_{12}) + a_{23} + a_{13}) & < & \frac{1}{2}(1 - \mu_2 + a_{13}) & \Longrightarrow & a_{12} & < & \frac{47}{72} - \frac{2}{9}a_{11} \\
\mathbf{10.} & a_{12} & < & a_{21} & \Longrightarrow & a_{12} & \leq & \frac{5}{8}
\end{array}$$

After comparison of the inequalities only 4. and 10. are remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_5} \quad = \quad \left\{ \left[0, \tfrac{1}{8}\right] \times \left(\tfrac{7}{12} + \tfrac{1}{3}a_{11}, \tfrac{5}{8}\right) \right\} \tag{98}$$

(See Figure 2)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{B}_5) \geq \frac{51}{96} - \epsilon \tag{99}$$

$\mathcal{B}_6$) Suppose $D_2 = (a_{12} + 4\epsilon, 1 - (a_{12} + 4\epsilon), 0)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is the best reaction for Player 3 and implies the correlated equilibrium $C_{13} \Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} + 4\epsilon & 1 - (a_{12} + 4\epsilon) & 0 \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 3 & 2^* \\ 3 & 1 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{100}$$

$$\begin{array}{lcllll} D_1 & = & (0.01 & 0.52 & 0.47) \\ D_2^{\mathcal{B}_2} & = & (0.52 + 4\epsilon & 0.48 - 4\epsilon & 0.0) \\ D_3 & = & (0.00 & 0.5 - 2\epsilon & 0.5 + 2\epsilon) \end{array}$$

This can be satisfied if the constraint set (i) of D_3^{cor} holds (by construction we have $\underline{x} = a_{12}$). The feasible set $\mathcal{S}_{\mathcal{B}_6}$ is then determined by

$$\begin{array}{lrcl}
1. & 2a_{22} & \geq & a_{12} & \Longrightarrow & a_{12} & < & \frac{2}{3} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & \geq & 2a_{11} \\
3. & \mu_2 & < & \underline{x} & \Longrightarrow & a_{12} & \geq & \frac{1}{2} \\
\mathbf{4.} & \mu_1 + 2a_{22} - a_{12} & \geq & \mu_2 & \Longrightarrow & a_{12} & < & \frac{3}{5} + \frac{a_{11}}{5} \\
5. & a_{23} & < & a_{13} & \Longrightarrow & a_{12} & < & 1 - a_{11} \\
6. & 1 - \mu_2 + a_{23} & \leq & 2a_{13} & \Longrightarrow & a_{12} & < & \frac{3}{4} - a_{11} \\
\mathbf{7.} & \frac{1}{2}(1 - \mu_2 + a_{13}) & > & 1 - \underline{x} & \Longrightarrow & a_{12} & > & \frac{1}{2} + a_{11}
\end{array}$$

After comparison of the inequalities, only 4. and 7. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_6} \quad = \quad \left\{ \left[0, \tfrac{1}{8}\right] \times \left[\tfrac{1}{2} + a_{11}, \tfrac{3}{5} + \tfrac{1}{5}a_{11}\right) \right\} \tag{101}$$

(See Figure 2)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{B}_6) \geq \frac{1}{2} - \epsilon \tag{102}$$

$\mathcal{B}_7$) Suppose $D_2 = (1, 0, 0)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is the best reaction fir Player 3 and a single-proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{103}$$

$$\begin{array}{lcl}
D_1 & = & (0.01 \;\; 0.86 \;\; 0.03) \\
D_2^{\mathcal{B}_2} & = & (1.00 \;\; 0.00 \;\; 0.00) \\
D_3 & = & (0.00 \;\; 0.43 \;\; 0.57)
\end{array}$$

This can be satisfied if the constraints of D_3^{μ} hold. The feasible set $\mathcal{S}_{\mathcal{B}_7}$ is then determined by

$$\begin{array}{lrcl}
1. & \mu_2 & < & \mu_1 & \Longrightarrow & a_{12} & < & 1 + a_{11} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{11} & < & \frac{1}{2} \\
3. & a_{23} & \leq & a_{13} & \Longrightarrow & a_{12} & \leq & 1 - a_{11} \\
4. & 1 - \mu_2 + a_{23} & > & 2a_{13} & \Longrightarrow & a_{12} & > & \frac{2}{3} - \frac{4}{3}a_{11} \\
\mathbf{5.} & \mu_2 & > & \frac{3}{8} & \Longrightarrow & a_{12} & > & \frac{3}{4}
\end{array}$$

After comparison of the inequalities only 5. remains.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_7} \quad = \quad \left\{ \left[0, \tfrac{1}{4}\right] \times \left(\tfrac{3}{4}, 1 - a_{11}\right] \right\} \tag{104}$$

(See Figure 2)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{B}_7) > \frac{3}{8} \tag{105}$$

Thus we have shown that $\forall\ D_1 \in \mathcal{A} \cup \mathcal{B}\ \exists$ a proposal D_2^s of Player 2, such that ρ is symmetric and $\pi_2 > \frac{3}{8}$.

■

7.2.2 Proof of Corollary 3

1. Suppose $D_1 \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5$ and D_2 is such that ρ is non-symmetric and the best reaction for Player 2, then $\pi_2 \geq \frac{1}{2}$, because if D_2 is such that ρ is symmetric Player 2 obtains at least $\pi_2 = \frac{1}{2}$ (see proof of Proposition 3). Further, we have $\pi_3 > \frac{1}{4}$ from the proof of Proposition 2. Together with the resource constraint we obtain $1 = \pi_1 + \pi_2 + \pi_3 > \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$ $\lightning$

2. Suppose $D_1 \in (\mathcal{A} \cup \mathcal{B}) \setminus \{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5\}$ and D_2 is such that ρ is non-symmetric and $\pi_1 \geq \frac{1}{4}$.

   First note that $\pi_1 \geq \frac{1}{4}$ is only possible if the constraints for D_3^o or $D_3^{L^\epsilon}$ hold, because otherwise Proposition 2 implies $\pi_1 = 0$ and $\pi_2 \geq \frac{3}{8}$, since the minimum payoff with D_2 such that ρ is symmetric is $\pi_2 = \frac{3}{8}$.

   (a) Suppose D_2 is such that the constraints of D_3^o hold $\Longrightarrow a = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$
   $\Longrightarrow a_{21} + a_{22} \leq \underline{x}$ and $a_{21} = \pi_1 \geq \frac{1}{4}$, $a_{22} = \pi_2 \geq \frac{3}{8} \Longrightarrow \frac{5}{8} \leq \underline{x} \leq \frac{1}{2}$ $\lightning$

   (b) Suppose D_2 is such that the constraints of $D_3^{L^\epsilon}$ hold $\Longrightarrow a = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$
   and $D_3 = (0, 2a_{12} - a_{22} + \epsilon, 1 - (2a_{12} - a_{22} + \epsilon))$ is the best reaction for Player 3 and $\pi_1 = \frac{1}{3}(a_{11} + a_{21})$ with $\pi_2 = \frac{1}{3}(2a_{12} - a_{22} + \epsilon + a_{12} + a_{22}) = a_{12} + \frac{\epsilon}{3}$.

      i. Suppose $D_2 \in \mathcal{A} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_4) \Longrightarrow a_{12} \leq \frac{1}{4} \Longrightarrow$
      $\pi_2 = \frac{1}{4} + \frac{\epsilon}{3} < \frac{3}{8}$ $(\epsilon < \frac{3}{8})$ $\lightning$

      ii. Suppose $D_2 \in \mathcal{A}_4 \Longrightarrow \pi_1 = \frac{1}{6}$ $\lightning$

      iii. Suppose $D_2 \in \mathcal{B} \setminus (\mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5) \Longrightarrow a_{11} \leq \frac{3}{8}$ and $2a_{21} < a_{11}$ (constraints of $D_3^{L^\epsilon}$) $\Longrightarrow \pi_1 < \frac{1}{3}(\frac{3}{8} + \frac{3}{16}) = \frac{3}{16}$ $\lightning$

   (c) Suppose D_2 is such that the constraints of $D_3^{L^\epsilon}$ hold $\Longrightarrow a = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$
   and $D_3 = (2a_{11} - a_{21} + \epsilon, 0, 1 - (2a_{11} - a_{21} + \epsilon))$ with $\pi_2 = \frac{1}{3}(a_{12} + a_{22})$

      i. Suppose $D_2 \in \mathcal{A} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_4) \Longrightarrow a_{12} \leq \frac{1}{6}$ and $\pi_2 \leq \frac{1}{9}$
      since $a_{22} \leq a_{12}$. $\lightning$

      ii. Suppose $D_2 \in \mathcal{A}_4 \Longrightarrow \pi_2 \leq \frac{1}{3} < \pi_2(\mathcal{A}_4) = \frac{1}{2} - \epsilon$ $\lightning$

iii. Suppose $D_2 \in \mathcal{B}\backslash(\mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5) \implies a_{12} \leq \frac{3}{4}$ and $\pi_2 < \frac{3}{8}$ since $2a_{22} < a_{12}$.

■

7.2.3 Proof of Proposition 4 and Corollary 4

Suppose $D_1 \in \mathcal{C}$

$\mathcal{C}_1$) $a_{11} + a_{12} \leq \frac{1}{2}$ and $D_2 = (1 - a_{22}, a_{22}, 0)$ with $1 - (2a_{22} - a_{12} + \epsilon) = \frac{1}{2}(1 - a_{11} - a_{12}) \implies a_{22} = \frac{1}{4}(1+3a_{12}+a_{11}-2\epsilon)$ and $D_3 = (0, 2a_{22}-a_{12}+\epsilon', 1-(2a_{22}-a_{12}+\epsilon'))$ is the best reaction for Player 3 and a single-proposal equilibrium. This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1 - a_{22} & a_{22} & 0 \\ 0 & 2a_{22} - a_{12} + \epsilon' & 1 - (2a_{22} - a_{12} + \epsilon') \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2^* \end{pmatrix} \tag{106}$$

$$\begin{array}{rcllll} D_1 & = & (0.10 & 0.10 & 0.8) \\ D_2^{\mathcal{C}_1} & = & (0.65 + \frac{\epsilon}{2} & 0.35 - \frac{\epsilon}{2} & 0.0 \\ D_3 & = & (0.00 & 0.6 + \epsilon + \epsilon' & 0.4 - (\epsilon + \epsilon')) \end{array}$$

This can be satisfied if the constraint set (i) of D_3^3 hold with $a_{22} = \underline{x}$. The feasible set $\mathcal{S}_{\mathcal{C}_1}$ is then determined by

$$\begin{array}{lrcllcl} \mathbf{1}. & a_{11} + a_{12} & \leq & a_{22} & \implies & a_{12} & < & 1 - 3a_{11} \\ 2. & 2a_{11} & < & a_{21} & \implies & a_{12} & \leq & 1 - 3a_{11} \\ \mathbf{3}. & 2a_{12} & < & a_{22} & \implies & a_{12} & < & \frac{1}{5} + \frac{1}{5}a_{11} \\ 4. & 2a_{22} - a_{12} & < & 1 & \implies & a_{12} & \leq & 1 - a_{11} \\ 5. & 1 - (2a_{22} - a_{12} + \epsilon') & > & \frac{1}{2}(a_{13} + a_{23}) & \implies & \epsilon' & < & \epsilon \\ 6. & a_{22} & \leq & a_{21} & \implies & a_{12} & \leq & \frac{1}{3} - \frac{1}{3}a_{11} \\ 7. & 2a_{21} - a_{11} & > & 2a_{22} - a_{12} & \implies & a_{12} & \leq & \frac{1}{2} - a_{11} \end{array}$$

After comparison of the inequalities, only 1. and 3. remain.

$$\implies \quad \mathcal{S}_{\mathcal{C}_1} \quad = \quad \left\{ \left(0, \tfrac{1}{4}\right] \times \left[0, \tfrac{1}{5} + \tfrac{1}{5}a_{11}\right) \cup \left[\tfrac{1}{4}, \tfrac{1}{3}\right) \times [0, 1 - 3a_{11}) \right\} \tag{107}$$

(See Figure 3)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{C}_1) > \frac{1}{2} \tag{108}$$

$\mathcal{C}_2$) $D_2 = (1-a_{22}, a_{22}, 0)$ with $2a_{21} - a_{11} = 1 \Longrightarrow a_{21} = \frac{1}{2}(1+a_{11})$ and $a_{22} = \frac{1}{2}(1-a_{11})$. $D_3 = (0, 2a_{22} - a_{12} + \epsilon, 1 - (2a_{22} - a_{12} + \epsilon))$ is the best reaction for Player 3 and a single-proposal equilibrium. This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1 - a_{22} & a_{22} & 0 \\ 0 & 2a_{22} - a_{12} + \epsilon & 1 - (2a_{22} - a_{12} + \epsilon) \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2^* \end{pmatrix} \tag{109}$$

$$\begin{array}{lcl} D_1 & = & (0.08 \quad 0.30 \quad\quad 0.62) \\ D_2^{\mathcal{C}_2} & = & (0.54 \quad 0.46 \quad\quad 0.00 \\ D_3 & = & (0.00 \quad 0.62 + \epsilon \quad 0.38 - \epsilon) \end{array}$$

This can be satisfied if the constraint set (ii) of D_3^3 holds with $a_{22} = \underline{x}$. The feasible set $\mathcal{S}_{\mathcal{C}_2}$ is then determined by

$$\begin{array}{lrclcrcl} \mathbf{1}. & a_{11} + a_{12} & \leq & a_{22} & \Longrightarrow & a_{12} & \leq & \frac{1}{2} - \frac{3}{2}a_{11} \\ 2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{11} & < & \frac{1}{3} \\ \mathbf{3}. & 2a_{12} & \geq & a_{22} & \Longrightarrow & a_{12} & \geq & \frac{1}{4} - \frac{1}{4}a_{11} \\ 4. & 2a_{22} - a_{12} & < & 1 & \Longrightarrow & a_{12} & > & -a_{11} \\ 5. & 2a_{21} - a_{11} & \geq & 1 & \Longrightarrow & 1 & \geq & 1 \\ \mathbf{6}. & 1 - (2a_{22} - a_{12} + \epsilon) & \geq & \frac{1}{2}(a_{13} + a_{23}) & \Longrightarrow & a_{12} & > & \frac{1}{3} - a_{11} \\ 7. & a_{22} & \leq & a_{21} & \Longrightarrow & a_{11} & \geq & 0 \\ 8. & 2a_{22} - a_{12} & \geq & \frac{1}{2} & \Longrightarrow & a_{12} & \leq & \frac{1}{2} - a_{11} \end{array}$$

After comparison of the inequalities, only 1., 3. and 6. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{C}_2} \quad = \quad \left\{ \left[0, \tfrac{1}{9}\right] \times \left(\tfrac{1}{3} - a_{11}, \tfrac{1}{2} - \tfrac{3}{2}a_{11}\right] \cup \left(\tfrac{1}{9}, \tfrac{1}{5}\right] \times \left[\tfrac{1}{4} - \tfrac{1}{4}a_{11}, \tfrac{1}{2} - \tfrac{3}{2}a_{11}\right] \right\} \tag{110}$$

(See Figure 3)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{C}_2) > \frac{1}{2} \tag{111}$$

$\mathcal{C}_3$) $D_2 = (1 - a_{22}, a_{22}, 0)$ with $2a_{22} - a_{12} = \frac{1}{2} \Longrightarrow a_{21} = \frac{1}{4}(3 - 2a_{12})$ and $a_{22} = \frac{1}{4}(1 + 2a_{12})$. $D_3 = (0, 2a_{22} - a_{12} + \epsilon, 1 - (2a_{22} - a_{12} + \epsilon))$ is the best reaction for Player 3 and a single-proposal equilibrium. This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1 - a_{22} & a_{22} & 0 \\ 0 & \frac{1}{2} + \epsilon & \frac{1}{2} + \epsilon) \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2^* \end{pmatrix} \tag{112}$$

$$\begin{array}{lcl} D_1 & = & (0.02 \quad 0.24 \quad\quad 0.74) \\ D_2^{\mathcal{C}_3} & = & (0.63 \quad 0.37 \quad\quad 0.0 \\ D_3 & = & (0.00 \quad 0.5 + \epsilon \quad 0.5 - \epsilon) \end{array}$$

This can be satisfied if the constraint set (ii) of D_3^3 holds with $a_{22} = \underline{x}$. The feasible set $\mathcal{S}_{\mathcal{C}_3}$ is then determined by

$$
\begin{array}{lrcllcl}
\mathbf{1}. & a_{11}+a_{12} & \leq & a_{22} & \Longrightarrow & a_{12} & \leq & \frac{1}{2}-2a_{11} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & < & \frac{3}{2}-4a_{11} \\
\mathbf{3}. & 2a_{12} & \geq & a_{22} & \Longrightarrow & a_{12} & \geq & \frac{1}{6} \\
4. & 2a_{22}-a_{12} & < & 1 & \Longrightarrow & \frac{1}{2} & < & 1 \\
5. & 2a_{21}-a_{11} & \geq & 1 & \Longrightarrow & a_{12} & \leq & \frac{1}{2}-a_{11} \\
6. & 1-(2a_{22}-a_{12}+\epsilon) & \geq & \frac{1}{2}(a_{13}+a_{23}) & \Longrightarrow & a_{12} & > & -a_{11} \\
7. & a_{22} & \leq & a_{21} & \Longrightarrow & a_{12} & \leq & \frac{1}{2}
\end{array}
$$

After comparison of the inequalities, only 1. and 3. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{C}_3} \quad = \quad \left\{\left[0,\tfrac{1}{6}\right] \times \left[\tfrac{1}{6},\tfrac{1}{2}-2a_{11}\right]\right\} \tag{113}$$

(See Figure 3)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{C}_3) > \frac{1}{2} \tag{114}$$

$\mathcal{C}_4$) $D_2 = (1-a_{22}, a_{22}, 0)$ with $2a_{22}-a_{12}+\epsilon = 2a_{21}-a_{11} \Longrightarrow a_{21} = \frac{1}{4}(2+a_{11}-a_{21}+\epsilon)$ and $a_{21} = \frac{1}{4}(2-a_{11}+a_{21}-\epsilon)$. $D_3 = (0, 2a_{22}-a_{12}+\epsilon', 1-(2a_{22}-a_{12}+\epsilon'))$ is the reaction for Player 3 and a single-proposal equilibrium. This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1-a_{22} & a_{22} & 0 \\ 0 & 2a_{22}-a_{12}+\epsilon' & 1-(2a_{22}-a_{12}+\epsilon') \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \tag{115}$$

$$
\begin{array}{lcllll}
D_1 & = & (0.06 & 0.10 & 0.84) \\
D_2^{\mathcal{C}_4} & = & (0.49+\epsilon & 0.51-\epsilon & 0.0 \\
D_3 & = & (0.00 & 0.92-\frac{\epsilon}{2}+\epsilon' & 0.08+\frac{\epsilon}{2}-\epsilon')
\end{array}
$$

This can be satisfied if the constraint set (i) of $D_3^{T_\epsilon}$ holds with $a_{21} = \underline{x}$. The feasible set $\mathcal{S}_{\mathcal{C}_4}$ is then determined by

$$
\begin{array}{lrcllcl}
1. & a_{11}+a_{12} & \leq & a_{21} & \Longrightarrow & a_{12} & \leq & \frac{2}{5}-\frac{3}{5}a_{11} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & \leq & 2-7a_{11} \\
\mathbf{3}. & 2a_{12} & < & a_{22} & \Longrightarrow & a_{12} & < & \frac{2}{7}-\frac{1}{7}a_{11} \\
4. & 2a_{22}-a_{12} & < & 1 & \Longrightarrow & a_{12} & \geq & -a_{11} \\
5. & 1-(2a_{22}-a_{12}+\epsilon') & < & \frac{1}{2}(a_{13}+a_{23}) & \Longrightarrow & a_{12} & < & \frac{1}{2}-a_{11} \\
\mathbf{6}. & a_{21} & \leq & a_{22} & \Longrightarrow & a_{12} & > & a_{11} \\
7. & 2a_{21}-a_{11} & > & 2a_{22}-a_{12} & \Longrightarrow & \epsilon & > & 0
\end{array}
$$

After comparison of the inequalities, only 3. and 6. remain.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{C}_4} \quad = \quad \left\{ \left[0, \tfrac{1}{4}\right] \times \left(a_{11}, \tfrac{2}{7} - \tfrac{1}{7}a_{11}\right) \right\} \tag{116}$$

(See Figure 3)

Minimum payoff for Player 2:

$$\pi_2^{min}(\mathcal{C}_4) > \frac{1}{2} - \epsilon \tag{117}$$

$\mathcal{C}_5$) In the remaining part $\mathcal{S}_{\mathcal{C}_5} = \left\{ \mathcal{C} \setminus \bigcup_{k=1}^{4} \mathcal{S}_{\mathcal{C}_k} \right\}$ (see Figure 3), we show that, given $(a_{11}, a_{12}) \in \mathcal{S}_{\mathcal{C}_5}$ the share of Player 1 is $\pi_1 = 0$ if Player 2's best proposal implies ρ to be non-symmetric or the share of Player 2 is $\pi_2 > \frac{3}{8}$ if Player 2's best proposal implies ρ to be symmetric.

In order to prove this, we divide $\mathcal{S}_{\mathcal{C}_5}$ into three subsets:

$$\begin{aligned} \mathcal{S}^1_{\mathcal{C}_5} &= \left\{ (a_{11}, a_{12}) \,|\, a_{11} + a_{12} > \tfrac{1}{2} \,,\, a_{11} < \tfrac{1}{2} \,,\, a_{12} < \tfrac{1}{2} \right\} \\ \mathcal{S}^2_{\mathcal{C}_5} &= \left\{ (a_{11}, a_{12}) \,|\, a_{11} + a_{12} \leq \tfrac{1}{2} \,,\, a_{12} \geq \tfrac{2}{7} - \tfrac{1}{7}a_{11} \,,\, a_{12} > \tfrac{1}{2} - \tfrac{3}{2}a_{11} \right\} \\ \mathcal{S}^3_{\mathcal{C}_5} &= \left\{ (a_{11}, a_{12}) \,|\, a_{11} + a_{12} \leq \tfrac{1}{2} \,,\, a_{12} \geq 1 - 3a_{11} \,,\, a_{11} \geq 0 \right\} \end{aligned} \tag{118}$$

(a) Given $(a_{11}, a_{12}) \in \mathcal{S}^1_{\mathcal{C}_5} \Longrightarrow$ with $D_2 = (\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon, 0)$, the inequalities of $D_3^{\underline{x}^\epsilon}$ hold and $D_3 = (0, \frac{1}{2} - \epsilon + \epsilon', \frac{1}{2} + \epsilon - \epsilon')$ is the best proposal for Player 3 with D_3 being a proposal equilibrium including $\pi_1 = 0$ and $\pi_2 = \frac{1}{2} - \epsilon + \epsilon'$.

  i. Suppose Player 2 constructs his proposal D_2 in such a way that ρ is non-symmetric and $\pi_1 > 0$, then the inequalities of $D_3^{L^\epsilon}$ or D_3^o must hold ($a_{11} = y_1$, $a_{12} = y_2$). But $y_1 + y_2 \leq \underline{x}$ cannot hold since $\underline{x} \leq \frac{1}{2}$.

  ii. Suppose Player 2 constructs his proposal D_2 in such a way that ρ is symmetric and $\pi_2 \leq \frac{3}{8}$, then this cannot be his best proposal as proposing $D_2 = (\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon, 0)$ with non-symmetric ρ and $\epsilon < \frac{1}{8}$ ensures him a share of $\pi_2 > \frac{3}{8}$.

(b) Given $(a_{11}, a_{12}) \in \mathcal{S}^2_{\mathcal{C}_5} \cup \mathcal{S}^3_{\mathcal{C}_5} \Longrightarrow$ with $D_2 = (1 - (a_{11} + a_{12} - \epsilon), a_{11} + a_{12} - \epsilon, 0)$ the inequalities of $D_3^{\underline{x}^\epsilon}$ hold and $D_3 = (0, a_{11} + a_{12} - \epsilon + \epsilon', 1 - (a_{11} + a_{12} - \epsilon + \epsilon')$ is the best proposal for Player 3 with D_3 being a proposal equilibrium including $\pi_1 = 0$ and $\pi_2 = a_{12} + a_{11} - \epsilon + \epsilon'$.

  i. Suppose Player 2 constructs his proposal D_2 in such a way that ρ is non-symmetric, $\pi_1 > 0$, and the inequalities of D_3^o hold, with $\pi_2 = a_{12}$. This cannot be the best reaction for Player 2 as $D_2 = (1 - (a_{11} + a_{12} - \epsilon), a_{11} + a_{12} - \epsilon, 0)$ with $\epsilon < a_{11}$ ensures him a share of $\pi_2 = a_{11} + a_{12} - \epsilon + \epsilon' > a_{12}$.

  ii. Suppose Player 2 constructs his proposal D_2 in such a way that ρ is non-symmetric, $\pi_1 > 0$ and the inequalities of $D_3^{L^\epsilon}$ hold with $D_3 = (2a_{21} - a_{11} + \epsilon, 0, 1 - (2a_{21} - a_{22} + \epsilon))$ is the best proposal for Player 3 ($a_{11} = y_1$, $a_{12} = y_2$) $\Longrightarrow$ This cannot be the best reaction for Player 2 because his share $\pi_2 = \frac{1}{3}(x_2 + y_2)$ must exceed his share $\pi_2 = y_1 + y_2 + \epsilon - \epsilon'$ generated by $D_2 = (0, \frac{1}{2} - \epsilon + \epsilon', \frac{1}{2} + \epsilon - \epsilon')$. But

$$\tfrac{1}{3}(x_2 + y_2) \quad \geq \quad y_1 + y_2 \quad \overset{y_2 \geq 1 - 3y_1 \geq \frac{1}{2} - \frac{3}{2}y_1}{\Longrightarrow} \quad x_2 = 1 \overset{x_1 \leq 1 - x_2}{\Longrightarrow} x_1 = 0 = \underline{x} \quad \lightning \tag{119}$$

since $a_{12} \geq \frac{1}{4}$ and $a_{11} + a_{12} \leq \underline{x}$

iii. Suppose Player 2 constructs his proposal D_2 in such a way that ρ is symmetric and $\pi_2 \leq \frac{3}{8} \Longrightarrow$
   - $a_{11} + a_{12} \leq \frac{3}{8}$ and
   - $D_3 \in \{D_3^{cor}, D_3^L\}$ must hold.

   Otherwise Player 2 can obtain a share of $\pi_2 = a_{11} + a_{12} - \epsilon + \epsilon' > \frac{3}{8} \geq a_{11} > a_{12}$ by choosing $\epsilon < a_{11} + a_{12} - \frac{3}{8}$ and proposing $D_2 = (1-(a_{11}+a_{12}-\epsilon), a_{11}+a_{12}-\epsilon, 0)$ with non-symmetric ρ. This also implies $D_3 \in \{D_3^{cor}, D_3^L\}$ following from Corollary 5, because if $(a_{11}, a_{12}) \in \{(\mathcal{S}^2_{\mathcal{C}_5} \cup \mathcal{S}^3_{\mathcal{C}_5}) \cap [0, \frac{3}{8}] \times [0, \frac{3}{8} - a_{11}]\} = \{[\frac{5}{16}, \frac{1}{3}] \times [1 - 3a_{11}, \frac{3}{8} - a_{11}] \cup [\frac{1}{3}, \frac{3}{8}] \times [0, \frac{3}{8} - a_{11}]\}$ and given symmetric ρ, we have $\underline{x} \leq a_{11} \leq \frac{3}{8} \vee \underline{x} \leq a_{12} \leq \frac{1}{16}$.

   α. Suppose Player 2 constructs his proposal D_2 in such a way that ρ is symmetric and $D_3 = D_3^{cor}$ is the best proposal for Player 3 leading to a correlated equilibrium $C_{23} \Longrightarrow$
   This cannot be the best reaction for Player 2 because his share $\pi_2(D_3^{cor}) = \frac{1}{2}(\mu_2 + x_2) = \frac{1}{2}(\frac{1}{2}(a_{22} + a_{12}) + a_{22}) = \frac{3}{4}a_{22} + \frac{1}{4}a_{12}$ must exceed $\pi_2(D_3^{\underline{x}^\epsilon}) = a_{11} + a_{12} - \epsilon + \epsilon'$, which implies

$$\frac{3}{4}a_{22} + \frac{1}{4}a_{12} \geq a_{11} + a_{12} \Longrightarrow a_{22} \geq \frac{4}{3}a_{11} + a_{12} \underset{\substack{a_{11} \geq \frac{5}{16} \\ a_{12} \geq 0}}{\geq} \frac{5}{12} \tag{120}$$

   , But this is a contradiction of

$$\hat{R}(D_3^{cor}) = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 2^* \\ 1 & 2^* & 3 \end{pmatrix} \tag{121}$$

   which implies

$$a_{23} \overset{a_{11}+a_{12} \leq \frac{3}{8}}{\geq} \frac{5}{8} \Longrightarrow a_{22} \leq \frac{3}{8} \tag{122}$$

   β. Suppose Player 2 constructs his proposal D_2 in such a way that ρ is symmetric and $D_3 = D_3^{cor}$ is the best proposal for Player 3 leading to a correlated equilibrium $C_{13} \Longrightarrow$
   This cannot be the best reaction for Player 2 since his share is given by $\pi_2 = \frac{1}{2}(a_{12} + a_{32}) \leq a_{12}$ because $\Phi(a_{32}) \leq 2$.

   γ. Suppose Player 2 constructs his proposal D_2 in such a way that ρ is symmetric and $D_3 = D_3^L$ is the best proposal for Player 3 leading to a selection of the proposal by drawing lots $\Longrightarrow$
   This cannot be the best reaction for Player 2 because if $a_{32} = 2a_{12} - a_{22} + \epsilon$ we have $\pi_2(D_3^L) = \frac{1}{3}(a_{12} + a_{22} + a_{32}) = a_{12} + \frac{\epsilon}{3} < \pi_2(D_3^{\underline{x}^\epsilon})$ and if $a_{32} = 0$, $\pi_2(D_3^L) > \pi_2(D_3^{\underline{x}^\epsilon})$ would imply

$$\frac{1}{3}(a_{12} + a_{22}) \geq a_{11} + a_{12} \Longrightarrow a_{22} \geq 3a_{11} + 2a_{12} \underset{\substack{a_{11} \geq 0 \\ a_{12} \geq 1 - 3a_{11} \\ a_{12} \leq \frac{3}{8} - a_{11}}}{\Longrightarrow} a_{22} \geq 1 \tag{123}$$

But then we have $a_{23}=0$, $a_{13}\geq\frac{5}{8}$, and $a_{33}\leq 1$, which cannot lead to a decision by drawing lots with $\Phi(a_{33})=3$.

iv. Suppose $(a_{11},a_{12})\in\mathcal{S}^2_{\mathcal{C}_5}$ and Player 2 constructs his proposal D_2 in such a way that ρ is non-symmetric, $\pi_1>0$, and the inequalities of $D_3^{L^\epsilon}(i)$ hold with $D_3=(0,2a_{22}-a_{12}+\epsilon,1-(2a_{22}-a_{12}+\epsilon))$ is the best proposal for Player 3 ($a_{11}=y_1$, $a_{12}=y_2$) $\Longrightarrow$

$$\begin{array}{ccccc} 3y_2 & \overset{2y_2<x_2}{<} & 2x_2-y_2 & \overset{2x_2-y_2<2x_1-y_1}{<} & 2x_1-y_1 \\ & \overset{x_1\leq 1-x_2}{\leq} & 2(1-x_2)-y_1 & \overset{2y_2<x_2}{<} & 2-4y_2-y_1 \end{array} \Longrightarrow y_2<\frac{2}{7}-\frac{1}{7}y_1 \quad \lightning \tag{124}$$

since $a_{12}\geq\frac{2}{7}-\frac{1}{7}a_{11}$ must hold in $\mathcal{S}^2_{\mathcal{C}_5}$.

v. Suppose $(a_{11},a_{12})\in\mathcal{S}^2_{\mathcal{C}_5}$ and Player 2 constructs his proposal D_2 in such a way that ρ is non-symmetric, $\pi_1>0$, and the inequalities of $D_3^{L^\epsilon}(ii)$ hold with $D_3=(0,2a_{22}-a_{12}+\epsilon,1-(2a_{22}-a_{12}+\epsilon))$ is the best proposal for Player 3 ($a_{11}=y_1$, $a_{12}=y_2$) $\Longrightarrow$

$$\begin{array}{ccccc} 1 & \leq & 2x_1-y_1 & \overset{x_1\leq 1-x_2}{\leq} & 2-2x_2-y_1 \\ & \overset{x_2\geq\underline{x}}{\leq} & 2-2\underline{x}-y_1 & \overset{y_1+y_2\leq\underline{x}}{\leq} & 2-3y_1-2y_2 \end{array} \Longrightarrow y_2\leq\frac{1}{2}-\frac{3}{2}y_1 \quad \lightning \tag{125}$$

since $a_{12}\leq\frac{1}{2}-\frac{3}{2}a_{11}$ must hold in $\mathcal{S}^2_{\mathcal{C}_5}$.

vi. Suppose $(a_{11},a_{12})\in\mathcal{S}^3_{\mathcal{C}_5}$ and Player 2 constructs his proposal D_2 in such a way that ρ is non-symmetric, $\pi_1>0$, and the inequalities of $D_3^{L^\epsilon}$ hold with $D_3=(0,2a_{22}-a_{12}+\epsilon,1-(2a_{22}-a_{12}+\epsilon))$ is the best proposal for Player 3 ($a_{11}=y_1$, $a_{12}=y_2$) $\Longrightarrow$

$$\begin{array}{ccccc} y_1+y_2 & \leq & \underline{x} & \overset{x_2\geq\underline{x}}{<} & x_2 \\ & \overset{x_2\leq 1-x_1}{\leq} & 1-x_1 & \overset{2y_1<x_1}{<} & 1-2y_1 \end{array} \Longrightarrow y_2<1-3y_1 \quad \lightning \tag{126}$$

since $a_{12}\geq 1-3a_{11}$ must hold in $\mathcal{S}^3_{\mathcal{C}_5}$.

Thus we have proved Proposition 4 and Corollary 4.

$\blacksquare$

7.2.4 Proof of Proposition 5

In the following, we only give binding constraints.

If $a=\begin{pmatrix}0&1\\1&0\end{pmatrix}\Longrightarrow\pi_2=\frac{1}{4}$ is already shown in Corollary 1.

1. Suppose $a=\begin{pmatrix}0&1\\0&a_{22}\end{pmatrix}\Longrightarrow D_3=(\epsilon,0,1-\epsilon)$ is the best reaction for Player 3 with D_3 being a single-proposal equilibrium $\Longrightarrow\pi_2=0$

The solution set is given by

$$\mathcal{S}_1 \;=\; \{[0,0]\times[0,1]\} \tag{127}$$

(See Figure 4)

2. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrcll} 1. & a_{22} & < & \frac{1}{2} & \\ 2. & 1-\mu_1 & > & 2a_{23} \implies a_{22} & > \frac{1}{2}-\frac{3}{4}a_{21} \end{array} \tag{128}$$

then the constraints of D_3^{μ} hold and $D_3^{\mu} = (\mu_1, 0, 1-\mu_1)$ is the best reaction for Player 3 with D_3^{μ} being a single-proposal equilibrium. $\implies \pi_2 = 0$

The solution set is given by

$$\mathcal{S}_2 \;=\; \left\{ \begin{array}{l} \left[0,\frac{1}{2}\right]\times\left(\frac{1}{2}-\frac{3}{4}a_{21},\frac{1}{2}\right)\cup \\ \left(\frac{1}{2},\frac{2}{3}\right]\times\left(\frac{1}{2}-\frac{3}{4}a_{21},1-a_{21}\right]\cup \\ \left(\frac{2}{3},1\right)\times\left[0,1-a_{21}\right]\right\} \end{array}\right. \tag{129}$$

(See Figure 4)

3. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrclcrcl} 1. & a_{22} & < & \frac{1}{2} & & & & \\ 2. & 1-\mu_1 & \le & 2a_{23} & \implies & a_{22} & \le & \frac{1}{2}-\frac{3}{4}a_{21} \\ 3. & \frac{1}{2}(1-\mu_1+a_{23}) & \le & 1-a_{21} & \implies & a_{22} & \ge & \frac{1}{2}a_{21} \end{array} \tag{130}$$

then the constraints (ii) of $D_3^{\underline{x}}$ hold and $D_3^{\underline{x}} = (\underline{x}, 0, 1-\underline{x})$ is the best reaction for Player 3 with $D_3^{\underline{x}}$ being a single-proposal equilibrium. $\implies \pi_2 = 0$

The solution set is given by

$$\mathcal{S}_3 \;=\; \left\{\left(0,\frac{2}{5}\right]\times\left[\frac{1}{2}a_{21},\frac{1}{2}-\frac{3}{4}a_{21}\right]\right\} \tag{131}$$

(See Figure 4)

4. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrclcrcl} 1. & 1-\mu_1 & \le & 2a_{23} & \implies & a_{22} & \le & \frac{1}{2}-\frac{3}{4}a_{21} \\ 2. & \frac{1}{2}(1-\mu_1+a_{23}) & > & 1-a_{21} & \implies & a_{22} & < & \frac{1}{2}a_{21} \end{array} \tag{132}$$

then the constraints (a) of D_3^{cor} hold and $D_3^{cor} = (\mu_1, 0, 1-\mu_1)$ is the best reaction for Player 3 with C_{23} being a correlated equilibrium. $\implies \pi_2 = \frac{1}{2}a_{22} < \frac{1}{10}$

The solution set is given by

$$\mathcal{S}_4 \;=\; \left\{\left(0,\frac{2}{5}\right]\times\left[0,\frac{1}{2}a_{21}\right)\cup\left(\frac{2}{5},\frac{2}{3}\right]\times\left[0,\frac{1}{2}-\frac{3}{4}a_{21}\right)\right\} \tag{133}$$

(See Figure 4)

5. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrcl} 1. & a_{22} & \geq & \frac{1}{2} \\ 2. & \mu_1 + 2a_{22} - 1 & < & a_{21} \implies a_{22} < \frac{1}{2} + \frac{1}{4}a_{21} \\ 3. & 1 - \mu_1 - (2a_{22} - 1) & > & 2a_{23} \implies a_{21} > 0 \end{array} \tag{134}$$

then the constraints (i) of $D_3^{\mu_i}$ hold and $D_3^{\mu_i} = (\mu_1, 2a_{22}-1+\epsilon, 1-(\mu_1+2a_{22}-1+\epsilon))$ is the best reaction for Player 3 with $D_3^{\mu_i}$ being a single-proposal equilibrium. $\implies$ $\pi_2 = 2a_{22} - 1 + \epsilon < \frac{1}{5} + \epsilon$

The solution set is given by

$$\mathcal{S}_5 \;=\; \left\{\left(0, \tfrac{2}{5}\right] \times \left[\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{4}a_{21}\right) \cup \left(\tfrac{2}{5}, \tfrac{1}{2}\right] \times \left[\tfrac{1}{2}, 1 - a_{21}\right]\right\} \tag{135}$$

(See Figure 4)

6. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrcl} 1. & a_{21} & > & 0 \\ 2. & \mu_1 + 2a_{22} - 1 & \geq & a_{21} \implies a_{22} \geq \frac{1}{2} + \frac{1}{4}a_{21} \end{array} \tag{136}$$

then the constraints (i) of $D_3^{\underline{x}}$ hold and $D_3^{\underline{x}} = (\underline{x}, 0, 1 - \underline{x})$ is the best reaction for Player 3 with $D_3^{\underline{x}}$ being a single proposal equilibrium. $\implies \pi_2 = 0$

The solution set is given by

$$\mathcal{S}_6 \;=\; \left\{\left(0, \tfrac{2}{5}\right] \times \left[\tfrac{1}{2} + \tfrac{1}{4}a_{21}, 1 - a_{21}\right]\right\} \tag{137}$$

(See Figure 4)

Altogether we have shown that $D_2 = (1, 0, 0)$ is the best reaction for Player 2 given $D_1 = (0, 1, 0)$

■

8 Literature

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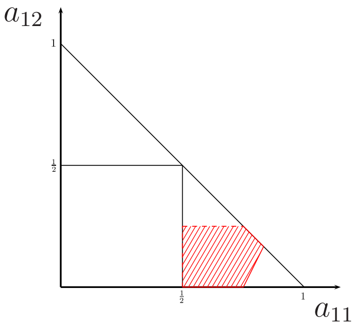
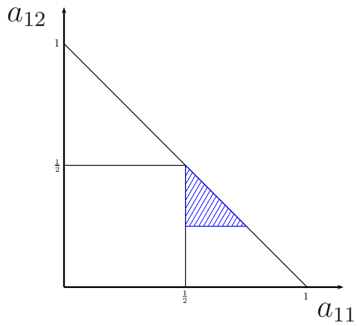


Feasible set $\mathcal{S}_{\mathcal{A}_1}$



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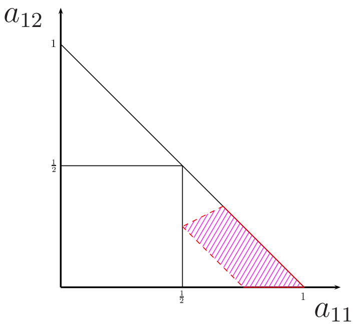

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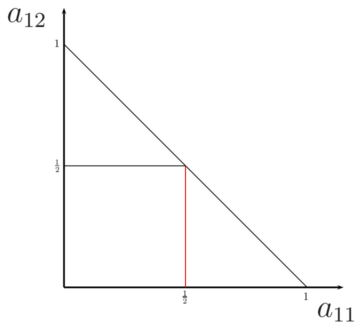
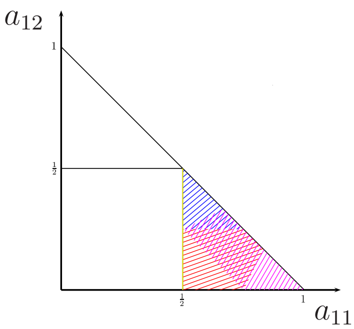
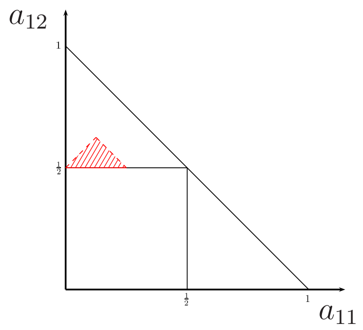


Feasible set $\mathcal{S}_{\mathcal{A}_4}$

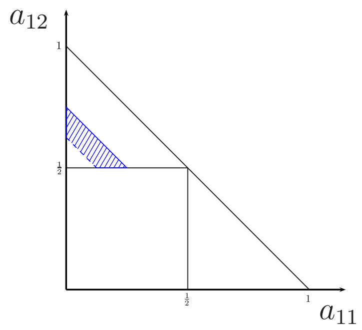


Feasible sets $\mathcal{S}_{\mathcal{A}_1}$–$\mathcal{S}_{\mathcal{A}_4}$

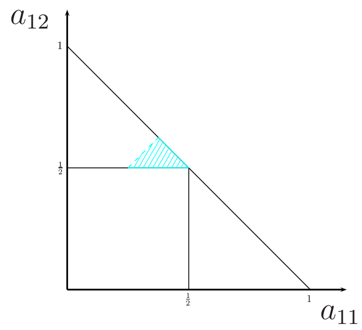
Figure 1: Feasible set $\mathcal{S}_{\mathcal{A}}$.



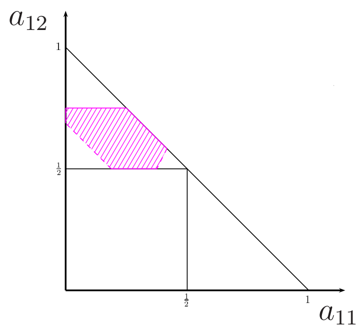
Feasible set $\mathcal{S}_{\mathcal{B}_1}$



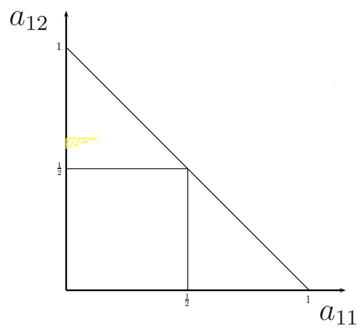
Feasible set $\mathcal{S}_{\mathcal{B}_2}$



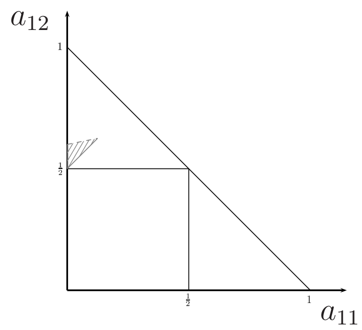
Feasible set $\mathcal{S}_{\mathcal{B}_3}$



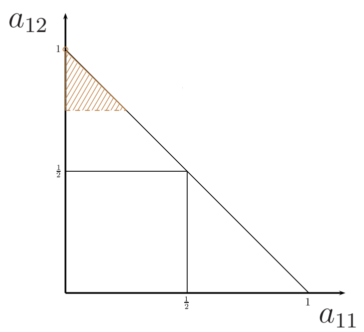
Feasible set $\mathcal{S}_{\mathcal{B}_4}$



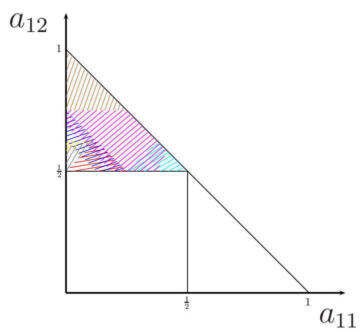
Feasible set $\mathcal{S}_{\mathcal{B}_5}$



Feasible set $\mathcal{S}_{\mathcal{B}_6}$



Feasible set $\mathcal{S}_{\mathcal{B}_7}$



Feasible sets $\mathcal{S}_{\mathcal{B}_1}$–$\mathcal{S}_{\mathcal{B}_7}$

Figure 2: Feasible set $\mathcal{S}_{\mathcal{B}}$.

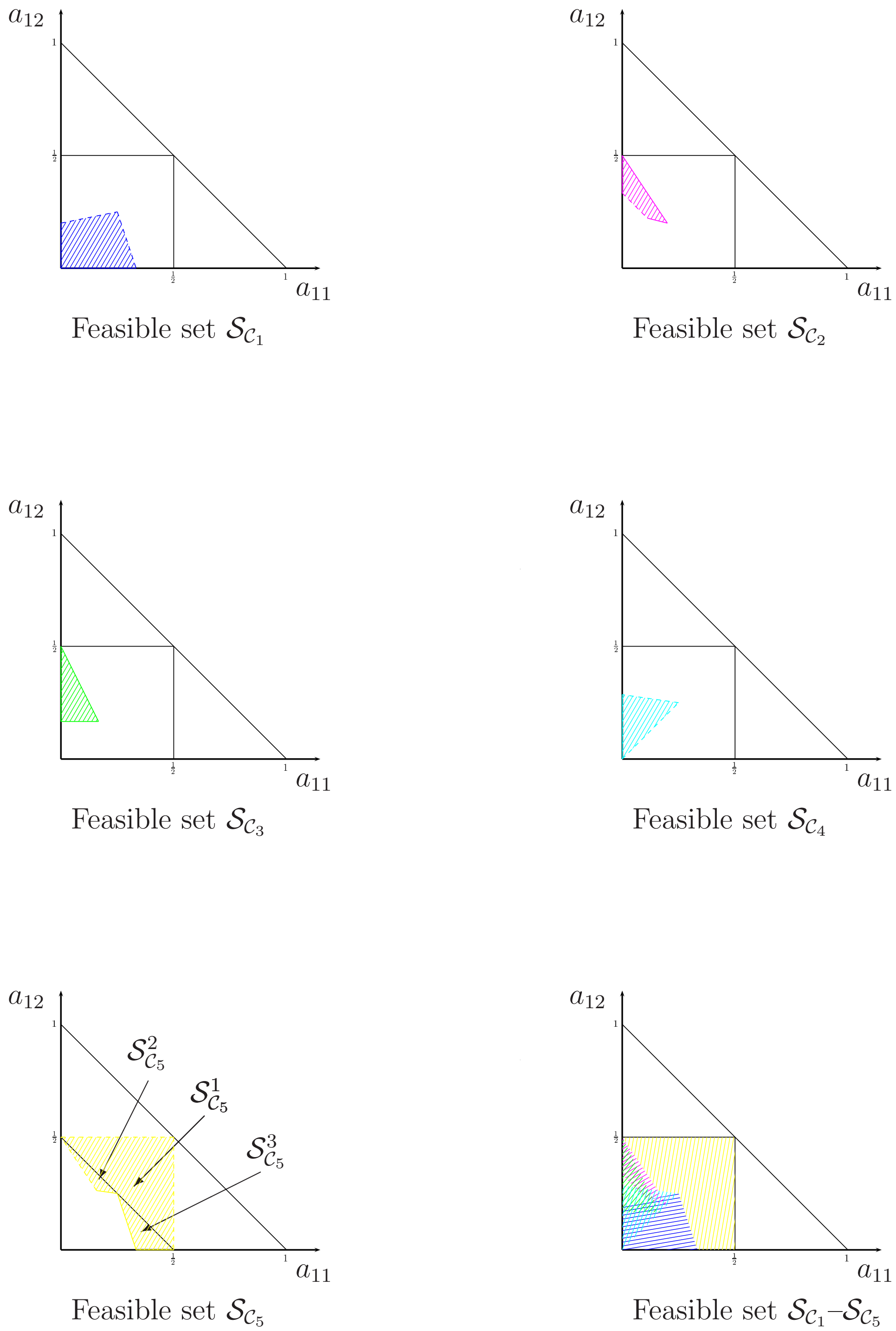


Figure 3: Feasible set $\mathcal{S}_{\mathcal{C}}$.

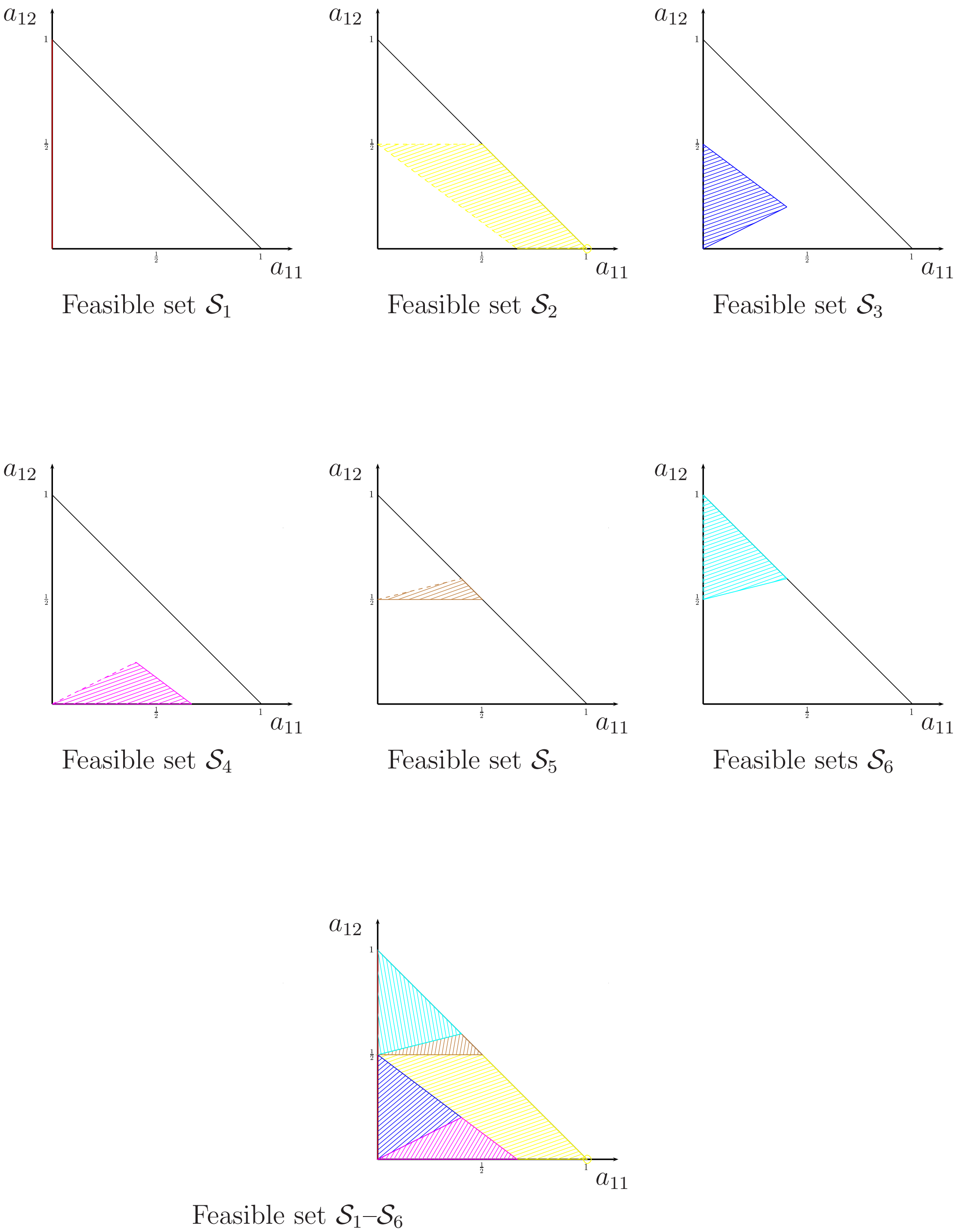


Figure 4: Feasible set $\mathcal{S}$.

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