THE FISHER/COBB-DOUGLAS PARADOX, FACTOR SHARES, AND COINTEGRATION

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Abstract

This note uses insights from cointegration analysis to reexamine two separate but related issues concerning the estimation of production function parameters. Fisher (1971) documented a paradox in estimating substitution elasticities -- the puzzling divorce between the technology underlying his simulated data and the technology estimated from these data. This note both resolves the Paradox and, based on this resolution, raises important questions about estimation strategies (pioneered by Caballero, 1994) that rely on cointegration to recover production function parameters.

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The Fisher/Cobb-Douglas Paradox, Factor Shares, And Cointegration

I. The Fisher/Cobb-Douglas Paradox

This note is motivated by a paradox first noted by Franklin Fisher in a 1971 article. His study was concerned with estimating aggregate production function parameters from simulated unit-level data. Fisher found that, when aggregate factor shares were nearly constant, an aggregate Cobb-Douglas technology provided the best fit even though no aggregate production function could be constructed from the underlying unit-level production functions.[1] The Fisher/Cobb-Douglas Paradox is "that an aggregate Cobb-Douglas will continue to work well so long as labor's share continues to be roughly constant, even though that rough constancy is not itself a consequence of the economy having a technology that is truly summarized by an aggregate Cobb-Douglas" (Fisher, 1971, p. 307). Fisher's results were based on unit-level Cobb-Douglas production functions. Similar results were obtained by Fisher, Solow, and Kearl (1977) when the study was extended to unit-level Constant Elasticity of Substitution (CES) production functions. The Paradox uncovered by both studies is the puzzling divorce between the technology underlying the simulated data and the technology estimated from these data.

This note takes the Fisher/Cobb-Douglas Paradox as its point of departure and uses insights from cointegration analysis to understand the results from these two simulation studies.[2] Our analysis resolves the Paradox for simulated data and, based on this resolution, sheds new light on estimation strategies with "real" data. The second contribution of this note is to raise important questions about estimation strategies that rely on cointegration (pioneered by Caballero, 1994) to recover production function parameters.

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[1] The Fisher study was open on the definition of "unit-level" and hence whether the aggregation was from units-as-firms to an industry aggregate or from units-as-industries to a macroeconomic aggregate. In a private correspondence, Fisher has noted that the appropriate interpretation of the aggregate depends on the assumed homogeneity of output, an assumption that may be more plausible in the case of an industry aggregate. This assumption is not important for the resolution of the paradox proposed in this note.

[2] The "Fisher/Cobb-Douglas Paradox" needs to be distinguished from the "Fisher Paradox" that occupies a prominent place in macroeconomics. The latter paradox is that, while there is a strong theoretical case for the constancy of real rates and hence a one-for-one movement of nominal rates with expected inflation, this theoretical relation is usually not found empirically (Carmichael and Stebbing, 1983).

II. Resolving The Fisher/Cobb-Douglas Paradox

Fisher (1971) and Fisher, Solow, and Kearl (1977) (hereafter F+FSK) document that, when their simulated data had nearly constant factor shares, the best fit is obtained by a Cobb-Douglas technology. This result can be cast in terms of the CES production function and its substitution elasticity between capital and labor, σ. The Fisher/Cobb-Douglas Paradox is thus why the estimated σ equals unity independent of the underlying production technologies generating the simulated data.

We begin by defining factor shares and specifying a cointegrating relation. First, the share in income of factor X is defined as $FS_t^X \equiv (P_t^X X)/(P_t^Y Y)$, where P_t^X and P_t^Y are the prices of factor X and output Y, respectively. It will prove convenient to analyze the factor share in terms of logarithms (denoted by lower case letters),

$$fs_t^x \equiv p_t^x + x_t - p_t^y - y_t \quad . \tag{1}$$

No restrictions are imposed on the time-series properties of fs_t^x. Second and <u>independent</u> of equation (1), we specify a statistical relation between the factor/output ratio, $(x_t - y_t)$, and the relative factor price, $(p_t^x - p_t^y)$. Specifically, we assume that these two series are I(1) and cointegrated, properties implied by the Solow growth model when the factor is labor. It is important to note that we are not imposing any structural interpretation on the relation between $(x_t - y_t)$ and $(p_t^x - p_t^y)$; rather, the assumed relation is merely statistical. Thus, these series can be modeled as the following cointegrating relation,

$$(x_t - y_t) = \alpha - \beta(p_t^x - p_t^y) + e_t \quad , \tag{2}$$

where e_t is a white-noise error term. Under these assumptions, long-run movements in the factor/output ratio and the relative factor price dominate and deliver super-consistent estimates of the cointegrating vector, (α, β). The estimated value of β equals the price elasticity of capital. Under a CES technology with constant returns and holding output constant, this elasticity equals σ.

The factor share definition and cointegrating relation are useful in resolving the Fisher/Cobb-Douglas Paradox. Before proceeding to that analysis, a caveat is in order. The I(1) and cointegration properties are not imposed on the simulated data. However, all of the series are influenced by deterministic time trends, are quite persistent over a relatively short time span, and may best be modeled as integrated processes even if the series are not literally I(1).[3] Given this "slippage" between the assumed and simulated properties of the series, the resolution presented here can only be suggestive. With this caveat noted, we insert equation (2) into (1) and obtain the following equation for the factor share,

$$fs_t^x = (1-\beta)(p_t^x - p_t^y) + \alpha + e_t \ . \tag{3}$$

The Fisher/Cobb-Douglas Paradox is based on constant factor shares. (In terms of time-series analysis, such constancy is equivalent to factor shares being I(0).) As equation (3) makes clear, constancy of the factor share (up to a white-noise error) can obtain if and only if the influence of relative prices is eliminated. In this case, β must equal unity. It is important to note that this result is driven by the assumptions of cointegration and constant factor shares and is independent of the underlying production technology. As long as these two assumptions are imposed, $\beta = 1$; if the production technology is CES, this estimate of β implies a Cobb-Douglas production function. It is thus not surprising that F+FSK are drawn to the Cobb-Douglas technology when factor shares are nearly constant in their simulations.[4]

Fisher offers a heuristic explanation (presented in his Section VII) of the Paradox that relies on (i) a good estimate of equation (2) and (ii) the near constancy of the factor share. His explanation can be understood in terms of the analysis of integrated and cointegrated time

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[3] Blough (1992) documents the difficulties with differentiating in finite samples time series that are highly persistent but I(0) from time series that are I(1).

[4] An alternative explanation has been advanced by Shaikh (1974) and Felipe and Holz (2001), who argue that the accounting identities inherent in value added data necessarily lead to a Cobb-Douglas production function. This argument has been forcefully challenged by Solow (1974, 1987) on empirical and theoretical grounds. Regarding the latter point, Solow (1987, p. 20) shows that, when factor shares are constant, any production function can be represented as the product of a Cobb-Douglas production function and the production function suitably normalized.

series -- the super-consistency of the parameters in equation (2) (point (i)) and the I(0) property of the factor share (point (ii)).

III. Reconsidering The Cointegration Model

While the resolution of the Paradox is of independent interest, it also sheds insights on estimation strategies for production function parameters that apply cointegration methods to "real" data. In innovative papers, Caballero (1994, 1999) and Caballero, Engle, and Haltiwanger (1995) exploit the cointegration relation in equation (2) to generate super-consistent estimates of β.[5] This estimation strategy emphasizes the long-run movements in the data and hence has the decided benefit of using the variation that is most germane to production relations. These authors focus on capital as the factor and obtain aggregate estimates of σ close to the Cobb-Douglas value of unity. This Cointegration Model provides an elegant solution to the problem of estimating the substitution elasticity from data subject to short-run deviations from long-run values.

However, the resolution of the Fisher/Cobb-Douglas Paradox implies that this estimation framework will not be informative in an important case. A well-accepted stylized fact, since at least Kaldor (1961) and Klein and Kosobud (1961, 295-299) and confirmed more recently by King and Rebelo (1999, pp. 940-941), is that aggregate factor shares are constant in the long-run. As shown in equation (3), the combination of cointegrating properties and constant long-run factor shares necessarily yields $\beta = \sigma = 1$. This result does not challenge the validity of estimating the cointegrating relation in equation (2), just that it is uninformative about production technologies.

Several theoretical models are consistent with constant factor shares and values of σ that differ from unity. Acemoglu (2003) examines the tension between fluctuations in income shares, the value of σ, and balanced growth. He develops a model in which technical change is both labor-augmenting and capital-augmenting and shows that, along the balanced growth path, all technical change will be labor-augmenting. If $\sigma < 1$, technical change stabilizes income shares,

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[5] Deviations between long-run and observed values bias parameter estimates in the cointegration model and are accounted for with the Stock and Watson (1993) correction that adds leads and lags (of the first-difference of the regressor) to the estimating equation and that has a substantial influence on the estimated β's.

and the balanced growth path is stable and unique. Antrás (2004) shows that, if σ differs from unity, factor shares can be constant if technical change is biased. The stochastic endogenous growth model of Turnovsky and Smith (2006) and the two-sector neoclassical growth model of Eicher and Turnovsky (1999) exhibit balanced growth for values of σ less than, equal to, or greater than unity. While the Cobb-Douglas technology is a sufficient condition for constant factor shares, it is not necessary.

Are there conditions under which the Cointegration Model delivers reliable estimates of the key production function parameter, $\beta = \sigma$? There are four cases to consider -- the presence or absence of cointegrating relations between the factor/output ratio and the relative factor price and the constancy or non-constancy of factor shares. For Case (i), assume that the cointegration relation holds -- as would be reasonable if labor is the factor -- and factor shares are constant. Per the resolution of the Paradox, equation (2) will not be useful for estimating technology parameters because the assumed conditions necessarily generate a unitary elasticity independent of the underlying technology.

Case (ii) also assumes that the cointegration relation holds but that factor shares vary (i.e., I(1)). Varying factor shares imply that $\beta \neq 1$ in equations (2) or (3). While Case (ii) allows for a range of estimated β's, the assumption that factor shares are I(1) seems empirically implausible and thus calls into question the relevance of Case (ii).

Cases (iii) and (iv) assume that cointegration does not hold, a situation that can occur for a variety of reasons. For example, when capital is the factor, the Solow growth model implies that the capital/output ratio and the relative factor price are I(0). Hence, a necessary condition for cointegration does not hold, and the parameter estimates from equation (2) are inconsistent. Alternatively, the variables in equation (2) may be I(1) but driven by different underlying processes. In this situation, cointegration does not obtain, equation (2) suffers from a spurious regression problem, and the estimated β, which may differ from unity, is suspect. We conclude that, while using long-run variation to estimate production function parameters is a desirable estimation strategy, the Cointegration Model will not be informative for estimating substitution elasticities.

References

Acemoglu, Daron, “Labor- And Capital-Augmenting Technical Change,” *Journal of the European Economic Association* 1 (2003), 1-37.

Antrás, Pol, "Is the U.S. Aggregate Production Function Cobb-Douglas?: New Estimates of the Elasticity of Substitution," *Contributions to Macroeconomics* 4 (2004), Article 4.

Blough, Stephen R., "The Relationship between Power and Level for Generic Unit Root Tests in Finite Samples," *Journal of Applied Econometrics* 7 (July/September 1992), 295-308.

Caballero, Ricardo J., “Small Sample Bias and Adjustment Costs,” *The Review Of Economics And Statistics* 76 (February 1994), 52-58.

Caballero, Ricardo J., “Aggregate Investment,” in John B. Taylor and Michael Woodford (eds.), *Handbook Of Macroeconomics*, Volume 1B (Amsterdam: Elsevier North-Holland), 1999), 813-862.

Caballero, Ricardo J., Engel, Eduardo M.R.A., and Haltiwanger, John C., "Plant-Level Adjustment and Aggregate Investment Dynamics," *Brookings Papers on Economic Activity* (1995:2), 1-54.

Carmichael, Jeffrey, and Stebbing, Peter W., "Fisher's Paradox and the Theory of Interest," *American Economic Review* 83 (September 1983), 619-630.

Eicher, Theo S., and Turnovsky, Stephen J., "Non-Scale Models of Economic Growth," *Economic Journal* 109 (July 1999), 394-415.

Felipe, Jesus, and Holz, Carsten A., "Why Do Aggregate Production Functions Work?: Fisher's Simulations, Shaikh's Identity, and Some New Results," *International Review of Applied Economics* 15 (2001), 261-285.

Fisher, Franklin M., "Aggregate Production Functions and the Explanation of Wages: A Simulation Experiment," *Review of Economics and Statistics* 53 (November 1971), 305-325. Reprinted in John Monz (ed.), *Aggregation: Aggregate Production Functions and Related Topics* (Cambridge, MIT Press, 1992), 173-214.

Fisher, Franklin M., Solow, Robert M., and Kearl, James M., "Aggregate CES Production Functions: Some CES Experiments," *Review of Economic Studies* 44 (June 1977), 305-320. Reprinted in John Monz (ed.), *Aggregation: Aggregate Production Functions and Related Topics* (Cambridge, MIT Press, 1992), 215-237

Kaldor, Nicholas, "Capital Accumulation and Economic Growth," in Freidrich A. Lutz and Douglas C. Hague (eds.), *The Theory of Capital* (New York: St. Martin's Press, 1961), 177-222.

King, Robert G., and Rebelo, Sergio T., “Resuscitating Real Business Cycles,” in John B. Taylor and Michael Woodford (eds.), *Handbook Of Macroeconomics,* Volume 1B (Amsterdam: Elsevier North-Holland), 1999), 927-1007.

Klein, Lawrence R., and Kosobud, Richard F., "Some Econometrics of Growth: Great Ratios of Economics," *Quarterly Journal of Economics* 75 (May 1961), 173-198. Reprinted in Jaime Marquez (ed.), *Economic Theory and Econometrics* (Philadelphia: University of Pennsylvania Press, 1985), 288-313.

Shaikh, Anwar, "Laws of Production and Laws of Algebra: The Humbug Production Function," *Review of Economics and Statistics* 56 (February 1974), 115-120.

Solow, Robert M., "Laws of Production and Laws of Algebra: The Humbug Production Function: A Comment," *Review of Economics and Statistics* 56 (February 1974), 121.

Solow, Robert M., "Second Thoughts on Growth Theory," in Alfred Steinherr and Daniel Weiserbs (eds.), Employment and Growth: Issues for the 1980s (Dordrecht and Boston: Kluwer Academic Publishers, 1987), 13-28.

Stock, James H., and Watson, Mark W., “A Simple MLE of Cointegrating Vectors in Higher Order Integrated Systems,” *Econometrica* 61 (July 1993), 783-820.

Turnovsky, Stephen J., and Smith, William T., "Equilbrium Consumption and Precautionary Savings in a Stochastically Growing Economy," *Journal of Economic Dynamics & Control* 30 (February 2006), 243-278.

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