Friedman Rule in a Model with Endogenous Growth and Cash-in-advance Constraint

Firouz Gahvari

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Abstract

This paper introduces money into an overlapping generations model with endogenous growth. The model, due to Docquier *et al.* (2007), exhibits a positive intergenerational externality which precludes its laissez-fair equilibrium to be optimal even if the government can control the level of physical capital and set it to satisfy the modified golden rule. The main message of the paper is that, as long as the modified golden rule is attained, Friedman rule is optimal. The result holds regardless of the ability of the government to internalize the externality and control the level of human capital. Other results include: (i) violation of Friedman rule for a different second-best environment wherein human capital accumulation is controlled but not physical capital accumulation; (ii) existence of a negative relationship between money growth rate and the economy's endogenous growth rate, and (iii) non-uniqueness of Friedman rule.

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Keywords: Friedman rule, endogenous growth, investment in human capital, overlapping generations, second best.

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1 Introduction

For over four decades, the optimal money supply literature has studied the environ-
ments under which the Friedman rule may or may not hold. One interesting result, in
the context of overlapping-generations models à la Diamond (1965), is the importance
of attaining the (modified) golden rule for the application of Friedman rule. Weiss
(1980) had originally argued that Friedman rule does not hold in overlapping genera-
tions models. Later, Able (1987) and Gahvari (1988) showed that Weiss's result was
due to the generic failure of the laissez-faire equilibrium of overlapping generations mod-
els to deliver the (modified) golden rule. Introducing generational lump-sum tax and
transfers, or a debt policy, to control capital accumulation allows the economy to attain
the (modified) golden rule and restores the optimality of Friedman rule.[1]

One important feature of Diamond's (1965) model is that it turns it into a first-best
environment through the control of economy's physical capital: The model contains no
other intrinsic sources of market failures (such as externalities). Extending the model to
include some form of market failure destroys this particular feature of Diamond's model.
This opens up the question of optimality of Friedman rule even if the (modified) golden
rule is attained. van der Ploeg and Alogoskoufis (1994), for example, demonstrate that
Friedman rule is violated in an overlapping generations model that exhibits endogenous
growth.[2] However, although van der Ploeg and Alogoskoufis allow for lump-sum tax-
ation and debt policy, they do not set the fiscal instruments in such a way as to fully
control the capital stock of the economy. Indeed, in their model, it is the non-neutrality

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[1]Gahvari (1988) went further and showed that the optimality of Friedman rule does not rest on the
attainment of the modified golden rule. If a switch to Friedman rule is accompanied by generation
specific lump-sum taxes that neutralize the ensuing intergenerational wealth transfers, Friedman rule
becomes optimal given any initial steady state laissez faire equilibrium. This finding, and the importance
of intergenerational wealth transfers appear to have gone unnoticed in the subsequent literature dealing
with the Friedman rule until its rediscovery by Bhattacharya *et al.* (2005). See, e.g., Freeman (1993),
and Smith (1991, 2002). An exception is Ireland (2005); this issue has been discussed in Gahvari (2007).

[2]They consider a setup as in Weil (1989) wherein individuals are infinitely-lived but that new gen-
erations are born every period. They also rationalize money by putting real balances in the utility
function.

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of monetary policy that leads to the breakdown of Friedman rule.

Two questions arise regarding the optimality of Friedman rule when physical capital is fully controlled. One concerns an environment wherein the government has additional fiscal instruments to offset the other sources of distortion in the economy. The second environment allows for no additional instruments. The latter environment is particularly interesting in light of Phelps (1973) who raised one of the earliest objections to Friedman rule. Phelps' argument relied on an application of the general theory of second best; see Lipsey and Lancaster (1956). This well-known proposition in public finance teaches us that familiar first-best prescriptions often do not hold in second-best environments. Put differently, if there already exists a distortion in the economy, adding another source of distortion can enhance welfare. Applied to a monetary economy, the lesson is that the violation of Friedman rule may in fact be welfare increasing in a second-best environment.

The current paper addresses both of these questions and studies if the control of physical capital ensures the optimality of Friedman rule in overlapping generations model that exhibit an inherent source of distortion—with and without correcting the distortion. The model that I use for this purpose is due to Docquier *et al.* (2007) who have recently extended Diamond's model to allow for endogenous growth that emanates from building up one's human capital. This model is interesting not only because it allows for (per capita) growth that does not exist in Diamond (1965), but also the fact that its laissez-faire equilibrium exhibits an additional source of market failure through an intergenerational externality. The externality is due to the positive effect of investment in education on the human capital of not just the investor, but his children as well. It arises when one's human capital is determined partly through education and partly through the human capital one inherits from his parents. However, in deciding how much to spend on education, individuals ignore the effect that their decision has on the human capital of their children.

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To address these questions, the paper introduces money in Docquier *et al.*'s (2007) model and rationalize it through a cash-in-advance constraint.[3] It derives the dynamics of the model and characterize its balanced growth path under laissez faire. Subsequently, it derives the first-best allocations of this economy and shows that they are not affected by the introduction of money.[4] It proves that the implementation of the first-best, i.e. when both physical and human capital are fully controlled, requires Friedman rule to be satisfied. This result generalizes the earlier result of Able (1987) and Gahvari (1988) to overlapping generations models with endogenous growth. It also puts van der Ploeg and Alogoskoufis' result to the contrary in the right perspective.

Subsequently, the paper proves that Friedman rule holds even if one cannot control human capital accumulation and internalize the externality of education. On the face of it, this result appears to be surprising in the light of Phelps (1973) and the second-best logic. The key to understanding it is that introducing another distortion in the economy via the violation of Friedman rule does nothing to alleviate the existing distortion in human capital accumulation, as long as one fully controls the economy's stock of physical capital. The paper highlights this point.

Other grounds covered by this paper include: (i) A comparison between the values that other economic variables (besides the nominal interest rate) assume in the first- and second-best environments studied. Whereas the value of the nominal interest rate is zero in both environments, the second-best entails a lower endogenous rate of growth and a lower monetary growth rate. (ii) The study of a different second-best environment; one in which human capital accumulation can be controlled but not physical capital accumulation. The paper proves that this setting calls for the violation of Friedman rule. In this case, the distortion due to the violation of Friedman rule does alleviate the distortion due to the lack of physical capital accumulation. (iii) The existence of a

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[3] This is done along the lines of Hahn and Solow (1995). See also Crettez *et al.* (1999, 2002), and Michel and Wigniolle (2005).

[4] Docquier *et al.*'s (2007) result applies and the relationship between laissez faire and first-best solutions remain the same as in their paper.

negative relationship between money growth rate on the one hand and the endogenous growth rate of the economy on the other. (iv) Non-uniqueness of Friedman rule: A continuum of monetary growth rates and tax rates on consumption during working years result in a zero opportunity cost of holding money.[5]

The paper is related to the vast literature on Friedman rule particularly those written in the context of endogenous growth and/or within the overlapping generations framework. van der Ploeg and Alogoskoufis (1994) referred to earlier is one. Paal and Smith (2000) discuss the suboptimality of Friedman rule in a monetary growth model where spatial separation and limited communication rationalize money holding for transaction purposes. There are no (offsetting) fiscal instruments in their model and the suboptimality of Friedman rule is caused by the impact of bank portfolio reallocations on the real economy. Other similar papers that discusses the effects of monetary growth on the real economy within an endogenous growth framework include Wang and Yip (1992), Gomme (1993), van der Ploeg and Alogoskoufis (1994), Jones and Manuelli (1995), Chang and Lai (2000), Dotsey and Sarte (2000), Chang *et al.* (2007), and Bhattacharya *et al.* (2008).

2   The model and its laissez-faire equilibrium

Consider the Diamond's (1965) two-period overlapping generations model wherein individuals work in the first period supplying one unit of labor, derive utility from consuming a composite consumption good c in the first period, and d in the second period. There is no bequest motive, and population grows at a constant rate, n. Append to this model (i) money holdings and (ii) human capital accumulation as modeled by Docquier *et al.* (2007). To stay within a two-period formulation, assume that the decisions on human capital accumulation, production, and consumption are all undertaken at the beginning

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[5] The results in (iii) and (iv) generalize those for Diamond's overlapping generations model with cash-in-advance constraint but in the absence of endogenous growth. They are opposite the results obtained with money in the utility function [see Gahvari (1988, 2007)].

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of period one and in the order stated.[6] Output of each period can be used for consumption in the same period, or retained with no depreciation, to be used next period as an input either in the educational process of the young or the production process.

2.1 Education

At the beginning of time t, the young start life with a given amount of human capital, h_{t-1}, that they have inherited from their parents. I shall refer to persons born in calendar time t as members of generation t. Thus those who are old at time t are members of generation time $t-1$. Education combined with one's inherited human capital determines a worker's *effective* labor. Thus, investing e_t in the young's education raises their effective labor from h_{t-1} to h_t. The human capital formation technology is characterized by the linear homogeneous function $\Phi(e_t, h_{t-1})$. That is,

$$h_t = \Phi(e_t, h_{t-1}) = h_{t-1}\varphi\left(\frac{e_t}{h_{t-1}}\right), \tag{1}$$

where $\varphi(\cdot) \equiv \Phi(\cdot, 1)$. I assume that $\varphi(\cdot)$ is positive, increasing, and strictly concave; it also satisfies the Inada conditions: $\varphi'(0) = \infty$ and $\varphi'(\infty) = 0$.[7]

When deciding on how much to invest in their education, the young have no resources. They thus borrows e_t each for their own education from the old of the previous generation. In this way, each young person raises his effective labor from h_{t-1} to h_t. In deciding how much to educate himself, the individual choose e_t to maximize his net earnings. Let w_t denote the real wage at time t (measured in units of composite consumption good), and r_t the real interest rate. The individual chooses e_t to maximize $w_t h_t - e_t(1 + r_t)$ subject to (1). This yields a solution for e_t characterized by

$$\varphi'\left(\frac{e_t}{h_{t-1}}\right) = \frac{1 + r_t}{w_t}. \tag{2}$$

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[6]This formulation is a "short cut" to stay within a two-period overlapping generations model. A more realistic formulation allows for three periods, the first of which is dedicated to the education of children.

[7]This process is also a "short-cut" in that one does not model the precise nature of the contribution of a parent's human capital to that of his children and the resources that may have to be spent in this process.

Observe that $\varphi(e_t/h_{t-1}) > 1$; otherwise there will be no investment in education.

2.2 Production

Production takes place after education. The production technology, which also exhibits constant returns to scale, uses capital, K_t, and *effective* labor, H_t, to produce a composite output, $Y_t = F(K_t, H_t)$. Let N_t denote the number of young persons—equivalently workers—at time t, and define output, capital, and effective labor per worker according to $y_t = Y_t/N_t$, $k_t = K_t/N_t$, $h_t = H_t/N_t$. The production function can then be presented by $y_t = F(k_t, h_t)$. Assuming a competitive setting w_t and r_t are determined according to

$$w_t = F_h(k_t, h_t), \tag{3}$$

$$r_t = F_k(k_t, h_t). \tag{4}$$

At the beginning of period t, prior to the educational and production decisions, the *sum* of aggregate capital to be used for educational investment of the young, $N_t e_t$, and for production, K_t, is pre-determined from the savings decisions of the old at time $t-1$ when they were young. Moreover, given that the number of young individuals N_t is also exogenously given, $e_t + k_t$ is given at time t. With $e_t + k_t$ given, the additional four equations (1), (2), (3), and (4) determine the equilibrium values for k_t, e_t, h_t, w_t, and as functions of h_{t-1}.

2.3 Money and the current price level

Holdings of money is rationalized through a Clower cash-in-advance constraint. I assume that it takes the form of

$$M_t \geqq \alpha N_{t-1} p_t d_t, \tag{5}$$

where M_t is the aggregate stock of money at time t, p_t is the price level at time t and $\alpha < 1$ is the proportion of the old's consumption that has to be financed through cash.

The old's money holdings at t consist of two components: their own cash savings from the previous period and a lump-sum money transfer from (or to) the government, s_t. All money injections to (or subtractions from) the economy are in the form of these lump-sum transfers. The lump-sum transfer to each old person is set equal to a fraction, θ, of his money holdings from the previous period which he carries into the current period. That is,

$$s_t = \theta \frac{M_{t-1}}{N_{t-1}},$$

and money stock grows at the rate of θ per period according to

$$M_t = M_{t-1} + N_{t-1} s_t = (1 + \theta) M_{t-1}. \tag{6}$$

At the end of the production process, each young person receives $w_t h_t$ out of which he has to pay his educational loan to the old plus interest, $e_t(1 + r_t)$, and use the rest for his consumption and savings that will be channeled into the purchase of all real assets and the existing cash in the economy. The resources of the old consist of the educational loan they had given the young plus the associated interest, the capital stock they brought into production process plus interest, their cash holdings, and their lump-sum money transfers. Their budget constraint is given by

$$p_t d_t = \frac{p_t \left(K_t + N_t e_t\right)\left(1 + r_t\right) + M_{t-1}}{N_{t-1}} + s_t. \tag{7}$$

Assume that the cash constraint (5) is binding so that $M_t/N_{t-1} p_t = \alpha d_t$. Substituting for d_t from this relationship into (7), while also substituting $\theta M_{t-1}/N_{t-1}$ for s_t. Rearranging the terms yields,

$$\frac{M_t}{N_t} \left(\frac{1 - \alpha}{\alpha}\right) = p_t \left(k_t + e_t\right)\left(1 + r_t\right). \tag{8}$$

Equation (8) then determines the price level, p_t. Observe that p_t depends on M_t, and through (6), on θ.

2.4 Consumption and saving

Observe first that consumption level of the current old, d_t, is determined through equation (7). It depends on the value of p_t (which itself depends on θ). Turning to the young, their preferences are represented by

$$u = u(c_t, d_{t+1}). \tag{9}$$

The utility function $u(\cdot, \cdot)$ is strictly quasi-concave and twice differentiable. The young consume c_t each, buy the existing money stock, M_t, and save the rest of their income, $K_{t+1} + N_{t+1}e_{t+1}$. Their budget constraint is given by

$$p_t c_t + \frac{p_t\left(K_{t+1} + N_{t+1}e_{t+1}\right) + M_t}{N_t} = p_t w_t h_t - p_t e_t(1 + r_t). \tag{10}$$

They choose the values of their current consumption,money holdings,and real savings based on their expectation for the future interest rate, r_{t+1}^e, and the price level, p_{t+1}^e. In doing so, the young realize that their expected future consumption d_{t+1}^e will have be financed from their savings according to their budget constraint when old

$$p_{t+1}^e d_{t+1}^e = \frac{p_{t+1}^e\left(K_{t+1} + N_{t+1}e_{t+1}\right)\left(1 + r_{t+1}^e\right) + M_t}{N_t} + s_{t+1}. \tag{11}$$

Eliminating $(K_{t+1} + N_{t+1}e_{t+1})$ between equation (10) and (11) results in the young's intertemporal budget constraint

$$c_t + \frac{d_{t+1}^e}{1 + r_{t+1}^e} = w_t h_t - e_t(1 + r_t) + \left[\frac{p_t}{p_{t+1}^e(1 + r_{t+1}^e)} - 1\right]\frac{M_t}{N_t p_t} + \frac{s_{t+1}}{p_{t+1}^e(1 + r_{t+1}^e)}. \tag{12}$$

Now to consume d_{t+1}^e, the young must hold an amount of cash that satisfies the Clower cash constraint

$$\alpha\, d_{t+1}^e = \frac{M_{t+1}}{N_t p_{t+1}^e} = \frac{(1 + \theta)\, M_t}{N_t p_t}\frac{p_t}{p_{t+1}^e}. \tag{13}$$

Denote the inflation rate by π_t, and the nominal interest rate by i_t so that

$$\frac{p_{t+1}}{p_t} \equiv 1 + \pi_{t+1}, \tag{14}$$

$$1 + i_{t+1} \equiv (1 + r_{t+1})(1 + \pi_{t+1}). \tag{15}$$

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Using the above notation, applied to expectations, substituting for $M_t/(N_t p_t)$ from (13) into (12), and simplifying, one arrives at

$$c_t + \frac{1}{1 + r_{t+1}^e}\left(1 + \frac{\alpha\, i_{t+1}^e}{1 + \theta}\right) d_{t+1}^e = w_t h_t - e_t(1 + r_t) + \frac{s_{t+1}}{p_{t+1}^e(1 + r_{t+1}^e)}. \tag{16}$$

Observe that not all of the above equations are independent from one another. Moreover, the Clower cash constraint has previously been used in the determination of $k_t, e_t, h_t, w_t, r_t,\ M_t$, and p_t. The independent equations that remain for the determination of the laissez-faire equilibrium are (10), (14), (15), and (16).

The problem of the young can then be formulated as that maximizing the utility function (9) subject to the intertemporal budget constraint (16) and the human capital formation function (1), when s_{t+1} is treated as lump sum. This results in the first-order condition

$$\frac{u_d(c_t, d_{t+1}^e)}{u_c(c_t, d_{t+1}^e)} = \frac{1 + \alpha\, i_{t+1}^e/(1 + \theta)}{1 + r_{t+1}^e}, \tag{17}$$

which is another independent equation. Two of these five equations, (16) and (17), determine c_t, and d_{t+1}^e as functions of the already determined variables, e_t, h_t, w_t, r_t, p_t, as well as the policy parameters, s_{t+1}, θ, and the expected variables, r_{t+1}^e, p_{t+1}^e, and i_{t+1}^e. Using equations (14) and (15), one can eliminate i_{t+1}^e from the list of expected variables. A perfect foresight assumption, governing the determination of r_{t+1}^e, and p_{t+1}^e ($r_{t+1}^e = r_{t+1}$ and $p_{t+1}^e = p_{t+1}$), closes the model.

As a final observation, note that the determination of d_{t+1}^e fixes, via (10), the value of $(K_{t+1} + N_{t+1}e_{t+1})$, the aggregate amount of capital to be used for educational investment and for production in period $t + 1$. This confirms my previous assumption, when discussing education and production decisions, that this sum is pre-determined at the beginning of each period from the decisions of the previous period.

2.5 Dynamics

First, to simplify the exposition, rewrite the variables of the model as a fraction of effective labor. Denote $\widehat{e}_t \equiv e_t/h_{t-1}$ and thus rewrite equations (1) and (2) as[8]

$$
\begin{aligned}
h_t &= h_{t-1}\varphi\left(\widehat{e}_t\right), \\
\varphi'\left(\widehat{e}_t\right) &= \frac{1+r_t}{w_t}.
\end{aligned}
$$

Similarly, denote $\widehat{y}_t = Y_t/H_t = y_t/h_t$ and $\widehat{k}_t = K_t/H_t = k_t/h_t$. This allows the production function to be represented by $\widehat{y}_t = f(\widehat{k}_t)$, where $f(\cdot)$ is positive, increasing, and strictly concave. One can then rewrite equations (3) and (4) as

$$
\begin{aligned}
w_t &= f(\widehat{k}_t) - \widehat{k}_t f'(\widehat{k}_t), \\
r_t &= f'(\widehat{k}_t).
\end{aligned}
$$

Observe that the values of $\widehat{e}_t$ and $\widehat{k}_t$ will be determined as soon as e_t and k_t are determined.

Now, to examine the dynamic evolution of this economy, consider the following three equations: (10), (16), and (17), where I have replaced the variables in expectation by their "actual" values and dropped all superscripts T. First, from the Clower cash constraint, substitute $\alpha\, d_t$ for $M_t/\left(N_{t-1}p_t\right)$ in equation (10), then divide it by $p_t h_t$, and substitute in it $\widehat{e}_{t+1}$ for e_{t+1}/h_t, $\varphi\left(\widehat{e}_{t+1}\right)$ for h_{t+1}/h_t, and $\varphi\left(\widehat{e}_t\right)$ for h_t/h_{t-1}. Simplifying yields

$$
(1+n)\left[\widehat{k}_{t+1}\varphi\left(\widehat{e}_{t+1}\right) + \widehat{e}_{t+1}\right] + \frac{\alpha}{1+n}\frac{\widehat{d}_t}{\varphi\left(\widehat{e}_t\right)} = w_t - (1+r_t)\frac{\widehat{e}_t}{\varphi\left(\widehat{e}_t\right)} - \widehat{c}_t, \qquad (18)
$$

where $\widehat{c}_t \equiv c_t/h_t$ and $\widehat{d}_t \equiv d_t/h_{t-1}$.

Second, turning to (16), substitute $\theta M_t/N_t$ for s_{t+1}, $M_{t+1}/\left(1+\theta\right)$ for M_t, $p_{t+1}(1+r_{t+1})$ for $p_t(1+i_{t+1})$, $\alpha\, d_{t+1}$ for $M_{t+1}/\left(p_{t+1}N_t\right)$, to rewrite it as

$$
c_t + \left[1 + \frac{\alpha(i_{t+1}-\theta)}{1+\theta}\right]\frac{d_{t+1}}{1+r_{t+1}} = w_t h_t - (1+r_t)e_t.
$$

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[8]From now on whenever a "hat" appears over a variable, it denotes that the variable is expressed per unit of effective labor.

Divide this equation by h_t, and substitute $\varphi(\widehat{e}_t)$ for h_t/h_{t-1} to get

$$\widehat{c}_t + \left[1 + \frac{\alpha(i_{t+1} - \theta)}{1 + \theta}\right] \frac{\widehat{d}_{t+1}}{1 + r_{t+1}} = w_t - (1 + r_t)\frac{\widehat{e}_t}{\varphi(\widehat{e}_t)}. \tag{19}$$

Finally, assuming that the utility function (9) is homothetic, the marginal rate of substitution between $\widehat{c}_t$ and $\widehat{d}_{t+1}$ and between c_t and d_{t+1} will be the same. It then follows from the first-order condition (17) that

$$\frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_c\left(\widehat{c}_t, \widehat{d}_{t+1}\right)} = \frac{1 + \alpha\, i_{t+1}/(1 + \theta)}{1 + r_{t+1}}. \tag{20}$$

Now eliminate $\widehat{c}_t$ between equations (18), (19), (20), and solve the remaining two equations for $\widehat{k}_{t+1}$ and $\widehat{d}_{t+1}$. Noting that i_{t+1} depends on π_{t+1} and r_{t+1}, w_t and r_t on $\widehat{k}_t$, and r_{t+1} on $\widehat{k}_{t+1}$, one will get: $\widehat{k}_{t+1} = \widehat{k}_{t+1}\left(\widehat{k}_t, \widehat{d}_t, \widehat{e}_{t+1}, \widehat{e}_t, \pi_{t+1}\right)$ and $\widehat{d}_{t+1} = \widehat{d}_{t+1}\left(\widehat{k}_t, \widehat{d}_t, \widehat{e}_{t+1}, \widehat{e}_t, \pi_{t+1}\right)$. Then observe, from the Clower cash constraint (13), that

$$\frac{1 + \theta}{1 + n} \frac{d_t}{d_{t+1}} = 1 + \pi_{t+1}.$$

Multiply the numerator by h_{t-1}/h_{t-1} and the denominator of this relationship by h_t/h_t to rewrite it as

$$\frac{1 + \theta}{1 + n} \frac{h_{t-1}d_t/h_{t-1}}{h_t d_{t+1}/h_t} = \frac{1 + \theta}{1 + n} \frac{1}{\varphi(\widehat{e}_t)} \frac{\widehat{d}_t}{\widehat{d}_{t+1}} = 1 + \pi_{t+1} \tag{21}$$

Using (21) to eliminate π_{t+1} in $\widehat{k}_{t+1} = \widehat{k}_{t+1}(\widehat{k}_t, \widehat{d}_t, \widehat{e}_{t+1}, \widehat{e}_t, \pi_{t+1})$ and $\widehat{d}_{t+1} = \widehat{d}_{t+1}(\widehat{k}_t, \widehat{d}_t, \widehat{e}_{t+1}, \widehat{e}_t, \pi_{t+1})$, one gets two new expressions for $\widehat{k}_{t+1}$ and $\widehat{d}_{t+1}$ which I show, with an abuse of notation, by $\widehat{k}_{t+1} = \widehat{k}_{t+1}\left(\widehat{k}_t, \widehat{d}_t, \widehat{e}_{t+1}, \widehat{e}_t\right)$ and $\widehat{d}_{t+1} = \widehat{d}_{t+1}\left(\widehat{k}_t, \widehat{d}_t, \widehat{e}_{t+1}, \widehat{e}_t\right)$. Finally, recall that the first-order condition (2) determines $\widehat{e}_t$ as a function of $\widehat{k}_t$, and thus $\widehat{e}_{t+1}$ as a function of $\widehat{k}_{t+1}$. Using these relationships in $\widehat{k}_{t+1} = \widehat{k}_{t+1}\left(\widehat{k}_t, \widehat{d}_t, \widehat{e}_{t+1}, \widehat{e}_t\right)$ and $\widehat{d}_{t+1} = \widehat{d}_{t+1}\left(\widehat{k}_t, \widehat{d}_t, \widehat{e}_{t+1}, \widehat{e}_t\right)$ leads to a pair of first-order difference equations of the form

$$\widehat{k}_{t+1} = \Psi\left(\widehat{k}_t, \widehat{d}_t\right), \tag{22}$$
$$\widehat{d}_{t+1} = \Phi\left(\widehat{k}_t, \widehat{d}_t\right). \tag{23}$$

The system of difference equations (22)–(23) determines the dynamic path of the economy. Moreover, the economy tends to a balanced growth path if equations (22)–(23) have a steady state solution and if it is stable. Under the latter two assumptions, the steady state values of $\left(\widehat{k}_t, \widehat{d}_t\right)$ will be the rest point of equations (22)–(23).[9]

To examine the (local) stability properties of the steady-state solution, I linearize the system of difference equations (22)–(23) around the steady-state solution $\left(\widehat{k}, \widehat{d}\right)$ according to

$$\left(\begin{array}{c} \widehat{k}_{t+1} - \widehat{k} \\ \widehat{d}_{t+1} - \widehat{d} \end{array} \right) = \left(\begin{array}{cc} \Psi_k\left(\widehat{k}, \widehat{d}\right) & \Psi_d\left(\widehat{k}, \widehat{d}\right) \\ \Phi_k\left(\widehat{k}, \widehat{d}\right) & \Phi_d\left(\widehat{k}, \widehat{d}\right) \end{array} \right) \left(\begin{array}{c} \widehat{k}_t - \widehat{k} \\ \widehat{d}_t - \widehat{d} \end{array} \right) \equiv \Omega \left(\begin{array}{c} \widehat{k}_t - \widehat{k} \\ \widehat{d}_t - \widehat{d} \end{array} \right), \qquad (24)$$

where $\Psi_k, \Psi_d, \Phi_k,$ and Φ_d denote the partial derivatives of $\Psi(\cdot)$ and $\Phi(\cdot)$ with respect to k and d. The dynamic path given by (24) converges to a steady state as t increases (i.e., $\widehat{k}_t - \widehat{k}$ and $\widehat{d}_t - \widehat{d}$ tend to zero), if at time $t = 0$, the initial values of $\widehat{k}_0$ and $\widehat{d}_0$ are such that $\left(\widehat{k}_0 - \widehat{k}, \widehat{d}_0 - \widehat{d}\right)$ is in the space spanned by the eigenvectors of Ω that are associated with the eigenvalues of Ω with modulus smaller than one. Now, at $t = 0$, the value of $\widehat{k}_0$ is pre-determined. However, the value of $\widehat{d}_0$ depends on p_0. Consequently, the system will be stable for any value of p_0 if Ω possesses two eigenvalues with modulus less than one. If there is only one eigenvalue with a modulus less than one, then there will be one value for p_0 and a unique path which leads to the steady state.

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[9]The regularity conditions one imposes on preferences and the technology ensure only that positive values exist for the variables along a solution path. There is no guarantee, however, that there exists a steady-state solution to which the system is driven.

2.6 Balanced growth

At the steady state, equations (20), (19), (18), (21), (15), (2), (4), and (3) are simplified to

$$\frac{u_d\left(\widehat{c},\widehat{d}\right)}{u_c\left(\widehat{c},\widehat{d}\right)} = \frac{1+\alpha\,i/\left(1+\theta\right)}{1+r}, \tag{25}$$

$$\widehat{c} + \left[1 + \frac{\alpha\left(i-\theta\right)}{1+\theta}\right]\frac{\widehat{d}}{1+r} = w - \left(1+r\right)\frac{\widehat{e}}{\varphi\left(\widehat{e}\right)}, \tag{26}$$

$$\left(1+n\right)\left[\widehat{k}\,\varphi\left(\widehat{e}\right)+\widehat{e}\right] + \frac{\alpha}{1+n}\frac{\widehat{d}}{\varphi\left(\widehat{e}\right)} = w - \left(1+r\right)\frac{\widehat{e}}{\varphi\left(\widehat{e}\right)} - \widehat{c}, \tag{27}$$

$$1+\pi = \frac{1+\theta}{\left(1+n\right)\varphi\left(\widehat{e}\right)}, \tag{28}$$

$$1+i = \left(1+r\right)\frac{1+\theta}{\left(1+n\right)\varphi\left(\widehat{e}\right)}, \tag{29}$$

$$\varphi'\left(\widehat{e}\right) = \frac{1+r}{w}, \tag{30}$$

$$r = f'(\widehat{k}), \tag{31}$$

$$w = f(\widehat{k}) - \widehat{k}f'(\widehat{k}). \tag{32}$$

These equations determine the steady state values of real variables $\widehat{c},\widehat{d},\widehat{e},\widehat{k},w,r$, and monetary variables i, and π. Observe that while these values remain unchanged, the values of the other variables of the model grow over time at a constant rate. This is the balanced growth path of the economy. Specifically, let $g \equiv \varphi\left(\widehat{e}\right) - 1$. Then, $e_t = (1+g)e_{t-1}$, $h_t = (1+g)h_{t-1}$, $k_t = (1+g)k_{t-1}$, $c_t = (1+g)c_{t-1}$, $d_t = (1+g)d_{t-1}$, and $K_t = (1+n)(1+g)K_{t-1}$. Utility also grows over time. In particular, if the utility function is homogeneous of degree β, $u(c_t, d_{t+1}) = u\left(h_t\widehat{c}_t, h_t\widehat{d}_{t+1}\right) = h_t^\beta u\left(\widehat{c}_t, \widehat{d}_{t+1}\right)$. On the balanced growth path, $u(c_{t+1}, d_{t+2}) = h_{t+1}^\beta u\left(\widehat{c}_{t+1}, \widehat{d}_{t+2}\right) = (h_{t+1}/h_t)^\beta h_t^\beta u\left(\widehat{c}_t, \widehat{d}_{t+1}\right) = (1+g)^\beta u(c_t, d_{t+1})$.[10] Thus all per capita terms, h_t, e_t, k_t, c_t, d_t, increase by a factor of

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[10]With $s_{t+1} = \theta M_t / N_t$,

$$\frac{s_{t+1}}{s_t} = \frac{\theta M_t / N_t}{\theta M_{t-1}/N_{t-1}} = \frac{M_t}{M_{t-1}}\frac{N_{t-1}}{N_t} = \frac{1+\theta}{1+n}.$$

$(1 + g)$ per period and the utility increases by a factor of $(1 + g)^{\beta}$. Note that $\beta > 0$.[11]

The following proposition summarizes these results:

Proposition 1 *Consider a version of Diamond's (1965) overlapping-generations wherein each generation's human capital is determined via the level they inherit from their parents and their own educational attainment. Assume further that money is an alternative asset to physical capital and required for second period transactions. On a balanced growth path, per capita educational expenditures, human capital, physical capital, consumption during working years, and consumption during retirement all grow at a constant rate g determined by the human capital production function. The monetary variables of the economy, the inflation rate and the nominal rate of interest, remain constant and vary with the rate of money growth, θ. Equations (25)–(32) characterize the laissez faire balanced growth path of the economy.*

3 First best

Let the social welfare function be presented by the discounted sum of the average of all generations' lifetime utilities. The first best is then characterized by maximizing this function subject to every generation's human capital formation and resource constraint. The human capital formation is given by equation (1); the resource constraint for the generation born at time t is given by

$$K_t + F(K_t, H_t) = N_t c_t + N_{t-1} d_t + K_{t+1} + N_{t+1} e_{t+1},$$

Observe also that the above implies, using (21),

$$\frac{s_{t+1}/p_{t+1}h_t}{s_t/p_t h_{t-1}} = \frac{s_{t+1}}{s_t} \frac{p_t}{p_{t+1}} \frac{h_{t-1}}{h_t} = \frac{1+\theta}{1+n} \frac{1}{1+\pi_{t+1}} \frac{1}{\varphi(\widehat{e}_t)} = \frac{\widehat{d}_{t+1}}{\widehat{d}_t}.$$

Now with $\widehat{d}_{t+1}/\widehat{d}_t = 1$ in the steady state, $s_{t+1}/p_{t+1}h_t$ remains constant over time.

[11]Otherwise, with $g > 0$, we have c_t and d_{t+1} increasing per period, while $u(c_t, d_{t+1})$ remains the same or is decreasing.

which one can alternatively write, by dividing it by N_t, as

$$k_t + h_t f\left(\frac{k_t}{h_t}\right) = c_t + \frac{1}{1+n}d_t + (1+n)\left(k_{t+1} + e_{t+1}\right).$$

The Lagrangian expression for this optimization problem can then be written as,

$$\pounds \equiv \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t}\left\{ u\left(c_t, d_{t+1}\right) + \mu_t \left[h_t\varphi\left(\frac{e_{t+1}}{h_t}\right) - h_{t+1} \right]\right.$$
$$\left. + \lambda_t \left[k_t + h_t f\left(\frac{k_t}{h_t}\right) - c_t - \frac{1}{1+n}d_t - (1+n)\left(k_{t+1} + e_{t+1}\right) \right] \right\} \qquad (33)$$

where $\rho \in (0,1)$ is the discount factor reflecting the "planner's" social time preference, and $\mu_t/(1+\rho)^t$ and $\lambda_t/(1+\rho)^t$ are the multipliers associated with the resource constraint and the human capital equation at time t. The first order conditions with respect to $c_t, d_{t+1}, h_{t+1}, k_{t+1}$, and e_{t+1}, are

$$u_c(c_t, d_{t+1}) = \lambda_t, \qquad (34)$$

$$u_d(c_t, d_{t+1}) = \frac{\lambda_{t+1}}{(1+n)(1+\rho)}, \qquad (35)$$

$$(1+\rho)\mu_t = \lambda_{t+1}\left[f(\widehat{k}_{t+1}) - \widehat{k}_{t+1}f'(\widehat{k}_{t+1}) \right] + \mu_{t+1}\left[\varphi(\widehat{e}_{t+2}) - \widehat{e}_{t+2}\varphi'(\widehat{e}_{t+2}) \right], \quad (36)$$

$$\lambda_t(1+\rho)(1+n) = \lambda_{t+1}\left[1 + f'(\widehat{k}_{t+1}) \right], \qquad (37)$$

$$\mu_t\varphi'(\widehat{e}_{t+1}) = \lambda_t(1+n). \qquad (38)$$

Finally, the transversality condition is:

$$\lim_{t\to\infty}\frac{\lambda_t k_{t+1} + \mu_t h_{t+1}}{(1+\rho)^t} = 0. \qquad (39)$$

In characterizing the optimal values of the economic variables, I originally do not mix the implications of either (37) or (38) with those of other first-order conditions of the problem. This facilitates the characterization of second-best outcomes without the instruments for controlling k_t and e_t which I shall discuss later. Thus derive an expression for λ_{t+1}/λ_t once from (35) by writing it for $t+1$ and t and dividing one by

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the other, and once from dividing (35) by (34). One gets the following two equations

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{u_d(c_t, d_{t+1})}{u_d(c_{t-1}, d_t)} = \frac{h_t^{\beta-1} u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{h_{t-1}^{\beta-1} u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)} = [\varphi\left(\widehat{e}_t\right)]^{\beta-1} \frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)}, \tag{40}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = (1+n)(1+\rho)\frac{u_d(c_t, d_{t+1})}{u_c(c_t, d_{t+1})} = (1+n)(1+\rho)\frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_c\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}. \tag{41}$$

It then follows from (40)–(41) that

$$\frac{u_d(c_t, d_{t+1})}{u_c(c_t, d_{t+1})} = \frac{[\varphi\left(\widehat{e}_t\right)]^{\beta-1}}{(1+n)(1+\rho)} \frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)}. \tag{42}$$

Next, substituting from (4) in (37) yields $\lambda_{t+1}/\lambda_t = (1+\rho)(1+n)/(1+r_{t+1})$. Equating this to the expression for λ_{t+1}/λ_t in (41) and using (42)

$$1 + r_{t+1} = \frac{(1+n)(1+\rho)}{[\varphi\left(\widehat{e}_t\right)]^{\beta-1}} \frac{u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)}{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}. \tag{43}$$

Finally, use equation (38) to substitute for μ_t and μ_{t+1} into (36). Divide the resulting equation by λ_t, substitute for λ_{t+1}/λ_t from (40), and manipulate to arrive at

$$\varphi'\left(\widehat{e}_{t+1}\right) = \frac{1+n}{w_{t+1}} \left\{ [\varphi\left(\widehat{e}_t\right)]^{1-\beta}(1+\rho)\frac{u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)}{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)} - \frac{\varphi'\left(\widehat{e}_{t+1}\right)}{\varphi'\left(\widehat{e}_{t+2}\right)} \left[\varphi(\widehat{e}_{t+2}) - \widehat{e}_{t+2}\varphi'\left(\widehat{e}_{t+2}\right)\right] \right\}. \tag{44}$$

3.1   Balanced growth

On the balanced growth path, $\widehat{c}_t$ and $\widehat{d}_{t+1}$ remain constant over time so that equations (42), (43), and (44) simplify to

$$\frac{u_d\left(\widehat{c}, \widehat{d}\right)}{u_c\left(\widehat{c}, \widehat{d}\right)} = \frac{1}{(1+n)(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}}, \tag{45}$$

$$1 + r = (1+n)(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}, \tag{46}$$

$$\varphi'\left(\widehat{e}\right) = \frac{1+n}{w}\left\{(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta} - \left[\varphi(\widehat{e}) - \widehat{e}\varphi'\left(\widehat{e}\right)\right]\right\}. \tag{47}$$

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Now, given that on a balanced-growth path the transversality condition (39) reduces to[12]

$$\frac{[\varphi(\widehat{e})]^{\beta}}{1+\rho} < 1, \tag{48}$$

in what follows I assume $\beta < 1$.[13]

Equation (45) shows the first-best marginal rate of intertemporal substitution in consumption. It is the counterpart of equation (17) for the laissez faire. For the two be the same, not only (46) must hold but $i = 0$ so that the opportunity cost of holding money is zero (as required by Friedman rule). Equation (46) is the modified golden rule condition; this is generally not satisfied in the laissez faire.

Substituting from equation (46) into equation (47), one gets

$$\varphi'(\widehat{e}) = \frac{1+r}{w} - \frac{(1+n)[\varphi(\widehat{e}) - \widehat{e}\varphi'(\widehat{e})]}{w}. \tag{49}$$

This is the counterpart of equation (2). The two differ in that the laissez faire equation does not include the right-hand side terms in braces. This reflects the externality that parents bestow on their children by educating themselves. Recall that education increases one's human capital which also enhances the human capital of one's children. Parents do not take this externality into account when deciding on their own educational attainment. Observe also that this is a positive externality. The above equation is in accordance with this in that with $\varphi(\cdot)$ being concave, $\varphi(\widehat{e}) - \widehat{e}\varphi'(\widehat{e}) > 0$ so that $\varphi'(\widehat{e}) < [1+r]/w$.

With the first-best allocations being the same as in Docquier *et al.* (2007), it is plain that the introduction of money leaves the relationship between laissez faire and first-best solutions intact as in their model. That is, comparison of the first-best values of $(\widehat{k}, \widehat{e})$, which vary with the social planner's discount rate, with their laissez-faire counterparts reveals three possible regimes: First-best values are larger for both $\widehat{k}$ and $\widehat{e}$; first-best

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[12]Using (46), condition $[\varphi(\widehat{e})]^{\beta} < 1 + \rho$ also implies $\varphi(\widehat{e}) < (1+r)/(1+n)$ so that at the first best, $r > n$.

[13]This assumption makes the satisfaction of (48) easier; it is neither necessary nor sufficient for it.

value of $\widehat{k}$ is smaller or equal, and first-best value of $\widehat{e}$ is larger; and (iii) first-best values are smaller for both $\widehat{k}$ and $\widehat{e}$ (with the first-best values moving from (i) to (iii) as the utility of future generations is discounted more and more).

3.2   Decentralization

Decentralization of the first-best requires the use of a number of fiscal instruments, in particular, generation-specific lump-sum taxes and a subsidy on educational expenditures. Let z_t^y and z_t^o denote lump sum taxes imposed at time t on the young of generation T and the old of generation $T - 1$, and σ the rate of subsidy on educational expenditures. These taxes are related through the government's budget constraint so that

$$N_t z_t^y + N_{t-1} z_t^o - N_t \sigma (1 + r_t) e_t = 0.$$

Or, dividing by N_{t-1},

$$(1 + n) \left[z_t^y - \sigma (1 + r_t) e_t \right] + z_t^o = 0. \tag{50}$$

Next dividing (50) by h_t yields

$$(1 + n) \left[\widehat{z}_t^y - \sigma (1 + r_t) \frac{\widehat{e}_t}{\varphi(\widehat{e}_t)} \right] + \frac{\widehat{z}_t^o}{\varphi(\widehat{e}_t)} = 0, \tag{51}$$

where $\widehat{z}_t^y = z_t^y / h_t$ and $\widehat{z}_t^o = z_t^o / h_{t-1}$. Observe also that $\widehat{z}_t^y$ and $\widehat{z}_{t+1}^o$ remain invariant on the balanced growth path.so that $z_t^y = (1 + g) z_{t-1}^y$ and $z_t^o = (1 + g) z_{t-1}^o$.

Allowing for the above tax rates, equations (26), (27), and (29) change to

$$\widehat{c} + \left[1 + \frac{\alpha (i - \theta)}{1 + \theta} \right] \frac{\widehat{d}}{1 + r} = w - (1 - \sigma)(1 + r) \frac{\widehat{e}}{\varphi(\widehat{e})} - \widehat{z}^y - \frac{\widehat{z}^o}{1 + r}, \tag{52}$$

$$(1 + n) \left[\widehat{k} \varphi(\widehat{e}) + \widehat{e} \right] + \frac{\alpha}{1 + n} \frac{\widehat{d}}{\varphi(\widehat{e})} = w - (1 - \sigma)(1 + r) \frac{\widehat{e}}{\varphi(\widehat{e})} - \widehat{c} - \widehat{z}^y, \tag{53}$$

$$\varphi'(\widehat{e}) = \frac{(1 - \sigma)(1 + r)}{w}. \tag{54}$$

The remaining steady-state equations in (25)–(32) do not change. Thus the market solutions for $\widehat{c}, \widehat{d}, \widehat{e}, \widehat{k}, w, r, i$, and π are now found from equations (25), (28), (29), (31),

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(32), and (52)–(54). To have these equations lead to the first-best balanced growth path characterized by (45)–(46), the policy instruments must be set as follows. First, lump-sum taxes z^y, z^o must be set such that the modified golden rule (46) is attained.[14] This requirement is commonplace in overlapping generations models. Second, to have the first best condition (47), or (49), satisfied, a subsidy is required on the purchase of e. The subsidy rate is found from (47) and (54) to be[15]

$$\sigma = \frac{1 + r - (1 + n)(1 + \rho) [\varphi(\widehat{e})]^{1-\beta}}{1 + r} + \frac{(1 + n) [\varphi(\widehat{e}) - \widehat{e}\varphi'(\widehat{e})]}{1 + r},$$

where $\widehat{e}$ and r take their first-best values. Observe also that in the first best, with condition (46) satisfied, the above relationship is simplified to

$$\sigma = \frac{(1 + n) [\varphi(\widehat{e}) - \widehat{e}\varphi'(\widehat{e})]}{1 + r}. \tag{55}$$

Finally, attaining (45) is predicated on the well-known result in the money literature that at the optimum there should be no opportunity cost in holding money; that is that i must be equal to zero. To have (45) satisfied, one must set $i = 0$ in equation (25). The implication of this for the rate of money growth is straightforward. Substituting $i = 0$ in equation (15) yields $(1 + r)(1 + \pi) = 1$ so that from (46)

$$1 + \pi = \frac{1}{(1 + \rho)(1 + n) [\varphi(\widehat{e})]^{1-\beta}}$$

It the follows from (28) that

$$\theta = \frac{[\varphi(\widehat{e})]^{\beta}}{1 + \rho} - 1 < 0, \tag{56}$$

where $\widehat{e}$ is set at its first-best value. The sign of θ follows from the transversality condition (48).

The following proposition summarizes the results of this section.

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[14] To be precise, one needs to set either z^y or z^o; the other is determined through the government's budget constraint (51).

[15] In the presence of the subsidy, the young choose $\widehat{e}_t$ at t to maximize $w_t\varphi(\widehat{e}_t) - (1 + r_t)(1 - \sigma)\widehat{e}_t$. This yields, in the steady state,

$$\varphi'(\widehat{e}) = \frac{(1 + r)(1 - \sigma)}{w}.$$

Proposition 2 *The decentralization of the first-best balanced growth path requires generation specific lump-sum taxes, a subsidy on educational expenditures given by equation (55), and the satisfaction of Friedman rule that requires the opportunity cost of holding money to be zero.*

4 Second best with control of physical capital

Assume now that educational subsidies are unavailable so that the level of investment in education is not optimal. The question I want to address is if the Friedman rule continues to be optimal as long as the government can control the level of physical capital in the economy. To answer this question, I first characterize the second-best allocation when k can be controlled but not e. The formulation of this problem is exactly as that in the first best with the sole difference of not being able to optimize over e. The optimization problem is summarized by the Lagrangian (33) where the optimization is done with respect to c_t, d_{t+1}, k_{t+1}, and h_{t+1}. The corresponding first-order conditions are (34), (35), (37) and (36). Specifically, first-order condition (38) with respect to e_{t+1} does not hold which means equation (44) does not hold. The equations that characterize this second best are (42) and (43) and their corresponding balanced growth versions, equations (45) and (46). This also means that, as with the first-best, the second-best solution satisfies the no intertemporal consumption distortion, despite the fact that the choice of e is distorted, and the modified golden rule.

Interestingly, as long as one can control $\widehat{k}$ and thus $r = f'(\widehat{k})$, the condition for an undistorted intertemporal consumption decision remains $i = 0$. Given any $\widehat{e}$, setting $r = (1 + n)(1 + \rho) [\varphi(\widehat{e})]^{1-\beta} - 1$ ensures that the modified golden rule condition (46) is satisfied. Setting $i = 0$ on top of this ensures that the intertemporal consumption decision remains undistorted. This indicates the optimality of the Friedman rule in the second-best as well. We have:

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Proposition 3 *Friedman rule holds in a second-best environment wherein the levels of education and human capital are suboptimal, as long as physical capital can be controlled and the modified golden rule is satisfied.*

4.1 Why Friedman rule?

Given our second-best environment, one may wonder why introducing an additional distortion through the violation of the Friedman rule does not improve welfare. The answer lies in the fact the violation of the Friedman rule, in either direction, will not alleviate the existing distortion due to the non-optimality of educational. To demonstrate this, first observe first that in the second-best the relationship between $\widehat{e}$ and $\widehat{k}$ is governed by the laissez-fair condition (30) and in the first-best by equation (49). These relationships imply that, *for any given same value of* $\widehat{k}$,

$$\text{In the SB: } \varphi'\left(\widehat{e}\right) = \frac{1 + f'(\widehat{k})}{w}, \tag{57}$$

$$\text{In the FB: } \varphi'\left(\widehat{e}\right) = \frac{1 + f'(\widehat{k})}{w} - \frac{(1+n)\left[\varphi(\widehat{e}) - \widehat{e}\varphi'\left(\widehat{e}\right)\right]}{w} < \frac{1 + f'(\widehat{k})}{w}, \tag{58}$$

where the inequality sign follows from the concavity of $\varphi(\cdot)$. It then follows from equations (57)–(58) that $\varphi'\left(\widehat{e}^{FB}\left(\widehat{k}\right)\right) < \varphi'\left(\widehat{e}^{SB}\left(\widehat{k}\right)\right)$ where superscripts FB and SB denote first best and second best. The concavity of $\varphi(\cdot)$ then implies $\widehat{e}^{FB}\left(\widehat{k}\right) > \widehat{e}^{SB}\left(\widehat{k}\right)$ so that the graph of $\widehat{e}^{FB}\left(\widehat{k}\right)$ is *above* graph $\widehat{e}^{SB}\left(\widehat{k}\right)$.

Next, differentiate equation (57) with respect to $\widehat{k}$. This yields

$$\frac{d\widehat{e}}{d\widehat{k}} = \frac{w + (1+r)\widehat{k}}{w^2} \frac{f''\left(\widehat{k}\right)}{\varphi''\left(\widehat{e}\right)} > 0. \tag{59}$$

Intuitively, as $\widehat{k}$ increases, $(1+r)/w$ declines and this lowers the cost of borrowing to educate oneself relative to returns to education. This, in turn, increases the demand for education. It follows from equation (59) that the graph of (57) slopes upward. Similarly, one can easily establish, through differentiation, that the graph of (58), relating $\widehat{e}$ and $\widehat{k}$ in the first best, also slopes upward. On the other hand, differentiation establishes that
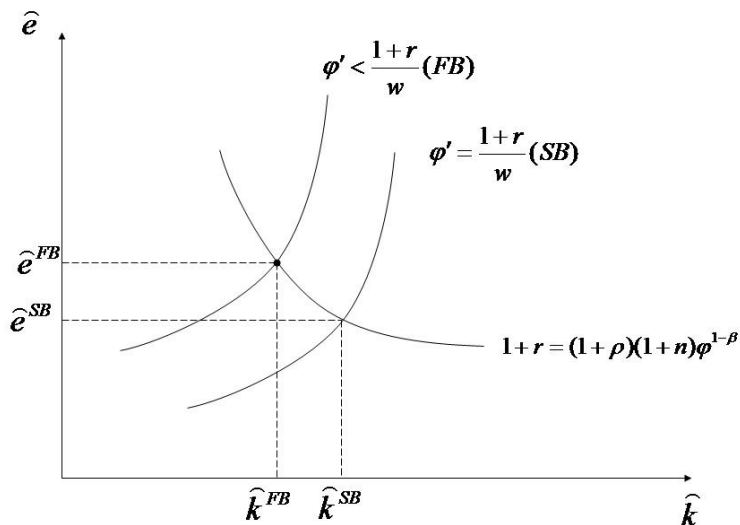
Figure 1:

the graph of the modified golden rule condition (46) which holds both in the second best as well as first best, slopes downward. Diagram 1 depicts these three graphs.

The upshot of all this is that the first-best value of $\widehat{e}$ exceeds its second-best value, while the first-best value of $\widehat{k}$ is smaller than its second-best value. We have

$$
\begin{aligned}
\widehat{e}^{SB} &< \widehat{e}^{FB}, \\
\widehat{k}^{SB} &> \widehat{k}^{FB}.
\end{aligned}
$$

These inequalities show that whereas the second-best value of $\widehat{e}$ must increase, the second-best value of $\widehat{k}$ must decrease in order to move the second-best equilibrium "closer" to the first-best equilibrium.

Now observe that the laissez-fair condition (30), which governs the relationship between $\widehat{e}$ and $\widehat{k}$ in the second best, is *independent* of θ. Specifically, one has $d\widehat{e}/d\theta =$

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$\left(d\widehat{e}/d\widehat{k}\right)\left(d\widehat{k}/d\theta\right)$. This implies that while a change in θ changes the equilibrium *values* of $\widehat{e}$ and $\widehat{k}$, it will leave the relationship between the two intact. In terms of Diagram 1, a change in θ moves the second-best equilibrium along the graph of $\varphi'\left(\widehat{e}\right) = \left(1 + f'(\widehat{k})\right)/w$. Put differently, changing θ moves the equilibrium values of $\widehat{e}$ and $\widehat{k}$ *in the same direction.* Either it increases both $\widehat{e}$ and $\widehat{k}$ or decreases them. This means that violating the Friedman rule has two offsetting effects. It alleviates one of the existing distortions in the variables $\widehat{e}$ or $\widehat{k}$, but exacerbates the other one. Considering that the violation of the Friedman rule is itself an added source of distortion, in one's intertemporal choice, there is no obvious net benefit to it.

4.2 First versus second best

The finding that $\widehat{e}^{SB} < \widehat{e}^{FB}$ and $\widehat{k}^{SB} > \widehat{k}^{FB}$ allows one to compare the first-best and second-best values of the other variables of interest. Specifically, with $\widehat{k}^{SB} > \widehat{k}^{FB}$, it follows that $r^{SB} < r^{FB}$ and $w^{SB} > w^{FB}$. However, at any given time t, it is not w that matters to the individual but $w\,h_t$. Now h_t grows at the rate of $g^{FB} = \varphi(\widehat{e}^{FB}) - 1$ in the first best and $g^{SB} = \varphi(\widehat{e}^{SB}) - 1$ in the second best, with $g^{SB} < g^{FB}$. Finally, observe that while $i = 0$ in both first and second best, the corresponding money growth rate that induces this differs across the two equilibria. Because condition (43) holds in the second best as well, the corresponding monetary growth rate continues to be given by equation (46)

$$\theta = \frac{[\varphi\left(\widehat{e}\right)]^{\beta}}{1 + \rho} - 1.$$

With $\widehat{e}^{SB} < \widehat{e}^{FB}$, the optimal monetary growth rate—the Friedman rule—calls for a smaller monetary growth rate. That is

$$\theta^{SB} < \theta^{FB} < 0.$$

These results are summarized as:

Proposition 4 *Let $\widehat{k}$ denote physical capital to human capital ratio, g denote the growth rate of the economy, θ denote the money growth rate, and superscripts FB and SB denote first- and second-best solutions. Then, $\widehat{k}^{SB} > \widehat{k}^{FB}$, $\widehat{e}^{SB} < \widehat{e}^{FB}$, $g^{SB} < g^{FB}$, and $\theta^{SB} < \theta^{FB} < 0$.*

5   Second best without control of physical capital

Assume now that differential lump-sum tax and transfers are not feasible and the government cannot control the level of physical capital in the economy. However, educational subsidies are available and $\widehat{e}$ is set optimally. To characterize this second-best equilibrium, one can again formulate the problem as in the first best with the exception of not optimizing over k. Specifically, the optimization problem is summarized by the Lagrangian (33) where the optimization is done with respect to c_t, d_{t+1}, h_{t+1}, and e_{t+1}. The first-order condition that does not hold in this case is (37) and thus (43). The equations that characterize this second-best allocation are then the same as (45) and (47): One continues to require an undistorted intertemporal consumption decision and an undistorted decision concerning educational expenditures.

Unlike the previous two cases, the Friedman rule of $i = 0$ no longer ensures that equation (45), requiring an undistorted intertemporal consumption decision, is satisfied. This arises because the modified golden rule condition (46) does not hold in this second best. Put differently, bringing about an undistorted intertemporal consumption decision when the modified golden rule is not satisfied, requires that Friedman rule is violated. Indeed, from equations (25) and (29), condition (45) is satisfied if

$$i = \frac{(1+r) - (1+\rho)(1+n)\,[\varphi\,(\widehat{e})]^{1-\beta}}{\alpha\,(1+r)\,(1+\rho)\,[\varphi\,(\widehat{e})]^{-\beta} - \left\{(1+r) - (1+\rho)(1+n)\,[\varphi\,(\widehat{e})]^{1-\beta}\right\}}, \qquad (60)$$

$$\theta = -1 + \frac{\alpha(1+\rho)(1+n)\,[\varphi\,(\widehat{e})]^{1-\beta}}{\alpha\,(1+r)\,(1+\rho)\,[\varphi\,(\widehat{e})]^{-\beta} - \left\{(1+r) - (1+\rho)(1+n)\,[\varphi\,(\widehat{e})]^{1-\beta}\right\}}. (61)$$

In the first best, when $\widehat{k}$ is set at its optimal value, $(1+r) = (1+\rho)(1+n)\,[\varphi\,(\widehat{e})]^{1-\beta}$

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and equation (60) simplifies to $i = 0$; similarly equation (61) reduces to $\theta = -1 + (1+n)\,\varphi\left(\widehat{e}\right)/(1+r) = -1 + \left[\varphi\left(\widehat{e}\right)\right]^{\beta}/(1+\rho) < 0$ which is the first-best value of θ derived in equation (56).

Observe also that $1 + \theta$ must always be positive. It then follows that denominator on the right-hand side of (61) and thus (60) is positive. This implies that i has the same sign as $(1+r) - (1+\rho)(1+n)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$. Put differently, when second-best optimality requires $i > 0$, it means that $1 + r > (1+\rho)(1+n)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$ and the equilibrium is characterized by a "low" physical-capital to human-capital ratio. On the other hand, when $i < 0$ is optimal the equilibrium is characterized by a "high" physical-capital to human-capital ratio. This makes sense. A positive level of i is associated with a "high" level of θ which depresses $\widehat{k}$. The following proposition summarizes the results of this section.

Proposition 5 *(i) The second-best balanced growth environment wherein physical-capital cannot be controlled is characterized by equations (45) and (47).*

(ii) Friedman rule does not hold in this environment.

(iii) The optimal nominal interest rate is given by equation (60) and the optimal monetary growth rate by (61).

(iv) A positive (negative) nominal interest rate is associated with a "low" ("high") value of physical-capital to human-capital ratio.

6 Money, growth, and Friedman rule

This section examines how the laissez-faire economy responds to changes in the rate of monetary growth. In doing so, it also determines the precise nature of the change in the second-best values of $\widehat{e}$ and $\widehat{k}$ discussed previously, as the inclusion of lump-sum taxes $\widehat{z}^{y}$ and $\widehat{z}^{o}$ in the steady-state equations does not change the results. It also revisits what one means in this model by "Friedman rule".

Substitute for i from equation (29) into equations (25)–(26) and simplify to get

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$$\frac{u_d\left(\widehat{c},\widehat{d}\right)}{u_c\left(\widehat{c},\widehat{d}\right)} = \frac{1}{1+r}\left(1 - \frac{\alpha}{1+\theta}\right) + \frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}, \tag{62}$$

$$\widehat{c} + \left[\frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]\widehat{d} = w - (1+r)\frac{\widehat{e}}{\varphi\left(\widehat{e}\right)}. \tag{63}$$

With equations (31)–(32) determining w and r as functions of $\widehat{k}$, equations (62)–(63) determine $\widehat{c}$ and $\widehat{d}$ as functions of $\widehat{e}, \widehat{k}$ and θ. Moreover, equation (30) relates $\widehat{e}$ to w and r and thus to $\widehat{k}$. Substituting for $\widehat{e}$ as a function $\widehat{k}$ in (62)–(63) then yields a solution for $\widehat{c}$ and $\widehat{d}$ as functions of $\widehat{k}$ and θ only: $\widehat{c}\left(\widehat{k},\theta\right)$ and $\widehat{d}\left(\widehat{k},\theta\right)$. Now substitute $\widehat{c}\left(\widehat{k},\theta\right)$ and $\widehat{d}\left(\widehat{k},\theta\right)$ into equation (27) and differentiate the resulting equation totally with respect to θ. I show in the Appendix that the result is

$$\frac{d\widehat{k}}{d\theta} = -\frac{1}{\Gamma}\left[\frac{\partial\widehat{c}}{\partial\theta} + \frac{\alpha}{(1+n)\varphi\left(\widehat{e}\right)}\frac{\partial\widehat{d}}{\partial\theta}\right], \tag{64}$$

where I also present the expression for Γ noting that it is positive.

To determine the expression for the right-hand side of (64), partially differentiate the system of equations (62)–(63) with respect to θ. Define

$$\Delta \equiv \left(\widehat{u}_{cd} - \frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc}\right)\left[\frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right] - \left(\widehat{u}_{dd} - \frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd}\right). \tag{65}$$

I prove in the Appendix that $\Delta > 0$ and that

$$\frac{\partial\widehat{c}}{\partial\widehat{\theta}} = \frac{1}{\Delta}\left[\frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]\frac{\alpha\,\widehat{u}_c}{(1+r)(1+\theta)^2} > 0. \tag{66}$$

$$\frac{\partial\widehat{d}}{\partial\widehat{\theta}} = \frac{1}{\Delta}\frac{-\alpha\,\widehat{u}_c}{(1+r)(1+\theta)^2} < 0. \tag{67}$$

Intuitively, as θ increases the relative price of future goods to present consumption increases so that consumers respond by lowering their consumption of future goods and increasing their consumption of present goods.

Substituting for $\partial\widehat{c}/\partial\theta$ and $\partial\widehat{d}/\partial\theta$ from (66)–(67) into the bracketed expression on the right-hand side of (64) yields

$$\frac{\partial\widehat{c}}{\partial\theta} + \frac{\alpha}{(1+n)\varphi\left(\widehat{e}\right)}\frac{\partial\widehat{d}}{\partial\theta} = \frac{1}{\Delta}\frac{\alpha\left(1-\alpha\right)\widehat{u}_c}{(1+r)^2(1+\theta)^2} > 0.$$

Consequently,

$$\frac{d\widehat{k}}{d\theta} < 0.$$

This result may at first appear to be counter-intuitive. An increase in θ increases the cost of holding money. As a result, consumers should want to lower their cash holdings and switch into holdings of real capital. The intuition lies in the model's rationalization for holding money. With a Clower cash constraint, people's future consumption is equal to a multiple of their cash holdings. Thus a reduction in cash holdings implies a *higher* reduction in future consumption. This, in turn, requires a smaller level of real savings to be carried into the future.

Next, recall that $d\widehat{e}/d\theta = \left(d\widehat{e}/d\widehat{k}\right)\left(d\widehat{k}/d\theta\right)$ with $d\widehat{e}/d\widehat{k} > 0$ from (59). Consequently, $d\widehat{e}/d\theta < 0$. Now with the growth rate of the economy being $g = \varphi\left(\widehat{e}\right) - 1$ and $\varphi'\left(\widehat{e}\right) > 0$, g moves positively with $\widehat{e}$ and

$$\frac{dg}{d\theta} < 0.$$

This result is summarized as:

Proposition 6 *An increase in the money growth rate leads to a reduction in the balanced growth path values of the physical capital per human capital ratio and the rate of growth of the economy.*

6.1 Non-uniqueness of Friedman rule

That Friedman rule of $i = 0$ ensures the satisfaction of the optimality condition (45) is predicated on the stipulation that there are no consumption taxes in the economy.

Assume instead that the government imposes a tax on the consumption of the young, c, at the rate of τ. This changes the market equilibrium condition (25) to[16]

$$\frac{u_d\left(\widehat{c},\widehat{d}\right)}{u_c\left(\widehat{c},\widehat{d}\right)} = \frac{1+\alpha i/\left(1+\theta\right)}{\left(1+\tau\right)\left(1+r\right)}.$$

This equation will be reduced to the optimality condition (45) as long as

$$i = \frac{\tau\left(1+\theta\right)}{\alpha}. \tag{68}$$

Moreover, one can easily see from (29) that to have (68) satisfied, all that is needed is to set θ, for *any given* τ, according to

$$\theta = \frac{\alpha\left(1+n\right)\varphi\left(\widehat{e}\right)}{\alpha\left(1+r\right)-\tau\left(1+n\right)\varphi\left(\widehat{e}\right)} - 1 = \frac{\alpha}{\alpha(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{-\beta}-\tau} - 1, \tag{69}$$

where $\widehat{e}$ takes its first-best value.

To sum, as long as there are no commodity taxes ($\tau = 0$), i must be equal to zero in the first best, requiring the monetary authority to set θ according to (56). However, if the fiscal authority levies a tax on the first-period consumption goods at a rate equal to τ, monetary authority should set θ according to (69). Put differently, optimality condition (45) can be attained as a competitive equilibrium not just for $i = 0$, and thus for a monetary growth rate governed by (56); but for a continuum of i and a continuum of monetary growth rates and tax rates on c that satisfy (69). That is, the Friedman requirement for a zero opportunity cost of money holding will be satisfied by a continuum

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[16]Observe also that allowing for a tax rate on c changes the market conditions (52)–(53) to

$$(1+\tau)\widehat{c}+\left[1+\frac{\alpha\left(i-\theta\right)}{1+\theta}\right]\frac{\widehat{d}}{1+r} = w-\left(1-\sigma\right)\left(1+r\right)\frac{\widehat{e}}{\varphi\left(\widehat{e}\right)}-\widehat{z}^y-\frac{\widehat{z}^o}{1+r},$$

$$(1+n)\left[\widehat{k}\varphi\left(\widehat{e}\right)+\widehat{e}\right]+\frac{\alpha}{1+n}\frac{\widehat{d}}{\varphi\left(\widehat{e}\right)} = w-\left(1-\sigma\right)\left(1+r\right)\frac{\widehat{e}}{\varphi\left(\widehat{e}\right)}-\left(1+\tau\right)\widehat{c}-\widehat{z}^y.$$

It also changes the government's budget constraint (51) to

$$(1+n)\left[\widehat{z}^y_t+\tau\widehat{c}_t-\sigma(1+r_t)\frac{\widehat{e}_t}{\varphi\left(\widehat{e}_t\right)}\right]+\frac{\widehat{z}^o_t}{\varphi\left(\widehat{e}_t\right)} = 0.$$

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of (τ, θ). Intuitively, what is important here is the relative intertemporal price. In this model, the two policy instruments τ and θ control this single price (through i). In other words, we have one instrument too many.[17]

Proposition 7 *Friedman rule can be satisfied by a continuum of monetary growth rates and tax rates on consumption during working years with the two being related according to equation (69) resulting in a nominal interest rate that is characterized by (68). Only if there are no taxes on consumption during working years, Friedman rule requires the nominal interest rate to be zero. In this case, the monetary growth rate is negative and given by equation (56).*

7 Conclusion

This paper has studied the implications of introducing money into an overlapping-generations model with endogenous growth where the rationalization for money holding comes from the Clower's cash-in-advance constraint. It has considered an economy populated with finitely-lived individuals whose human capital is determined partly through education and partly through their inherited human capital. Throughout the paper, to emphasize allocative efficiency, the paper has assumed no government tax requirements. With an external revenue requirement, one can always reject Friedman rule by suitably ruling out certain fiscal instruments.[18]

The main message of the paper has been that, as long as the government can fully control the level of physical capital in the economy and sets it to satisfy the modified golden rule, Friedman rule remains optimal. The result holds regardless of the ability of the government to control the level of human capital. With the control of human capital, controlling physical capital yields a first-best environment. Given this perspec-

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[17] This does not arise in monetary models with money in the utility function. There, real balances also enter the utility function and the two instruments τ and θ are needed to control the relative intertemporal price and real balances.

[18] The stark example is provided by the case when there are no fiscal instruments.

tive, the result has generalized the earlier result of Abel (1987) and Gahvari (1988) derived for overlapping-generations models without endogenous growth. Without the control of human capital, the economy is in a second-best environment. The result thus contradicts one's intuition based on the second-best theory. The paper has clarified this apparent contradiction by demonstrating that introducing an additional distortion in the economy, via the violation of Friedman rule, does nothing to alleviate the existing distortion in human capital accumulation as long as one can fully control physical capital accumulation. Furthermore, the paper has demonstrated that notwithstanding a zero nominal interest rate in both first- and second-best environments, the monetary growth rate and the economy's endogenous rate of growth are both smaller in the second best.

The paper has also studied another second-best environment; one in which human capital accumulation can be controlled but not physical capital. It has shown that in this setting the distortion due to the violation of Friedman rule does alleviate the distortion due to the lack of physical capital accumulation. Here, second-best optimality calls for the violation of Friedman rule.

Another interesting result is the existence of a negative relationship between money growth rate on the one hand and the endogenous growth rate of the economy on the other. The reason for this is that, in a cash-in-advance constraint model, people carry cash to finance a fraction of their future consumption. Put the other way, their future consumption is equal to a multiple of their cash holdings. Consequently, an increase in the growth rate of money supply, which increases the cost of holding cash and reduces money holdings, leads to a reduction in future consumption that is higher than the original reduction in the individuals' money holdings. This, in turn, requires a smaller level of real savings to be carried into the future. Finally, the paper has also shown that, in the context of a cash-in-advance constraint model, Friedman rule is not unique. A continuum of monetary growth rates and tax rates on consumption during working years result in a zero opportunity cost of holding money.

Appendix

Per human capital notation used in the paper:

$$\widehat{e}_t \equiv e_t/h_{t-1}, \widehat{k}_t = k_t/h_t, \widehat{c}_t \equiv c_t/h_t, \widehat{d}_t \equiv d_t/h_{t-1}, \varphi(\widehat{e}_t) \equiv h_t/h_{t-1}.$$

A useful property: Rewrite equation (47) as

$$[w - (1+n)\,\widehat{e}]\,\varphi'(\widehat{e}) = (1+n)\,[\varphi(\widehat{e})]^{1-\beta}\left[(1+\rho) - [\varphi(\widehat{e})]^{\beta}\right]$$

Then transversality condition (46) $[\varphi(\widehat{e})]^{\beta} < 1 + \rho$ also implies that

$$w - (1+n)\,\widehat{e} > 0. \tag{A1}$$

Derivation of (64): Substitute $\widehat{c}\left(\widehat{k},\theta\right)$ and $\widehat{d}\left(\widehat{k},\theta\right)$, the solution found from equations (62)–(63), into equation (27) to get

$$(1+n)\left[\widehat{k}\,\varphi\left(\widehat{e}\left(\widehat{k}\right)\right) + \widehat{e}\left(\widehat{k}\right)\right] + \frac{\alpha}{1+n}\frac{\widehat{d}\left(\widehat{k},\theta\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)} = \tag{A2}$$

$$w - (1+r)\frac{\widehat{e}\left(\widehat{k}\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)} - \widehat{c}\left(\widehat{k},\theta\right).$$

Move the expressions on the right-hand side of (A2) to its left-hand side and let

$$\Theta\left(\widehat{k},\theta\right) \equiv (1+n)\left[\widehat{k}\,\varphi\left(\widehat{e}\left(\widehat{k}\right)\right) + \widehat{e}\left(\widehat{k}\right)\right] + \frac{\alpha}{1+n}\frac{\widehat{d}\left(\widehat{k},\theta\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)}$$

$$-w + (1+r)(1+r)\frac{\widehat{e}\left(\widehat{k}\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)} + \widehat{c}\left(\widehat{k},\theta\right).$$

Differentiating $\Theta\left(\widehat{k},\theta\right) = 0$ totally with respect to θ yields

$$\frac{\partial\Theta\left(\widehat{k},\theta\right)}{\partial\theta} + \frac{\partial\Theta\left(\widehat{k},\theta\right)}{\partial\widehat{k}}\frac{d\widehat{k}}{d\theta} = -\left[\frac{\partial\widehat{c}\left(\widehat{k},\theta\right)}{\partial\theta} + \frac{\alpha}{1+n}\frac{1}{\varphi(\widehat{e})}\frac{\partial\widehat{d}\left(\widehat{k},\theta\right)}{\partial\theta}\right]$$

Rearranging and defining $\Gamma \equiv \partial\Theta\left(\widehat{k},\theta\right)/\partial\widehat{k}$ yields equation (64) in the text.

Now observe that one can think of the left-hand side of (A2) as the demand for $\widehat{k}$ and its right-hand side as the supply of $\widehat{k}$. The static stability condition then requires that the excess demand function $\Theta\left(\widehat{k},\theta\right)$ to be downward-sloping in r or upward-sloping in $\widehat{k}$. Consequently, $\Gamma = \partial\Theta\left(\widehat{k},\theta\right)/\partial\widehat{k} > 0$.

Derivation of (66)–(67): Differentiate equations (62)–(63) partially with respect to θ. The resulting equations are, in matrix notation,

$$\begin{pmatrix} \widehat{u}_{cd} - \frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc} & \widehat{u}_{dd} - \frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd} \\ 1 & \frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\varphi(\widehat{e})} \end{pmatrix} \begin{pmatrix} \frac{\partial\widehat{c}}{\partial\theta} \\ \frac{\partial\widehat{d}}{\partial\theta} \end{pmatrix} = \begin{pmatrix} \frac{\alpha\,\widehat{u}_c}{(1+r)(1+\theta)^2} \\ 0 \end{pmatrix}. \qquad \text{(A3)}$$

Pre-multiplying (A3) by the inverse of the 2×2 matrix that appears in its left-hand side yields

$$\begin{pmatrix} \frac{\partial\widehat{c}}{\partial\theta} \\ \frac{\partial\widehat{d}}{\partial\theta} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\varphi(\widehat{e})} & -\widehat{u}_{dd} + \frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd} \\ -1 & \widehat{u}_{cd} - \widehat{u}_{cc}\frac{\widehat{u}_d}{\widehat{u}_c} \end{pmatrix} \begin{pmatrix} \frac{\alpha\,\widehat{u}_c}{(1+r)(1+\theta)^2} \\ 0 \end{pmatrix}, \qquad \text{(A4)}$$

where Δ is defined in the text by equation (65). Equations (66)–(67) allow immediately from (A4).

Proof of $\Delta > 0$: To determine the sign of Δ, rewrite its expression from (65) as

$$\begin{aligned} \Delta &= -\left[\frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\,\varphi\,(\widehat{e})}\right]\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc} - \widehat{u}_{dd} + \left[\frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\,\varphi\,(\widehat{e})} + \frac{\widehat{u}_d}{\widehat{u}_c}\right]\widehat{u}_{cd} \\ &= \left[\frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\,\varphi\,(\widehat{e})}\right]\left(\widehat{u}_{cd} - \frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc}\right) + \left(\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd} - \widehat{u}_{dd}\right). \end{aligned}$$

Assuming d and c are normal goods, one can easily show that $\widehat{u}_{cd} - (\widehat{u}_d/\widehat{u}_c)\,\widehat{u}_{cc} > 0$ and $(\widehat{u}_d/\widehat{u}_c)\,\widehat{u}_{cd} - \widehat{u}_{dd} > 0$ (where due to quasi-concavity of preferences, $-(\widehat{u}_d/\widehat{u}_c)^2\,\widehat{u}_{cc} - \widehat{u}_{dd} + 2(\widehat{u}_d/\widehat{u}_c)\,\widehat{u}_{cd} > 0$). Consequently, $\Delta > 0$.

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