

# Effort Maximization in Asymmetric N-Person Contest Games

JÖRG FRANKE  
CHRISTIAN KANZOW  
WOLFGANG LEININGER  
ALEXANDRA VÄTH

CESIFO WORKING PAPER NO. 2744  
CATEGORY 2: PUBLIC CHOICE  
AUGUST 2009

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## Abstract

This paper provides existence and characterization of the optimal contest success function under the condition that the objective of the contest designer is total effort maximization among  $n$  heterogeneous players. Heterogeneity of players makes active participation of a player in equilibrium endogenous with respect to the specific contest success function adopted by the contest designer. Hence, the aim of effort maximization implies the identification of those players who should be excluded from making positive efforts. We give a general proof for the existence of an optimal contest success function and provide an algorithm for the determination of the set of actively participating players. This in turn allows to determine optimal efforts in closed form. An important general feature of the solution is that maximization of total effort requires at least three players to be active.

JEL Code: C72, D72.

Keywords: effort maximization, existence of solution, asymmetric contests, participation constraints.

*Jörg Franke*  
*University of Dortmund (TU)*  
*Department of Economics*  
*Vogelpothsweg 87*  
*Germany – 44227 Dortmund*  
*Joerg.Franke@tu-dortmund.de*

*Christian Kanzow*  
*University of Würzburg*  
*Institute of Mathematics*  
*Am Hubland*  
*Germany – 97074 Würzburg*  
*kanzow@mathematik.uni-wuerzburg.de*

*Wolfgang Leininger*  
*University of Dortmund (TU)*  
*Department of Economics*  
*Vogelpothsweg 87*  
*Germany – 44227 Dortmund*  
*Wolfgang.Leininger@tu-dortmund.de*

*Alexandra Väth*  
*University of Würzburg*  
*Institute of Mathematics*  
*Am Hubland*  
*Germany – 97074 Würzburg*  
*alexandra.vaeth@mathematik.uni-wuerzburg.de*

# 1 Introduction

Competitive social situations are frequently modeled as contest games where contestants compete for a fixed prize by exerting effort which increases their chances to win the prize. The prominent application in this literature is rent-seeking which was introduced by Tullock in his seminal paper [18]. Based on this framework an extensive literature followed, compare Congleton et al. [4] for a collection of related works. Moreover, this framework is also successfully applied in different contexts, like conflict, lobbying, patent races, sports tournaments, etc., see Konrad [9] for a recent survey.

An important question in this literature is related with the aggregated effort level that is exerted by contestants in equilibrium. If the objective of the contest designer is effort maximization, as suggested by the mentioned examples, then the optimal design of the contest becomes crucial, see for instance, Gradstein and Konrad [8] for the optimal number of stages in multistage contests, Amegashie [1] for the optimal seeding of contestants, and Dasgupta and Nti [6] for the optimal contest rule. This literature is based on the assumption that contestants are homogeneous, or that there are only two contestants. While Dasgupta and Nti [6] search for the optimal contest success function for  $n$  identical contestants, Nti [12] does so for two heterogeneous players.

Heterogeneity affects aggregated equilibrium effort because contestants with low valuation will exert less effort to win the less valued prize. This is anticipated by contestants with higher valuation such that overall effort in equilibrium is relatively low. In these cases the design of the contest rule, or contest success function, is of importance because balancing the heterogeneity by biasing the contest success function in favor of weak players might induce higher aggregated equilibrium effort in comparison to an unbiased contest success function.

Extending this analysis to contest games with more than two heterogeneous players is far from obvious. This is due to the fact that favoring very weak players might be too costly because the incentives of strong players are distorted as well such that overall equilibrium effort might be decreased. Hence, it might be profitable to exclude specific contestants, a question which cannot arise in the two-player case studied by Nti [12].

The analysis of the  $n$ -player case is the focus of our study, i.e., our objective is to determine the optimal contest success function that induces maximal aggregated equilibrium effort among  $n$  heterogeneous contestants. From a technical perspective this extension is not trivial because in the  $n$ -player contest the equilibrium is usually not interior, i.e., there are contestants that prefer to remain inactive, see Stein [16]. The set of active contestants depends on the distribution of the heterogeneous parameters but also on the respective contest success function. Hence, the derivation of the optimally designed contest success function is a complex issue because not only individual effort is affected by this contest rule but also the set of active contestants which has an additional impact on equilibrium effort.

However, our results for the  $n$ -player contest game are straightforward: We can show that there exists an optimal contest rule that maximizes aggregated equilibrium effort given the distribution of heterogeneous cost parameters and valuations. As in the two-player contest, weak players are favored, however, not all contestants will be active. The crucial step in the analysis is the characterization of the optimal set of active contestants. We provide an algorithm that describes how this subset is determined for a given distribution of player characteristics and dis-

cuss its properties based on numerical examples. Once this set is uncovered, the specification of the optimal contest success function follows automatically. An important by-product of this algorithmic construction is the result that at least three players are active in the effort maximizing equilibrium, if  $n > 2$  holds. Hence, the asymmetric two-player case and its solution properties turn out to be special and do not generalize to more than two asymmetric contestants.

Biased contest rules are clearly relevant in practical applications. For instance, bid preferences in public procurement auctions, affirmative action policies in selection tournaments, biased rules in sport tournaments in order to restore competitive balancedness, and political, e.g. national biases in international trade provide rich examples.

From a mathematical point of view, the problem we are dealing with in this paper is a bilevel program or, more precisely, a mathematical program with equilibrium constraints. The latter is of the general form

$$\max_{x,\alpha} f(x,\alpha) \quad \text{subject to} \quad x \in \mathcal{S}(\alpha),$$

where  $x, \alpha$  denote the variables and  $\mathcal{S}(\alpha)$  is the solution set of another optimization or (Nash) equilibrium problem, typically called the lower-level problem, cf. Luo et al. [10], Outrara et al. [14], Dempe [7] and references therein for an extensive discussion. Note that this lower level problem depends on  $\alpha$ . In our case, the lower level problem is the contest game and has a unique solution  $x(\alpha)$  (depending on  $\alpha$ ) for which also an analytic expression is known. This allows us to follow the so-called implicit programming approach from [13, 14] and to replace the unknown  $x$  in the objective function  $f$  by the unique solution  $x(\alpha)$  of the lower level problem. We then obtain an unconstrained (but typically nonsmooth) optimization problem

$$\max_{\alpha} \tilde{f}(\alpha) := f(x(\alpha), \alpha)$$

depending on  $\alpha$  only. Standard solvers for the solution of such a kind of (usually nonconcave) optimization problem find local maxima or certain stationary points of the objective function, but not necessarily a global maximum. Here we exploit the special structure of our effort maximization problem in order to derive a very simple algorithm for the computation of a global maximum.

The remainder of the paper is structured as follows. In the next section we introduce our  $n$ -player contest game with heterogeneous players. The contest game is based on a contest success function (CSF) with weighted efforts that is a simplified version of an asymmetric CSF as axiomatized in Clark and Riis [3]. We derive some properties of the equilibrium in the underlying contest game that are helpful for the subsequent analysis in Section 2. In Section 3 we prove that there exists a vector of weighting factors that yields a global maximum for aggregated equilibrium effort. In Section 4 we present an algorithm to characterize the optimal set of active contestants that determines the optimal set of weighting factors.

Notation:  $x^v \in \mathbb{R}$  denotes the variable of player  $v$ ,  $x := (x^1, \dots, x^n)$  is the vector of all decision variables. In order to emphasize the role of player  $v$  within this vector, we sometimes write  $x = (x^v, x^{-v})$ , where the symbol  $x^{-v}$  subsumes the variables of all other players. We further denote by  $B(x; \delta)$  the Euclidian ball of radius  $\delta > 0$  around a given point  $x \in \mathbb{R}^n$ .

## 2 The Underlying Contest Game

The contest game to be considered here is a special instance of a Nash equilibrium problem and defined as follows: Let  $N := \{1, \dots, n\}$  be the set of players. Furthermore, let

$$\theta_\nu(x^\nu, x^{-\nu}) := \begin{cases} \frac{\alpha_\nu x^\nu}{\sum_{\mu=1}^n \alpha_\mu x^\mu} - \beta_\nu x^\nu, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

be the expected payoff or utility of player  $\nu \in N$ , where  $\alpha_\nu, \beta_\nu$  are positive constants for all  $\nu \in N$  which are assumed to be fixed throughout this section. Then each player  $\nu \in N$  chooses a strategy  $x^{*,\nu}$  from his strategy space  $X_\nu := [0, +\infty)$  in such a way that

$$\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \geq \theta_\nu(x^\nu, x^{*,-\nu}) \quad \forall x^\nu \in X_\nu$$

holds for all  $\nu \in N$ , i.e. player  $\nu$  tries to maximize his utility function  $\theta_\nu$  with respect to his own strategy  $x^\nu$ , whereas the strategies of all other players are fixed (at their optimal values).

The utility functions  $\theta_\nu$  can be interpreted in the following way: All players take part in a lottery, where a price with the value  $V = 1$  can be won. Every player  $\nu$  can increase his probability of winning, which is given by the contest success function  $\frac{\alpha_\nu x^\nu}{\sum_{\mu=1}^n \alpha_\mu x^\mu}$ , by increasing his effort  $x^\nu$ , but he has to have in mind that this also increases his costs  $\beta_\nu x^\nu$ . We could extend our model to the case where different players have different valuations  $V_\nu > 0$  of the price. This would lead to the following utility function

$$\tilde{\theta}_\nu(x^\nu, x^{-\nu}) := \begin{cases} \frac{\alpha_\nu x^\nu}{\sum_{\mu=1}^n \alpha_\mu x^\mu} V_\nu - \tilde{\beta}_\nu x^\nu, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

In this case, we can obtain utility functions of the form  $\theta_\nu$  by multiplying the functions  $\tilde{\theta}_\nu$  with the factor  $\frac{1}{V_\nu}$  and defining  $\beta_\nu := \frac{\tilde{\beta}_\nu}{V_\nu}$ . Note that re-scaling the function  $\tilde{\theta}_\nu(\cdot, x^{*,-\nu})$  with a positive multiplier does not change the location of its maximum, i.e. it does not change the optimal effort  $x^{*,\nu}$ . Hence, the case of inhomogeneous valuations of the price is included in our model.

We now summarize a number of properties of this Nash equilibrium problem. The following existence and uniqueness result for the above problem was established in Cornes and Hartley [5], Szidarovsky and Okuguchi [17], and Stein [16].

**Theorem 2.1** *The above Nash equilibrium problem has a unique solution  $x^*$ .*

Note that the previous result holds for any fixed parameters  $\alpha_\nu$  and  $\beta_\nu$ , but that, of course, the solution depends on the exact values of these parameters. More precisely, we have the following representation, see Cornes and Hartley [5] and Stein [16] for a proof of these statements.

**Theorem 2.2** *Let  $x^*$  be the unique solution of the above equilibrium problem, let  $K := \{\nu \in N \mid x^{*,\nu} > 0\}$  be the corresponding set of active players, and let  $k := |K|$  the number of active players. Then:*

- (a)  *$K$  consists of at least two elements.*

(b) The active players can be characterized as follows:

$$v \in K \iff (k-1) \frac{\beta_v}{\alpha_v} < \sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}. \quad (1)$$

(c) The components  $x^{*,v}$  of the solution satisfy the relation

$$x^{*,v} = \max \left\{ 0, \frac{1}{\alpha_v} \left( 1 - \frac{\beta_v}{\alpha_v} \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}} \right) \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}} \right\}$$

for all  $v \in N$ .

Theorem 2.2 characterizes the equilibrium based on an implicit description of the set  $K$  of active players in (1). It should be noted that also the expression for the unique solution  $x^*$  of the Nash equilibrium problem is implicit because it depends on the set  $K$  of active players.

From expression (1) it is clear that the set  $K$  of active players consists of those players with the lowest combined parameters  $\frac{\beta_v}{\alpha_v}$ . Hence, the following result due to Stein [16] allows an explicit calculation of the set  $K$ . Together with Theorem 2.2, we are then in a position to compute the unique solution of our Nash equilibrium problem.

**Theorem 2.3** Assume without loss of generality that the players  $v$  are ordered in such a way that

$$\frac{\beta_1}{\alpha_1} \leq \frac{\beta_2}{\alpha_2} \leq \dots \leq \frac{\beta_n}{\alpha_n}. \quad (2)$$

Furthermore, let  $x^*$  be the unique solution of the Nash equilibrium problem. Then the corresponding set  $K$  of active players is given by

$$K = \left\{ v \in N \mid (v-1) \frac{\beta_v}{\alpha_v} < \sum_{\mu=1}^v \frac{\beta_\mu}{\alpha_\mu} \right\}.$$

A simple consequence of the previous result is the following corollary that will be used later in the proof of Lemma A.1.

**Corollary 2.4** Assume without loss of generality that the players  $v$  are ordered as in (2). Let  $x^*$  be the unique Nash equilibrium and  $K$  be the corresponding index set of active players. Then

$$K \subseteq \left\{ v \in N \mid \frac{\beta_v}{\alpha_v} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right\}.$$

**Proof.** Assumption (2) implies that  $K = \{1, \dots, k\}$ . Now, the inequality  $\frac{\beta_v}{\alpha_v} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}$  obviously holds for  $v = 1, 2$ . For  $v = 3, \dots, k$ , this inequality follows inductively taking into account the inequality

$$(v-1) \frac{\beta_v}{\alpha_v} < \sum_{\mu=1}^v \frac{\beta_\mu}{\alpha_\mu} \quad \text{for } v = 3, \dots, k,$$

from Theorem 2.3. □

Note that the inclusion in this corollary can be an equality but, in general, will be a strict inclusion as the following example illustrates.

**Example 2.5** Consider a game with four players and  $\alpha = (1, 1, 1, 1)^T$ .

(a) If  $\beta = (2, 3, 3.5, 4)^T$ , we have

$$K = \{1, 2, 3, 4\} = \left\{ v \in N \mid \frac{\beta_v}{\alpha_v} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} = 5 \right\}.$$

(b) If, however,  $\beta = (2, 3, 3.5, 4.5)^T$ , we have

$$K = \{1, 2, 3\} \subsetneq \{1, 2, 3, 4\} = \left\{ v \in N \mid \frac{\beta_v}{\alpha_v} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} = 5 \right\}.$$

◇

We next provide a characterization of the unique Nash equilibrium in terms of the subsets  $K \subseteq N$ . This characterization will turn out to be useful for our analysis in the subsequent sections.

**Theorem 2.6** Let  $L, M \subseteq N$  be subsets with  $l := |L| \geq 2$ ,  $m := |M| \geq 2$ , and the property that

$$\left( (l-1) \frac{\beta_v}{\alpha_v} < \sum_{\mu \in L} \frac{\beta_\mu}{\alpha_\mu} \iff v \in L \right) \quad \text{and} \quad \left( (m-1) \frac{\beta_v}{\alpha_v} < \sum_{\mu \in M} \frac{\beta_\mu}{\alpha_\mu} \iff v \in M \right).$$

Then  $L = M$ , hence the index set of active players corresponding to the unique Nash equilibrium is the only subset of  $N$  with the properties mentioned above.

**Proof.** Let  $x^*$  be the unique Nash equilibrium. Then we know from Theorem 2.2 (a), (b), that the index set of active players

$$K := \{v \in N \mid x^{*,v} > 0\}$$

has the postulated properties. Hence we only have to verify that  $L = M$  follows for all sets  $L, M \subseteq N$  with these properties. Assume now that there are two such sets with  $L \neq M$ . If we assume without loss of generality that the players are ordered according to (2), this implies

$$L = \{1, \dots, l\} \quad \text{and} \quad M = \{1, \dots, m\}.$$

Since  $L \neq M$ , we can assume without loss of generality that  $l > m$ . Then  $l \notin M$ , and together with (2) it follows that

$$(l-1) \frac{\beta_l}{\alpha_l} = (l-m) \frac{\beta_l}{\alpha_l} + (m-1) \frac{\beta_l}{\alpha_l} \geq \sum_{\mu=m+1}^l \frac{\beta_\mu}{\alpha_\mu} + \sum_{\mu=1}^m \frac{\beta_\mu}{\alpha_\mu} = \sum_{\mu=1}^l \frac{\beta_\mu}{\alpha_\mu},$$

a contradiction to  $l \in L$ . Consequently, we have  $L = M$ . □

We summarize the previous results in the following note which, basically, says that we have a Nash equilibrium if and only if we are able to find a set  $K$  satisfying the requirements (a) and (b) from Theorem 2.2.

**Remark 2.7** The following statements hold:

- (a) If  $x^*$  is the unique Nash equilibrium and  $K$  the corresponding set of active players, then  $K$  has at least two elements and satisfies the conditions from (1).
- (b) Conversely, if we have a subset  $K \subseteq N$  with at least two elements such that (1) holds, then  $K$  is the set of active players corresponding to the unique Nash equilibrium of our game.

### 3 Effort Maximization: Existence of Solution

We consider once again the Nash equilibrium problem from Section 2. Recall that this problem depends on two sets of parameters  $\alpha_v, \beta_v$  which were assumed to be positive. For the rest of this paper, we still view the parameters  $\beta_v > 0$  as being fixed, whereas the parameters  $\alpha_v > 0$  will be viewed as variables. Since the unique solution  $x^*$  depends on  $\alpha_v$  (and  $\beta_v$  which, however, are fixed), we now write  $x(\alpha)$  for this solution as well as  $K(\alpha)$  for the corresponding set of active players. Moreover, let  $k(\alpha) := |K(\alpha)|$  be the number of active players. In view of Theorem 2.2, the components  $x^v(\alpha)$  satisfy

$$x^v(\alpha) = \max \left\{ 0, \frac{1}{\alpha_v} \left( 1 - \frac{\beta_v}{\alpha_v} \frac{k(\alpha) - 1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \right) \frac{k(\alpha) - 1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \right\} \quad \forall v \in N,$$

whereas the characterization

$$v \in K(\alpha) \iff (k(\alpha) - 1) \frac{\beta_v}{\alpha_v} < \sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu} \quad (3)$$

holds for the set  $K(\alpha)$ .

The problem that we deal with in this and the next section is the following one:

$$\max \sum_{v=1}^n x^v(\alpha) \quad \text{s.t.} \quad \alpha \in (0, \infty)^n. \quad (4)$$

Recall that  $x^v(\alpha)$  is the (Nash) *equilibrium* effort of player  $v$ ,  $v = 1, \dots, n$ , if the contest success function uses the vector of weights  $\alpha$ . A contest administrator - or more general, mechanism designer - can now mediate the contest by *choice* of the weights  $\alpha$  in order to elicit maximal total effort from the  $n$  potential contestants (some of whom may choose to stay inactive). Hence, it is the contest designer's problem that is described by (4). Note also that the  $\beta$ -parameters describe



personal characteristics of the contestants, which consequently cannot be altered neither by the contestants themselves nor the contest designer.

Taking into account the previous representation of  $x^y(\alpha)$ , the objective function of (4) can be written in the following form:

$$f(\alpha) := \sum_{v=1}^n x^v(\alpha) = \sum_{v \in K(\alpha)} x^v(\alpha) = \frac{k(\alpha) - 1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \left( \sum_{\mu \in K(\alpha)} \frac{1}{\alpha_\mu} - \frac{k(\alpha) - 1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu^2} \right). \quad (5)$$

The aim of this section is to show that the maximization problem (4) has a solution. This is not clear a priori since the feasible set for  $\alpha$  is both unbounded and open. The unboundedness turns out to be less serious (and will be dealt with in Lemma 3.1), the really crucial part is the fact that the objective function  $f$  is not defined as soon as  $\alpha_v = 0$  for one player  $v$ .

We begin with some results to show that both  $f(\alpha)$  and  $K(\alpha)$  remain unchanged under certain manipulations of  $\alpha$ . A first result of this kind is the following lemma which shows that both  $f(\alpha)$  and  $K(\alpha)$  are homogeneous of degree zero in  $\alpha$ . This is not surprising since the utility functions  $\theta_v$  themselves are homogeneous of degree zero in  $\alpha$ , but the lemma can also be proven directly using the definitions of  $f(\alpha)$  and  $K(\alpha)$ .

**Lemma 3.1** *For all  $\alpha \in (0, \infty)^n$  and all  $c > 0$ , we have*

$$K(c\alpha) = K(\alpha) \quad \text{and} \quad f(c\alpha) = f(\alpha).$$

**Proof.** First note that  $c\alpha$  is feasible (i.e., belongs to  $(0, \infty)^n$ ) for all feasible  $\alpha$ . The characterization (3) together with the uniqueness of the set  $K(\alpha)$  by Remark 2.7 then immediately implies  $K(c\alpha) = K(\alpha)$ . Taking this into account and calculating  $f(c\alpha)$  gives precisely the same value as  $f(\alpha)$  since the factor  $c$  can be cancelled.  $\square$

Another manipulation of  $\alpha$  which leaves the function value  $f(\alpha)$  unchanged is presented in the following result which basically says that, given a fixed parameter  $\alpha^*$ , we can replace the components  $\alpha_v^*$  of  $\alpha^*$  corresponding to the inactive players by arbitrary small numbers  $\alpha_v$  and still have  $K(\alpha^*) = K(\alpha)$  and  $f(\alpha^*) = f(\alpha)$ .

**Lemma 3.2** *Let  $\alpha^* \in (0, \infty)^n$  be arbitrarily given. Then  $K(\alpha^*) = K(\alpha)$  and  $f(\alpha^*) = f(\alpha)$  hold for all  $\alpha \in (0, \infty)^n$  satisfying the following properties:*

(a) *For all  $v \in K(\alpha^*)$ , we have*

$$\alpha_v = \alpha_v^*.$$

(b) *For all  $v \notin K(\alpha^*)$ , we have*

$$\alpha_v \in \left( 0, \frac{(k(\alpha^*) - 1)\beta_v}{\sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*}} \right].$$

**Proof.** Choose  $\alpha \in (0, \infty)^n$  in such a way that the two properties (a) and (b) hold. Then Remark 2.7 shows that the corresponding index set  $K(\alpha)$  is uniquely defined. Using property (a), we obtain for all  $\nu \in K(\alpha^*)$

$$(k(\alpha^*) - 1) \frac{\beta_\nu}{\alpha_\nu} = (k(\alpha^*) - 1) \frac{\beta_\nu}{\alpha_\nu^*} < \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu}.$$

On the other hand, property (b) implies for all  $\nu \notin K(\alpha^*)$

$$(k(\alpha^*) - 1) \frac{\beta_\nu}{\alpha_\nu} \geq \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu}.$$

The uniqueness of  $K(\alpha)$  therefore gives  $K(\alpha) = K(\alpha^*)$ . Together with property (a) we then obtain  $f(\alpha) = f(\alpha^*)$ .  $\square$

So far, we are not able to prove the existence of a solution for the maximization problem (4). However, Lemmas 3.1 and 3.2 show that such a solution (if it exists) is certainly not unique. In order to verify the existence of a solution, we first verify the continuity of the function  $f$  on  $(0, \infty)^n$ . Note that this continuity is not completely obvious since the index set  $K(\alpha^*)$  may change in points arbitrarily close to  $\alpha^*$ .

**Theorem 3.3** *The objective function  $f$  is continuous on  $(0, \infty)^n$ . Moreover, this function is continuously differentiable in an open neighbourhood of any vector  $\alpha^* \in (0, \infty)^n$  having the following property:*

$$\nu \notin K(\alpha^*) \implies (k(\alpha^*) - 1) \frac{\beta_\nu}{\alpha_\nu^*} > \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*}. \quad (6)$$

**Proof.** The statement regarding the continuous differentiability is clear since condition (6) guarantees that, locally, the index set  $K(\alpha^*)$  is constant, hence  $K(\alpha) = K(\alpha^*)$  for all  $\alpha$  from a sufficiently small neighbourhood of  $\alpha^*$ . In particular,  $f$  is continuous in these points.

In order to verify the continuity of  $f$  on the whole set  $(0, \infty)^n$ , it therefore remains to consider a point  $\alpha^* \in (0, \infty)^n$  such that the index set

$$L := \left\{ \nu \in N \mid (k(\alpha^*) - 1) \frac{\beta_\nu}{\alpha_\nu^*} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*} \right\}$$

is nonempty. Now, it is not difficult to see that there is a neighbourhood  $U \subseteq (0, \infty)^n$  of  $\alpha^*$  such that

$$K \subseteq K(\alpha) \subseteq K \cup L \quad \forall \alpha \in U,$$

where, for simplicity of notation, we use the abbreviation  $K := K(\alpha^*)$ . Let us further write  $k := |K|$  und  $l := |L|$ . Then, for each  $\alpha \in U$ , we have  $K(\alpha) = M$  for one of the  $2^l$  sets  $M$  satisfying  $K \subseteq M \subseteq K \cup L$ . Setting  $m := |M|$  and using

$$\frac{\beta_\nu}{\alpha_\nu^*} = \frac{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu^*}}{k - 1}$$

for all  $\nu \in M \setminus K$ , we obtain for all these index sets  $M$

$$\begin{aligned}
f_M(\alpha^*) &:= \frac{m-1}{\sum_{\mu \in M} \frac{\beta_\mu}{\alpha_\mu^*}} \left( \sum_{\mu \in M} \frac{1}{\alpha_\mu^*} - \frac{m-1}{\sum_{\mu \in M} \frac{\beta_\mu}{\alpha_\mu^*}} \sum_{\mu \in M} \frac{\beta_\mu}{(\alpha_\mu^*)^2} \right) \\
&= \frac{m-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu^*} \cdot \left(1 + \frac{m-k}{k-1}\right)} \left[ \sum_{\mu \in K} \frac{1}{\alpha_\mu^*} + \sum_{\mu \in M \setminus K} \frac{\sum_{\lambda \in K} \frac{\beta_\lambda}{\alpha_\lambda^*}}{(k-1)\beta_\mu} \right. \\
&\quad \left. - \frac{m-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu^*} \cdot \left(1 + \frac{m-k}{k-1}\right)} \left( \sum_{\mu \in K} \frac{\beta_\mu}{(\alpha_\mu^*)^2} + \sum_{\mu \in M \setminus K} \frac{\left(\sum_{\lambda \in K} \frac{\beta_\lambda}{\alpha_\lambda^*}\right)^2}{(k-1)^2 \beta_\mu} \right) \right] \\
&= \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu^*}} \left[ \sum_{\mu \in K} \frac{1}{\alpha_\mu^*} + \frac{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu^*}}{(k-1)} \sum_{\mu \in M \setminus K} \frac{1}{\beta_\mu} \right. \\
&\quad \left. - \frac{k-1}{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu^*}} \left( \sum_{\mu \in K} \frac{\beta_\mu}{(\alpha_\mu^*)^2} + \left( \frac{\sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu^*}}{k-1} \right)^2 \sum_{\mu \in M \setminus K} \frac{1}{\beta_\mu} \right) \right] \\
&= f(\alpha^*).
\end{aligned}$$

Since the  $2^l$  functions  $f_M$  are continuous in  $\alpha^*$ , we obtain for an arbitrary  $\varepsilon > 0$  and all  $M$  a suitable  $\delta_M > 0$  such that  $|f_M(\alpha) - f_M(\alpha^*)| < \varepsilon$  for all  $\alpha \in B(\alpha^*, \delta_M)$ . Define  $\delta := \min\{\delta_M \mid K \subseteq M \subseteq K \cup L\}$ . Then we obtain for all  $\alpha \in B(\alpha^*, \delta)$  that  $K(\alpha) = M$  for one of the above index sets  $M$  and, therefore,

$$|f(\alpha) - f(\alpha^*)| = |f_M(\alpha) - f_M(\alpha^*)| < \varepsilon.$$

This proves continuity of  $f$  in  $\alpha^*$ .  $\square$

So far, we know that  $f$  is continuous on  $(0, \infty)^n$ . However, this set is still unbounded and open. Based on the following argument the problem of unboundedness becomes irrelevant in our context: Consider an arbitrary  $\alpha \in (0, \infty)^n$ . Then Lemma 3.1 implies

$$f(\alpha) = f\left(\frac{1}{\sum_{\mu=1}^n \alpha_\mu} \alpha\right).$$

Therefore, defining the set

$$A := \left\{ \alpha \in (0, \infty)^n \mid \sum_{\mu=1}^n \alpha_\mu = 1 \right\}, \quad (7)$$

we obtain  $f((0, \infty)^n) = f(A)$ , and the function  $f$  attains a global maximum on  $(0, \infty)^n$  if and only if it has a maximizer on the bounded set  $A$ . But this set  $A$  is still not closed, hence not compact. However, Theorem 3.4 below shows that the function  $f$  can be extended continuously onto the

closure  $\bar{A}$  of  $A$ . This continuous extension of  $f$  then attains a maximum on  $\bar{A}$  by a standard compactness argument. We then show that this, in turn, implies the existence of a maximizer of  $f$  on its original domain  $(0, \infty)^n$ .

In order to simplify our notation, let us define the index set

$$J(\alpha) := \{\nu \in N \mid \alpha_\nu = 0\}$$

corresponding to a given  $\alpha \in [0, \infty)^n$ . Then the following existence result holds.

**Theorem 3.4** *The function  $f : A \rightarrow \mathbb{R}$  has a continuous extension onto the closure  $\bar{A}$  of  $A$  and, therefore, attains a global maximum on  $\bar{A}$ . Moreover, no vector  $\alpha \in \bar{A}$  with  $|J(\alpha)| = n - 1$  is a global maximum.*

**Proof.** The fact that  $f$  can be continuously extended from  $A$  onto  $\bar{A}$  follows from Lemmas A.1 and A.2 in the appendix, where, in particular, it is shown that this extension satisfies  $f(\alpha) = 0$  for all  $\alpha \in \bar{A}$  with  $|J(\alpha)| = n - 1$ , hence none of these vectors is a global maximum of  $f$  since  $f$  attains positive values on  $A$ . The existence of a global maximum then follows immediately from the fact that  $\bar{A}$  is a compact set.  $\square$

The global maximizer from Theorem 3.4 might belong to the set  $\bar{A} \setminus A$ . However, the feasible set of our original maximization problem is the set  $A$  or (without scaling) the set  $(0, \infty)^n$ . Using a suitable scaling together with a small perturbation, we now obtain the existence of a maximizer for our original problem from (4). Note that the following result shows that we can choose the maximizer in such a way that it also has some additional differentiability properties that will be exploited in Section 4.

**Corollary 3.5** *The function  $f$  attains a global maximum in  $(0, \infty)^n$ . Moreover, this global maximum can be chosen in such a way that condition (6) from Theorem 3.3 holds.*

**Proof.** By Theorem 3.4, the function  $f$  attains a global maximum in  $\bar{A}$ , and this maximum necessarily belongs to the set

$$\{\alpha \in \bar{A} \mid |J(\alpha)| \in \{0, \dots, n - 2\}\}.$$

However, as a consequence of Lemma 3.1, we have  $f(c\alpha) = f(\alpha)$  for all  $\alpha$  from this set and for all scalars  $c > 0$ . Consequently, the function  $f$  attains a global maximum  $\alpha^*$  on the set

$$\{\alpha \in [0, \infty)^n \mid |J(\alpha)| \in \{0, \dots, n - 2\}\}.$$

If, for this maximum, we have  $|J(\alpha^*)| \in \{1, \dots, n - 2\}$ , i.e.  $\alpha^* \notin (0, \infty)^n$ , Lemma 3.2 shows that there is a point  $\alpha^{**} \in (0, \infty)^n$  with the same function value so that  $\alpha^{**}$  is also a global maximizer. Consequently,  $f$  has a global maximum in the set  $(0, \infty)^n$ , too. If this maximum does not satisfy condition (6) from Theorem 3.3, we can apply Lemma 3.2 once more and get another point in  $(0, \infty)^n$  with the same function value (which, therefore, is also a maximum) such that (6) holds.  $\square$

## 4 Effort Maximization: Computation of Solution

Corollary 3.5 shows that there exists at least one global maximum  $\alpha^*$  of the optimization problem from (4). Moreover, this result tells us that the maximum can be chosen in such a way that the objective function  $f$  is differentiable in a neighbourhood of this maximum. Since the feasible set  $(0, \infty)^n$  is open, it follows that each such maximizer satisfies  $\nabla f(\alpha^*) = 0$ .

Basically, the aim of this section is to compute a global maximum by looking for all possible solutions of the nonlinear system of equations  $\nabla f(\alpha) = 0$  at those points  $x$  where the derivative of  $f$  exists. The previous discussion shows that this technique will eventually provide us a global maximum of (4). Moreover, our derivation will result in a particular algorithm for the computation of such a global maximum.

Unfortunately, computing the zeros of  $\nabla f(\alpha) = 0$  is not an easy task, especially since the derivative with respect to  $\alpha$  leads to complicated formulas. In order to simplify our calculations, we therefore use the transformation  $\gamma : (0, \infty)^n \rightarrow (0, \infty)^n$  defined by

$$\gamma_v(\alpha) := \frac{\beta_v}{\alpha_v} \quad (8)$$

for all  $v \in N$ . Since  $\beta \in (0, \infty)^n$ , the mapping  $\gamma$  is a diffeomorphism from  $(0, \infty)^n$  onto  $(0, \infty)^n$ . We further write  $\gamma = \frac{\beta}{\alpha}$  for the vector whose components are given by  $\frac{\beta_v}{\alpha_v}$ . For some  $\gamma \in (0, \infty)^n$ , we also write

$$K(\gamma) := \left\{ v \in N \mid (k(\gamma) - 1)\gamma_v < \sum_{\mu \in K(\gamma)} \gamma_\mu \right\}$$

with  $k(\gamma) := |K(\gamma)|$ . Using Theorem 2.1, it follows that for each  $\gamma \in (0, \infty)^n$ , there is precisely one such set  $K(\gamma)$ . Based on the set  $K(\gamma)$ , we now define the function  $g : (0, \infty) \rightarrow \mathbb{R}$  by

$$g(\gamma) := \frac{k(\gamma) - 1}{\sum_{\mu \in K(\gamma)} \gamma_\mu} \left( \sum_{\mu \in K(\gamma)} \frac{\gamma_\mu}{\beta_\mu} - \frac{k(\gamma) - 1}{\sum_{\mu \in K(\gamma)} \gamma_\mu} \sum_{\mu \in K(\gamma)} \frac{\gamma_\mu^2}{\beta_\mu} \right).$$

Since

$$K(\gamma(\alpha)) = K(\alpha)$$

for all  $\alpha \in (0, \infty)^n$ , we have  $g = f \circ \gamma^{-1}$ . Hence, for all global maxima  $\alpha^*$  of the function  $f$  satisfying condition (6) of Theorem 3.3, the function  $g$  has a global maximum in  $\gamma^* := \frac{\beta}{\alpha^*}$  and is continuously differentiable in a neighbourhood of  $\gamma^*$ , since

$$(k(\gamma^*) - 1)\gamma_v^* = (k(\alpha^*) - 1)\frac{\beta_v}{\alpha_v^*} > \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*} = \sum_{\mu \in K(\gamma^*)} \gamma_\mu^* \quad \forall v \notin K(\gamma^*).$$

Conversely, if  $\gamma^*$  denotes a global maximum of  $g$  with the property

$$v \notin K(\gamma^*) \implies (k(\gamma^*) - 1)\gamma_v^* > \sum_{\mu \in K(\gamma^*)} \gamma_\mu^*, \quad (9)$$

then  $\alpha^* := \frac{\beta}{\gamma^*}$  is a global maximum of  $f$  such that condition (6) of Theorem 3.3 holds. Hence we have the following result.

**Lemma 4.1**  $\alpha^* \in (0, \infty)^n$  is a global maximum of  $f$  satisfying property (6) of Theorem 3.3 if and only if  $\gamma^* = \frac{\beta}{\alpha^*}$  is a global maximum of  $g$  satisfying condition (9).

Consequently, if we find all global maxima of  $g$  satisfying (9), then a simple re-transformation gives us all the global maxima of  $f$  satisfying condition (6) from Theorem 3.3.

If a global maximum  $\gamma^*$  satisfies (9), then there is a neighbourhood of  $\gamma^*$  such that  $K(\gamma) \equiv K(\gamma^*)$  for all  $\gamma$  from this neighbourhood and, hence,  $g$  is continuously differentiable in this neighbourhood. Since  $\gamma^*$  is a maximum, we therefore have  $\nabla g(\gamma^*) = 0$ . Consequently, we have to compute the zeros of  $\nabla g$ . To this end, we first state two simple properties of the function  $g$  whose analogues were already shown for the function  $f$ .

**Lemma 4.2** (a) For all  $\gamma \in (0, \infty)^n$  and all  $c > 0$ , we have  $K(\gamma) = K(c\gamma)$  and  $g(\gamma) = g(c\gamma)$ .

(b) Let  $\gamma^* \in (0, \infty)^n$  be arbitrary. Then  $K(\gamma^*) = K(\gamma)$  and  $g(\gamma^*) = g(\gamma)$  hold for all  $\gamma \in (0, \infty)^n$  satisfying

$$\gamma_v = \gamma_v^* \quad \forall v \in K(\gamma^*) \quad \text{and} \quad \gamma_v \geq \frac{1}{k(\gamma^*) - 1} \sum_{\mu \in K(\gamma^*)} \gamma_\mu^* \quad \forall v \notin K(\gamma^*).$$

Lemma 4.2 shows that it suffices to compute maxima  $\gamma^*$  of  $g$  such that  $\sum_{\mu \in K(\gamma^*)} \gamma_\mu^* = 1$ . The following result summarizes some properties of global maxima satisfying this additional condition.

**Theorem 4.3** Let  $\gamma^* \in (0, \infty)^n$  be a global maximum of the function  $g$  satisfying  $\sum_{\mu \in K(\gamma^*)} \gamma_\mu^* = 1$  and (9). Then the following statements hold:

(a) For all  $v \in K(\gamma^*)$ , we have

$$\gamma_v^* = \frac{1}{2(k(\gamma^*) - 1)} \left[ 1 + (k(\gamma^*) - 2) \frac{\beta_v}{\sum_{\mu \in K(\gamma^*)} \beta_\mu} \right].$$

(b) For all  $v \notin K(\gamma^*)$ , we have

$$\gamma_v^* > \frac{1}{k(\gamma^*) - 1}.$$

(c) For all  $v \in K(\gamma^*)$ , we have

$$(k(\gamma^*) - 2)\beta_v < \sum_{\mu \in K(\gamma^*)} \beta_\mu.$$

(d) We have

$$g(\gamma^*) = \frac{1}{4} \left[ \sum_{\mu \in K(\gamma^*)} \frac{1}{\beta_\mu} - \frac{(k - 2)^2}{\sum_{\mu \in K(\gamma^*)} \beta_\mu} \right].$$

**Proof.** Since the maximum  $\gamma^*$  satisfies condition (9), there is a neighbourhood  $U$  of  $\gamma^*$  with

$$K(\gamma) = K(\gamma^*) =: K \quad \text{and} \quad k(\gamma) = k(\gamma^*) =: k.$$

Hence  $g$  is continuously differentiable in this neighbourhood of  $\gamma^*$  and, therefore, being an (essentially unconstrained) global maximum, we have  $\nabla g(\gamma^*) = 0$ .

The only statement we obtain for the components  $\gamma_\nu$  with  $\nu \notin K$  follows from (9):

$$\gamma_\nu^* > \frac{1}{k-1} \sum_{\mu \in K} \gamma_\mu^* = \frac{1}{k-1}.$$

This shows that statement (b) holds.

Moreover, for all  $\nu \in K$ , we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial \gamma_\nu} g(\gamma^*) \\ &= -\frac{k-1}{\left(\sum_{\mu \in K} \gamma_\mu^*\right)^2} \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} + \frac{k-1}{\sum_{\mu \in K} \gamma_\mu^*} \frac{1}{\beta_\nu} + \frac{2(k-1)^2}{\left(\sum_{\mu \in K} \gamma_\mu^*\right)^3} \sum_{\mu \in K} \frac{(\gamma_\mu^*)^2}{\beta_\mu} - \frac{2(k-1)^2}{\left(\sum_{\mu \in K} \gamma_\mu^*\right)^2} \frac{\gamma_\nu^*}{\beta_\nu} \\ &= -(k-1) \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} + (k-1) \frac{1}{\beta_\nu} + 2(k-1)^2 \sum_{\mu \in K} \frac{(\gamma_\mu^*)^2}{\beta_\mu} - 2(k-1)^2 \frac{\gamma_\nu^*}{\beta_\nu}. \end{aligned} \quad (10)$$

Summing up equation (10) over all  $\nu \in K$ , we get

$$-(k-1) \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} + 2(k-1)^2 \sum_{\mu \in K} \frac{(\gamma_\mu^*)^2}{\beta_\mu} = \frac{1}{k} \left( 2(k-1)^2 \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} - (k-1) \sum_{\mu \in K} \frac{1}{\beta_\mu} \right).$$

Inserting this into (10) and cancelling the factor  $k-1$ , we obtain for all  $\nu \in K$ :

$$\begin{aligned} \frac{2(k-1)}{k} \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} - \frac{1}{k} \sum_{\mu \in K} \frac{1}{\beta_\mu} + \frac{1}{\beta_\nu} - 2(k-1) \frac{\gamma_\nu^*}{\beta_\nu} &= 0 \\ \iff \gamma_\nu^* - \frac{1}{k} \sum_{\mu \in K} \frac{\beta_\nu}{\beta_\mu} \gamma_\mu^* &= \frac{1}{2(k-1)} \left[ 1 - \frac{\beta_\nu}{k} \sum_{\mu \in K} \frac{1}{\beta_\mu} \right]. \end{aligned}$$

Consequently, the vector  $\gamma_K^* := (\gamma_\nu^*)_{\nu \in K}$  is a solution of the linear system of equations

$$\left( I_{k \times k} - \frac{1}{k} \begin{pmatrix} \beta_\nu \\ \beta_\mu \end{pmatrix}_{\nu, \mu \in K} \right) (\gamma_\mu)_{\mu \in K} = \frac{1}{2(k-1)} \left( 1 - \frac{\beta_\nu}{k} \sum_{\lambda \in K} \frac{1}{\beta_\lambda} \right)_{\nu \in K}.$$

Using the abbreviations  $\beta_K := (\beta_\nu)_{\nu \in K}$  and  $\beta_K^{-1} := (\frac{1}{\beta_\nu})_{\nu \in K}$ , the matrix of this linear system can be written as

$$I_{k \times k} - \frac{1}{k} \begin{pmatrix} \beta_\nu \\ \beta_\mu \end{pmatrix}_{\nu, \mu \in K} = I_{k \times k} - \frac{1}{k} \beta_K (\beta_K^{-1})^T =: M.$$

This matrix  $M$  is singular, more precisely, it has rank  $k - 1$  and its null space (kernel) is given by  $\ker(M) = \text{span}\{\beta_K\}$  (this singularity reflects the fact that the function value  $g(\gamma)$  is independent of the scaling of  $\gamma$ , cf. Lemma 4.2, hence  $M$  cannot be expected to be nonsingular at an arbitrary stationary point). Now it is easy to see that one particular solution of the above linear system of equations is the vector from the right-hand side:

$$\tilde{\gamma}_v = \frac{1}{2(k-1)} \left( 1 - \frac{\beta_v}{k} \sum_{\lambda \in K} \frac{1}{\beta_\lambda} \right) \quad \forall v \in K.$$

Therefore, the vector  $\gamma_K^*$  is of the form  $\gamma_K^* = \tilde{\gamma}_K + c\beta_K$ , where  $c \in \mathbb{R}$  has to be chosen in such a way that  $\sum_{\mu \in K} \gamma_\mu^* = 1$ . It follows that

$$c = \frac{1}{2(k-1)} \left[ \frac{k-2}{\sum_{\mu \in K} \beta_\mu} + \frac{1}{k} \sum_{\mu \in K} \frac{1}{\beta_\mu} \right]$$

and, therefore,

$$\gamma_v^* = \frac{1}{2(k-1)} \left[ 1 + (k-2) \frac{\beta_v}{\sum_{\mu \in K} \beta_\mu} \right] (> 0)$$

for all  $v \in K$ . Hence statement (a) holds.

By the definition of  $K = K(\gamma^*)$ , we have for all  $v \in K$ :

$$(k-1)\gamma_v^* < \sum_{\mu \in K} \gamma_\mu^* = 1 \iff (k-2)\beta_v < \sum_{\mu \in K} \beta_\mu.$$

This verifies statement (c). Inserting the representation of  $\gamma_K^*$  gives the desired representation of  $g(\gamma^*)$  from assertion (d).  $\square$

Based on Theorem 4.3 (a) a re-transformation of  $\gamma_v^*$  yields the optimal parameter for the original set up:

$$\alpha_v^* = 2(k-1) \left[ \frac{1}{\beta_v} + \frac{k-2}{\sum_{\mu \in K(\gamma^*)} \beta_\mu} \right]^{-1}, \quad (11)$$

which is clearly increasing in  $\beta_v$ . This implies that in a global maximum players with high costs are favored relatively more than players with low costs. For a global maximum the heterogeneity between active players is therefore reduced. A closer look at the formula from Theorem 4.3 (a) reveals, however, that the quotient

$$\frac{\beta_v}{\alpha_v^*} = \gamma_v^* = \frac{1}{2(k-1)} \left[ 1 + (k-2) \frac{\beta_v}{\sum_{\mu \in K(\gamma^*)} \beta_\mu} \right].$$

is also increasing in  $\beta_v$ , whenever there are more than two players active in the optimum. We will show in Theorem 4.9 that this condition is met for all sets  $N$  consisting of more than two players. Hence, the 2-player-case is the only one in which heterogeneity between the active players is totally removed.



Note also that Theorem 4.3 (c) does not say that the inequality

$$(k(\gamma^*) - 2)\beta_\nu < \sum_{\mu \in K(\gamma^*)} \beta_\mu.$$

is violated for all  $\nu \notin K(\gamma^*)$ . The next lemma shows that, in some sense, the converse of Theorem 4.3 also holds.

**Lemma 4.4** *Let  $K \subseteq N$  be arbitrarily given, let  $k := |K| \geq 2$  and suppose that*

$$(k - 2)\beta_\nu < \sum_{\mu \in K} \beta_\mu \quad \forall \nu \in K. \quad (12)$$

*Define the vector  $\gamma^* \in (0, \infty)^n$  in such a way that  $\gamma_\nu^* > \frac{1}{k-1}$  is arbitrary for all  $\nu \notin K$  and*

$$\gamma_\nu^* = \frac{1}{2(k-1)} \left[ 1 + (k-2) \frac{\beta_\nu}{\sum_{\mu \in K} \beta_\mu} \right] \quad \forall \nu \in K.$$

*Then the following statements hold:*

- (a)  $\sum_{\mu \in K} \gamma_\mu^* = 1$ .
- (b)  $K(\gamma^*) = K$  and  $\gamma^*$  satisfies condition (9).
- (c) The function  $g$  is continuously differentiable in a neighbourhood of  $\gamma^*$  with  $\nabla g(\gamma^*) = 0$ .
- (d)  $g(\gamma^*) = \frac{1}{4} \left[ \sum_{\mu \in K} \frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu \in K} \beta_\mu} \right]$ .

**Proof.** Statement (a) can be verified easily using the definition of  $\gamma_\mu^*$  for  $\mu \in K$ . Assertions (c) and (d), on the other hand, follow in essentially the same way as in the proof of Theorem 4.3 since our *definition* of  $\gamma_\nu^*$  is exactly the same as the *representation* of  $\gamma_\nu^*$  obtained in Theorem 4.3 for  $\gamma_\nu^*$  ( $\nu \in K$ ). To see that statement (b) holds, we verify that

$$(k-1)\gamma_\nu^* < \sum_{\mu \in K} \gamma_\mu^* = 1 \iff \nu \in K. \quad (13)$$

The definition of the index set  $K(\gamma^*)$  together with the uniqueness of this index set then shows  $K = K(\gamma^*)$ . Now, for  $\nu \in K$ , we obtain from the definition of  $\gamma_\nu^*$  together with (12) that

$$(k-1)\gamma_\nu^* = \frac{1}{2} \left[ 1 + (k-2) \frac{\beta_\nu}{\sum_{\mu \in K} \beta_\mu} \right] < \frac{1}{2} [1 + 1] = 1,$$

hence the implication “ $\Leftarrow$ ” holds in (13). On the other hand, for  $\nu \notin K$ , we have  $(k-1)\gamma_\nu^* > 1$  which, by contraposition, shows that also the implication “ $\Rightarrow$ ” holds in (13).  $\square$

Lemma 4.4 and Theorem 4.3 are the foundation of the following algorithm for the computation of all global maxima that satisfy the additional conditions from Theorem 4.3. Using the variations

from Lemma 4.2, we obtain all global maxima of  $g$  in  $(0, \infty)^n$  since each global maximum can be modified by these variations in such a form that the conditions from Theorem 4.3 hold.

For a better understanding of our algorithm, note that Theorem 4.3 allows the following interpretation: If  $\gamma^*$  is a global maximum of  $g$  satisfying  $\sum_{\mu \in K(\gamma^*)} \gamma_\mu^* = 1$  and (9), then we necessarily have

$$|K(\gamma^*)| \geq 2 \quad \text{and} \quad (k(\gamma^*) - 2)\beta_\nu < \sum_{\mu \in K(\gamma^*)} \beta_\mu \quad \forall \nu \in K(\gamma^*)$$

by statement (c). (Assertions (a) and (b) only give the structure of the maximizer  $\gamma^*$ , whereas statement (d) calculates the corresponding function value  $g(\gamma^*)$ .) Now, Lemma 4.4 takes an arbitrary index set  $K \subseteq N$  with

$$k := |K| \geq 2 \quad \text{and} \quad (k - 2)\beta_\nu < \sum_{\mu \in K} \beta_\mu \quad \forall \nu \in K, \quad (14)$$

defines corresponding values for  $\gamma_\nu^*$  ( $\nu \in N$ ) and then states that, in particular, we have  $K = K(\gamma^*)$  and that  $\gamma_\nu^*$  satisfies  $\sum_{\mu \in K(\gamma^*)} \gamma_\mu^* = 1$  as well as condition (9). Note that there are only finitely many index sets  $K \subseteq N$  with (14), hence we necessarily get a global maximum among these candidate points by comparing the corresponding function values  $g(\gamma^*)$ . Since the  $\gamma^*$  are specified by the given index set  $K$  in Lemma 4.4, we define the function

$$h(K) := \frac{1}{4} \left[ \sum_{\mu \in K} \frac{1}{\beta_\mu} - \frac{(k - 2)^2}{\sum_{\mu \in K} \beta_\mu} \right]$$

for all  $K \subseteq N$  with  $k := |K| \geq 2$  and  $(k - 2)\beta_\nu < \sum_{\mu \in K} \beta_\mu$  for all  $\nu \in K$ . We next state our algorithm formally.

**Algorithm 4.5** (S.0) Set  $h_{\max} := 0$  and  $L := \emptyset$ .

(S.1) Check for all  $K \subseteq N$  with  $k := |K| \geq 2$  and  $(k - 2)\beta_\nu < \sum_{\mu \in K} \beta_\mu$  for all  $\nu \in K$  the following conditions:

- (a) If  $h(K) > h_{\max}$ , then set  $h_{\max} = h(K)$  and  $L = \{K\}$ .
- (b) If  $h(K) = h_{\max}$ , then set  $L = L \cup \{K\}$ .

Basically, Algorithm 4.5 compares all sets  $K$  that could belong to a global maximum and remembers those with the highest value  $h(K)$  found so far. In the end, the set  $L$  contains all sets  $K$  such that the corresponding vector  $\gamma^*$  from Lemma 4.4 is a global maximum. This procedure is completely justified by Theorem 4.3 and Lemma 4.4. Via the transformation (8), we then obtain a solution  $\alpha_\nu^* := \beta_\nu / \gamma_\nu^*$  of the original effort maximization problem from (4). Recall, however, that the solution of (4) is always nonunique, and that any positive multiple of  $\alpha^*$  is also a solution of the effort maximization game.

Note that Algorithm 4.5 reduces the original maximization problem (4), where the maximization is taken over infinitely many values  $\alpha \in (0, \infty)^n$ , to a finite procedure over certain subsets  $K$  of  $N$ . In the worst case, the number of subsets that have to be checked in Algorithm 4.5 is

large, depending exponentially on the number of players, but in practice it seems to work very efficiently at least for  $n \leq 25$ .

Now, we want to apply our results to two special cases that have already been discussed in the literature, cf. Nti [12], Dasgupta and Nti [6].

**Example 4.6** In the 2-player case, the set of active players in the global maximum is  $K = \{1, 2\}$  and Theorem 4.3 implies that the optimal parameters are

$$\gamma_v^* = \frac{\beta_v}{\alpha_v^*} = \frac{1}{2}, \quad \text{hence} \quad \alpha_v^* = 2\beta_v \quad \forall v = 1, 2.$$

Therefore, heterogeneity between the players is completely removed in the optimum. The optimal set of weighting parameters yields the following equilibrium results:

$$\begin{aligned} x^{*,v} &= \frac{1}{4\beta_v} \quad \forall v = 1, 2; \\ f(\alpha^*) &= \frac{\beta_1 + \beta_2}{4\beta_1\beta_2}; \\ \theta_v(x^{*,v}, x^{*, -v}) &= \frac{1}{4} \quad \forall v = 1, 2. \end{aligned}$$

The complete removal of heterogeneity is also reflected by the fact that expected payoff in equilibrium is identical for both players.  $\diamond$

**Example 4.7** In the homogeneous  $n$ -player case, where  $\beta_v = \beta_\mu (= \beta)$  for all  $v, \mu \in N$ , all subsets  $K \subseteq N$  with  $k := |K| \geq 2$  are feasible for Algorithm 4.5 and the set of active players in the global maximum is the one that maximizes

$$h(K) = \frac{1}{4} \left[ \sum_{\mu \in K} \frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu \in K} \beta_\mu} \right] = \frac{k-1}{k\beta}.$$

Thus, the set of active players in the global maximum is  $K = N$ , and Theorem 4.3 shows that the corresponding optimal parameters are

$$\gamma_v^* = \frac{\beta_v}{\alpha_v^*} = \frac{1}{n}, \quad \text{hence} \quad \alpha_v^* = n\beta_v \quad \forall v \in N.$$

In particular, all players are active in the optimum. Equilibrium results are the following:

$$x^{*,v} = \frac{n-1}{n^2\beta}; \quad f(\alpha^*) = \frac{n-1}{n\beta}; \quad \theta_v(x^{*,v}, x^{*, -v}) = \frac{1}{n^2} \quad \forall v \in N.$$

$\diamond$

As we mentioned above, the number of subsets that have to be checked in Algorithm 4.5 can be quite large. It is, however, possible to restrict the candidates for  $L$  further. This follows from the subsequent result.

**Proposition 4.8** Let  $K, M \subseteq N$  be feasible subsets for Algorithm 4.5, i.e., let  $k := |K| \geq 2, m := |M| \geq 2$ , and suppose that

$$(k-2)\beta_\nu < \sum_{\mu \in K} \beta_\mu \quad \forall \nu \in K \quad \text{and} \quad (m-2)\beta_\nu < \sum_{\mu \in M} \beta_\mu \quad \forall \nu \in M.$$

Then, if  $M \subsetneq K$ , we have  $h(M) < h(K)$ .

**Proof.** Using the well-known inequality between the arithmetic and harmonic mean together with some elementary calculations, we obtain

$$\begin{aligned} & h(K) - h(M) \\ &= \frac{1}{4} \left[ \sum_{\mu \in K \setminus M} \frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu \in K} \beta_\mu} + \frac{(m-2)^2}{\sum_{\mu \in M} \beta_\mu} \right] \\ &\geq \frac{1}{4} \left[ \frac{(k-m)^2}{\sum_{\mu \in K \setminus M} \beta_\mu} - \frac{(k-m)^2 + 2(k-m)(m-2) + (m-2)^2}{\sum_{\mu \in K \setminus M} \beta_\mu + \sum_{\mu \in M} \beta_\mu} + \frac{(m-2)^2}{\sum_{\mu \in M} \beta_\mu} \right] \\ &= \frac{1}{4} \left[ \frac{\left( (k-m) \sum_{\mu \in M} \beta_\mu - (m-2) \sum_{\mu \in K \setminus M} \beta_\mu \right)^2}{\sum_{\mu \in K \setminus M} \beta_\mu \sum_{\mu \in K} \beta_\mu \sum_{\mu \in M} \beta_\mu} \right] \\ &\geq 0, \end{aligned}$$

and equality  $h(K) = h(M)$  holds if and only if

$$\sum_{\mu \in K \setminus M} \frac{1}{\beta_\mu} = \frac{(k-m)^2}{\sum_{\mu \in K \setminus M} \beta_\mu} \quad \text{and} \quad \sum_{\mu \in M} \beta_\mu = (m-2) \frac{\sum_{\mu \in K \setminus M} \beta_\mu}{k-m}.$$

Since all  $\beta_\mu$  ( $\mu \in K \setminus M$ ) are positive, the harmonic mean and the arithmetic mean coincide if and only if all  $\beta_\mu$  ( $\mu \in K \setminus M$ ) coincide, i.e. if  $\beta_\mu = \beta$  for all  $\mu \in K \setminus M$  and a suitable  $\beta > 0$ . Hence, the second equation implies that, for all  $\mu \in K \setminus M$ , we have

$$(k-2)\beta_\mu = (k-m)\beta + (m-2)\beta = \sum_{\mu \in K \setminus M} \beta_\mu + \sum_{\mu \in M} \beta_\mu = \sum_{\mu \in K} \beta_\mu,$$

which, however, is a contradiction to the feasibility of  $K$  for Algorithm 4.5.  $\square$

Proposition 4.8 says that strict subsets of feasible sets for Algorithm 4.5 cannot be optimal. Thus, all elements of  $K \in L$  have to be “maximal” subsets of  $N$ .

Now, consider the case of  $N \geq 3$  players. Further note that every subset  $M \subseteq N$  consisting of two players is feasible for Algorithm 4.5. Then, take an arbitrary element  $\nu \in N \setminus M$  and define  $K := M \cup \{\nu\}$ . The so defined set  $K$  consists of three players containing  $M$  as a strict subset and is feasible for Algorithm 4.5. In view of Proposition 4.8, it follows that  $M$  cannot be an optimal set. Hence we obtain the following result.

**Theorem 4.9** Consider the effort maximization problem (4) with  $|N| \geq 3$ . Then there are at least three active players in every solution  $K \in L$ .

Note that the previous argument cannot be used in order to show that four or more players will be active at a solution of the maximization problem (4). In fact, Example 4.12 below shows that there exist instances of our problem with precisely three players being active in the optimum, hence, from this point of view, Theorem 4.9 is optimal. The last Theorem is remarkable: it not only improves on previous knowledge as summarized in Theorem 2.2 (a); it is also in marked contrast to well-established results from contests, which are modeled as all-pay auctions; i.e. the contest success function is such that the highest effort wins with certainty (in case of  $m$  highest bids each wins with probability  $\frac{1}{m}$ ). Then the equilibrium of the  $n$ -player complete information contest is generically unique and exhibits *precisely* two active players. Moreover, in the non-generic case with multiple equilibria, total effort in equilibrium is highest in the equilibrium with only two active players (see Baye et al. [2]). Hence allowing for free entry into the contest cannot improve the competitiveness of the contest as the equilibrium strategies of the two active players do not depend on the number and identity of inactive players. This is not true in our model: a third player can always improve on the effort levels obtained in a two-player contest from the contest designer's point of view.

Exploiting Theorem 4.9, we are now able to give an analytic solution also for the case of three players. In particular, the optimality of evenness (Example 4.6) only applies to two-player contests.

**Example 4.10** In the 3-player case, Theorem 4.9 shows that the set of active players in the global maximum is  $K = \{1, 2, 3\}$ , and Theorem 4.3 gives the corresponding optimal parameters

$$\gamma_v^* = \frac{\beta_v}{\alpha_v^*} = \frac{1}{4} \left[ 1 + \frac{\beta_v}{\sum_{\mu=1}^3 \beta_\mu} \right], \quad \text{hence} \quad \alpha_v^* = 4 \left[ \frac{1}{\beta_v} + \frac{1}{\sum_{\mu=1}^3 \beta_\mu} \right]^{-1} \quad \forall v = 1, 2, 3.$$

Hence, heterogeneity between the players is not completely removed in the optimum, but is relaxed. This is also indicated by the fact that expected equilibrium payoff is not identical among players:

$$\begin{aligned} x^{*,v} &= \frac{1}{4} \left( \frac{1}{\beta_v} - \frac{\beta_v}{(\sum_{\mu=1}^3 \beta_\mu)^2} \right) \quad \forall v = 1, 2, 3; \\ f(\alpha^*) &= \frac{1}{4} \left( \sum_{\mu=1}^3 \frac{1}{\beta_\mu} - \frac{1}{\sum_{\mu=1}^3 \beta_\mu} \right); \\ \theta_v(x^{*,v}, x^{*, -v}) &= \frac{1}{4} \left( 1 - \frac{\beta_v}{\sum_{\mu=1}^3 \beta_\mu} \right)^2 \quad \forall v = 1, 2, 3. \end{aligned}$$

◇

One of the “maximal” subsets mentioned above is

$$K^* := \left\{ v \in N \mid (|K^*| - 2)\beta_v < \sum_{\mu \in K^*} \beta_\mu \right\}.$$

One can prove analogously to Theorem 2.6 that for all  $\beta$  there is at most one subset of  $N$  satisfying this condition. Conversely, a simple calculation shows that there is always at least one such set. If, furthermore,  $\beta_1 \leq \dots \leq \beta_n$ , then one can show similarly to Theorem 2.3 that

$$K^* = \left\{ \nu \in N \mid (\nu - 2)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\}.$$

Numerical tests indicate the following conjecture:

**Conjecture 4.11** *For every  $\beta$ , the set  $L$  generated by Algorithm 4.5 is  $L = \{K^*\}$ , hence  $K^*$  is the unique set such that the corresponding vector  $\gamma^*$  from Lemma 4.4 is a global maximum.*

Finally, let us consider some (nontrivial) examples.

**Example 4.12** We applied Algorithm 4.5 to different instances of our effort maximization problem from (4). To this end, we consider problems with 7 players and take different values for the parameters  $\beta_\nu$ . Table 1 contains the precise values for  $\beta_\nu$  together with the optimal set  $L$  computed by our method. The corresponding values of the solution  $\alpha^*$  are calculated using (11) for the active players. For the inactive players, the maximal possible values of  $\alpha_\nu^*$  according to Theorem 4.3 are given. In the last column, the total effort  $f(\alpha^*)$  is given. Note that, in all the instances computed in Table 1 as well as in many further calculations that we did (with up to 25 players), our conjecture holds.  $\diamond$

$\beta$	$L$	$\alpha^*$	$f(\alpha^*)$
$(1, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3)$	$\{(1, 2, 3)\}$	$(3.8919, 26.1818, 61.7143, 128, 250, 432, 686)$	0.2836
$(1, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2)$	$\{(1, 2, 3)\}$	$(3.7\bar{3}, 12.\bar{4}, 21.9130, 32, 50, 72, 98)$	0.3224
$(1, 2, 3, 4, 5, 6, 7)$	$\{(1, 2, 3, 4)\}$	$(5, 8.5714, 11.25, 13.\bar{3}, 15, 18, 21)$	0.4208
$(1, 1, 1, 1, 1, 1, 1)$	$\{(1, 2, 3, 4, 5, 6, 7)\}$	$(7, 7, 7, 7, 7, 7, 7)$	0.8571
$(\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1})$	$\{(1, 2, 3, 4, 5)\}$	$(0.8209, 0.9148, 1.0329, 1.1860, 1.3925, 2, 4)$	4.1912
$(\frac{1}{7^2}, \frac{1}{6^2}, \frac{1}{5^2}, \frac{1}{4^2}, \frac{1}{3^2}, \frac{1}{2^2}, \frac{1}{1})$	$\{(1, 2, 3, 4)\}$	$(0.0964, 0.1218, 0.1568, 0.2050, \frac{1}{3}, \frac{3}{4}, 3)$	24.8637
$(\frac{1}{7^3}, \frac{1}{6^3}, \frac{1}{5^3}, \frac{1}{4^3}, \frac{1}{3^3}, \frac{1}{2^3}, \frac{1}{1})$	$\{(1, 2, 3)\}$	$(0.0098, 0.0143, 0.0211, \frac{1}{32}, \frac{2}{27}, \frac{1}{4}, 2)$	154.9177
$(\frac{1}{7^4}, \frac{1}{6^4}, \frac{1}{5^3}, \frac{1}{4^4}, \frac{1}{3^4}, \frac{1}{2^4}, \frac{1}{1})$	$\{(1, 2, 3)\}$	$(0.0014, 0.0024, 0.0041, \frac{1}{128}, \frac{2}{81}, \frac{1}{8}, 2)$	990.8331

Table 1: Numerical illustration of Algorithm 4.5

## 5 Final Remarks

We have analyzed contest games with players characterized by heterogeneous valuations as well as heterogeneous cost functions, which are based on asymmetric variants of the arguably most popular contest success function in the literature: Tullock's [17] lottery model, which determines a contestant's success probability to be equal to the ratio of his effort and total efforts.

These variants apply weight biases to the individual efforts in the original Tullock specification. We summarize known existence results and further develop equilibrium characterizations in order to address the general problem of total effort maximization from a contest designer's point of view, who has control over the bias weights; i.e., the choice of the contest success function. The designer's choice of the optimal contest success function leads to a nonsmooth optimization problem as the number of contestants contributing to total efforts is endogenous and depends on the weights (given valuation and cost parameters). Nevertheless, we are able to show that an optimal contest success function always exists; i.e., for *any* given distribution of valuation and cost parameters there is a solution to the contest designer's problem. Moreover, this solution is easily computable once the set of active players in the effort maximizing equilibrium is determined. The latter is the crucial task; i.e., we infer the optimal contest success function from the set of in the effort maximizing equilibrium active players, not vice versa. The determination of this set of players is a finite search problem, which can be solved algorithmically. It is shown that our algorithm cannot end at a set containing only two players; hence the optimal contest success function always activates at least three players in the total efforts maximizing equilibrium. An open question is whether the optimal weights  $\alpha_\nu$  of the active players are unique up to a positive scalar factor; i.e., whether *relative* weights  $\frac{\alpha_\nu}{\alpha_\mu}$ ,  $\nu, \mu \in K(\alpha)$ , are uniquely determined by the aim of effort maximization.

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## A Proof of Theorem 3.4

Here we give a proof of Theorem 3.4 which is the central existence result from Section 3. In particular, we have to show that the function  $f : A \rightarrow \mathbb{R}$  from (5) has a continuous extension from the set  $A$  defined in (7) onto its closure

$$\bar{A} = \left\{ \alpha \in [0, \infty)^n \mid \sum_{\mu=1}^n \alpha_{\mu} = 1 \right\}.$$

To this end, we first recall the definition of the index set

$$J(\alpha) := \{v \in N \mid \alpha_v = 0\}$$

for a given  $\alpha \in [0, \infty)^n$ . We already know from Theorem 3.3 that  $f$  is a continuous function on  $A$ , i.e.  $f$  is continuous at any point  $\alpha \in \bar{A}$  such that  $|J(\alpha)| = 0$ . In a first step, we will show in Lemma A.1 that  $f$  has a continuous extension to all  $\alpha \in \bar{A}$  with  $|J(\alpha)| \leq n - 2$ . Then, we will prove in Lemma A.2 that  $f$  can also be extended continuously to all points  $\alpha \in \bar{A}$  such that  $|J(\alpha)| = n - 1$  by defining  $f(\alpha) := 0$  in these points. Since the case  $|J(\alpha)| = n$  cannot occur for  $\alpha \in \bar{A}$ , this yields Theorem 3.4.

Here is our first result regarding the extension of  $f$  to points  $\alpha$  with  $|J(\alpha)| \leq n - 2$ .



**Lemma A.1** *The function  $f$ , viewed as a mapping from  $A$  to  $\mathbb{R}$ , can be extended continuously onto the set*

$$\{\alpha \in \bar{A} \mid |J(\alpha)| \leq n - 2\}.$$

**Proof.** Recall from the proof of Theorem 3.3 that  $f$  is continuous on the set

$$A = \{\alpha \in \bar{A} \mid |J(\alpha)| = 0\}$$

(in fact, it is continuous on  $(0, \infty)^n$ ). Now, let  $\alpha^* \in \bar{A}$  with  $|J(\alpha^*)| \in \{1, \dots, n - 2\}$  be arbitrarily given. Then let us define the set of players  $N^* := N \setminus J(\alpha^*)$ . Since we have  $|N^*| \geq 2$ , it follows that the Nash game with the set of players  $N^*$  replacing the set of players  $N$  has all the properties that were already shown. Consequently, if we let

$$f^*(\alpha) := \sum_{\nu \in N^*} x^\nu(\alpha)$$

be the objective function of this new game, we, in particular, obtain from Theorem 3.3 that  $f^*$  is continuous in a sufficiently small neighbourhood of  $\alpha^*$  simply since we eliminated the critical players  $\nu$  with  $\alpha_\nu^* = 0$  from the set  $N$ . We will show in the next paragraph that, for all  $\alpha$  from a sufficiently small neighbourhood  $U$  of  $\alpha^*$ , we have  $K(\alpha) \subseteq N^*$ . This then implies  $f(\alpha) = f^*(\alpha)$  for all  $\alpha \in U$  and, in this way, we obtain the desired continuous extension of  $f$  in  $\alpha^*$ .

To verify the above claim, we have to find a sufficiently small neighbourhood  $U$  of  $\alpha^*$  such that  $K(\alpha) \subseteq N^*$  for all  $\alpha \in U$ , i.e., for all  $\alpha \in U$  and all indices  $\nu$  with  $\nu \in K(\alpha)$ , we necessarily have  $\alpha_\nu > 0$ . By contraposition, this is equivalent to showing that, for all  $\alpha \in U$  and all indices  $\nu$  with  $\alpha_\nu = 0$ , we have  $\nu \notin K(\alpha)$ .

To see this, we first choose a sufficiently small neighbourhood of  $\alpha^*$  such that  $|J(\alpha)| \in \{0, 1, \dots, n - 2\}$  for all  $\alpha \in U$ . We then define a function  $c(\alpha)$  on  $U$  as the sum of the two smallest quotients  $\frac{\beta_\mu}{\alpha_\mu}$  ( $\mu \in N$ ). Then  $c(\alpha)$  is continuous and finite. Moreover, Corollary 2.4 shows that we always have  $K(\alpha) \subseteq \{\nu \in N \mid \frac{\beta_\nu}{\alpha_\nu} < c(\alpha)\}$ . By taking a possibly smaller neighbourhood  $U$ , we may assume by continuity that  $c(\alpha) < 2c(\alpha^*)$  for all  $\alpha \in U$  and, in addition, that  $\frac{\beta_\nu}{\alpha_\nu} > 2c(\alpha^*)$  for all  $\nu \in J(\alpha^*)$ . This implies the desired claim since, now, we obtain  $\frac{\beta_\nu}{\alpha_\nu} > 2c(\alpha^*) > c(\alpha)$  for all  $\alpha \in U$  and all  $\nu \in J(\alpha^*)$ , hence  $\nu \notin K(\alpha)$ .  $\square$

It remains to consider the case  $|J(\alpha)| = n - 1$ . This is done in the following result.

**Lemma A.2** *The function  $f$ , viewed as a mapping from  $\{\alpha \in \bar{A} \mid |J(\alpha)| \leq n - 2\}$  to  $\mathbb{R}$ , can be extended continuously onto the set  $\bar{A}$  by setting  $f(\alpha^*) = 0$  for all  $\alpha^* \in \bar{A}$  with  $|J(\alpha^*)| = n - 1$ .*

**Proof.** We begin with some preliminary comments. In order to verify our claim, we have to show that, given an arbitrary vector  $\alpha^* \in \bar{A}$  with  $|J(\alpha^*)| = n - 1$  as well as a sequence  $\{\alpha\} \rightarrow \alpha^*$  with  $\alpha \in \bar{A}$  satisfying  $|J(\alpha)| \leq n - 2$  for all  $\alpha$ , we have  $f(\alpha) \rightarrow f(\alpha^*)$ . Now, for all  $\alpha \in A$  (so all components of  $\alpha$  are positive), we have the representation

$$f(\alpha) = \sum_{\nu \in K(\alpha)} x^\nu(\alpha) \tag{15}$$

of our objective function, where  $K(\alpha)$  is the set of active players, cf. (5). On the other hand, if one or more (at most  $n - 2$ ) components of  $\alpha$  are equal to zero, we obtained  $f$  by a continuous extension in the proof of Lemma A.1, hence the representation (15) does not necessarily hold in this case. However, we showed in the proof of Lemma A.1 that  $K(\alpha) \cap J(\alpha) = \emptyset$  so that players  $\nu$  with  $\alpha_\nu = 0$  are certainly not active. This means that for all  $\alpha \in \bar{A}$  with  $|J(\alpha)| \leq n - 2$ , the representation (15) is still valid, and we will work with it throughout this proof.

Now, take an arbitrary  $\alpha^* \in \bar{A}$  with  $|J(\alpha^*)| = n - 1$ , i.e.  $\alpha^* = e_j$  for some  $j \in \{1, \dots, n\}$ . Then we obtain for all  $\alpha \in \bar{A} \setminus \{\alpha^*\}$  sufficiently close to  $\alpha^*$  that, on the one hand,  $|J(\alpha)| \in \{0, \dots, n - 2\}$  and, on the other hand,

$$\frac{\beta_j}{\alpha_j} = \min_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu},$$

hence  $j \in K(\alpha)$ . Consider an arbitrary sequence  $\{\alpha\} \subset \bar{A} \setminus \{\alpha^*\}$  with  $\alpha \rightarrow \alpha^*$ . We can divide the sequence into finitely many subsequences such that, within each subsequence, the set  $K(\alpha)$  is constant. We verify the statement for each of these subsequences which then, obviously, implies that the statement holds for the entire sequence. We now consider one of these subsequences and call it, once again,  $\{\alpha\}$ . In view of the previous remark, we have  $K(\alpha) \equiv K$  and  $k(\alpha) \equiv k$  for all  $\alpha$ . We now verify the limit  $f(\alpha) = \sum_{\nu \in K} x^\nu(\alpha) \rightarrow 0$  by showing that  $x^\nu(\alpha) \rightarrow 0$  holds for all  $\nu \in K$ . For  $\nu = j$ , this follows immediately from

$$x^j(\alpha) = \left( 1 - \frac{\beta_j(k-1)}{\sum_{\mu \in K} \beta_\mu \frac{\alpha_j}{\alpha_\mu}} \right) \frac{(k-1)}{\sum_{\mu \in K} \beta_\mu \frac{\alpha_j}{\alpha_\mu}} \rightarrow (1-0)0 = 0.$$

Moreover, for  $k = 2$ , the statement also follows easily for  $\nu \in K \setminus \{j\}$ :

$$x^\nu(\alpha) = \left( 1 - \frac{\beta_\nu}{\beta_\nu + \beta_j \frac{\alpha_\nu}{\alpha_j}} \right) \frac{1}{\beta_\nu + \beta_j \frac{\alpha_\nu}{\alpha_j}} \rightarrow (1-1) \frac{1}{\beta_\nu} = 0.$$

It therefore remains to verify  $x^\nu(\alpha) \rightarrow 0$  for all  $\nu \in K \setminus \{j\}$  in the case  $k \geq 3$ . To this end, we show that, for all  $k = 3, 4, \dots$  and all  $\nu, \mu \in K \setminus \{j\}$  with  $\nu \neq \mu$ , we have

$$\lim_{\alpha \rightarrow e_j} \frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu}{\beta_\mu}. \quad (16)$$

Using (16), we then obtain for all  $\nu \in K \setminus \{j\}$  and all  $k \geq 3$

$$x^\nu(\alpha) = \left( 1 - \frac{(k-1)}{\sum_{\mu \in K} \frac{\beta_\mu \alpha_\nu}{\beta_\nu \alpha_\mu}} \right) \frac{(k-1)}{\beta_\nu \sum_{\mu \in K} \frac{\beta_\mu \alpha_\nu}{\beta_\nu \alpha_\mu}} \rightarrow (1-1) \frac{1}{\beta_\nu} = 0$$

and therefore the desired statement. To verify (16), it suffices to show that, for all  $k = 3, 4, \dots$  and all  $\nu, \mu \in K \setminus \{j\}$  with  $\nu \neq \mu$ , we have

$$\limsup_{\alpha \rightarrow e_j} \frac{\alpha_\nu}{\alpha_\mu} \leq \frac{\beta_\nu}{\beta_\mu}. \quad (17)$$

Exchanging the roles of  $\nu$  and  $\mu$  then yields (16).

To verify (17), we first consider the case  $k = 3$ . Therefore, let  $\nu, \mu \in K \setminus \{j\}$  be given with  $\nu \neq \mu$ . We then obtain for an arbitrary  $\alpha$ , exploiting the characteristic property (1) of  $\mu \in K$ , that

$$\frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu \alpha_\nu \beta_\mu}{\beta_\mu \beta_\nu \alpha_\mu} < \frac{\beta_\nu \alpha_\nu}{\beta_\mu \beta_\nu} \frac{1}{2} \left( \frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu} \right).$$

Rewriting this expression gives

$$\frac{\alpha_\nu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu} \left( \frac{\alpha_\nu \beta_j}{\beta_\nu \alpha_j} + 1 \right).$$

Taking into account  $\alpha \rightarrow e_j$ , we obtain (17).

Next, consider the case  $k = 4$ . To this end, choose arbitrary  $\nu, \mu \in K \setminus \{j\}$  with  $\nu \neq \mu$ , and let  $K = \{j, \nu, \mu, \lambda\}$ . Using  $\lambda \in K$ , we have

$$\frac{\beta_\lambda}{\alpha_\lambda} < \frac{1}{3} \sum_{\rho \in K} \frac{\beta_\rho}{\alpha_\rho} \iff \frac{\beta_\lambda}{\alpha_\lambda} < \frac{1}{2} \left( \frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu} \right). \quad (18)$$

Exploiting once again (1), we obtain from  $\mu \in K$  the inequality

$$\frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu \alpha_\nu \beta_\mu}{\beta_\mu \beta_\nu \alpha_\mu} < \frac{\beta_\nu \alpha_\nu}{\beta_\mu \beta_\nu} \frac{1}{3} \left( \frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu} + \frac{\beta_\lambda}{\alpha_\lambda} \right).$$

Estimating the right-hand side by using (18) and rearranging the resulting terms, we obtain the same inequality

$$\frac{\alpha_\nu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu} \left( \frac{\alpha_\nu \beta_j}{\beta_\nu \alpha_j} + 1 \right)$$

as above, so that  $\alpha \rightarrow e_j$  also yields (17) for the case  $k = 4$ . For  $k = 5, 6, \dots$ , the statement can be verified in an analogous way.  $\square$

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