

Outside versus Inside Bonds: A Modigliani-Miller Type Result for Liquidity Constrained Economies

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Abstract

When agents are liquidity constrained, two options exist — sell assets or borrow. We compare the allocations arising in two economies: in one, agents can sell government (outside) bonds and in the other they can borrow by issuing (inside) bonds. All transactions are voluntary, implying no taxation or forced redemption of private debt. We show that any allocation in the economy with inside bonds can be replicated in the economy with outside bonds but that the converse is not true. However, the optimal policy in each economy makes the allocations equivalent.

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1 Introduction

In monetary economies, households often face binding liquidity constraints. In such situations, they can acquire additional liquidity by selling assets or by borrowing. Several papers have studied the case where households can sell nominal government bonds (outside bonds) for money while others allow households to borrow money (issue inside bonds) to finance consumption.¹ These different methods for relaxing liquidity constraints raise the following question: following the logic of Modigliani-Miller, do these alternative financing arrangements of household consumption lead to equivalent allocations? Our focus in this paper is to address this question.

Within a common monetary framework, we consider two economies: one in which households trade outside bonds and one in which they trade inside bonds. Two main results emerge from our analysis. First, any allocation in the inside bond economy can be replicated in the outside bond economy. The converse is not true. Second, if government policy is set optimally in both economies, then the allocations are the same.

The key assumption for attaining these results is that all trades among private agents and between private agents and the government must be voluntary. For the inside bond economy, this implies that redemption of inside bonds must be voluntary. For the outside bond economy, it means that the government cannot levy taxes so private agents must be willing to pay a nominal ‘fee’ to receive government services. This implies that the government is constrained in how much revenue it can generate to redeem outstanding government debt. In short, for both economies individual participation constraints must be taken into account.²

The key feature that makes the allocations equivalent across the two economies is the cost associated with participating in the bond market. In the inside bond economy, if a household defaults on its debt, it is excluded from trading in the bond market until it repays its debt. In the outside bond economy, we assume the government can charge a fee to participate in a bond market that it operates. If a household does not pay the fee, it is denied access to this bond market.³ In this way, households face a similar participation decision in either economy: whether they should incur a cost today (repay loans or pay the fee) to have access to bond markets.

We show that for an arbitrary money growth rate, the allocation in the inside bond economy can be replicated in the outside bond economy by an appropriate choice of the fee. In general, the converse is not true. Hence, allocations can differ across the two economies. We then show that in the outside bond economy it is optimal to have the government charge the maximum fee – one that makes an individual indifferent between participating in the bond market or not. Under this policy, the allocation in the outside bond economy will always be equivalent to that in the inside

¹Examples of the first method include [12], [17] and [19, 20]. Examples of the latter include [1], [4], [5] and [7].

²In a recent paper, [13] emphasizes that many results in the literature rely on asymmetric collection powers of private and government entities. To eliminate this asymmetry, we assume that all trades must be voluntary. With this assumption we ensure that any differences in allocations that arise are not the result of inherent differences in the collection powers across public and private entities.

³In short, the government provides a particular financial service for private agents who have to pay for this service. This idea is motivated in part by [3], who looks at voluntary payment of fees to receive interest on money in the Lagos-Wright framework.

bond economy. Finally, we show that for sufficiently high discount factors, the optimal policy for both economies is to have a positive inflation rate.

At first glance our result that the government should charge the maximum fee to trade in the outside bond market seems counter-intuitive; most economists would probably argue that imposing a fee to participate in the bond market would inhibit trade and lower welfare, not raise it. However, the result is actually quite intuitive. Assume the participation constraint in the outside bond economy is not binding. In this case, marginally raising the fee does not deter agents from participating in the outside bond market, yet it allows the government to extract money from the economy. This reduces the inflation tax on money thereby raising its return and improving welfare. In short, this result is an application of standard public finance theory: If lump-sum ‘taxes’ are available to the government, then it is optimal to use them to reduce distortionary taxes. Since the participation fee is a lump-sum payment, the government can improve welfare by using it to the fullest extent in order to reduce the distortionary tax on money.

Finally, our result that the optimal inflation rate is positive may also be surprising. For the inside economy, those who default on their bond redemptions must carry more cash to trade. Inflation taxes those additional balances and makes the punishment for default worse. This loosens borrowing constraints and leads to an increase in the nominal interest rate that can be paid on ‘idle’ balances. As a result, the real value of money and welfare increase. For the outside bond economy a similar line of reasoning holds. Those who do not pay the fee must carry more cash and inflation taxes those balances, which makes them more willing to pay the fee. Greater fee collection allows the government to pay a higher nominal interest rate on its bonds, which in turn makes money more valuable since excess holdings can be used to acquire bonds and this improves the allocation.

The structure of the paper is as follows: Section 2 contains a brief review of related literature. Section 3 describes the environment. Sections 4 and 5 examine the economies with outside bonds and inside bonds, respectively, and Section 6 compares the allocations of the two economies. Section 7 concludes. All proofs are in the Appendix.

2 Related Literature

Our equivalence result is reminiscent of Wallace’s Modigliani-Miller type result [24] for open market operations. In an overlapping generation model, Wallace shows that the method for financing government spending, either by issuing money or holding interest-bearing real assets, does not affect the equilibrium allocation. A critical element for proving his result is that the government has access to lump-sum taxation.

Our equivalence result is also related to recent papers by Kocherlakota [13] and Hellwig and Lorenzoni [9]. Kocherlakota considers various models of asset trade. In these models, households can trade a privately issued one-period bond, a publicly issued one-period bond, or publicly issued money. He proves that the allocations for these economies are equivalent.⁴ As noted by

⁴In an earlier paper, [22], a related equivalence result between money and credit is derived.

Kocherlakota, for these results to hold, the government and private households must have the same enforcement powers, implying the government has access to lump-sum taxes and private lenders can force some repayment of loans. However, in Kocherlakota's model money plays no role in transactions. We obtain our equivalence result for economies with limited enforcement and for environments where trade requires a medium of exchange.

Hellwig and Lorenzoni assume the same enforcement structure as we do. They compare two economies: one with inside bonds and no enforcement of repayment and the other with unbacked government debt (outside bonds). In the latter environment, the government cannot force households to pay taxes and households cannot force the government to redeem debt in real goods. They show that the allocations in the two economies are equivalent – any allocation in the inside bond economy can be replicated in the outside bond economy and vice versa. This is driven by the fact that unbacked government debt in Hellwig and Lorenzoni's model is equivalent to fiat money, which means that fiat money and government bonds are identical assets. In our framework, money and government debt have different liquidity properties and hence they are not identical assets.

Several other papers are related to what we do here. Kehoe and Levine [11] compare allocations in a dynamic economy when households can acquire consumption goods in one case by selling their capital holdings and in another case by issuing debt subject to a borrowing constraint. They show that if households are sufficiently patient, the allocations are the same in a deterministic environment, but if they are sufficiently impatient, then the economy where agents issue debt leads to a better allocation. Note that they study trade in real assets while we analyze trade in nominal assets. Furthermore, they do not examine government policy in their economies, whereas we do. Shi [20] examines the implications of illiquid bonds in a monetary search model where there are legal restrictions preventing bonds from being used as a medium of exchange in some transactions but not in others. The legal restrictions make outside bonds less liquid than money. He finds that having legal restrictions can be welfare improving. In [17] households with idle money holdings can buy illiquid outside bonds directly from the government who finances the interest payment through lump-sum taxes. The authors show that the opportunity to buy interest-bearing bonds is strictly welfare improving because it allows households with idle money to save.⁵ A closely related paper is [7] who study the effects of monetary policy on privately supplied credit in the Bewley [6] economy. Like in our paper, money is needed for transaction purposes and agents who default on their loans are excluded from participating in the credit market. The authors show that inflation increases the incentives to repay loans since it lowers the expected lifetime utility of an agent who is excluded from the private bond market. Finally, the paper is also related to [3] and [10], who analyze the impact of participation constraints on allocations arising in the Lagos-Wright framework.

⁵Furthermore, there are a number of papers that study the coexistence of money and bonds (e.g., [8], [21], and [23]). The key difference with our work is that they never compare the allocative effects of different bonds.

3 The environment

Our basic framework is the divisible money model developed in [16]. This model is useful because it allows us to introduce heterogeneous preferences while still keeping the distribution of money balances analytically tractable.⁶ Time is discrete, and in each period there are three perfectly competitive markets that open sequentially. The first market is a bond market where agents trade money for bonds. The second market is a goods market where they trade money for goods. In the third market, agents produce and consume and readjust their portfolios.

The economy is populated by two types of infinitely lived agents: households and sellers. Each type of agent has measure 1. There are two non-storable and perfectly divisible consumption goods at each date: market 2 goods and market 3 goods. Non-storable means that they cannot be carried from one market to the next. Households consume in market 2 and consume and produce in market 3. Sellers produce in market 2 and consume in market 3. The common discount factor across periods is $\beta = (1 + r)^{-1} < 1$, where r is the time rate of discount.

At the beginning of each period, before the bond market opens, the household receives a preference shock ε . The preference shock ε has a continuous distribution $F(\varepsilon)$ with support $[0, \varepsilon_H]$, is iid across households, and is serially uncorrelated. Given ε , the period utility of a household is

$$\varepsilon u(q_\varepsilon) + U(x_\varepsilon) - h_\varepsilon,$$

where $\varepsilon u(q_\varepsilon)$ is the household's consumption utility in market 2, $U(x_\varepsilon)$ is the household's consumption utility in market 3, and h_ε is the disutility of working h_ε hours in market 3. The utility function in market 2 satisfies $u'(q) > 0$, $u''(q) < 0$, and $u'(0) = +\infty$. Furthermore, we assume that the coefficient of relative risk aversion, $R \equiv -qu''/u'$, is constant.⁷ The utility function in market 3 satisfies $U'(x) > 0$, $U''(x) < 0$, $U'(0) = \infty$, and $U'(+\infty) = 0$. Households can also produce market 3 goods with a constant returns to scale production technology where one unit of the consumption good is produced with one unit of labor h_ε generating one unit of disutility.⁸

Sellers produce in market 2 with a constant returns to scale production technology where one unit of the consumption good q_s is produced with one unit of labor h generating one unit of disutility. Their utility of consuming y in market 3 satisfies $U(y) = y$. Accordingly, the period utility of a seller is

$$y - q_s.$$

⁶An alternative framework would be [18] which we could amend with preference and technology shocks to generate the same results.

⁷The last restriction on preferences is only used in the proofs of Proposition 3 and 9. All other proofs do not require a constant relative risk aversion.

⁸The idiosyncratic preference shock ε plays a role similar to random matching and bargaining in [16]. Depending on the realization of ε , households will spend different amounts of money in the goods market. Then, without quasilinear preferences and unbounded hours in market 3, the preference shocks would generate a nondegenerate distribution of money holdings since the money holdings of individual households would depend on their history of shocks. For tractability, we therefore impose assumptions, as in [16], that make the portfolio choices of households at the end of the settlement market independent of their trading histories.

Note that the sellers do not play an important role in our analysis. What we get from the sellers is a first-order condition that pins down the relative price between market 2 and market 3 goods. Furthermore, since they have linear disutility of producing in market 2 and linear utility of consuming in market 3 (with equal margins), they are irrelevant for the welfare calculations when we derive optimal policy.⁹

3.1 First-best allocation

We assume without loss of generality that the planner treats all sellers symmetrically. He also treats all households experiencing preference shock ε symmetrically. Given this assumption, the weighted average of expected steady-state lifetime utility of households and sellers can be written as follows:

$$(1 - \beta) \mathcal{W} = \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) + U(x_\varepsilon) - h_\varepsilon] dF(\varepsilon) + y - q_s, \quad (1)$$

where h_ε is hours worked by an ε -household in market 3 and q_ε is consumption of an ε -household in market 2. The planner maximizes (1) subject to the feasibility constraints

$$\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) - q_s \leq 0, \quad (2)$$

$$\int_\varepsilon^{\varepsilon_H} (x_\varepsilon - h_\varepsilon) dF(\varepsilon) + y \leq 0. \quad (3)$$

The first-best allocation satisfies

$$\begin{aligned} U'(x_\varepsilon^*) &= 1 \text{ and} \\ \varepsilon u'(q_\varepsilon^*) &= 1 \text{ for all } \varepsilon. \end{aligned} \quad (4)$$

These are the quantities for the households chosen by a social planner who is able to dictate production and consumption. The quantities for the sellers then are $y = q_s = \int_0^{\varepsilon_H} q_\varepsilon^* dF(\varepsilon)$.

3.2 Information frictions, money, and credit

To motivate a role for fiat money, search models of money typically impose three assumptions on the exchange process [20]: a double coincidence problem, anonymity, and costly communication. First, our preference structure creates a single-coincidence problem in market 2 since buyers do not have a good desired by sellers. Second, agents in market 2 are anonymous, which rules out trade credit between individual buyers and sellers. Third, there is no public communication of individual trading outcomes (public memory), which in turn eliminates the use of social punishments in support of gift-giving equilibria. The combination of these frictions implies that sellers require immediate

⁹In an earlier version of this paper we had just one type of household; at the beginning of the period, an idiosyncratic shock determined whether the household became a buyer or a seller in the goods market. Furthermore, the buyer experienced the same idiosyncratic preference shock ε as in the current version of the paper. This earlier version generated the same results as the current version, but its exposition was considerably less elegant.

compensation from buyers. In short, there must be immediate settlement through the exchange of some durable asset and money is the only durable asset. These are the micro-frictions that make money essential for trade in market 2. In contrast, in market 3 all agents can produce for their own consumption or use money balances acquired earlier. In this market, money is not essential for trade.¹⁰

3.3 Outside bonds versus inside bonds

We analyze equilibria of the model under two different bond markets – a market for outside bonds and one for inside bonds. Outside bonds are nominal government debt obligations, whereas inside bonds are private debt obligations.

Outside bond economy In the outside bond economy, we assume a government exists that controls the supply of fiat currency and issues one-period, nominal bonds. These bonds are perfectly divisible, payable to the bearer, and default free.¹¹ One bond pays off one unit of currency at maturity. The government is assumed to have a record-keeping technology over bond trades and bonds are book-keeping entries – no physical object exists. This implies that households are not anonymous to the government. Nevertheless, despite using the record-keeping technology to track bond trades, the government has no record-keeping technology to track goods trades.

At time t , the government sells one-period, nominal discount bonds in market 3 and redeems bonds that were sold in $t - 1$. At the start of $t + 1$, the idiosyncratic shocks ε are revealed. Then households trade bonds and money in a secondary bond market. The government acts as the intermediary for all bond trades, by recording purchases/sales of bonds, and redistributing money receipts. Trading in the secondary bond market is what matters for our comparison to the inside bond economy since it allows households to readjust their portfolios trades after the idiosyncratic shocks are realized.

Private households are anonymous to each other and thus are incapable of arranging and honoring intertemporal promises. Since bonds are intangible objects, they are incapable of being used as media of exchange in market 2 and hence are illiquid. Since households are anonymous and cannot commit, a household's promise in market 2 to deliver outside bonds to a seller in market 3 is not credible. Consequently, fiat money is essential for trade in market 2.

Inside bond economy Inside bonds are financial claims on private households, issued in a private bond market. Consequently, issuing inside bonds is equivalent to receiving credit. Almost by definition, credit requires record-keeping over private trading histories, and borrowers must reveal their identity. It is exactly this tension that makes it difficult to have money and credit coexist in microfounded models. Thus we follow [5] and assume that a limited record-keeping technology exists in market 1 that can track financial transactions. Thus, while households are

¹⁰One can think of agents being able to barter perfectly in this market. Obviously in such an environment, money is not needed.

¹¹The government has no incentive to default since it redeems its bonds by printing money at no cost.

anonymous to each other, they are not anonymous to financial intermediaries.¹² In market 1, after the idiosyncratic shocks are realized, the intermediaries acquire nominal debt obligations from borrowers and issue nominal debt obligations on themselves to depositors, which are securitized by their acquired claims. In market 3 all debt obligations are settled. We assume that intermediaries, like the government, can commit to honor their debt obligations. Thus, the key financial trades in both economies occur in market 1 after the idiosyncratic shocks have been realized.

We assume perfect competition among financial intermediaries. The nominal interest rate on loans is denoted by i . We assume that any funds borrowed or lent in market 1 are repaid in market 3. Given the discrete time aspect of the model, loans are technically ‘intra-period’ loans, whereas in reality they can be thought of as an inter-period loan. For example, consider a loan taken out at 23:59 on December 31 or one taken out at 00:01 on January 1 with both being repaid the following December 31. The first is an ‘inter-period’ loan and the latter is an ‘intra-period’ loan. While technically different, there is no serious economic difference between the two loans. Thus, our intra-period loans should be thought of this way: funds are borrowed early in the period and repaid late in the period.

One can show that due to the quasi-linearity of preferences in market 3 there is no gain from multi-period financial contracts. Furthermore, since the idiosyncratic shocks are revealed prior to contracting, the one-period nominal debt contracts that we consider are optimal.

Limited enforcement We consider economies where all trades must be voluntary. For the outside bond economy, this means that the government faces a constraint on how much revenue it can generate to redeem outstanding government debt. For the inside bond economy, this means that repayment of debt must be voluntary – creditors have no power to collect unpaid debts.

For the inside bond economy, unpaid debt has two consequences for a household. First, it receives no further loans until the debt is repaid. Second, it cannot save by acquiring nominal debt obligations from the financial intermediary, unless it repays any outstanding debt. These two assumptions imply that a household that defaults on its debt is excluded from participating in future bond markets. Thus, repayment of debt is the price for participating in future bond markets. Given these rules, we derive conditions to ensure voluntary redemption and show that this may involve binding borrowing constraints; i.e., credit rationing.

For the outside bond economy, the government can charge a participation fee for trading in the bond market.¹³ The fee has to be paid before the households learn the realization of the

¹²An example is a bank that accepts nominal deposits and makes nominal loans. While the bank knows who it trades with, borrowers do not know the identity of depositors and vice versa. This limited record-keeping technology is similar to an ATM machine – agents can identify themselves to the ATM machine and either borrow or deposit cash. Agents cannot borrow or deposit consumption goods at the ATM. These cash transactions can be recorded and interest is charged to borrowers and paid to depositors. Thus, while there is record-keeping over financial transactions, the ATM machine has no idea what a borrower does with the borrowed cash; i.e., there is no record of how the cash is used for buying consumption goods in market 2.

¹³In an environment, where all trades must be voluntary, lump-sum taxes are not feasible. If lump-sum taxes were feasible in our environment, the government could implement the first-best allocation by running the Friedman rule. See [15] for the optimality and implementation of the Friedman rule in the search theory of money.

preference shock ε . Households that do not pay the fee cannot buy newly issued government bonds or trade them in the secondary bond market. The government can do this because outside bonds are intangible objects and the trades among private households in the secondary bond market are executed by the government, since it controls the record-keeping technology.¹⁴ As a result, paying the fee is similar to repaying one's debt: it is the price for participating in the bond market.¹⁵

3.4 Government policy and the money supply process

In this section we describe the evolution of the money stock for each economy. In both economies we assume that the government does not purchase any goods with money issuance or revenues received from bond sales. This is without loss of generality.

Outside bond economy Denote M_t as the per capita money stock and B_t as the per capita stock of newly issued bonds at the end of period t . Fiat currency pays no interest. Then M_{t-1} is the beginning-of-period money stock in period t . Let $\varphi_t M_{t-1}$ denote the nominal fee charged by the government in market 3 of period t to participate in the bond market. We define the nominal fee as being proportional to the aggregate money stock for mathematical ease. If $\varphi_t > 0$, the government collects a positive fee from households to access the bond market; if $\varphi_t < 0$, then the government is actually paying households to use the bond market. The change in the money stock in period t is given by

$$M_t = M_{t-1} - \mathbf{I}_t \varphi_t M_{t-1} + B_{t-1} - \eta_t B_t, \quad (5)$$

where $\mathbf{I}_t \in [0, 1]$ is the measure of households that choose to pay the fee in t . Given our assumptions that the government does not purchase goods or levy taxes, (5) is the government's temporal budget constraint. If $\mathbf{I}_t = 1$, then all households pay the fee and $\varphi_t > 0$ can be interpreted as the fraction of the aggregate money stock that is withdrawn from the economy from payment of fees. The total change in the money stock comprises two components: first, the net difference between the cash created to redeem bonds, B_{t-1} , and the net cash withdrawal from selling B_t units of bonds at the price η_t ; second, the cash withdrawn from households that pay the fee φ_t to access the secondary bond market.¹⁶ A government policy is a sequence $\{M_t, B_t, \varphi_t\}_{t=1}^{\infty}$ that satisfies (5) given the initial values $M_0, B_0 > 0$.

Inside bond economy In the model with inside bonds, the government controls only the amount of fiat currency in the economy. In this case, the government can only inject lump-sum transfers

¹⁴Note that there can be no pairwise deviations since agents are anonymous and cannot commit to honoring inter-temporal promises. For example, the following deviation is not possible: 1) agent i pays the fee, 2) i collects money from agent j to buy bonds while promising to pay back the value of the bonds in market 3. Agent i would always renege on the promise and j cannot force redemption. If such a deviation were possible, then money would not be essential for goods trades in market 2.

¹⁵One could consider imposing a fee to access the inside bond market as well. Since we assume free entry of intermediaries and zero operating cost, the fee would be driven to zero.

¹⁶We want to emphasize that we do not impose a lump-sum tax. The difference is that the payment of the fee is voluntary which limits the revenue a government can collect.

of money, $\tau_t M_{t-1}$, to households. As a result, the money stock evolves as

$$M_t = (1 + \tau_t) M_{t-1}. \quad (6)$$

We assume that these lump-sum transfers of cash are given only to households that participate in the bond market and the transfer is received in market 3. Since all exchange must be voluntary, a government policy is a sequence $\{\tau_t \geq 0\}_{t=1}^{\infty}$ given an initial value $M_0 > 0$.

4 Outside bonds

In this section, we analyze the economy with outside bonds. For notational ease, variables corresponding to the next period are indexed by $+1$, and variables corresponding to the previous period are indexed by -1 . The money price of goods in market 3 is P , implying that the goods price of money in market 3 is $\phi = 1/P$. Let p be the money price of goods in market 2, ρ the money price of bonds in market 1, and η the money price of newly issued bonds in market 3.

4.1 Seller choices

Sellers produce market 2 goods with linear cost $c(q_s) = q_s$ and consume in market 3, obtaining linear utility $U(y) = y$. It is straightforward to show that sellers are indifferent as to how much they sell in market 2 if

$$p\phi = 1. \quad (7)$$

Since we focus on a symmetric equilibrium, we assume that all sellers produce the same amount. With regard to bond holdings, it is straightforward to show that, in equilibrium, sellers are indifferent to holding any bonds if the Fisher equation holds and will not hold bonds if the yield on the bonds does not compensate them for inflation and time discounting. Thus, for brevity of analysis, we assume that sellers hold no bonds.

4.2 Household choices

Now we characterize (i) a household's choices under the assumption that it pays the fee φM_{-1} and therefore has access to the bond market and (ii) the optimal choices for a deviating household that does not pay the fee. This allows us to derive the set of fees for which it is individually rational to participate in the bond market.

Let $V(m, b)$ be the expected value from entering market 3 with m units of fiat money and b units of nominal government bonds at time t . Let q_ε denote the quantity of goods bought by a type ε household in market 2 and y_ε the quantity of government bonds *bought* by a household of

type ε in market 1. Then, in market 3, the problem of a representative household in period $t - 1$ is

$$V_{-1}(m_{-1}, b_{-1}) = \max_{x_{-1}, h_{-1}, m, b, \{q_\varepsilon, y_\varepsilon\}} U(x_{-1}) - h_{-1} + \beta \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) + V(m - \rho y_\varepsilon - p q_\varepsilon, b + y_\varepsilon)] dF(\varepsilon)$$

subject to constraints

$$x_{-1} + \phi_{-1}(m + \eta_{-1}b) = h_{-1} + \phi_{-1}(m_{-1} + b_{-1} - \varphi_{-1}M_{-2}) \quad (8)$$

$$m - \rho y_\varepsilon \geq 0 \quad \forall \varepsilon \quad (9)$$

$$b + y_\varepsilon \geq 0 \quad \forall \varepsilon \quad (10)$$

$$m - \rho y_\varepsilon - p q_\varepsilon \geq 0 \quad \forall \varepsilon. \quad (11)$$

Constraint (8) is the $t - 1$ budget constraint in market 3, constraints (9) and (10) are the period t short-selling constraints on money and bonds in market 1, and constraint (11) is the period t budget constraint for purchasing goods in market 2. Note that households choose m and b in $t - 1$ before the realization of the period t shock ε . Given these choices of m and b , households then choose the state-contingent values q_ε and y_ε for all ε .

Using the market 3 constraint to eliminate h_{-1} we get the following program:

$$V_{-1}(m_{-1}, b_{-1}) = \max_{x_{-1}, b, m, \{q_\varepsilon, y_\varepsilon\}} U(x_{-1}) - x_{-1} + \phi_{-1}(m_{-1} + b_{-1} - \varphi_{-1}M_{-2}) - \phi_{-1}(m + \eta_{-1}b) + \beta \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) + V(m - \rho y_\varepsilon - p q_\varepsilon, b + y_\varepsilon)] dF(\varepsilon) \quad (12)$$

s.t. (9) - (11).

The envelope conditions are

$$V_{-1}^{m-1}(m_{-1}, b_{-1}) = V_{-1}^{b-1}(m_{-1}, b_{-1}) = \phi_{-1} \quad (13)$$

Let $\beta\phi\mu_\varepsilon$, $\beta\phi\theta_\varepsilon$, and $\beta\phi\lambda_\varepsilon$ denote the multipliers on (9), (10), and (11), respectively. Using (7) and (13), the first-order conditions are

$$\begin{aligned} x_{-1} & : & 0 & = U'(x_{-1}) - 1 \\ b & : & 0 & = -\phi_{-1}\eta_{-1} + \phi\beta + \phi\beta \int_0^{\varepsilon_H} \theta_\varepsilon dF(\varepsilon) \\ m & : & 0 & = -\phi_{-1} + \phi\beta + \phi\beta \int_0^{\varepsilon_H} \mu_\varepsilon dF(\varepsilon) + \phi\beta \int_0^{\varepsilon_H} \lambda_\varepsilon dF(\varepsilon) \\ q_\varepsilon & : & 0 & = \varepsilon u'(q_\varepsilon) - 1 - \lambda_\varepsilon \quad \forall \varepsilon \\ y_\varepsilon & : & 0 & = 1 - \rho - \rho\mu_\varepsilon + \theta_\varepsilon - \rho\lambda_\varepsilon \quad \forall \varepsilon. \end{aligned}$$

It is straightforward to show that, in any monetary equilibrium, $\mu_\varepsilon = 0$ and $\lambda_\varepsilon > 0$ for all $\varepsilon > 0$.

This follows from the fact that households will never sell all their money for bonds or ever carry money (and forgo interest-bearing bonds) that will not be spent on market 2 goods. It then follows from these expressions that the remaining multipliers are

$$\lambda_\varepsilon = \varepsilon u'(q_\varepsilon) - 1 \quad \text{and} \quad \theta_\varepsilon = \rho \varepsilon u'(q_\varepsilon) - 1.$$

The last expression implies that for $\theta_\varepsilon > 0$, the ε household is constrained by its bond holdings; i.e., it sells all of its bonds for money to acquire goods in market 2. When $\theta_\varepsilon = 0$, the ε household trades off the interest payment on the bond to the marginal liquidity value of having an extra dollar in market 2. In short, it may sell some but not all of its bonds or actually buy bonds with some of its extra cash. Whether or not this constraint is binding for all households or only for a fraction of households drives the equilibrium allocation.

Using these expressions in the first-order conditions for b and m and rearranging yields the following:

$$\phi_{-1} \eta_{-1} / \rho = \phi \beta \int_0^{\varepsilon_H} \varepsilon u'(q_\varepsilon) dF(\varepsilon) \quad (14)$$

$$\phi_{-1} = \phi \beta \int_0^{\varepsilon_H} \varepsilon u'(q_\varepsilon) dF(\varepsilon). \quad (15)$$

With regard to consumption in market 3, we get $U'(x) = 1$ in all t . With regard to consumption in market 2, because a household's desired consumption is increasing in ε , there is a critical value for the taste index $\tilde{\varepsilon}$ such that if $\varepsilon \leq \tilde{\varepsilon}$, $\theta_\varepsilon = 0$ and if $\varepsilon \geq \tilde{\varepsilon}$, $\theta_\varepsilon \geq 0$. For $\varepsilon \leq \tilde{\varepsilon}$, q_ε solves

$$\rho \varepsilon u'(q_\varepsilon) = 1 \quad \forall \varepsilon \leq \tilde{\varepsilon}. \quad (16)$$

If $\varepsilon = \tilde{\varepsilon}$, the critical household sells all its bonds in market 1 and spends all its money in market 2 to acquire \tilde{q}_ε units of goods. It then follows that households with $\varepsilon \geq \tilde{\varepsilon}$ consume the same \tilde{q}_ε . Accordingly, in market 2 a household's consumption satisfies

$$q_\varepsilon = \begin{cases} u'^{-1}[1/(\rho\varepsilon)] & \text{if } \varepsilon \leq \tilde{\varepsilon} \\ u'^{-1}[1/(\rho\tilde{\varepsilon})] & \text{if } \varepsilon \geq \tilde{\varepsilon} \end{cases}. \quad (17)$$

Note from (16) that for those households that are unconstrained, the marginal utility of consumption is equalized. Given these consumption choices and the pricing conditions, we get the following bond demands:

$$\begin{aligned} y_\varepsilon &\in [-b, m/\rho] & \text{if } \varepsilon \leq \tilde{\varepsilon} \\ y_\varepsilon &= -b & \text{if } \varepsilon \geq \tilde{\varepsilon}. \end{aligned} \quad (18)$$

4.3 Equilibrium

We focus on symmetric stationary equilibria where households participate in the bond market and money is used as a medium of exchange. Such equilibria meet the following requirements: (i)

households' decisions solve the maximization problem (12); (ii) the decisions are symmetric across all households with the same preference shocks; (iii) the relative price between market 2 goods and market 3 goods satisfies (7); (iv) the goods and bond markets clear; (v) all real quantities are constant across time; (vi) the law of motion for the stock of money (5) holds in each period.

Point (v) requires that the real stock of money is constant; i.e., $\phi M_{-1} = \phi_{+1} M$. This implies that $\phi/\phi_{+1} = M/M_{-1} \equiv \gamma$, where γ is the gross steady-state money growth rate.¹⁷ Symmetry requires $m = M_{-1}$ and $b = B_{-1}$. The restriction that there is a positive demand for money and bonds requires that the following pricing relationship holds in equilibrium:

$$\eta_{-1} = \rho. \quad (19)$$

This relationship comes from (14) and (15). It implies that the bond price has to be the same between market 3 and market 1 in period +1. Moreover, in a stationary equilibrium the bonds price ρ has to be constant. This can be seen for example from (16), where a changing ρ involves a non-stationary path for consumption. A constant bond price then implies that the bond-money ratio has to be constant, and this can be achieved only when the growth rates of money and bonds are equal.¹⁸

We assume there are positive initial stocks of money M_0 and outside bonds B_0 .¹⁹ Assuming that all households pay the fee, the law of motion for money holdings (5) can be written as follows:

$$\frac{B_0}{M_0} = \frac{\gamma - (1 - \varphi)}{1 - \rho\gamma}. \quad (20)$$

From this equation the government has two independent policy instruments. We study the case where the government chooses the fee φ and the gross growth rate of the money supply γ which requires that the initial bonds ratio satisfies (20).

Market clearing in market 1 and market 2 requires

$$\int_0^{\varepsilon_H} y_\varepsilon dF(\varepsilon) = 0 \quad (21)$$

$$q_s - \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) = 0, \quad (22)$$

where q_s is aggregate production by sellers. Note that, since the entire stock of money is held by the households that then spend it all in market 2, in any equilibrium aggregate production in market 2 is equal to the real stock of money; i.e., $q_s = \phi M_{-1}$. To see this, note that bond demand

¹⁷Note that we consider the beginning-of-period nominal stock of money and deflate it by the end-of-period price of goods.

¹⁸To see this, consider the budget constraint of the critical household $p\tilde{q}_\varepsilon = m + \rho b$. Then, in equilibrium, $m = M_{-1}$, $b = B_{-1}$, and $\phi p = 1$, implying that $\tilde{q}_\varepsilon = \phi M_{-1} \left(1 + \rho \frac{B_{-1}}{M_{-1}}\right)$. Thus, since in a stationary equilibrium \tilde{q}_ε , ϕM_{-1} and ρ are constant, $\frac{B_{-1}}{M_{-1}}$ must be constant.

¹⁹Since the assets are nominal objects, the government can start the economy off by one-time injections of cash M_0 and bonds B_0 .

is $y_\varepsilon = m/\rho - pq_\varepsilon/\rho$. The market clearing condition (21) then implies that $M_{-1} - \int_0^{\varepsilon_H} pq_\varepsilon dF(\varepsilon) = 0$ since $m = M_{-1}$. Multiplying by ϕ , using (22) to substitute $\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$, and noting that $\phi p = 1$ yields $q_s = \phi M_{-1}$.

Finally, the requirement that households participate in the bond market imposes an upper bound on φ . The participation constraint requires that the difference between the expected discounted utility of a household that participates and the expected discounted utility of a household that does not participate in the bond market is non-negative. In the proof of Proposition 1 we show that this condition is summarized by the inequality

$$\Omega_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi) \geq 0, \quad (23)$$

where the function $\Omega_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi)$ depends only on policy (γ, φ) , the endogenous cutoff value $\tilde{\varepsilon}$, and the endogenous bond price ρ .

The equilibrium can be of two types. Either some households are constrained in market 1 (i.e., $\theta_\varepsilon > 0$ for $\varepsilon \geq \tilde{\varepsilon}$) or none are constrained.

Proposition 1 *For the outside bond economy, an unconstrained equilibrium is a policy (γ, φ) and endogenous variables $(\tilde{\varepsilon}, \rho)$ that satisfy (23) and*

$$\Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi) \geq 0 \quad (24)$$

$$\varepsilon_H - \tilde{\varepsilon} = 0 \quad (25)$$

$$\rho - \beta/\gamma = 0. \quad (26)$$

The term $\Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi)$ in (24) is derived from the budget constraint of the household with the largest preference shock. The inequality reflects the fact that an ε_H household must have enough funds to buy q_H , where q_H solves $\rho\varepsilon_H u'(q_H) = 1$. Equation (26) is obtained by using $\rho\varepsilon u'(q_\varepsilon) = 1$ for all ε in (15). To verify whether an unconstrained equilibrium exists for a given policy (γ, φ) , set the bond price to $\rho = \beta/\gamma$ and the critical value to $\tilde{\varepsilon} = \varepsilon_H$ and then calculate the participation constraint (23) and the feasibility constraint (24). If both hold, the equilibrium exists. All remaining endogenous variables can then be calculated as follows: From (17), consumption satisfies $q_\varepsilon = u'^{-1}[\gamma/(\beta\varepsilon)]$; from (22), production and the real stock of money is $q_s = \phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$. Finally, from the government budget constraint (20) one obtains ϕB_{-1} .

Proposition 2 *For the outside bond economy, a constrained equilibrium is a policy (γ, φ) and endogenous variables $(\tilde{\varepsilon}, \rho)$ that satisfy (23) and*

$$\Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi) = 0 \quad (27)$$

$$\varepsilon_H - \tilde{\varepsilon} > 0 \quad (28)$$

$$\frac{\gamma\rho - \beta}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1\right) dF(\varepsilon) = 0. \quad (29)$$

The term $\Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi)$ in (27) is derived from the budget constraint of the critical household with preference shock $\tilde{\varepsilon}$. It reflects the fact that all households with $\varepsilon \geq \tilde{\varepsilon}$ must have enough funds to buy \tilde{q}_{ε} , where \tilde{q}_{ε} solves $\rho\tilde{\varepsilon}u'(\tilde{q}_{\varepsilon}) = 1$. Equation (29) is obtained by using $\rho\varepsilon u'(q_{\varepsilon}) = 1$ for all $\varepsilon \leq \tilde{\varepsilon}$ and $\rho\tilde{\varepsilon}u'(\tilde{q}_{\varepsilon}) = 1$ for all $\varepsilon \geq \tilde{\varepsilon}$ in (15). To verify whether a constrained equilibrium exists for a given policy (γ, φ) , one first derives ρ and $\tilde{\varepsilon}$ by solving (27) and (29). Then one needs to check the participation constraint (23) and the equilibrium condition $\tilde{\varepsilon} < \varepsilon_H$. Other endogenous variables can then be derived from (17), (20), and (22).

An interesting result is the different interest rates prevailing in each equilibrium. In the unconstrained equilibrium, the nominal interest rate satisfies the Fisher equation, $1 + i = \gamma/\beta = (1 + \pi)(1 + r)$. In the constrained equilibrium, the interest rate on bonds, $1 + i = 1/\rho < \gamma/\beta$, is lower than the value satisfying the Fisher equation. This implies that bonds in the constrained equilibrium are ‘bad’ stores of value; i.e., no household would buy one in market 3 with the intention of simply holding it to the next market 3. In short, the marginal liquidity value of bonds from relaxing households’ cash constraints increases the bond’s price and hence reduces its return below the risk-free rate.²⁰

From Propositions 1 and 2 it is evident that the government’s choice of γ and φ affects which equilibrium occurs. Given φ , define $\bar{\gamma}(\varphi)$ as the value of γ such that $\tilde{\varepsilon} = \varepsilon_H$. We then have the following proposition:

Proposition 3 *Consider a policy (γ, φ) such that the participation constraint (23) holds. Then, there exists a $0 \leq \Theta \leq 1$, where Θ is defined in the proof, such that the following is true. If $\beta > \Theta$, there exists a unique $1 - \varphi < \bar{\gamma}(\varphi) < \infty$ such that the following is true. If $\gamma \geq \bar{\gamma}(\varphi)$, then a unique unconstrained equilibrium exists, and if $\gamma \leq \bar{\gamma}(\varphi)$, a unique constrained equilibrium exists. If $\beta < \Theta$, a unique constrained equilibrium exists.*

The essence of this proposition is that for sufficiently low inflation rates, high ε households will face binding constraints on bond sales, and so $\rho\varepsilon u'(q_{\varepsilon}) > 1$. In contrast, for sufficiently high inflation rates, all households are unconstrained, implying $\rho\varepsilon u'(q_{\varepsilon}) = 1$ for all ε .

Essential Illiquid Bonds Note that, if $\rho < 1$, then illiquid outside bonds are essential since they improve the allocation relative to the money-only economy. This follows from two features of the equilibrium allocation. First, at $\rho = 1$, from (16) we have $\varepsilon u'(q_{\varepsilon}) = 1$, so unconstrained households are consuming the first-best quantity while constrained households consume less than the first-best quantities. By reducing ρ marginally, the consumption of the unconstrained households falls since they sell some of their real balances for interest-bearing bonds. But the first-order welfare loss from this reduction in consumption is zero due to standard envelope arguments. By shifting real balances to constrained households, their consumption increases and this generates a first-order welfare gain. Second, from (29), we see that a reduction in ρ from 1 causes $\tilde{\varepsilon}$ to increase. This

²⁰An econometrician observing the interest rate of an constrained economy would infer that the risk-free rate is too low and conclude that there is a risk-free rate puzzle. A similar point has been made by [14].

means fewer households are constrained, so the marginal utility of consumption is equated across more households. Thus, the distribution of consumption is improved. As a result, these two effects imply that welfare is higher when bonds are illiquid and $\rho < 1$. This confirms that Kocherlakota's [12] result can be extended to stationary, inflationary economies.

5 Inside Bonds

In this section, we analyze the model with inside bonds. In market 1, low ε households can use their idle cash balances to acquire nominal bonds from the financial intermediary, which are redeemed in market 3. High ε households can issue nominal bonds to the financial intermediary and redeem them in market 3. Inside bonds are perfectly divisible, and one inside bond pays off 1 unit of fiat currency in market 3. Let ρ denote the market 1 price of these inside bonds.²¹

5.1 Household choices

Let $V(m, y)$ be the expected value from entering market 3 with m units of fiat money and y units nominal bonds at time t . Let q_ε denote the quantity consumed by a type ε household in market 2 and y_ε the quantity of inside bonds *bought* by a household of type ε in market 1. Let b denote the maximal amount of bonds that a household can issue in market 1. Then, in the third market, the problem of a representative household in period $t - 1$ is

$$V_{-1}(m_{-1}, y_{-1}) = \max_{x_{-1}, m, \{q_\varepsilon, y_\varepsilon\}} U(x_{-1}) - x_{-1} - \phi_{-1}m + \phi_{-1}(m_{-1} + y_{-1}) + \tau_{-1}M_{-2} \quad (30)$$

$$+ \beta \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) + V(m - \rho y_\varepsilon - p q_\varepsilon, y_\varepsilon)] dF(\varepsilon)$$

subject to constraints

$$m - \rho y_\varepsilon \geq 0 \quad \forall \varepsilon \quad (31)$$

$$b + y_\varepsilon \geq 0 \quad \forall \varepsilon \quad (32)$$

$$m - \rho y_\varepsilon - p q_\varepsilon \geq 0 \quad \forall \varepsilon. \quad (33)$$

Constraint (31) is the period t short-selling constraint on money; constraint (32) is the borrowing constraint; and constraint (33) is the period t money constraint for purchasing goods in market 2. Note that households choose m in $t-1$ before the realization of the period t shock ε . Given the choice of m households then choose the state-contingent values $\{q_\varepsilon, y_\varepsilon\}$. Except for the choice of outside bonds in market 3, the two maximization problems (12) and (30) are equivalent. Consequently, the first-order conditions (15)-(18) continue to hold in the inside bond economy.

Nevertheless, problem (30) differs from problem (12) in one important way. In the inside bond economy, the borrowing constraint (32) limits the amount of credit that a household can get.

²¹One-period contracts are optimal here due to the quasi-linearity of preferences. In short, linearity of utility in hours worked means there are no welfare gains from smoothing market 3 labor across time to repay current debt.

Although the households take this constraint as exogenous, in equilibrium it is endogenously determined. The corresponding constraint in the outside bond economy is the short-selling constraint (10). The crucial difference is that, in the outside bond economy, b is a choice variable of the household; whereas, in the inside bond economy, b is determined by the financial intermediary, which calculates the maximal loan that a household is willing to pay back in market 3.

5.2 Stationary equilibria

We focus on symmetric stationary equilibria where households participate in the bond market and money is used as a medium of exchange. Such an equilibrium meets the following requirements: (i) households' decisions solve the maximization problems specified above; (ii) the decisions are symmetric across all households with the same preference shocks; (iii) the goods and bond markets clear; (iv) all real quantities are constant across time; (v) the government budget constraint (6) holds in each period.

As for the outside bonds economy, point (iv) requires that the real stock of money is constant, implying $\phi/\phi_{+1} = M/M_{-1} = (1 + \tau) \equiv \gamma$. Symmetry requires $m = M_{-1}$. Market clearing in market 1 and market 2 requires (21) and (22) to hold. Note also that since the entire stock of money is held by the households who then spend it all in market 2, aggregate production in market 2 is equal to the real stock of money; i.e., $q_s = \phi M_{-1}$.

Finally, the requirement that households participate in the bond market imposes a lower bound on y_ε (recall that $y_\varepsilon < 0$ means that a household borrows money). Existence of an equilibrium with credit requires that the difference between the expected discounted utility of a household that repays and the expected discounted utility of a household that does not repay is non-negative. In the proof of Proposition 4 we show that this condition is summarized by the inequality

$$\Omega_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) \geq 0, \quad (34)$$

where the function $\Omega_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) \geq 0$ depends only on policy γ , the endogenous cutoff value $\tilde{\varepsilon}$, and the endogenous bonds price ρ .

The equilibrium can be of two types. Either some households are constrained in market 1 or none are constrained.

Proposition 4 *For the inside bond economy, an unconstrained equilibrium is a policy γ and endogenous variables $(\tilde{\varepsilon}, \rho)$ that satisfy (34) and*

$$\Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) \geq 0 \quad (35)$$

$$\varepsilon_H - \tilde{\varepsilon} = 0 \quad (36)$$

$$\rho - \beta/\gamma = 0. \quad (37)$$

Inequality (35) is a feasibility constraint. The term $\Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma)$ is derived from the budget constraint of the household with the largest preference shock. The inequality (35) reflects the fact

that an ε_H household must have enough funds to buy q_H where q_H solves $\rho\varepsilon_H u'(q_H) = 1$. Equation (37) is obtained by using $\rho\varepsilon u'(q_\varepsilon) = 1$ for all ε in (15).

Proposition 5 *For the inside bond economy, a constrained equilibrium is a policy γ and endogenous variables $(\tilde{\varepsilon}, \rho)$ that satisfy (34) and*

$$\Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) = 0 \quad (38)$$

$$\varepsilon_H - \tilde{\varepsilon} > 0 \quad (39)$$

$$\frac{\gamma\rho - \beta}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) = 0. \quad (40)$$

The term $\Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma)$ in (38) is derived from the budget constraint of the critical household with preference shock $\tilde{\varepsilon}$. The equation (38) reflects the fact that all households $\varepsilon \geq \tilde{\varepsilon}$ must have enough funds to buy \tilde{q}_ε . Equation (40) is obtained by using $\rho\varepsilon u'(q_\varepsilon) = 1$ for all $\varepsilon \leq \tilde{\varepsilon}$ and $\rho\tilde{\varepsilon} u'(\tilde{q}_\varepsilon) = 1$ for all $\varepsilon \geq \tilde{\varepsilon}$ in (15).

In any equilibrium with $\rho < 1$, illiquid inside bonds are essential. The reasoning is the same as in [5]; interest-bearing inside bonds allow households to earn interest on "idle" money balances. This makes money more valuable, thereby raising ϕ and consumption.

As was the case in the outside bond economy, the nominal interest rate satisfies the Fisher equation, $1 + i = \gamma/\beta = (1 + \pi)(1 + r)$, in the unconstrained equilibrium. When households are credit-constrained, the interest rate on inside bonds, $1 + i = 1/\rho < \gamma/\beta$, is lower than the value satisfying the Fisher equation. In short, when households are credit-constrained, interest rates must be low to induce repayment. This result is similar to that found by [2] and [9].

6 Inside versus outside bonds

We now state our first proposition that relates the set of feasible allocations in the two economies. Define φ_{\max} as the value of φ such that the participation constraint (23) is binding.

Proposition 6 *Any equilibrium allocation in the inside bond economy can be replicated in the outside bonds economy by choosing the same money growth rate γ and charging φ_{\max} . The converse is not true.*

Proposition 6 states that, for a given γ , the set of equilibrium allocations in the inside bonds economy is a subset of the set of equilibrium allocations in the outside bonds economy. The reason for this result is as follows. In the inside bond economy, the money growth rate is the only policy instrument. The multiplicity of policy instruments in the outside bond economy is what drives the converse part of the proposition: in the inside bond economy the government has only γ as a policy instrument to affect the allocation. So in general, it is not possible to replicate the allocation occurring in the outside bond economy via a choice of γ alone.

Proposition 7 below contains a welfare ranking of these sets.

Proposition 7 *For a given $\gamma \geq 1$, the set of allocations that can be implemented in the outside bond economy is weakly dominated in terms of welfare by the equilibrium allocation under γ in the inside bond economy.*

The proof in the Appendix first shows that in the outside bonds economy for a given value of γ welfare is strictly increasing in φ in the constrained equilibrium and constant in the unconstrained equilibrium for any $\varphi < \varphi_{\max}$. Then, since we can replicate the allocation of the inside bonds economy by charging φ_{\max} , it has to be the case that an equilibrium in the outside bonds economy with $\varphi < \varphi_{\max}$ is strictly dominated in terms of welfare by the equilibrium allocation under the same γ in the inside bond economy.

Proposition 7 is just an application of standard public finance theory: if lump-sum ‘taxes’ are available to the government, then it is optimal to use them. Since the participation fee is effectively a lump-sum tax, the government can improve welfare by using it to the fullest extent.²² From a policy point of view, this suggests that the private sector (the inside bonds economy) and the public sector (the outside bonds economy) are equivalent mechanisms for providing insurance against idiosyncratic liquidity shocks if the fee φ is chosen optimally.

In the proof of Proposition 7 we also show that in a constrained equilibrium $\frac{dq_\varepsilon}{d\varphi} < 0$, $\frac{\partial \tilde{\varepsilon}}{\partial \varphi} > 0$, and $\frac{dq_{\tilde{\varepsilon}}}{d\varphi} > 0$. Thus, the welfare improving role of increasing φ works as follows. First, it lowers consumption by households with $\varepsilon < \tilde{\varepsilon}$, which has a negative effect on welfare. It increases the critical value $\tilde{\varepsilon}$, which means that more households have their marginal utility equalized. Finally, it increases consumption $q_{\tilde{\varepsilon}}$ of the constrained agents. The last two effects are welfare increasing and dominate the first effect. Note also that $\frac{d\rho}{d\varphi} < 0$; i.e., increasing the fee increases the nominal interest paid in the secondary market. The higher interest rate allows agents with "idle" balances to earn interest on it, which increases the demand for money and its value.

Proposition 8 *The optimal value of γ is the same in both economies. Consequently, the allocations are the same in both economies under optimal policies.*

To prove Proposition 8, since $\gamma \geq 1$ is required in the inside bonds economy, it is sufficient to prove that no equilibrium exists in the outside bonds economy if $\gamma < 1$. Then, if $\gamma^* \geq 1$ denotes the optimal policy in the inside bonds economy, we can replicate the corresponding allocation in the outside bonds economy by choosing the policy $(\gamma^*, \varphi_{\max})$.

We end this section with a further characterization of the optimal policy. There are two inefficiencies in this economy that policy must try to overcome. First, when $\tilde{\varepsilon} < \varepsilon_H$, there is an inefficient allocation of consumption across households since some households are constrained while others are not. As a result, the marginal utilities of consumption are not equalized. This is an ex-

²²Nevertheless, it is important to make a distinction between a fee and a lump-sum tax. First, households are not required to pay the fee. They are willing to do so only if they receive a utility-increasing service for it that compensates them for the disutility of working to acquire the money to pay the fee. A tax has to be paid even if one gets nothing for it. Second, since paying the fee is voluntary, the size of the fee is limited, which puts a constraint on the policies that the government can implement.

tensive margin inefficiency. Second, due to the time cost of holding money, the quantities consumed by all households are inefficiently low if $\gamma > \beta$. This is an intensive margin inefficiency.

For our equivalence result, it is crucial that there is a secondary market for government bonds and that this market opens after households learn their type. Without a secondary market or if such a market opens before idiosyncratic uncertainty is resolved, trading in these bonds would not improve the allocation in our model. Thus, a key difference between inside and outside bonds is that trading of inside bonds is contingent to household's idiosyncratic risk, whereas trading in outside bonds is not unless a secondary market opens which allow agents to trade bonds for money after observing their type. Our secondary market for government bonds therefore enables these bonds to mimic the properties of a contingent asset.

Keeping in mind these two inefficiencies, we now characterize the optimal policy in both economies.

Proposition 9 *In either economy, it is optimal to set γ such that $\tilde{\varepsilon} < \varepsilon_H$. Furthermore, if $\beta \geq \tilde{\beta}$, where $1/2 \leq \tilde{\beta} < 1$ is defined in the proof, it is optimal to set $\gamma > 1$.*

The proof that it is optimal to set γ such that $\tilde{\varepsilon} < \varepsilon_H$ is a straightforward application of the envelope theorem. In the unconstrained equilibrium, the marginal utility of consumption is equalized across all households. It then follows that the only inefficiency is from q_ε being too low when $\gamma > \beta$. Conjecture a policy γ such that $\tilde{\varepsilon} = \varepsilon_H$ implying that all households are unconstrained. Now consider a marginal reduction in γ causing $\tilde{\varepsilon} < \varepsilon_H$. The loss from reducing $\tilde{\varepsilon}$ below ε_H is zero, while there is a first-order gain from lowering inflation and raising q_ε for all households. Hence, it is optimal to choose γ such that $\tilde{\varepsilon} < \varepsilon_H$. The second part of the Proposition states that it is optimal to have some inflation if $\beta \geq \tilde{\beta}$. At $\gamma = 1$, $\rho = 1$ and the allocation is the same as the one in a money only economy with $\gamma = 1$. By raising γ and thereby reducing ρ , the government allows for better risk sharing but at the same time lowers consumption for the unconstrained households. Whether this is beneficial depends on the distribution $F(\varepsilon)$ and the discount factor β . In the proof of Proposition 9 we show how the critical value $\tilde{\beta}$ depends on the distribution. Finally, note that $1/2 \leq \tilde{\beta} < 1$ is a sufficient condition but not necessary for $\gamma > 1$ being optimal.

7 Conclusion

When households are liquidity-constrained, two options exist to relax this constraint: sell assets or issue debt. We have analyzed and compared the welfare properties of these two options in a model where households can either issue nominal inside bonds or sell nominal outside bonds. The key assumption of our analysis is the absence of collection powers by private households and the government. The following results emerged from our analysis: First, for any positive inflation rate, illiquid bonds are essential in both economies and thus generate societal benefits. Second, any allocation attained in the economy with inside bonds can be replicated in the economy with outside bonds. The converse is not true. Finally, under the optimal policies, the allocations in the

two economies are the same, as are the optimal money growth rates. We also showed that the key element responsible for these two economies having equivalent allocations is a cost to participating in bond markets. Thus, in a manner similar to the results of [3], [9] and [10], participation constraints have important ramifications for analyzing allocations arising in monetary models.

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8 Appendix

Allocation for a household that does not participate in the bond market

For many of the proofs that follow, we need to know the allocation of an agent who does not participate in the bond market. Throughout the Appendix we indicate the choice variables of a deviating household by a " $\hat{\cdot}$ ".

It is straightforward to show that the quantities consumed by an agent who does not participate in the bond market satisfy (41) and that the first-order condition for the choice of money holdings satisfies (42):

$$\hat{q}_\varepsilon = \begin{cases} u'^{-1}(1/\varepsilon) & \text{if } \varepsilon \leq \hat{\varepsilon} \\ u'^{-1}(1/\hat{\varepsilon}) & \text{if } \varepsilon \geq \hat{\varepsilon} \end{cases} \quad (41)$$

$$\phi_{-1} = \phi\beta \int_0^{\varepsilon_H} \varepsilon u'(\hat{q}_\varepsilon) dF(\varepsilon), \quad (42)$$

where $0 \leq \hat{\varepsilon} \leq \varepsilon_H$ is the critical cutoff for a household that does not participate and \hat{q}_ε are the quantities it consumes. Dividing (42) by $\phi\beta$ and using (41), we can write (42) as follows:

$$\frac{\gamma - \beta}{\beta} = \int_{\hat{\varepsilon}}^{\varepsilon_H} (\varepsilon/\hat{\varepsilon} - 1) dF(\varepsilon).$$

The right-hand side is decreasing in $\hat{\varepsilon}$ and approaches ∞ as $\hat{\varepsilon} \rightarrow 0$. The left-hand side is a constant larger than 0 for $\gamma > \beta$. Accordingly, for any $\gamma > \beta$ there exists a unique $\hat{\varepsilon}(\gamma) < \varepsilon_H$. Finally, note that a deviator brings in $\phi\hat{m} = \hat{q}_{\hat{\varepsilon}}$ (we use the notation $\hat{q}_{\hat{\varepsilon}}$ to indicate consumption of a deviator with $\varepsilon \geq \hat{\varepsilon}$) units of money into a period and that expected consumption $\int_0^{\varepsilon_H} \hat{q}_\varepsilon(\gamma) dF(\varepsilon)$ and expected utility $\int_0^{\varepsilon_H} \varepsilon u[\hat{q}_\varepsilon(\gamma)] dF(\varepsilon)$ depend on γ only (through their interaction with $\hat{\varepsilon}(\gamma)$). We will use these results to derive the participation constraints in the following proofs.

Proof of Proposition 1. The proof involves two steps. We first derive the participation constraint. We then derive the equilibrium conditions (24) and (26).

STEP 1: Participation constraint

To derive the participation constraint, consider a household of type ε that enters market 3 in t and that pays the fee in every period for all t . Its expected payoff in market 3 is

$$EV = U(x^*) - h_\varepsilon + \frac{\beta}{1-\beta} \left\{ \int_0^{\varepsilon_H} \varepsilon u(q_\varepsilon) dF(\varepsilon) + U(x^*) - Eh \right\},$$

where h_ε are hours worked in the current period in market 3 if it pays the fee and Eh is expected hours worked in future periods. Suppose a household deviates by not paying the fee in the current and all future periods (we could also use the one-step deviation principle to arrive at the same participation constraint). Since $\hat{x} = x^*$, a deviator's expected discounted utility is

$$E\hat{V} = U(x^*) - \hat{h}_\varepsilon + \frac{\beta}{1-\beta} \left\{ \int_0^{\varepsilon_H} \varepsilon u(\hat{q}_\varepsilon) dF(\varepsilon) + U(x^*) - E\hat{h} \right\}.$$

It then follows that the participation constraint satisfies $EV \geq E\hat{V}$, which requires

$$h_\varepsilon - \hat{h}_\varepsilon \leq \frac{\beta}{1-\beta} \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - \varepsilon u(\hat{q}_\varepsilon)] dF(\varepsilon) + \frac{\beta}{1-\beta} (E\hat{h} - Eh). \quad (43)$$

Deriving h_ε : On the equilibrium path, an ε household arrives in market 3 with $m - \rho y_\varepsilon - pq_\varepsilon$ money and $b + y_\varepsilon$ bonds that pay off one unit of money. It leaves market 3 with m_{+1} money and b_{+1} bonds. Accordingly, current hours worked on the equilibrium path are

$$h_\varepsilon = x^* + \phi(m_{+1} + \rho b_{+1}) + \varphi \phi M_{-1} - \phi(m - \rho y_\varepsilon - pq_\varepsilon) - \phi(b + y_\varepsilon). \quad (44)$$

Deriving \hat{h}_ε : On the equilibrium path, an ε household arrives in market 3 with $m - \rho y_\varepsilon - pq_\varepsilon$ money and $b + y_\varepsilon$ bonds. If the household deviates by not paying the fee, it leaves market 3 with \hat{m}_{+1} money and no bonds. Accordingly, current hours worked by a deviator are

$$\hat{h}_\varepsilon = x^* + \phi \hat{m}_{+1} - \phi(m - \rho y_\varepsilon - pq_\varepsilon) - \phi(b + y_\varepsilon).$$

The difference in current hours worked $h_\varepsilon - \hat{h}_\varepsilon$ is

$$h_\varepsilon - \hat{h}_\varepsilon = \varphi \phi M_{-1} + \phi(m_{+1} + \rho b_{+1}) - \phi \hat{m}_{+1}. \quad (45)$$

Deriving $E(h)$: To derive $E(h)$ we integrate (44) to get

$$Eh = \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) = x^* + \phi(m_{+1} + \rho b_{+1} - m - b + \varphi M_{-1}) + \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon),$$

since market clearing implies $\int_0^{\varepsilon_H} y_\varepsilon dF(\varepsilon) = 0$ and $\phi p = 1$. Using the government's budget constraint (5), and noting that on the equilibrium path $m_{+1} = M$ and $b_{+1} = B$, yields

$$Eh_\varepsilon = \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) = x^* + \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon). \quad (46)$$

Deriving $E(\hat{h})$: In the future a deviator holds $\hat{m} - p\hat{q}_\varepsilon$ units of money arriving in market 3 and leaves with \hat{m}_{+1} . A deviator's market 3 hours are then

$$\hat{h}_\varepsilon = x^* + \phi(\hat{m}_{+1} - \hat{m}) + \hat{q}_\varepsilon.$$

So its expected hours worked are

$$E\hat{h} = \int_0^{\varepsilon_H} \hat{h}_\varepsilon dF(\varepsilon) = x^* + \phi(\hat{m}_{+1} - \hat{m}) + \int_0^{\varepsilon_H} \hat{q}_\varepsilon dF(\varepsilon).$$

Thus the difference in expected hours worked is

$$E\hat{h} - Eh = \phi(\hat{m}_{+1} - \hat{m}) + \int_0^{\varepsilon_H} \hat{q}_\varepsilon dF(\varepsilon) - \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon). \quad (47)$$

Using (45) and (47), we can write the participation constraint (43) as follows:

$$\varphi\phi M_{-1} + \phi(m_{+1} + \rho b_{+1}) - \phi\hat{m}_{+1} \leq \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\beta}{1-\beta}\phi(\hat{m}_{+1} - \hat{m}),$$

where $\Psi(q_\varepsilon, \hat{q}_\varepsilon) \equiv \int_0^{\varepsilon_H} \{[\varepsilon u(q_\varepsilon) - q_\varepsilon] - [\varepsilon u(\hat{q}_\varepsilon) - \hat{q}_\varepsilon]\} dF(\varepsilon)$. Use the deviator's critical consumption $\phi\hat{m} = \hat{q}_{\tilde{\varepsilon}}$, and the fact that on the equilibrium path $m_{+1} = M$ and $b_{+1} = B$, to get

$$\varphi\phi M_{-1} \leq -\phi\gamma(M_{-1} + \rho B_{-1}) + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\gamma-\beta}{1-\beta}\hat{q}_{\tilde{\varepsilon}}. \quad (48)$$

Rewrite the government budget constraint (20) to get $\phi B_{-1}(1-\rho\gamma) = \phi M_{-1}[\gamma - (1-\varphi)]$, use it to substitute ϕB_{-1} in (48), and rearrange to get

$$0 \leq -\frac{\varphi\phi M_{-1}}{1-\rho\gamma} + \frac{\gamma\phi M_{-1}(\rho-1)}{1-\rho\gamma} + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\gamma-\beta}{1-\beta}\hat{q}_{\tilde{\varepsilon}}.$$

Finally, in any equilibrium, $\phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$, which yields (23), which we replicate here:

$$0 \leq \Omega_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi) \equiv 0 \leq \frac{[-\varphi + \gamma(\rho-1)] \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)}{1-\rho\gamma} + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\gamma-\beta}{1-\beta}\hat{q}_{\tilde{\varepsilon}}.$$

Note that, from (17), q_ε depend on ρ and $\tilde{\varepsilon}$ only. Moreover, as shown at the beginning of the Appendix, \hat{q}_ε and $\hat{q}_{\tilde{\varepsilon}}$ depend on γ only. Accordingly, the right-hand side of (23) can be summarized by the function $\Omega_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi)$, which depends on policy (γ, φ) , the bond price ρ , and the critical cutoff value $\tilde{\varepsilon}$ only.

STEP 2: Equilibrium conditions

To derive (26), divide (15) by $\phi\beta$, substitute ϕ_{-1}/ϕ by γ , and substitute $\varepsilon u'(q_\varepsilon)$ by $1/\rho$ to get $\gamma\rho/\beta = 1$. Inequality (24) is derived from ε_H household's budget constraint $p q_H \leq M_{-1} + \rho B_{-1}$. If we multiply it by ϕ , we can write it as follows:

$$q_H \leq \phi M_{-1} \left(1 + \rho \frac{B_{-1}}{M_{-1}}\right).$$

We next use the government's budget constraint (20) to substitute B_{-1}/M_{-1} to get the following expression:

$$q_H \leq \phi M_{-1} \frac{1-\rho(1-\varphi)}{1-\rho\gamma}.$$

In any equilibrium, $\phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$. Use this to substitute ϕM_{-1} , divide by q_H , and use (17) to substitute all q_ε to get (24):

$$0 \leq \Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi) \equiv \frac{1-\rho(1-\varphi)}{1-\rho\gamma} \int_0^{\varepsilon_H} (q_\varepsilon/q_H) dF(\varepsilon) - 1.$$

Note that, from (17), the quantities q_ε depend on ρ and $\tilde{\varepsilon}$ only. Hence, the right-hand side of (24) can be summarized by the function $\Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi)$, which depends on policy (γ, φ) , the bond price

$\rho = \beta/\gamma$, and the critical cutoff value $\tilde{\varepsilon} = \varepsilon_H$ only. ■

Proof of Proposition 2. To derive (29), divide (15) by $\phi\beta$, substitute ϕ_{-1}/ϕ by γ , and use (17) to substitute $u'(q_\varepsilon)$ to get

$$\gamma/\beta = \int_0^{\tilde{\varepsilon}} (1/\rho) dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} (1/\rho)(\varepsilon/\tilde{\varepsilon}) dF(\varepsilon).$$

Finally, multiply by ρ and rewrite it to get (29).

The equilibrium condition (27) is derived from the critical household's budget constraint $p\tilde{q}_\varepsilon = M_{-1} + \rho B_{-1}$. If we multiply the budget constraint by ϕ , we can write it as follows:

$$\tilde{q}_\varepsilon = \phi M_{-1} \left(1 + \rho \frac{B_{-1}}{M_{-1}} \right).$$

We next use the government's budget constraint (20) to substitute $B_{-1}/M_{-1} = \frac{\gamma - (1 - \varphi)}{1 - \rho\gamma}$ to get

$$\tilde{q}_\varepsilon = \phi M_{-1} \frac{1 - \rho(1 - \varphi)}{1 - \rho\gamma}.$$

Use the market clearing condition $\phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$ to substitute ϕM_{-1} and divide by \tilde{q}_ε to get (27):

$$0 = \frac{1 - \rho(1 - \varphi)}{1 - \rho\gamma} \int_0^{\varepsilon_H} (q_\varepsilon / \tilde{q}_\varepsilon) dF(\varepsilon) - 1.$$

Note that, from (17), the quantities q_ε and \tilde{q}_ε depend on ρ and $\tilde{\varepsilon}$ only. Hence, the right-hand side of (27) can be summarized by the function $\Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi)$ which depends on policy (γ, φ) , the bond price ρ , and the critical cutoff value $\tilde{\varepsilon}$ only. ■

Proof of Proposition 3. The proof involves two steps. In the first step, we derive $\bar{\gamma}(\varphi)$. In the second step, we prove existence and uniqueness of the equilibrium.

STEP 1: Derivation of $\bar{\gamma}(\varphi)$

Consider a policy (γ, φ) such that the participation constraint (23) is satisfied. If all households are unconstrained, we have $p q_\varepsilon \leq M_{-1} + \rho B_{-1}$ for all ε . Then, $\bar{\gamma}(\varphi)$ is the value of γ that solves $p q_H = M_{-1} + \rho B_{-1}$. If we multiply the ε_H -household's budget constraint $p q_H \leq M_{-1} + \rho B_{-1}$ by ϕ , we can write it as follows:

$$q_H \leq \phi M_{-1} \left(1 + \rho \frac{B_{-1}}{M_{-1}} \right).$$

Use the government's budget constraint (20) to substitute B_{-1}/M_{-1} to get the following expression:

$$q_H \leq \phi M_{-1} \frac{1 - \rho(1 - \varphi)}{1 - \rho\gamma}.$$

In any equilibrium $\phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$. Replace ϕM_{-1} and $\rho = \beta/\gamma$ (recall that in an uncon-

strained equilibrium the Fisher equation holds), and rearrange to get

$$\frac{1 - \beta}{1 - (\beta/\gamma)(1 - \varphi)} \leq \int_0^{\varepsilon_H} (q_\varepsilon/q_H) dF(\varepsilon). \quad (49)$$

The right-hand side of (49) is independent of γ due to constant relative risk aversion (which we assume in the paper but is effectively only used to prove this proposition). Moreover, we have $0 \leq \int_0^{\varepsilon_H} (q_\varepsilon/q_H) dF(\varepsilon) \leq 1$. The left-hand side of (49) is decreasing in γ , and equal to 1 at $\gamma = 1 - \varphi$, and approaches $1 - \beta$ for $\gamma \rightarrow \infty$. Accordingly, if $\beta \geq \Theta \equiv 1 - \int_0^{\varepsilon_H} (q_\varepsilon/q_H) dF(\varepsilon)$, where $0 \leq \Theta \leq 1$, there exists a unique $1 - \varphi < \bar{\gamma}(\varphi) < \infty$ that solves (49). If $\beta < \Theta$, the unconstrained equilibrium does not exist.

The critical value $\bar{\gamma}(\varphi)$ satisfies

$$\bar{\gamma}(\varphi) = \frac{\beta(1 - \varphi)(1 - \Theta)}{\beta - \Theta}.$$

Accordingly, for a given policy (γ, φ) for which the participation constraint (23) is satisfied, if $\gamma \geq \bar{\gamma}(\varphi)$, the bond price ρ and the cutoff value $\tilde{\varepsilon}$ satisfy Proposition 1. If $\gamma < \bar{\gamma}(\varphi)$, the bond price ρ and the cutoff value $\tilde{\varepsilon}$ satisfy Proposition 2.

STEP 2: Existence and uniqueness

Note again that the maintained assumption is that policy (γ, φ) is such that the participation constraint (23) is satisfied.

Consider $\gamma \geq \bar{\gamma}(\varphi)$. Then, $\tilde{\varepsilon} = \varepsilon_H$ and $\rho = \beta/\gamma$ so existence and uniqueness follows trivially.

Consider $\gamma \leq \bar{\gamma}(\varphi)$. Rewrite equations (27) and (29) that solve for the bond price ρ and the critical value $\tilde{\varepsilon}$ as follows:²³

$$\frac{\gamma - (1 - \varphi)}{\rho^{-1} - (1 - \varphi)} - \int_0^{\tilde{\varepsilon}} [1 - (q_\varepsilon/\tilde{q}_\varepsilon)] dF(\varepsilon) = 0 \quad (50)$$

$$\frac{\rho\gamma - \beta}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1\right) dF(\varepsilon) = 0. \quad (51)$$

Equation (51) is decreasing in $(\tilde{\varepsilon}, \rho)$ space with $\rho = \beta/\gamma$ at $\tilde{\varepsilon} = \varepsilon_H$. Equation (50) is increasing in $(\tilde{\varepsilon}, \rho)$ space. Moreover, at $\tilde{\varepsilon} = \varepsilon_H$, we have $\rho \geq \beta/\gamma$. To see this, evaluate (50) at $\tilde{\varepsilon} = \varepsilon_H$ to get

$$\frac{\gamma - (1 - \varphi)}{\rho^{-1} - (1 - \varphi)} = \Theta. \quad (52)$$

If we solve this equation for ρ , we get

$$\rho = \frac{\Theta}{\gamma + (\Theta - 1)(1 - \varphi)}.$$

²³Rewrite (27) as follows $0 = \frac{1 - \rho(1 + \varphi)}{1 - \rho\gamma} \left[\int_0^{\tilde{\varepsilon}} (q_\varepsilon/\tilde{q}_\varepsilon) dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} 1 dF(\varepsilon) \right] - 1$. Then add and subtract $\int_0^{\tilde{\varepsilon}} dF(\varepsilon)$ to get $0 = \frac{1 - \rho(1 + \varphi)}{1 - \rho\gamma} \left\{ \int_0^{\tilde{\varepsilon}} [(q_\varepsilon/\tilde{q}_\varepsilon) - 1] dF(\varepsilon) + 1 \right\} - 1$ and rearrange to get (50).

Then, $\rho \geq \beta/\gamma$ implies

$$\gamma \leq \bar{\gamma}(\varphi) = \frac{\beta(1-\varphi)(1-\Theta)}{\beta-\Theta},$$

which is true since by assumption $\gamma \leq \bar{\gamma}(\varphi)$. Furthermore, in any constrained equilibrium, we have $\rho\gamma \leq 1$. To see this, from (52), we have

$$\frac{\gamma - (1-\varphi)}{\rho^{-1} - (1-\varphi)} = \Theta \leq 1,$$

implying $\rho\gamma \leq 1$. Hence, for $\gamma \leq \bar{\gamma}(\varphi)$ there exists a unique $(\rho, \tilde{\varepsilon})$ that solves (50) and (51) with $\rho \in [\beta/\gamma, 1]$ and $\tilde{\varepsilon} \leq \varepsilon_H$. ■

Proof of Proposition 4. The proof involves two steps. We first derive the borrowing constraint. We then derive the equilibrium conditions (35) and (37).

STEP 1: Borrowing constraint

To derive the maximal loan, consider a household of type ε that enters market 3 in t and repays the loan in every period for all t . Its expected payoff in market 3 is

$$EV = U(x^*) - h_\varepsilon + \frac{\beta}{1-\beta} \left\{ \int_0^{\varepsilon_H} \varepsilon u(q_\varepsilon) dF(\varepsilon) + U(x^*) - Eh \right\},$$

where h_ε are hours worked in the current period in market 3 if it repays the loan and Eh is expected hours worked in future periods. Suppose a household deviates by not repaying in the current and all future periods (we could also use the one-step deviation principle to arrive at the same participation constraint). Since $\hat{x} = x^*$ a deviator's expected discounted utility is

$$E\hat{V} = U(x^*) - \hat{h}_\varepsilon + \frac{\beta}{1-\beta} \left\{ \int_0^{\varepsilon_H} \varepsilon u(\hat{q}_\varepsilon) dF(\varepsilon) + U(x^*) - E\hat{h} \right\}.$$

It then follows that the participation constraint satisfies $EV \geq E\hat{V}$ which requires

$$h_\varepsilon - \hat{h}_\varepsilon \leq \frac{\beta}{1-\beta} \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - \varepsilon u(\hat{q}_\varepsilon)] dF(\varepsilon) + \frac{\beta}{1-\beta} (E\hat{h} - Eh). \quad (53)$$

Deriving h_ε : On the equilibrium path, an ε household arrives in market 3 with $m - \rho y_\varepsilon - pq_\varepsilon$ money and y_ε bonds. It receives the transfer τM_{-1} and it leaves market 3 with m_{+1} money. Accordingly, current hours worked on the equilibrium path are

$$h_\varepsilon = x^* + \phi m_{+1} - \phi \tau M_{-1} - \phi (m - \rho y_\varepsilon - pq_\varepsilon) - \phi y_\varepsilon. \quad (54)$$

Deriving \hat{h}_ε : On the equilibrium path, an ε household arrives in market 3 with $m - \rho y_\varepsilon - pq_\varepsilon$ money and y_ε bonds. If the household deviates by not repaying the loan, it leaves market 3 with \hat{m}_{+1} . Note that it gets no lump-sum transfer from the government.²⁴ Accordingly, current hours

²⁴We could assume that all lump-sum transfers are paid out in mkt 1 so that a necessary requirement to get the transfers is participation in financial markets. This assumption would generate the same borrowing constraint.

worked by a deviator are

$$\hat{h}_\varepsilon = x^* + \phi \hat{m}_{+1} - \phi(m - \rho y_\varepsilon - p q_\varepsilon).$$

The difference in current hours worked $h_\varepsilon - \hat{h}_\varepsilon$ is

$$h_\varepsilon - \hat{h}_\varepsilon = \phi(m_{+1} - \hat{m}_{+1}) - \tau \phi M_{-1} - \phi y_\varepsilon. \quad (55)$$

Deriving $E(h)$: To derive $E(h)$ we integrate (54) to get

$$Eh = \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) = x^* + \phi(m_{+1} - m - \tau M_{-1}) + \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon).$$

since market clearing implies $\int_0^{\varepsilon_H} y_\varepsilon dF(\varepsilon) = 0$ and $\phi p = 1$. In equilibrium, $m_{+1} = M$ and $m - \tau M_{-1} = M$, which yields

$$Eh = \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) = x^* + \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon).$$

Deriving $E(\hat{h})$: In the future a deviator holds $\hat{m} - p \hat{q}_\varepsilon$ units of money arriving in market 3 and leaves the market with \hat{m}_{+1} . A deviator's market 3 hours are then

$$\hat{h}_\varepsilon = x^* + \phi(\hat{m}_{+1} - \hat{m}) + \hat{q}_\varepsilon.$$

His expected hours worked are

$$E\hat{h} = \int_0^{\varepsilon_H} \hat{h}_\varepsilon dF(\varepsilon) = x^* + \phi(\hat{m}_{+1} - \hat{m}) + \int_0^{\varepsilon_H} \hat{q}_\varepsilon dF(\varepsilon).$$

Thus the difference in expected hours worked is

$$E\hat{h} - Eh = \phi(\hat{m}_{+1} - \hat{m}) + \int_0^{\varepsilon_H} \hat{q}_\varepsilon dF(\varepsilon) - \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon). \quad (56)$$

Using (55) and (56), we can write the borrowing constraint (53) as follows

$$\phi(m_{+1} - \hat{m}_{+1}) - \phi y_\varepsilon - \tau M_{-1} \leq \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\beta}{1 - \beta} \phi(\hat{m}_{+1} - \hat{m}).$$

Use the deviator's critical consumption $\hat{q}_\varepsilon = \phi \hat{m}$ to get

$$\phi m_{+1} - \phi y_\varepsilon - \tau M_{-1} \leq \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon.$$

Then, since $\phi m = \phi m_{+1} - \tau M_{-1}$ the maximal loan ϕb satisfies

$$-\phi y_\varepsilon \leq \phi b \equiv -\phi m + \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon. \quad (57)$$

Finally, use the budget constraint of any agent $q_\varepsilon = \phi m - \rho \phi y_\varepsilon$ to replace $-\phi y_\varepsilon$ to get

$$q_\varepsilon/\rho \leq (1/\rho - 1) \phi m + \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon.$$

Finally, replace $\phi m = \phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$ and replace q_ε by its largest value \tilde{q}_ε to get (34):

$$0 \leq \Omega_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) \equiv (1/\rho - 1) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{(\gamma - \beta)}{1 - \beta} \hat{q}_\varepsilon - (1/\rho) \tilde{q}_\varepsilon.$$

From (17), q_ε depend on ρ and $\tilde{\varepsilon}$ only. Moreover, as shown at the beginning of the Appendix, \hat{q}_ε and \tilde{q}_ε depend on γ only. Accordingly, the right-hand side of (34) can be summarized by the function $\Omega_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma)$ which depends on policy γ , the bond price ρ , and the critical cutoff value $\tilde{\varepsilon}$ only.

STEP 2: Equilibrium conditions

To derive (37), divide (15) by $\phi\beta$, substitute ϕ_{-1}/ϕ by γ , and substitute $\varepsilon u'(q_\varepsilon)$ by $1/\rho$ to get

$$\gamma\rho/\beta = 1.$$

To derive (35) note that in the unconstrained equilibrium, the ε_H household must have enough funds to pay for its consumption; i.e., $q_H \leq \phi m + \rho\phi b$. Use (57) to substitute ϕb to get

$$q_H \leq (1 - \rho) \phi m + \frac{\rho\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \rho\frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon.$$

Finally, replace $\phi m = \phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$ and divide by ρ to get (35):

$$0 \leq \Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) \equiv (1/\rho - 1) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{(\gamma - \beta)}{1 - \beta} \hat{q}_\varepsilon - (1/\rho) q_H.$$

From (17) q_ε depends on ρ and $\tilde{\varepsilon} = \varepsilon_H$ only. Moreover, as shown at the beginning of the Appendix, \hat{q}_ε depends on γ only. Accordingly, the right-hand side of (35) can be summarized by the function $\Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma)$ which depends on policy γ and bond price ρ only.

Note that in a unconstrained equilibrium $\Omega_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) = \Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma)$ which implies that if the borrowing constraint holds, then it must be the case that the budget constraint is satisfied as well.

■

Proof of Proposition 5. To derive (40), divide (15) by $\phi\beta$, substitute ϕ_{-1}/ϕ by γ , and substitute $\varepsilon u'(q_\varepsilon)$ by $1/\rho$ to get

$$\frac{\gamma\rho - \beta}{\beta} = \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon).$$

To derive (38) note that in the constrained equilibrium, the critical household's budget constraint holds with equality; i.e., $\tilde{q}_\varepsilon = \phi m + \rho\phi b$. Use (57) to substitute ϕb in the budget constraint to get

$$\tilde{q}_\varepsilon = (1 - \rho) \phi m + \frac{\rho\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \rho\frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon.$$

Finally, replace $\phi m = \phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$ and divide by ρ to get (38):

$$0 = \Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) \equiv (1/\rho - 1) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{(\gamma - \beta)}{1 - \beta} \hat{q}_\varepsilon - (1/\rho) \tilde{q}_\varepsilon.$$

Note that in a constrained equilibrium $\Omega_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) = \Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) = 0$ which implies that if the borrowing constraint holds at equality, then it must be the case that the budget constraint holds at equality as well. ■

Proof of Proposition 6. The proof involves three steps. In the first step, we show that any unconstrained equilibrium allocation in the inside bonds economy can be replicated in the outside bonds economy. In the second step, we prove that any constrained equilibrium allocation in the inside bonds economy can be replicated in the outside bonds economy. In both cases, we show that the allocations are replicated by choosing the same γ and charging φ_{\max} , where φ_{\max} is the value of φ such that the participation constraint (23) in the outside bond economy holds with equality for a given value of γ . In the third step, we provide an example of an allocation in the outside bonds economy that cannot be replicated in the inside bonds economy.

First step: Replication of the unconstrained equilibrium allocation

Consider a policy γ such that an unconstrained equilibrium exists in the inside bond economy. Then, the equilibrium allocation satisfies

$$\begin{aligned} \tilde{\varepsilon} &= \varepsilon_H, \rho = \beta/\gamma \text{ and} \\ 0 &\leq \Omega_{\mathbf{I}}(\varepsilon_H, \beta/\gamma, \gamma) = \Phi_{\mathbf{I}}(\varepsilon_H, \beta/\gamma, \gamma) = \frac{\gamma - \beta}{\beta} \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon - \frac{\gamma}{\beta} q_H. \end{aligned}$$

We now replicate this allocation in the outside bonds economy by choosing the same γ as in the inside bonds economy and by charging φ_{\max} . If φ_{\max} is charged, the participation constraint (48) holds at equality; i.e.,

$$\varphi_{\max} \phi M_{-1} = -\phi \gamma (M_{-1} + \rho B_{-1}) + \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon.$$

Then, use the government budget constraint (20) to replace $\varphi_{\max} \phi M_{-1}$ to get

$$\phi M_{-1} + \phi B_{-1} = \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon. \quad (58)$$

To derive the budget constraint $\Phi_{\mathbf{O}}(\tilde{\varepsilon}, \beta/\gamma, \gamma, \varphi_{\max})$ note that in any unconstrained equilibrium $q_H \leq \phi M_{-1} + \rho \phi B_{-1}$. Use (58) to substitute ϕB_{-1} to get

$$q_H \leq (1 - \rho) \phi M_{-1} + \rho \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \rho \frac{\gamma - \beta}{1 - \beta} \hat{q}_\varepsilon.$$

Divide by ρ , use $\phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$, and set $\rho = \beta/\gamma$ to get

$$0 \leq \Phi_{\mathbf{O}}(\tilde{\varepsilon}, \beta/\gamma, \gamma, \varphi_{\max}) \equiv \frac{\gamma - \beta}{\beta} \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_{\tilde{\varepsilon}} - \frac{\gamma}{\beta} q_H.$$

Evidently, we have

$$\Phi_{\mathbf{I}}(\varepsilon_H, \beta/\gamma, \gamma) = \Phi_{\mathbf{O}}(\varepsilon_H, \beta/\gamma, \gamma, \varphi_{\max}) \geq 0.$$

Then, since $\Phi_{\mathbf{O}}(\varepsilon_H, \beta/\gamma, \gamma, \varphi_{\max}) \geq 0$ and $\Omega_{\mathbf{O}}(\varepsilon_H, \beta/\gamma, \gamma, \varphi_{\max}) \geq 0$ the equilibrium allocation satisfies Proposition 1 with, $\rho = \beta/\gamma$ and $\tilde{\varepsilon} = \varepsilon_H$. Moreover, since γ is the same as in the inside bonds economy, consumption q_ε is the same for all ε .

Second step: Replication of the constrained equilibrium allocation. Now consider a policy γ such that a constrained equilibrium exists in the inside bond economy. The equilibrium allocation satisfies

$$\begin{aligned} \frac{\gamma\rho - \beta}{\beta} &= \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) \text{ and} \\ 0 &= \Omega_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) = \Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) \equiv (1/\rho - 1) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_{\tilde{\varepsilon}} - (1/\rho) \tilde{q}_\varepsilon, \end{aligned}$$

with $\tilde{\varepsilon} \leq \varepsilon_H$.

We now replicate this allocation in the outside bonds economy by choosing the same γ as in the inside bonds economy and by charging φ_{\max} . If φ_{\max} is charged, the participation constraint (48) holds at equality; i.e.,

$$\varphi_{\max} \phi M_{-1} = -\phi\gamma(M_{-1} + \rho B_{-1}) + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_{\tilde{\varepsilon}}.$$

Then, use (20) to replace $\varphi_{\max} \phi M_{-1}$ to get

$$\phi M_{-1} + \phi B_{-1} = \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_{\tilde{\varepsilon}}. \quad (59)$$

To derive the feasibility constraint $\Phi_{\mathbf{O}}(\tilde{\varepsilon}, \beta/\gamma, \gamma, \varphi_{\max})$, note that in any constrained equilibrium $\tilde{q}_\varepsilon = \phi M_{-1} + \rho \phi B_{-1}$. Use (59) to substitute ϕB_{-1} in this equality to get

$$\tilde{q}_\varepsilon = (1 - \rho) \phi M_{-1} + \rho \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \rho \frac{\gamma - \beta}{1 - \beta} \hat{q}_{\tilde{\varepsilon}}.$$

Finally, use $\phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$, divide by ρ and rearrange to get

$$0 = \Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi_{\max}) \equiv (1/\rho - 1) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{\gamma - \beta}{1 - \beta} \hat{q}_{\tilde{\varepsilon}} - (1/\rho) \tilde{q}_\varepsilon.$$

If the same γ is chosen, then

$$\Phi_{\mathbf{I}}(\tilde{\varepsilon}, \rho, \gamma) = \Phi_{\mathbf{O}}(\tilde{\varepsilon}, \rho, \gamma, \varphi_{\max}) = 0.$$

Since ρ and $\tilde{\varepsilon}$ solve the same two equations as in the inside bonds economy, consumption q_ε is the same for all ε in the outside bonds economy.

Third step: The converse is not true

The converse is not true because there are policies (γ, φ) in the outside bond economy that result in allocations that cannot be replicated in the inside bond economy. We prove this by example. Consider the case of $\varphi < \varphi_{\max}$ and a constrained equilibrium in the outside bonds economy $\gamma < \bar{\gamma}(\varphi)$. For this case, from (50) and (51), $\rho_{\mathbf{O}}$ and $\tilde{\varepsilon}_{\mathbf{O}}$ (for this proof we use the index \mathbf{O} for variables in the outside bonds economy and the index \mathbf{I} for the inside economy) satisfy

$$\frac{\gamma - (1 - \varphi)}{\rho_{\mathbf{O}}^{-1} - (1 - \varphi)} - \int_0^{\tilde{\varepsilon}_{\mathbf{O}}} [1 - (q_\varepsilon / \tilde{q}_\varepsilon)] dF(\varepsilon) = 0 \quad (60)$$

$$\frac{\gamma \rho_{\mathbf{O}} - \beta}{\beta} - \int_{\tilde{\varepsilon}_{\mathbf{O}}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}_{\mathbf{O}}} - 1 \right) dF(\varepsilon) = 0. \quad (61)$$

Note that $\rho_{\mathbf{O}}$ and $\tilde{\varepsilon}_{\mathbf{O}}$ depend on φ .

To replicate the above allocation in the inside bonds economy, the equilibrium in the inside bonds economy has to be constrained. Accordingly, assume that the chosen γ implements a constrained equilibrium in the inside bonds economy. Then, $\rho_{\mathbf{I}}$ and $\tilde{\varepsilon}_{\mathbf{I}}$ satisfy

$$(1/\rho_{\mathbf{I}} - 1) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1 - \beta} + \frac{(\gamma - \beta)}{1 - \beta} \hat{q}_\varepsilon - (1/\rho_{\mathbf{I}}) \tilde{q}_\varepsilon = 0 \quad (62)$$

$$\frac{\gamma \rho_{\mathbf{I}} - \beta}{\beta} - \int_{\tilde{\varepsilon}_{\mathbf{I}}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}_{\mathbf{I}}} - 1 \right) dF(\varepsilon) = 0. \quad (63)$$

It is straightforward to show that if we substitute $\varphi = \varphi_{\max}$, we can transform (60) into (62) which implies that for $\varphi = \varphi_{\max}$ the two mechanisms generate the same allocation. It is then also clear that $\rho_{\mathbf{I}}$ and $\tilde{\varepsilon}_{\mathbf{I}}$ do not replicate $\rho_{\mathbf{O}}$ and $\varepsilon_{\mathbf{O}}$ for $\varphi < \varphi_{\max}$ since, for example, the allocation in the inside bonds economy depends on the deviator's consumption \hat{q}_ε while the allocation in the outside bonds economy is independent of \hat{q}_ε . ■

Proof of Proposition 7. The proof involves showing that, for a given γ , it is optimal to increase φ to φ_{\max} . We first consider the case of the constrained equilibrium; i.e., assume that policy (γ, φ) is such that the equilibrium allocation satisfies Proposition 2. Furthermore, assume that $\varphi < \varphi_{\max}$. In this case the bond price ρ and the critical value $\tilde{\varepsilon}$ satisfy (50) and (51) which we replicate here for easier reference:

$$\frac{\gamma - (1 - \varphi)}{\rho^{-1} - (1 - \varphi)} - \int_0^{\tilde{\varepsilon}} [1 - (q_\varepsilon / \tilde{q}_\varepsilon)] dF(\varepsilon) = 0 \quad (64)$$

$$\frac{\rho \gamma - \beta}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) = 0. \quad (65)$$

Recall that (64) is increasing in $(\tilde{\varepsilon}, \rho)$ space and (65) is decreasing. Then, note from (64) that an

increase in φ shifts (64) down and to the right in $(\tilde{\varepsilon}, \rho)$ space.²⁵ Since (65) does not move, it follows that $\frac{d\rho}{d\varphi} < 0$ and that $\frac{\partial \tilde{\varepsilon}}{\partial \varphi} > 0$.

Welfare is given by

$$(1 - \beta) \mathcal{W} = \int_0^{\tilde{\varepsilon}} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon u(\tilde{q}_\varepsilon) - \tilde{q}_\varepsilon] dF(\varepsilon) + U(x^*) - x^*.$$

The FOC wrt to φ yields

$$(1 - \beta) \frac{d\mathcal{W}}{d\varphi} = \int_0^{\tilde{\varepsilon}} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\varphi} dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon u'(\tilde{q}_\varepsilon) - 1] \frac{d\tilde{q}_\varepsilon}{d\varphi} dF(\varepsilon). \quad (66)$$

For all households we have $\varepsilon u'(q_\varepsilon) \geq 1/\rho$ so the bracketed terms are positive. Thus, the sign of this derivative hinges on the signs of $dq_\varepsilon/d\varphi$ and $d\tilde{q}_\varepsilon/d\varphi$. For all $\varepsilon < \tilde{\varepsilon}$, $\rho \varepsilon u'(q_\varepsilon) = 1$ and for $\varepsilon = \tilde{\varepsilon}$, $\rho \tilde{\varepsilon} u'(\tilde{q}_\varepsilon) = 1$ which yields

$$\frac{dq_\varepsilon}{d\varphi} = \frac{q_\varepsilon}{R} \frac{d\rho}{d\varphi} \frac{1}{\rho} \quad (67)$$

$$\frac{d\tilde{q}_\varepsilon}{d\varphi} = \frac{\tilde{q}_\varepsilon}{R} \frac{d\rho}{d\varphi} \frac{1}{\rho} + \frac{\tilde{q}_\varepsilon}{R} \frac{d\tilde{\varepsilon}}{d\varphi} \frac{1}{\tilde{\varepsilon}}. \quad (68)$$

where $R(q_\varepsilon) \equiv -u''(q_\varepsilon) q_\varepsilon / u'(q_\varepsilon)$ (observe that for this proof we do not need a constant R). Note that $\frac{d\tilde{q}_\varepsilon}{d\varphi} > 0$ since $\frac{d\rho}{d\varphi} \frac{1}{\rho} + \frac{d\tilde{\varepsilon}}{d\varphi} \frac{1}{\tilde{\varepsilon}} > 0$. To see this, take the total derivative of (65) to get

$$\frac{\gamma}{\beta} \frac{d\rho}{d\varphi} = - \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} \right) dF(\varepsilon) \frac{d\tilde{\varepsilon}}{d\varphi} \frac{1}{\tilde{\varepsilon}}. \quad (69)$$

Use this expression to replace $\frac{d\tilde{\varepsilon}}{d\varphi} \frac{1}{\tilde{\varepsilon}}$ in the above inequality to get

$$\frac{d\rho}{d\varphi} \frac{1}{\rho} - \frac{1}{\int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} \right) dF(\varepsilon)} \frac{\gamma}{\beta} \frac{d\rho}{d\varphi} > 0.$$

Divide by $\frac{d\rho}{d\varphi} < 0$ and rewrite it as follows

$$\int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) - \int_0^{\tilde{\varepsilon}} dF(\varepsilon) < \frac{\rho\gamma}{\beta} - 1.$$

From (65), we get $-\int_0^{\tilde{\varepsilon}} dF(\varepsilon) < 0$. Hence, $\frac{dq_\varepsilon}{d\varphi} = \frac{\tilde{q}_\varepsilon}{R} \frac{d\rho}{d\varphi} \frac{1}{\rho} + \frac{\tilde{q}_\varepsilon}{R} \frac{d\tilde{\varepsilon}}{d\varphi} \frac{1}{\tilde{\varepsilon}} > 0$.

We now establish that $\frac{d\mathcal{W}}{d\varphi} > 0$. From (66), $\frac{d\mathcal{W}}{d\varphi} > 0$ if

$$\int_0^{\tilde{\varepsilon}} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\varphi} dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon u'(\tilde{q}_\varepsilon) - 1] \frac{d\tilde{q}_\varepsilon}{d\varphi} dF(\varepsilon) > 0.$$

²⁵To see this, define $\Gamma = \frac{\gamma - (1 - \varphi)}{\rho - 1 - (1 - \varphi)}$. Then $\frac{\partial \Gamma}{\partial \varphi} \simeq 1/\rho - \gamma \geq 0$ and $\frac{\partial \Gamma}{\partial \rho} \simeq \gamma - (1 - \varphi) \geq 0$ which establishes the result.

Use (67) and (68) to get

$$\int_0^{\tilde{\varepsilon}} [\varepsilon u'(q_\varepsilon) - 1] \frac{q_\varepsilon d\rho}{R d\varphi} \frac{1}{\rho} dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon u'(\tilde{q}_\varepsilon) - 1] \left[\frac{\tilde{q}_\varepsilon d\rho}{R d\varphi} \frac{1}{\rho} + \frac{\tilde{q}_\varepsilon d\tilde{\varepsilon}}{R d\varphi \tilde{\varepsilon}} \right] dF(\varepsilon) > 0.$$

Then, use (69) to replace $\frac{d\tilde{\varepsilon}}{d\varphi} \frac{1}{\tilde{\varepsilon}}$ to get

$$\int_0^{\tilde{\varepsilon}} [\varepsilon u'(q_\varepsilon) - 1] \frac{q_\varepsilon d\rho}{R d\varphi} \frac{1}{\rho} dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] \left[\frac{\tilde{q}_\varepsilon d\rho}{R d\varphi} \frac{1}{\rho} - \frac{\frac{\tilde{q}_\varepsilon \gamma d\rho}{R \beta}}{\int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon)} \right] dF(\varepsilon) > 0.$$

Divide by $\frac{q_\varepsilon d\rho}{R d\varphi} \frac{1}{\rho} < 0$ and multiply by $\int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon)$ to get

$$\int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon) \int_0^{\tilde{\varepsilon}} [\varepsilon u'(q_\varepsilon) - 1] q_\varepsilon dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] \tilde{q}_\varepsilon \left[\int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon) - \frac{\rho\gamma}{\beta} \right] dF(\varepsilon) < 0.$$

Then from (65) $\frac{\rho\gamma}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon) = 1 - \int_{\tilde{\varepsilon}}^{\varepsilon_H} dF(\varepsilon) = \int_0^{\tilde{\varepsilon}} dF(\varepsilon)$. Use this to write the above inequality as follows

$$\int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon) \int_0^{\tilde{\varepsilon}} [\varepsilon u'(q_\varepsilon) - 1] q_\varepsilon dF(\varepsilon) < \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] \tilde{q}_\varepsilon \int_0^{\tilde{\varepsilon}} dF(\varepsilon) dF(\varepsilon).$$

Next use (17) to replace $\varepsilon u'(q_\varepsilon)$ to get

$$\int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) (1/\rho - 1) dF(\varepsilon) \int_0^{\tilde{\varepsilon}} q_\varepsilon dF(\varepsilon) < \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon/(\tilde{\varepsilon}\rho) - 1] dF(\varepsilon) \int_0^{\tilde{\varepsilon}} \tilde{q}_\varepsilon dF(\varepsilon).$$

It is evident that $\int_0^{\tilde{\varepsilon}} q_\varepsilon dF(\varepsilon) < \int_0^{\tilde{\varepsilon}} \tilde{q}_\varepsilon dF(\varepsilon)$ since $q_\varepsilon < \tilde{q}_\varepsilon$ for $\varepsilon < \tilde{\varepsilon}$. Furthermore,

$$\int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) (1/\rho - 1) dF(\varepsilon) < \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon/(\tilde{\varepsilon}\rho) - 1] dF(\varepsilon),$$

since $\int_{\tilde{\varepsilon}}^{\varepsilon_H} dF(\varepsilon) < \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon)$. Hence, we have just established that $\frac{d\mathcal{W}}{d\varphi} > 0$ if policy (γ, φ) is such that the equilibrium allocation satisfies Proposition 2 and $\varphi < \varphi_{\max}$.

We next consider the case of the unconstrained equilibrium; i.e., assume that policy (γ, φ) is such that the equilibrium allocation satisfies Proposition 1. Again, assume that $\varphi < \varphi_{\max}$. In this case, the bond price satisfies $\rho = \beta/\gamma$ and $\tilde{\varepsilon} = \varepsilon_H$. It is straightforward to show that a change in φ , holding γ constant, does not affect consumption and production since it does not affect the bond price ρ . It only affects ϕB_{-1} via the government's budget constraint (20). While this changes the total wealth of the household, it does not affect consumption since in a unconstrained equilibrium no household spends all its wealth for consumption in market 2. Consequently, if policy (γ, φ) is such that the equilibrium allocation satisfies Proposition 1 and $\varphi < \varphi_{\max}$, then $\frac{d\mathcal{W}}{d\varphi} = 0$. ■

Proof of Proposition 8. The proof contains two steps. We first show that any equilibrium in the outside bonds economy requires that $\gamma \geq 1$. We then show that this implies that under optimal policies the allocations are the same.

STEP 1: In any equilibrium $\gamma \geq 1$.

The idea of the proof of this step is as follows. Using the participation constraint (23), we first derive an upper bound for the fee φ , called $\check{\varphi}_{\max}$, such that in any equilibrium $\varphi \leq \check{\varphi}_{\max}$. We then show that if $\gamma < 1$, $\check{\varphi}_{\max} < 0$. This implies that in any equilibrium with $\gamma < 1$, $\varphi \leq \check{\varphi}_{\max} < 0$. Then, from the government budget constraint, we have

$$\gamma - 1 + \varphi = \frac{B_{-1}}{M_{-1}}(1 - \rho\gamma).$$

Since in any equilibrium the left-hand side of the government budget constraint is non-negative, $\varphi > 0$ if $\gamma < 1$. Thus, we have $\varphi \leq \check{\varphi}_{\max} < 0 < \varphi$ which is a contradiction. Accordingly, for $\gamma < 1$, the participation constraint and the government budget constraint cannot possibly hold simultaneously.

We now derive the upper bound $\check{\varphi}_{\max}$. For this purpose, we assume that the deviator is punished more harshly than what we actually impose in the paper. Since the punishment is harsher, the maximal fee that the government can ask without violating the participation constraint with the new punishment must be larger than the maximal fee that the government can ask without violating the participation constraint under the original punishment; i.e., $\check{\varphi}_{\max} \geq \varphi_{\max}$, where $\check{\varphi}_{\max}$ is the value of φ that makes the participation constraint binding under this harsher punishment.

The harsher punishment that we impose is that a deviator is forced to consume the same quantities in the goods market as an household that pays the fee. Note that this is a harsher punishment because a deviator is not allowed to choose his consumption in the goods market optimally and must carry more cash than he desires. In what follows, we use the upper script " $\check{}$ " to indicate the quantities of a deviator under this harsher punishment. To calculate the expected discounted utility under this harsher punishment, we have to calculate \check{h}_ε and $E\check{h}_\varepsilon$, where \check{h}_ε is the hours worked in the last market in the period of the defection, and $E\check{h}_\varepsilon$ is the expected hours worked in any of the following periods.

Deriving \check{h}_ε : On the equilibrium path, an ε household arrives in market 3 with $m - \rho y_\varepsilon - pq_\varepsilon$ money and $b + y_\varepsilon$ bonds. If the household deviates by not paying the fee, it leaves market 3 with $\check{m}_{+1} = m_{+1} + \rho b_{+1}$ money. Note that in order to be able to consume the same quantities as an equilibrium agent, he has to bring in $m_{+1} + \rho b_{+1}$ units of money into the period. Accordingly, current hours worked by a deviator under this harsher punishment are

$$\check{h}_\varepsilon = x^* + \phi(m_{+1} + \rho b_{+1}) - \phi(m - \rho y_\varepsilon - pq_\varepsilon) - \phi(b + y_\varepsilon).$$

Then, using (44), the difference in current hours worked $h_\varepsilon - \check{h}_\varepsilon$ is

$$h_\varepsilon - \check{h}_\varepsilon = \check{\varphi}\phi M_{-1}. \tag{70}$$

Deriving $E(\check{h})$: In the future a deviator under this harsher punishment holds $\check{m} - p\check{q}_\varepsilon$ units

of money arriving in market 3 and leaves with \check{m}_{+1} . A deviator's market 3 hours are then

$$\check{h}_\varepsilon = x^* + \phi(\check{m}_{+1} - \check{m}) + \check{q}_\varepsilon.$$

Then, since $\check{q}_\varepsilon = q_\varepsilon$, its expected hours worked are

$$E\check{h} = \int_0^{\varepsilon_H} \check{h}_\varepsilon dF(\varepsilon) = x^* + \phi(\check{m}_{+1} - \check{m}) + \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon).$$

Finally, from (46), the difference in expected hours worked is

$$E\check{h} - Eh = \phi(\check{m}_{+1} - \check{m}). \quad (71)$$

Using (70) and (71), we can write the participation constraint (43) as follows:

$$\varphi\phi M_{-1} \leq \frac{\beta}{1-\beta} \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - \varepsilon u(\check{q}_\varepsilon)] dF(\varepsilon) + \frac{\beta}{1-\beta} \phi(\check{m}_{+1} - \check{m}),$$

Finally, replacing the quantities \check{q}_ε with q_ε , and noting that $\phi\check{m}_{+1} = \gamma\phi\check{m}$ and $\phi\check{m} = \phi(m + \rho b) = \phi(M_{-1} + \rho B_{-1})$, we can write the participation constraint as follows:

$$\varphi \leq \frac{\beta}{1-\beta} \left(1 + \rho \frac{B_{-1}}{M_{-1}}\right) (\gamma - 1).$$

Define, $\check{\varphi}_{\max}$ the value of φ that satisfies the above inequality at equality; i.e.,

$$\check{\varphi}_{\max} = \left(1 + \rho \frac{B_{-1}}{M_{-1}}\right) \frac{\beta}{1-\beta} (\gamma - 1). \quad (72)$$

Note that $\check{\varphi}_{\max} < 0$ if $\gamma < 1$. This implies that in any equilibrium of the outside bonds economy (with the original punishment) $\varphi \leq \check{\varphi}_{\max} < 0$ if $\gamma < 1$. Now, consider the budget constraint of the government:

$$\gamma - 1 + \varphi = \frac{B_{-1}}{M_{-1}} (1 - \rho\gamma). \quad (73)$$

Since in any equilibrium the left-hand side of (73) is non-negative, $\varphi > 0$ if $\gamma < 1$. Thus, we have $\varphi \leq \check{\varphi}_{\max} < 0 < \varphi$ which is a contradiction. Hence, any equilibrium in the outside bonds economy requires that $\gamma \geq 1$.

STEP 2: Under optimal policies the allocations are the same.

Denote $\gamma^* \geq 1$ the optimal policy in the inside bonds economy. Then, Propositions 6 and 7 imply that we can replicate the corresponding allocation in the outside bonds economy by choosing the policy $(\gamma^*, \varphi_{\max})$.

Note that the STEP 1 of the proof is important since it tells us that there is no allocation in the outside bonds economy with $\gamma < 1$ that dominates the allocation under γ^* in the inside bonds economy. ■

Proof of Proposition 9. The proof involves three steps. We first show that the critical value

$\bar{\gamma}(\varphi)$ satisfies $\bar{\gamma}(\varphi) > 1$. We then show that it is optimal to set $\gamma < \bar{\gamma}(\varphi)$. Finally, we show that it is optimal to set $\gamma > 1$ if $\beta \geq \tilde{\beta}$.

STEP 1: The critical value satisfies $\bar{\gamma}(\varphi) > 1$.

Consider a policy $(\gamma = \bar{\gamma}(\varphi), \varphi)$. Then, no household is constrained and $\rho = \beta/\gamma$. Assume - to the contrary - that $\bar{\gamma}(\varphi) \leq 1$. Then, from (72), we have $\varphi \leq \check{\varphi}_{\max} \leq 0$, and, from (73), we get $\bar{\gamma}(\varphi) - 1 + \varphi = \frac{B-1}{M-1}(1-\beta)$ (since $\rho = \beta/\gamma$) implying that $\varphi > 0$. Combining the two conditions, we get $\varphi \leq \check{\varphi}_{\max} \leq 0 < \varphi$ which is a contradiction. Hence, $\bar{\gamma}(\varphi) > 1$.

STEP 2: The optimal γ satisfies $\gamma < \bar{\gamma}(\varphi)$.

In equilibrium, welfare is given by

$$(1-\beta)\mathcal{W} = \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) + U(x^*) - x^*.$$

Consider the change in welfare from a marginal increase in γ above $\bar{\gamma}(\varphi_{\max})$:

$$(1-\beta) \frac{d\mathcal{W}}{d\gamma} \Big|_{\gamma=\bar{\gamma}(\varphi)} = \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\gamma} \Big|_{\gamma=\bar{\gamma}(\varphi)} dF(\varepsilon).$$

At $\gamma = \bar{\gamma}(\varphi)$, $\varepsilon u'(q_\varepsilon) = 1/\rho$ for all ε , so we have

$$(1-\beta) \frac{d\mathcal{W}}{d\gamma} \Big|_{\gamma=\bar{\gamma}(\varphi)} = \int_0^{\varepsilon_H} \left(\frac{1-\rho}{\rho} \right) \frac{dq_\varepsilon}{d\gamma} \Big|_{\gamma=\bar{\gamma}(\varphi)} dF(\varepsilon).$$

Since $\rho = \beta/\gamma < 1$ at $\gamma = \bar{\gamma}(\varphi)$, the sign of this derivative hinges on the sign of $dq_\varepsilon/d\gamma$. From the household's FOC we have $\varepsilon u'(q_\varepsilon) = \gamma/\beta$ with

$$\frac{dq_\varepsilon}{d\gamma} \Big|_{\gamma=\bar{\gamma}(\varphi)} = \frac{\beta}{\varepsilon u''(q_\varepsilon)} < 0. \quad (74)$$

It then follows that $\frac{d\mathcal{W}}{d\gamma} \Big|_{\gamma=\bar{\gamma}(\varphi)} < 0$ so lowering γ at $\gamma = \bar{\gamma}(\varphi)$ is welfare improving

STEP 3: The optimal γ satisfies $\gamma > 1$ if $\beta \geq \tilde{\beta}$.

Totally differentiate (1) with respect to γ and evaluate it at $\gamma = 1$ to get

$$(1-\beta) \frac{d\mathcal{W}}{d\gamma} \Big|_{\gamma=1} = \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\gamma} \Big|_{\gamma=1} dF(\varepsilon) > 0.$$

Since $\varepsilon u'(q_\varepsilon) - 1 = 0$ for all $\varepsilon \leq \tilde{\varepsilon}$ at $\gamma = 1$, and $q_\varepsilon = \tilde{q}_\varepsilon$ for all $\varepsilon \geq \tilde{\varepsilon}$, welfare will be increasing in γ if $\frac{d\tilde{q}_\varepsilon}{d\gamma} \Big|_{\gamma=1} > 0$. To obtain $\frac{d\tilde{q}_\varepsilon}{d\gamma} \Big|_{\gamma=1}$, totally differentiate $\frac{\rho\gamma-\beta}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) = 0$ and $\rho \tilde{\varepsilon} u'(\tilde{q}_\varepsilon) = 1$ to get

$$\frac{1}{\beta} d\gamma + \frac{1}{\beta} d\rho + \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}^2) dF(\varepsilon) d\tilde{\varepsilon} = 0 \text{ and } \tilde{\varepsilon} u'(\tilde{q}_\varepsilon) d\rho + u'(\tilde{q}_\varepsilon) d\tilde{\varepsilon} + \tilde{\varepsilon} u''(\tilde{q}_\varepsilon) d\tilde{q}_\varepsilon = 0.$$

Eliminate $d\tilde{\varepsilon}$ from these two expressions and rewrite the resulting equation as follows

$$\beta \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon) \left[1 + \frac{u''(\tilde{q}_\varepsilon)}{u'(\tilde{q}_\varepsilon)} \frac{d\tilde{q}_\varepsilon}{d\rho} \right] = \frac{d\gamma}{d\rho} + 1. \quad (75)$$

To obtain an expression for $d\gamma/d\rho$, totally differentiate the borrowing constraint (38) to get:

$$\begin{aligned} -1/\rho^2 \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) d\rho + (1/\rho - 1) \int_0^{\varepsilon_H} dq_\varepsilon dF(\varepsilon) \\ + \frac{\beta}{1-\beta} \left\{ \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dq_\varepsilon dF(\varepsilon) - \int_0^{\varepsilon_H} [\varepsilon u'(\hat{q}_\varepsilon) - 1] d\hat{q}_\varepsilon dF(\varepsilon) \right\} \\ + \frac{\hat{q}_{\tilde{\varepsilon}}}{1-\beta} d\gamma + \frac{(\gamma - \beta)}{1-\beta} d\hat{q}_{\tilde{\varepsilon}} + (\tilde{q}_\varepsilon/\rho^2) d\rho - (1/\rho) d\tilde{q}_\varepsilon = 0. \end{aligned}$$

Evaluate it at $\rho = 1$, $\gamma = 1$, and $\hat{q}_{\tilde{\varepsilon}} = \tilde{q}_\varepsilon$ to get

$$\tilde{q}_\varepsilon - \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{\beta}{1-\beta} \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] \left(\frac{dq_\varepsilon}{d\rho} - \frac{d\hat{q}_\varepsilon}{d\rho} \right) dF(\varepsilon) + \frac{\tilde{q}_\varepsilon}{1-\beta} \frac{d\gamma}{d\rho} + \frac{d\hat{q}_{\tilde{\varepsilon}}}{d\rho} - \frac{d\tilde{q}_\varepsilon}{d\rho} = 0.$$

Since $\varepsilon u'(q_\varepsilon) = 1$ for $\varepsilon \leq \varepsilon$, we can write this expression as follows

$$\tilde{q}_\varepsilon - \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \left(\frac{d\tilde{q}_\varepsilon}{d\rho} - \frac{d\hat{q}_{\tilde{\varepsilon}}}{d\rho} \right) \frac{\beta}{1-\beta} \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon} - 1) dF(\varepsilon) + \frac{\tilde{q}_\varepsilon}{1-\beta} \frac{d\gamma}{d\rho} - \left(\frac{d\tilde{q}_\varepsilon}{d\rho} - \frac{d\hat{q}_{\tilde{\varepsilon}}}{d\rho} \right) = 0.$$

Then, from $\frac{1-\beta}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\frac{\varepsilon}{\tilde{\varepsilon}} - 1) dF(\varepsilon) = 0$ we get

$$\frac{d\gamma}{d\rho} = (1-\beta) \left[\int_0^{\varepsilon_H} (q_\varepsilon/\tilde{q}_\varepsilon) dF(\varepsilon) - 1 \right] < 0$$

Use this result to substitute $\frac{d\gamma}{d\rho}$ in (75) and rearrange to get

$$\beta \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon) \left[1 + \frac{u''(\tilde{q}_\varepsilon)}{u'(\tilde{q}_\varepsilon)} \frac{d\tilde{q}_\varepsilon}{d\rho} \right] = \beta + (1-\beta) \int_0^{\varepsilon_H} (q_\varepsilon/\tilde{q}_\varepsilon) dF(\varepsilon)$$

For $\frac{d\tilde{q}_\varepsilon}{d\rho} \Big|_{\gamma=1} > 0$ we need $\frac{d\tilde{q}_\varepsilon}{d\rho} < 0$ since ρ is decreasing in γ . This requires that $\beta + (1-\beta) \int_0^{\varepsilon_H} (q_\varepsilon/\tilde{q}_\varepsilon) dF(\varepsilon) - \beta \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon) > 0$. Use $\frac{1-\beta}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\frac{\varepsilon}{\tilde{\varepsilon}} - 1) dF(\varepsilon) = 0$ to replace $\int_{\tilde{\varepsilon}}^{\varepsilon_H} (\varepsilon/\tilde{\varepsilon}) dF(\varepsilon)$ and rearrange to get

$$(1-\beta) \int_0^{\varepsilon_H} (q_\varepsilon/\tilde{q}_\varepsilon) dF(\varepsilon) > 1 - \beta - \beta \int_0^{\tilde{\varepsilon}} dF(\varepsilon)$$

Since in any constrained equilibrium $0 < \int_0^{\varepsilon_H} (q_\varepsilon/\tilde{q}_\varepsilon) dF(\varepsilon)$ a sufficient condition for $\frac{d\tilde{q}_\varepsilon}{d\rho} < 0$ is

$$\beta \geq \left[1 + \int_0^{\tilde{\varepsilon}} dF(\varepsilon) \right]^{-1} \quad (76)$$

Accordingly, if $\beta \geq \left[1 + \int_0^{\tilde{\varepsilon}} dF(\varepsilon) \right]^{-1}$, the optimal γ satisfies $\gamma > 1$. Finally, from $\frac{1-\beta}{\beta} - \int_{\tilde{\varepsilon}}^{\varepsilon_H} (\frac{\varepsilon}{\tilde{\varepsilon}} - 1) dF(\varepsilon) =$

0, we have $\frac{d\tilde{\mathcal{E}}}{d\beta} > 0$, which implies that the right-hand side of (76) is decreasing in β . Thus, there exists a critical value $1/2 \leq \tilde{\beta} < 1$, where $\tilde{\beta}$ is the value of β that solves (76) at equality, such that if $\beta \geq \tilde{\beta}$, $\left. \frac{d\mathcal{W}}{d\gamma} \right|_{\gamma=1} > 0$. ■