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Abstract

The conditional equity premium in the model with production is often approximated by assuming a jointly log-normal distribution of the marginal rate of substitution in consumption and the marginal productivity of capital. We show that, for standard parameterization, this premium is about one third less than that implied by a non-linear approximation of the Euler equations.

JEL-Code: G12, C63, E22, E32.

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1 Introduction

In the production economy, the Euler equation of the household is given by (see, for example, (10.75) in Altug and Labadie (2008)):

$$\Lambda_t = \beta \mathbb{E}_t \Lambda_{t+1} R_{t+1}, \tag{1.1}$$

where β, Λ_t, and R_{t+1} denote the discount factor, the marginal utility of consumption and the return of equity in period $t+1$. The risk-free interest rate r_t follows from:

$$r_t = \frac{\Lambda_t}{\beta \mathbb{E}_t \Lambda_{t+1}} - 1. \tag{1.2}$$

In the asset pricing literature such as in Jerman (1998) or Altug and Labadie (2008), the asset premium is computed by assuming that the marginal rate of substitutions, $M_{t+1} := \beta \frac{\Lambda_{t+1}}{\Lambda_t}$, and the equity return are distributed jointly log-normal implying the equity premium:[1]

$$\mathbb{E}(R_{t+1} - 1) - r_t \simeq -0.5\,\mathrm{var}(\ln R_{t+1}) - \mathrm{cov}(\ln M_{t+1}, \ln R_{t+1}). \tag{1.3}$$

We show for the standard model of the production economy that the equity premium computed with the help of (1.3) is one third less than that of a more exact non-linear approximation.

2 The Model

We consider a model with habit in consumption and adjustment costs in capital as in Jerman (1998) that is able to reproduce the empirically observed equity premium. We follow the description of this model in Herr and Maußner (2009). Time is discrete and denoted by t.

Households. A representative household supplies labor in a fixed amount of $N = 1$ at the real wage w_t. Besides labor income he receives dividends d_t per unit of share S_t he holds of the representative firm. The current price of shares in units of the consumption good is v_t. His current period utility function u depends on current and

[1] The log-normal pricing formula (1.3) is derived in the Technical Appendix.

past consumption, C_t and C_{t-1}, respectively. Given his initial stock of shares S_t the households maximizes

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \frac{(C_{t+s} - bC_{t+s-1})^{1-\eta} - 1}{1-\eta} \right\}, \quad \eta \geq 0,\, \beta \in (0,1)$$

subject to the sequence of budget constraints

$$v_t(S_{t+1} - S_t) \leq w_t + d_t S_t - C_t. \tag{2.1}$$

The operator $\mathbb{E}_t$ denotes mathematical expectations with respect to information as of period t. The first-order conditions of this problem are (1.1) and:

$$\Lambda_t = (C_t - bC_{t-1})^{-\eta} - \beta b \mathbb{E}_t (C_{t+1} - bC_t)^{-\eta}, \tag{2.2a}$$

$$R_t := \frac{d_t + v_t}{v_{t-1}}, \tag{2.2b}$$

where Λ_t is the Lagrange multiplier of the budget constraint.

Firms. The representative firm uses labor N_t and capital K_t to produce output Y_t according to the production function

$$Y_t = Z_t N_t^{1-\alpha} K_t^{\alpha}, \quad \alpha \in (0,1). \tag{2.3}$$

The level of total factor productivity Z_t is governed by the AR(1)-Process

$$\ln Z_t = \rho^Z \ln Z_{t-1} + \epsilon_t^Z, \quad \epsilon_t^Z \sim N\left(0, (\sigma^Z)^2\right). \tag{2.4}$$

The firm finances part of its investment I_t from retained earnings RE_t and issues new shares to cover the remaining part:

$$I_t = v_t(S_{t+1} - S_t) + RE_t. \tag{2.5}$$

It distributes the excess of its profits over retained earnings to the household sector:

$$d_t S_t = Y_t - w_t N_t - RE_t. \tag{2.6}$$

Investment increases the firm's next-period stock of capital according to:

$$K_{t+1} = \Phi(I_t/K_t)K_t + (1-\delta)K_t, \quad \delta \in [0,1], \tag{2.7}$$

where we parameterize the function Φ as

$$\Phi(I_t/K_t) := \frac{a_1}{1-\zeta}\left(\frac{I_t}{K_t}\right)^{1-\zeta} + a_2, \quad \zeta > 0. \tag{2.8}$$

The firm's ex-dividend value at the end of the current period t, V_t, equals the number of outstanding stocks S_{t+1} times the current stock price v_t.[2] The first-order conditions

[2] The derivation of the firm value and its beginning-of-period value is delegated to the Technical Appendix.

for maximizing the beginning-of-period value of the firm subject to (2.7) are:

$$w_t = (1-\alpha)Z_t N_t^{-\alpha} K_t^{\alpha}, \tag{2.9a}$$

$$q_t = \frac{1}{\Phi'(I_t/K_t)}, \tag{2.9b}$$

$$q_t = \mathbb{E}_t \varrho_{t+1} \Big\{ \alpha Z_{t+1} N_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} - (I_{t+1}/K_{t+1}) + q_{t+1}\big[\Phi(I_{t+1}/K_{t+1}) + 1 - \delta\big] \Big\}, \tag{2.9c}$$

with $\varrho_{t+s} = \frac{1}{R_{t+1}R_{t+2}\dots R_{t+s}}$. In addition, the transversality condition

$$\lim_{s\to\infty} \mathbb{E}_t \varrho_{t+s} q_{t+s} K_{t+s+1} = 0 \tag{2.9d}$$

must hold.

Market Equilibrium. Using equations (2.5) and (2.6), the household's budget constraint implies the economy's resource restriction:

$$Y_t = C_t + I_t. \tag{2.10}$$

In equilibrium, the labor market clears at the wage w_t so that $N_t = 1$ for all t. Furthermore, using (1.1), ϱ_{t+1} can be replaced by $\beta\Lambda_{t+1}/\Lambda_t$ so that at any date t the set of equations

$$q_t = \frac{1}{\Phi'(I_t/K_t)}, \tag{2.11a}$$

$$Y_t = Z_t K_t^{\alpha}, \tag{2.11b}$$

$$Y_t = C_t + I_t, \tag{2.11c}$$

$$\Lambda_t = (C_t - bC_{t-1})^{-\eta} - \beta b \mathbb{E}_t (C_{t+1} - bC_t)^{-\eta}, \tag{2.11d}$$

$$q_t = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda t} \Big\{ \alpha Z_{t+1} K_{t+1}^{\alpha-1} - (I_{t+1}/K_{t+1}) + q_{t+1}\big[\Phi(I_{t+1}/K_{t+1}) + 1 - \delta\big] \Big\} \tag{2.11e}$$

$$K_{t+1} = \Phi(I_t/K_t)K_t + (1-\delta)K_t, \tag{2.11f}$$

determines $(Y_t, C_t, I_t, K_{t+1}, \Lambda_{t+1}, q_{t+1})$ given (K_t, Λ_t, q_t).

3 Computation and Calibration

We use the parameter settings from Heer and Maußner (2009), Section 6.3.4. Table 3.1 displays the respective values. In particular, we set the discount factor β equal to 0.994 implying an annual risk free rate in the stationary equilibrium of 2.4 percent.[3]

[3]The Fortran computer programs can be downloaded from
`http://www.wiwi.uni-augsburg.de/vwl/maussner/lehrstuhl/pap/hm_ln.zip`.

Table 3.1
Benchmark calibration

Preferences	β=0.994	b=0.8	η=2	N=0.13
Production	α=0.27	δ=0.011	ρ^Z=0.90	σ^Z=0.0072
	ζ=1/0.23			

Equity Premium. The solution of the model are functions g^i, $i \in \{K, Y, C, I, \Lambda, q\}$, that determine K_{t+1}, Y_t, C_t, I_t, Λ_t, and q_t given the current period state variables K_t, C_{t-1}, and the log of the productivity shock $\ln Z_t$.

Since

$$\begin{aligned}\Lambda_{t+1} &= g^\Lambda(K_{t+1}, C_t, \ln Z_{t+1})\\ &= g^\Lambda(g^K(K_t, C_{t-1}, \ln Z_t), g^C(K_t, C_{t-1}, \ln Z_t), \varrho \ln Z_t + \epsilon^Z_{t+1})\\ &=: \tilde{g}^\Lambda(K_t, C_{t-1}, \rho \ln Z_t + \epsilon^Z_{t+1},)\end{aligned}$$

and ϵ^Z_{t+1} is normally distributed, the expected value of the Lagrange multiplier equals

$$\mathbb{E}_t\Lambda_{t+1} = \int_{-\infty}^{\infty} \tilde{g}^\Lambda(K_t, C_{t-1}, \rho \ln Z_t + \epsilon^Z_{t+1},)\frac{1}{\sigma^Z\sqrt{2\pi}} e^{\frac{-(\epsilon^Z_{t+1})^2}{(\sigma^Z)^2}} d\epsilon^Z_{t+1}.$$

We use the quadratic approximation of g^Λ at the stationary equilibrium and the Gauss-Hermite 6-point quadrature formula to approximate the integral on the right-hand-side of this equation.

The labor market equilibrium condition (2.9a) and equation (2.7) imply that the right-hand-side of (2.9c) can be written as

$$\begin{aligned}1 &= \beta\mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{Y_{t+1} - w_{t+1}N_{t+1} - I_{t+1} + q_{t+2}K_{t+2}}{q_t K_{t+1}},\\ &= \beta\mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{d_{t+1} + v_{t+1}}{v_t} = \beta\mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1},\end{aligned}$$

where the second equality follows from equations (2.5) and (2.6) and the observation that $q_t K_{t+1} = v_t S_{t+1}$ (see Heer and Maußner (2009), p. 317). Therefore, the gross rate of return on the shares of the representative firm equals[4]

$$R_{t+1} = \frac{\alpha Y_{t+1} - I_{t+1} + q_{t+1}K_{t+2}}{q_t K_{t+1}}. \tag{3.1}$$

We use a random number generator to compute a long artificial time series for $R_{t+1} - r_t$ for a time series of 1,000,000 observations. The average of this time series is our measure of the ex-post equity premium implied by the model.

[4]Note, $\alpha Y_{t+1} = Y_{t+1} - w_{t+1}N_{t+1}$.

4 Results

Computing the equity premium with the help of the non-linear approximation of (3.1), we find an average annual risk free rate of about 1.0 percent and an equity premium of 4.0 percent. Using the same data, equation (1.3) yields an annual risk premia of 2.66 percent, and, thus, about 1.3 percentage points smaller than the risk premia implied by (3.1) and (1.2).[5] In Table 2, we report sensitivity analysis for other parameter values $\delta = 2.5\%$, $\eta \in \{1, 4\}$, $\alpha = 0.36$ that are frequently used in the real-business-cycle literature. These values confirm our results that the use of the log-normal approximation may result in a significant lower value for the equity premium.

Table 4.1

Sensitivity Analysis

	equity premium using	
$\{\alpha, \delta, \eta\}$	(3.1)	(1.3)
$\{0.36, 0.011, 2.0\}$	2.81	1.80
$\{0.27, 0.025, 2.0\}$	3.06	1.99
$\{0.27, 0.011, 1.0\}$	2.30	1.66
$\{0.27, 0.011, 4.0\}$	5.97	3.69

References

Altug, Sumru, and Pamela Labadie 2008. Asset Pricing for Dynamic Economies. Cambridge, UK: Cambridge University Press.

Heer, Burkhard and Alfred Maußner. 2009. Dynamic Disequilibrium Modeling. 2nd Edition. Berlin: Springer

Jermann, Urban J. 1998. Asset Pricing in Production Economies. *Journal of Monetary Economics*. Vol. 41. pp. 257-275.

Stephens, M. A. 1974. EDF Statistics for Goodness of Fit and Some Comparisons. *Journal of the American Statistical Association*. Vol. 69. pp. 730-737.

[5] In the Technical Appendix, we test and confirm the assumption of log-normal distribution implicit in (1.3).

Technical Appendix

A.1 Derivation of the Firm Value and Equation (3.1)

The firm value is equal to the value of the outstanding shares implying:

$$V_t = v_t S_{t+1} \overset{(2.5)}{=} I_t + v_t S_t - RE_t \overset{(2.6)}{=} I_t + w_t N_t - Y_t + (v_t + d_t) S_t,$$
$$\overset{(2.2b)}{=} I_t + w_t N_t - Y_t + R_t V_{t-1}.$$

Rearranging and taking expectations as of period t, yields

$$V_t = \mathbb{E}_t \left\{ \frac{Y_{t+1} - w_{t+1} N_{t+1} - I_{t+1} + V_{t+1}}{R_{t+1}} \right\}$$

Iterating on this equation using the law of iterated expectations and assuming

$$\lim_{s \to \infty} \mathbb{E}_t \left\{ \frac{V_{t+s}}{R_{t+1} R_{t+2} \dots R_{t+s}} \right\} = 0$$

establishes that the end-of-period value of the firm equals the discounted sum of its future cash flows $CF_{t+s} = Y_{t+s} - w_{t+s} N_{t+s} - I_{t+s}$:

$$V_t = \mathbb{E}_t \sum_{s=1}^{\infty} \varrho_{t+s} CF_{t+s}, \quad \varrho_{t+s} = \frac{1}{R_{t+1} R_{t+2} \dots R_{t+s}} \tag{A.1}$$

The firm's objective is to maximize its beginning-of-period value, which equals $V_t^{bop} = V_t + CF_t$. Defining $\varrho_t = 1$ allows us to write

$$V_t^{bop} = \mathbb{E}_t \sum_{s=0}^{\infty} \varrho_{t+s} CF_{t+s}. \tag{A.2}$$

The first-order conditions for maximizing V_t^{bop} subject to (2.7) are given by (2.9a)-(2.9d).

A.2 Deterministic Stationary Equilibrium

Since our solution strategy rests on a second-order approximation of the model we must consider the stationary equilibrium of the deterministic counterpart of our model that we get if we put $\sigma^Z = 0$ so that Z_t equals its unconditional expectation $Z = 1$ for all t. In this case we can ignore the expectations operator $\mathbb{E}_t$. Stationarity implies $x_{t+1} = x_t = x$ for any variable in our model. As usual, we specify Φ so that adjustment costs play no role in the stationary equilibrium, i.e., $\Phi(I/K)K = \delta K$ and $q = \Phi'(\delta) = 1$. This requires that we choose

$$a_1 = \delta^{\zeta},$$
$$a_2 = \frac{-\zeta\delta}{1-\zeta}.$$

These assumptions imply via equation (2.11e) the stationary solution for the stock of capital:

$$K = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}}. \tag{A.3a}$$

Output, investment, consumption, and the stationary solution for Λ are then given by

$$Y = K^{\alpha}, \tag{A.3b}$$
$$I = \delta K, \tag{A.3c}$$
$$C = Y - I, \tag{A.3d}$$
$$\Lambda = C^{-\eta}(1-b)^{-\eta}(1-b\beta). \tag{A.3e}$$

A.3 Derivation of the Log-Normal Pricing Formula (1.3)

Instead of using (3.1) some authors compute the equity premia via (1.1) assuming that the marginal rate of substitution $M_{t+1} := \beta\Lambda_{t+1}/\Lambda_t$ and the gross return on equity R_{t+1} follow a log-normal distribution.

In order to use this approach, notice that equation (1.1) holds also unconditionally. Now, let $a := \ln M$ and $b := \ln R$ denote the natural logarithms of the marginal rate of substitution and the gross return on equity, respectively, and assume $a \sim N(\mu_a, \sigma_a^2)$ and $b \sim N(\mu_b, \sigma_b^2)$.[6] Then, $\mathbb{E}(a+b) = \mu_a + \mu_b$ and $\operatorname{var}(a+b) = \sigma_a^2 + \sigma_b^2 + 2\operatorname{cov}(a,b)$. Since $X := e^{a+b} = MR$, the formula for the expectation of a log-normally distributed variable X, $E(X) = e^{\mu_x + 0.5\sigma_x^2}$, implies

$$\mathbb{E}(MR) = e^{\mu_a + \mu_b + 0.5\sigma_a^2 + 0.5\sigma_b^2 + \operatorname{cov}(a,b)}.$$

According to (1.1) this expectation equals unity. Thus, by setting the log of the previous equation equal to zero (and by putting $\mu_a = \mathbb{E}(\ln M_{t+1})$, $\sigma_a^2 = \operatorname{var}(\ln M_{t+1})$ and analogously for μ_b and σ_b^2):

$$\begin{aligned}\mathbb{E}(\ln R_{t+1}) = &-\mathbb{E}(\ln M_{t+1}) - 0.5\operatorname{var}(\ln M_{t+1}) - 0.5\operatorname{var}(\ln R_{t+1})\\ &- \operatorname{cov}(\ln M_{t+1}, \ln R_{t+1}).\end{aligned}$$

[6] For ease of exposition, we drop the time indices momentarily.

Equation (1.2) implies a similar formula for the the gross risk free rate $(1+r_t)$, namely[7]

$$\ln(1 + r_t) = -\mathbb{E}(\ln M_{t+1}) - 0.5\,\mathrm{var}(\ln M_{t+1}).$$

Thus, the expected return on equity obeys[8]

$$\mathbb{E}(R_{t+1} - 1) - r_t \simeq \mathbb{E}(\ln R_{t+1}) - \ln(1 + r_t) = -0.5\,\mathrm{var}(\ln R_{t+1}) - \mathrm{cov}(\ln M_{t+1}, \ln R_{t+1}). \tag{A.4}$$

To use this equation, the variance and covariance term have to be approximated by time series averages obtained from simulations of the model. Thus, by the law of large numbers, we estimate unconditional moments. Analogously, if we use time series averages to compute the equity premia from $(1/T)\sum_{t=0}^{T} R_{t+1} - r_t$, we derive an estimate of the unconditional expected equity premia.

A.4 Assumption of Log-Normal Distribution

Figure A.1 illustrates that the distribution assumption with respect to the natural logs of the gross return on equity R_{t+1} and the marginal rate of substitution M_{t+1} is well justified. Empirical distribution tests (conducted with EViews 7.0) do not reject the null hypothesis of normality. Table A.2 reports several test statistics and their respective probability values as described, e.g., in Stephens (1974).

[7]Since $\mathrm{var}(\ln(1 + r_t)) = 0$ and $\mathrm{cov}(\ln M_{t+1}, \ln(1 + r_t)) = 0$, and $\mathbb{E}\, ln(1 + r_t) = \ln(1 + r_t)$.

[8]Since $\ln(1 + x) \simeq x$.

Table A.2
Tests of Normality

Statistic	Value	Probability
	Gross return on equity	
Lilliefors D	0.000654	> 0.1
Cramer-von Mises W^2	0.071529	0.2671
Watson U^2	0.067303	0.2675
Anderson-Darling A^2	0.472767	0.2432
	Marginal rate of substitution	
Lilliefors D	0.000783	> 0.1
Cramer-von Mises W^2	0.090827	0.1502
Watson U^2	0.081983	0.1659
Anderson-Darling A^2	0.570298	0.1393

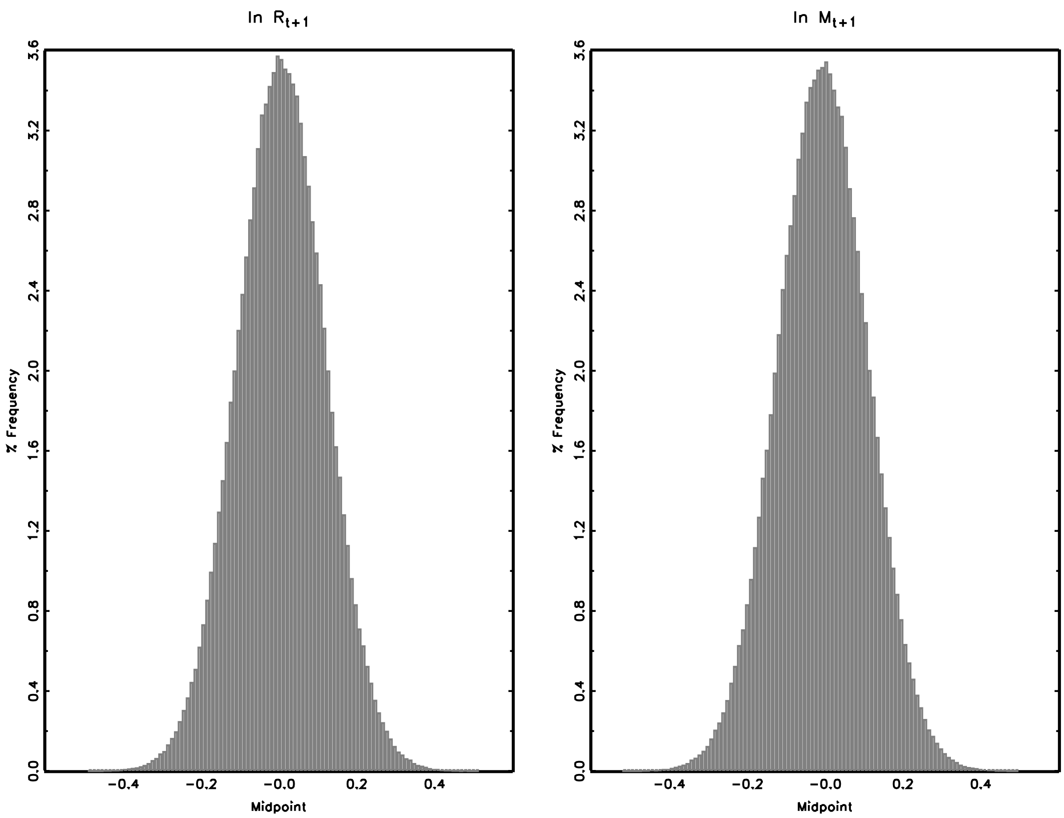

Figure A.1: Histograms of $\ln R_{t+1}$ and $\ln M_{t+1}$