The Euro/Dollar Exchange Rate: Chaotic or Non-Chaotic?

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Abstract

The aim of this paper is to develop a continuous time exchange rate model that allows for heterogeneity of the agents' beliefs, in order to explore non-linearities and possible chaotic behaviour. The theoretical model contains an intrinsic non-linearity that gives rise to a jerk differential equation, which is in principle capable of generating chaos. The model is econometrically estimated in continuous time with Euro/Dollar data and examined for the possible presence of chaotic motion. Our results indicate that the possibility of chaotic dynamics has to be rejected.

JEL-Code: F310, F370, C490, C610.

Keywords: exchange rate, chaos, jerk equation, continuous time econometrics.

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1 Introduction

The aim of this paper is to investigate the possibility of chaotic dynamics in the Euro/Dollar exchange rate. The interest of economists in chaos theory started in the 1980s, more than 20 years after the onset of this theory in physics, which is conventionally dated from 1963 when the meteorologist E. N. Lorenz published his paper on what became to be known as the Lorenz attractor (Lorenz, 1963).

A few words are in order to explain the interest of economists in chaos and the exchange rate. After the failure of the standard structural models of exchange rate determination in out-of-sample ex-post forecasts (the most notable empirical rejection is that by Meese and Rogoff, 1983a,b; for subsequent studies see Gandolfo et al., 1990, 1993, Gandolfo, 2002, and Rogoff and Stavrakeva, 2008), the exchange rate has come to be considered as a stochastic phenomenon, and exchange rate forecasting has come to rely on technical analysis and time series procedures, with no place for economic theory. Economic theory can be reintroduced in various ways, one of which is through a chaotic model. In fact, this would explain the apparently erratic behaviour of the exchange rate not through purely stochastic processes, but as due to a deterministic economic model capable of generating chaos. Another possibility would be to use a non-linear non-chaotic but stochastic structural model.

Furthermore, it has become evident that it is not possible to understand exchange rate behaviour by relying on models with representative agents. All forms of this simplifying approach have failed empirically (see Sarno and Taylor, 2002). There is now abundant evidence that market participants have quite heterogeneous beliefs on future exchange rates. These different expectations introduce non-linear features in the dynamics of the exchange rate. Heterogeneous agent models may create complex endogenous dynamics, including chaotic dynamics. This approach was initiated by Frankel and Froot (1987, 1990a,b). Further studies developed this line of research mainly in the context of stock markets (e.g. Kirman, 1991, Day and Huang, 1990; Brock and Hommes, 1997, 1998; Lux, 1998; Le Baron et al. 1999; Gaunersdorfer et al. 2003)[1].

The empirical evidence in favour of chaos in the exchange rate is not very strong. Sometimes chaos has been detected in the data (see Bajo-Rubio et al. 1992; De Grauwe et al. 1993; Chen, 1999; Bask, 2002; Brzozowska-Rup and Orlowski, 2004; Weston, 2007; Torkamani et al., 2007) but most often

[1] For surveys of this field of literature, see Hommes (2006), Chiarella et al. (2009), Hommes and Wagener (2009), Lux (2009) and Westerhoff (2009).

no such dynamics has been found (Brooks, 1998; Guillaume, 2000; Federici and Gandolfo, 2002; Serletis and Shahmoradi, 2004; Resende and Zeidan, 2008). In general, the empirical evidence for chaotic dynamics in economic time series is very fragile.

Studies aimed at detecting chaos in economic variables can be roughly classified into two categories.

I) On the one hand, there are studies that simply examine the data and apply various tests, such as the studies mentioned above (for applications to the exchange rate see Bajo-Rubio et al., 1992; Cuaresma, 1998; Guillaume, 2000, Chap. 3; Schwartz and Yousefi, 2003; Weston, 2007). These tests have been originally developed in the physics literature. This approach is not very satisfactory from our point of view, which aims at finding the dynamic model (if any) underlying the data. Besides, in the case of the investigation of individual time series to determine whether they are the result of chaotic or stochastic behaviour, the results could be inconclusive, as shown in the single blind comparative study of Barnett et al. (1997).

II) On the other hand, structural models are built and analysed. This analysis can in principle be carried out in several ways:

II.a) showing that plausible economic assumptions give rise to theoretical models having dynamic structures that fall into one of the mathematical forms known to give rise to chaotic motion;

II.b) building a theoretical model and then

II.b_1) giving plausible values to the parameters, simulating the model, and testing the resulting data series for chaos; or

II.b_2) estimating the parameters econometrically, and then proceeding as in b_1.

Existing chaotic exchange rate models (De Grauwe and Versanten, 1990; Reszat, 1992; De Grauwe and Dewachter, 1993a,b; De Grauwe, Dewachter, Embrechts, 1993; De Grauwe and Grimaldi, 2006a,b; Ellis, 1994; Szpiro, 1994; Chen, 1999: Da Silva, 2000, 2001; Moosa, 2000, Chap. 9) follow approaches (II.a) or (II.b_1). From the theoretical point of view, these models show that with orthodox assumptions (PPP, interest parity, etc.) and introducing nonlinearities in the dynamic equations, it is possible to obtain a dynamic system capable of giving rise to chaotic motion. However, none of these models is estimated, and the conclusions are based on simulations: the empirical validity of these models is not tested.

In the present paper, after a preliminary investigation of the data according to I)[2], we follow approach II.a) and approach II.b_2). For this purpose, we

[2]In order to detect the presence of chaos, in the first step, we use tick-by-tick Euro/Dollar exchange rate from January 2003 to December 2009 (one-minute and five-

develop a continuous-time exchange rate model that allows for heterogeneity of the agents' beliefs and possesses an *intrinsic* non-linearity, which is in principle capable of generating a chaotic motion.

After the analysis of its theoretical properties, the model is econometrically estimated in continuous time with Euro/Dollar data and examined for the possible presence of chaotic motion.

2 The model: formulation in terms of excess demands for foreign exchange

Our starting point is that the exchange rate is determined in the foreign exchange market through the demand for and supply of foreign exchange. This is a truism, but it should be complemented by the observation that, when all the sources of demand and supply—including the monetary authorities through their reaction function—are accounted for, that is, once one has specified behavioural equations for *all* the items included in the balance of payments, the exchange rate comes out of the solution of an implicit dynamical equation.

Let us then come to the formulation of the excess demands (demand minus supply) of the various agents. Our classification is functional. It follows that a commercial trader who wants to profit from the leads and lags of trade (namely, is anticipating payments for imports and/or delaying the collection of receipts from exports in the expectation of a depreciation of the domestic currency) is behaving like a speculator.

1) In the foreign exchange market non-speculators (commercial traders, etc.) are permanently present, whose excess demand only depends on the current exchange rate:

$$E_n(t) = g_n[r(t)], \;\; g_n' \gtreqless 0. \tag{1}$$

where $r(t)$ denotes the current spot exchange rate (price quotation system: number of units of domestic currency per unit of foreign currency). Possible transaction costs are subsumed under the non-linear function g_n. On the sign of g_n' see below, Sect. 3.1.

minute intervals). Similarly to many other papers, we study the exchange rate returns (the exchange rate return at time t is calculated as the log difference of two consecutive exchange rate levels). Tools from dynamical systems theory, such as the maximum Lyapunov exponent, are used. In addition we apply the reshuffled (surrogate) data procedure, which is unfortunately overlooked in most tests carried out in economic studies. The results of this analysis indicate that the data do not possess the features that are required to classify them as chaotic.

2) Let us now introduce speculators, who demand and supply foreign exchange in the expectation of a change in the exchange rate. According to a standard distinction, we consider two categories of speculators, fundamentalists and chartists[3].

2a) Fundamentalists hold regressive expectations, namely they think that the current exchange rate will move toward its "equilibrium" value. There are several ways to define such a value[4]; we believe that the most appropriate one is the NATREX (acronym of NATural Real EXchange rate), set forth by Stein (1990, 1995, 2001, 2002, 2006). It is based on a specific theoretical dynamic stock-flow model to derive the equilibrium real exchange rate. The equilibrium concept reflects the behaviour of the fundamental variables behind investment and saving decisions in the absence of cyclical factors, speculative capital movements and movements in international reserves. Two aspects of this approach are particularly worth noting. The first is that the hypotheses of perfect knowledge and perfect foresight are rejected: rational agents who efficiently use all the available information will base their intertemporal decisions upon a *sub-optimal feedback control* (SOFC) rule, which does not require the perfect-knowledge perfect-foresight postulated by the Representative Agent Intertemporally Optimizing Model, but only requires *current* measurements of the variables involved. The second is that expenditure is separated between consumption and investment, which are decided by different agents. The consumption and investment functions are derived according to SOFC, through dynamic optimization techniques with feedback control. Thus the NATREX approach is actually an intertemporal optimizing approach, though based on different optimization rules.

For a treatment of the NATREX, and for an empirical estimation of the \$/€ NATREX, see Belloc, Federici and Gandolfo (2008), and Belloc and Federici (2010). Let us call N_n the nominal NATREX . Then the excess demand by fundamentalists is given by the function

$$E_{sf}(t) = g_{sf}[N_n(t) - r(t)], \quad \mathrm{sgn} g_{sf}[...] = \mathrm{sgn}[...], \quad g'_{sf} > 0. \tag{2}$$

where N_n is the fundamental exchange rate, that we identify with the nominal NATREX, exogenously given and assumed known by fundamentalists. Transaction costs and the like are subsumed under the non-linear function g_{sf}, which is a sign-preserving function

2b) The excess demand by chartists is given by

$$E_{sc}(t) = g_{sc}[ER(t) - r(t)], \quad \mathrm{sgn} g_{sc}[...] = \mathrm{sgn}[...], \quad g'_{sc} > 0, \tag{3}$$

[3] For simplicity's sake we neglect the possibility of switch between the two categories.

[4] Typically in the literature the PPP value is used as a measure of the equilibrium exchange rate.

where $ER(t)$ denotes the expected spot exchange rate; the non-linear and sign-preserving function g_{sc} incorporates possible transaction costs. Chartists hold extrapolative expectations:

$$ER(t) = r(t) + h[\dot{r}(t), \ddot{r}(t)], h_1' > 0, h_2' > 0, \tag{4}$$

where the overdot denotes differentiation with respect to time, and $h[...]$ is a non-linear function. The assumed signs of the time derivatives mean that agents do not only extrapolate the current chnge ($h_1' > 0$) but also take account of the acceleration ($h_2' > 0$). It follows that

$$E_{sc}(t) = g_{sc}\left\{h[\dot{r}(t), \ddot{r}(t)]\right\}. \tag{5}$$

3) Finally, suppose that the monetary authorities are also operating in the foreign exchange market with the aim of influencing the exchange rate[5], account being taken of the NATREX, by using an integral policy *à la* Phillips. The authorities' excess demand $E_G(t)$ can be represented by the following function:

$$E_G(t) = G\left\{\int_0^t [N_n(t) - r(t)]\,dt\right\}, G' \gtrless 0. \tag{6}$$

where $G\{...\}$ is a non-linear function and the integral represents the sum of all the differences that have occurred, from time zero to the current moment, between the NATREX and the actual values of the exchange rate. The sign of G' depends on the policy stance of the monetary authorities. More precisely, if the aim is to stabilize the exchange rate around its NATREX value, then $G' > 0$. In fact, in such a case,

$$sgnG = sgn\left\{\int_0^t [N_n(t) - r(t)]\,dt\right\}, \tag{7}$$

because if the sum of the deviations is positive, this means that the NATREX has been on average greater than the actual exchange rate, so that the latter must increase (depreciate) to move towards the NATREX, hence a positive excess demand for foreign exchange. The opposite holds in the case of a negative sum. Thus the function G passes through zero when moving from negative to positive values, and $G'(0) > 0$.

But the authorities might wish to maintain or generate a situation of competitiveness, which occurs when the actual exchange rate has been on

[5] Central bank have often used direct interventions as a tool to stabilize short-run trends or to correct long term misalignments of the exchange rate. The large empirical literature on the impact and the effectiveness of these interventions provides mixed evidence (see Beine et al. 2009, Beine et al. 2007, Dominguez, 2006, Humpage, 2003 among others).

average greater than the NATREX, hence the integral is negative. To maintain or accentuate this situation, the authorities demand foreign exchange, so that

$$sgnG = -sgn\left\{\int_0^t [N_n(t) - r(t)]\, dt\right\}. \tag{8}$$

Thus the function G passes through zero when moving from positive to negative values, and $G'(0) < 0$.

Market equilibrium requires

$$E_n(t) + E_{sf}(t) + E_{sc}(t) + E_G(t) = 0. \tag{9}$$

In this way we have only one endogenous variable, $r(t)$, since the fundamentals are subsumed under the NATREX, which is known to both the authorities and the fundamentalists, and is considered exogenous in the present model.

Since the market equilibrium condition (9) holds instantaneously (given the practically infinite speed of adjustment of the FOREX market), we can differentiate Eq. (9) with respect to time, thus obtaining

$$\dot{E}_n(t) + \dot{E}_{sf}(t) + \dot{E}_{sc}(t) + \dot{E}_G(t) = 0. \tag{10}$$

By differentiating Eqs. (1), (2), (5), and (6)[6] with respect to time and substituting the result into Eq. (10) we obtain

$$g'_n \times \dot{r}(t) + g'_{sf} \times [\dot{N}_n(t) - \dot{r}(t)] + g'_{sc} \times [h'_1 \times \ddot{r}(t) + h'_2 \times \dddot{r}(t)] + G' \times [N_n(t) - r(t)] = 0. \tag{11}$$

Collecting terms we get

$$g'_{sc}h'_2 \times \dddot{r}(t) + g'_{sc}h'_1 \times \ddot{r} + (g'_n - g'_{sf}) \times \dot{r}(t) - G' \times r(t) = -G' \times N_n(t) + g'_{sf} \times \dot{N}_n(t), \tag{12}$$

whence, dividing through by $g'_{sc}h'_2 \neq 0$,

$$\dddot{r}(t) + \frac{h'_1}{h'_2}\ddot{r} + \frac{(g'_n - g'_{sf})}{g'_{sc}h'_2}\dot{r}(t) - \frac{G'}{g'_{sc}h'_2}r(t) = -\frac{G'}{g'_{sc}h'_2}N_n(t) + \frac{g'_{sf}}{g'_{sc}h'_2}\dot{N}_n(t). \tag{13}$$

This equation may seem linear, but it is not so. In fact, the derivative of a function is a function of the same arguments of the function, i.e.

$$g'_n = f_n[r(t)], g'_{sf} = f_{sf}[N_n(t) - r(t)], g'_{sc} = f_{sc}\left\{h[\dot{r}(t), \ddot{r}(t)]\right\}, \quad \text{etcetera}, \tag{14}$$

[6]Note that $\dot{E}_G(t) = G' \times [N_n(t) - r(t)]$.

so that the coefficients of Eq. (13) are to be considered as (non-linear) functions.

The model could be linearised and the resulting linear form analysed, but this would be uninteresting in the present context, since a linear model cannot give rise to chaos. The problem then arises of specifying the non-linearities of our model.

3 The intrinsic non-linearity of the model

When one abandons linearity (and related functional forms that can be reduced to linearity by a simple transformation of variables, such as log-linear equations), in general it is not clear which non-linear form one should adopt. Further to clarify the matter, let us distinguish between *purely qualitative* non-linearity and *specific* non-linearity.

By *purely qualitative* non-linearity we mean the situation in which we only know that a generic non-linear functional relation exists with certain *qualitative* properties, such as continuous first-order partial derivatives with a given sign and perhaps certain bounds. This is the aspect so far taken by our model, but it is hardly useful for our purposes, because the econometric estimation obviously requires specific functional forms.

By *specific* non-linearity we mean the situation in which we assume a specific non-linear functional relationship. Since in general it is not clear from the theoretical point of view which non-linear form one should adopt, the choice of a form is often arbitrary or made for convenience.

In our case, however, it is possible to introduce a non-linearity on sound economic grounds. This concerns the excess demand of non-speculators. To understand this point, a digression is called for on the derivation of the demand and supply schedules of these agents.

3.1 Derivation of the demand and supply schedules of non-speculators [7]

The main peculiarity of these demand and supply schedules for foreign exchange is the fact that they are *derived* or *indirect* schedules in the sense that they come from the underlying demand schedules for goods (demand for domestic goods by nonresidents and demand for foreign goods by residents). In other words, in the context we are considering, transactors do not directly demand and supply foreign exchange as such, but demand and supply it as

[7] For an in-depth treatment of this point see Sect. 7.3.1 of Gandolfo, 2002.

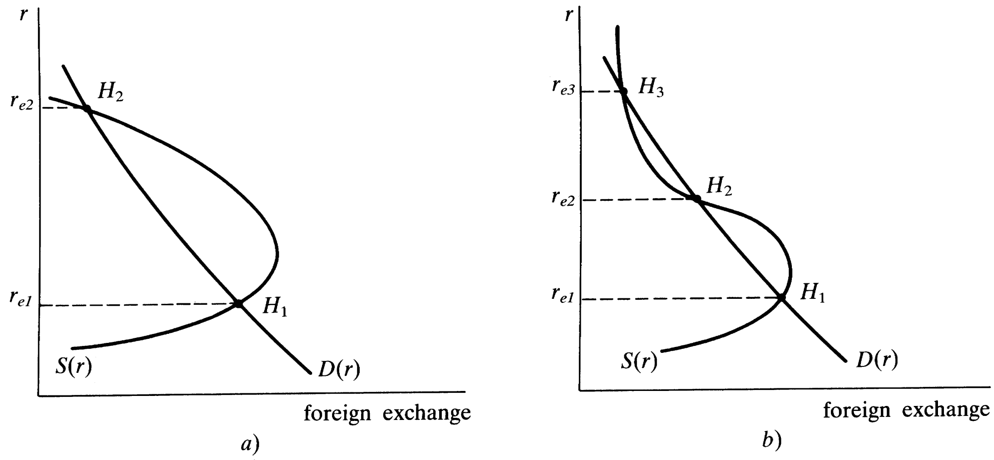

Figure 1: Non-linear supply functions

a consequence of the underlying demands for goods. Thus the demand for and supply of foreign exchange depend on the elasticities of the underlying demands for goods. Consider for example $S(r)$, the total revenue of foreign exchange from exports (determined by export demand), which depends on the elasticity of export demand. If the elasticity of exports is greater than one, an exchange-rate depreciation of, say, one per cent, causes an increase in the volume of exports greater than one per cent, which thus more than offsets the decrease in the foreign currency price of exports: total receipts of foreign exchange therefore increase. The opposite is true when the elasticity is lower than one.

Since a varying elasticity is the norm rather than an exception (a simple linear demand function has a varying elasticity), cases like those depicted in Fig. 1 are quite normal.

In the case depicted in Fig. 1a) the function $S(r)$ can be represented by a quadratic, while in the case of Fig. 1b) a cubic might do. Let us consider the simpler quadratic case, $S(r) = a + br + cr^2, a > 0, b > 0, c < 0$,where a, b, c are constants.[8]

What we propose to do is to introduce the above quadratic non-linearity *while assuming all the other functions to be linear and with constant coefficients.* Thus, assuming that $D(r)$ is linear ($D(r) = d_0 + d_1 r, d_0 > 0, d_1 < 0$,

[8]The quadratic function $a + br + cr^2$ as represented in the diagram implies $a > 0, b > 0, c < 0$.

where d_0, d_1 are constants), we can write

$$E_n(t) = D(r) - S(r) = (d_0 + d_1 r) - (a + br + cr^2) = (d_0 - a) + (d_1 - b)r - cr^2 \quad (15)$$

Given this, we have

$$\dot{E}_n(t) = \alpha \dot{r}(t) + \beta r(t)\dot{r}(t), \qquad \text{where } \alpha = (d_1 - b) < 0, \beta = -2c > 0. \quad (16)$$

Comparing Eqs. (15) and (1) we note that

$$g'_n = (d_1 - b) - 2cr(t) = \alpha + \beta r(t). \quad (17)$$

As regards the other excess demands, we set

$$\begin{aligned}
E_{sf}(t) &= m[N_n(t) - r(t)], m = g'_{sf} > 0 \\
E_{sc}(t) &= n[ER(t) - r(t)], n = g'_{sc} > 0 \\
ER(t) &= h[r(t), \dot{r}(t), \ddot{r}(t)] = r(t) + b_1\dot{r}(t) + b_2\ddot{r}(t), \quad (18) \\
h'_1 &= b_1 > 0, h'_2 = b_2 > 0; \text{ replacing in the previous equation we get} \\
E_{sc}(t) &= nb_1\dot{r}(t) + nb_2\ddot{r}(t) \\
E_G(t) &= g\left\{\int_0^t [N_n(t) - r(t)]\, dt\right\}, g = G' \gtrless 0
\end{aligned}$$

where m, n, b_1, b_2, g are all constants. Substituting Eq. (17), and the parameters defined in (18), into Eq. (13), and rearranging terms, we obtain

$$\dddot{r}(t) + \frac{b_1}{b_2}\ddot{r}(t) + [\frac{\alpha - m}{nb_2} + \frac{\beta}{nb_2}r]\dot{r}(t) - \frac{g}{nb_2}r(t) = \frac{-g}{nb_2}N_n(t) + \frac{m}{nb_2}\dot{N}_n(t) = 0, \quad (19)$$

or

$$\dddot{r}(t) = -\frac{b_1}{b_2}\ddot{r}(t) + [\frac{m - \alpha}{nb_2} - \frac{\beta}{nb_2}r(t)]\dot{r}(t) + \frac{g}{nb_2}r(t) + \varphi(t), \quad (20)$$

where

$$\varphi(t) \equiv -\frac{g}{nb_2}N_n(t) + \frac{m}{nb_2}\dot{N}_n(t). \quad (21)$$

The homogeneous part of the non-linear third-order differential equation (20) is a jerk function[9], and is known to possibly give rise to chaos for certain

[9] A jerk function has the general form

$$x''' = F(x'', x', x).$$

In physical terms, the jerk is the time derivative of the acceleration.

It seems that the denomination "jerk" came to the mind of a physics student traveling in a car of the New York subway some twenty years ago. When standing in a subway car it is easy to balance a slowly changing acceleration. But the subway drivers had a habit of accelerating erratically (possibly induced by the rudimentary controls then in use). The effect of this was to generate an extremely high jerk.

values of the parameters [Sprott, 1997, eq. (8)]. Besides, since the equation is non-autonomous, the dimension of the state space is increased by one. In fact, Eq. (19) can be easily rewritten as a *system* of first-order equations by defining new variables,

$$x_1 \equiv r, \quad x_2 \equiv \dot{r}, \quad x_3 \equiv \ddot{r}. \tag{22}$$

The resulting system consists of three first-order equations in the x_i, written as

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, 2, \\ \dot{x}_3 &= \tfrac{g}{nb_2}x_1 + \left[\tfrac{m-\alpha}{nb_2} - \tfrac{\beta}{nb_2}x_1\right]x_2 - \tfrac{b_1}{b_2}x_3 + \varphi(t). \end{aligned} \tag{23}$$

System (23) is obviously non-autonomous, like the original equation. It can be rewritten as an autonomous system at the expense of introducing an additional variable, say

$$x_4 = t. \tag{24}$$

In this case x_4 obeys the trivial equation

$$\dot{x}_4 = 1,$$

and system (23) becomes an autonomous system of *four* first order equations:

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, 2, \\ \dot{x}_3 &= \tfrac{g}{nb_2}x_1 + \left[\tfrac{m-\alpha}{nb_2} - \tfrac{\beta}{nb_2}x_1\right]x_2 - \tfrac{b_1}{b_2}x_3 + \varphi(x_4), \\ \dot{x}_4 &= 1. \end{aligned} \tag{25}$$

In any case, we are not interested in a general numerical analysis of our jerk equation or of its equivalent system, but in its analysis with the *estimated* values of its coefficients.

4 Estimation results

Estimates of the parameters were found by a Gaussian estimator of the non-linear model subject to all constraints inherent in the model by using Wymer's software for the estimation of continuous time non-linear dynamic models. We use daily observations of the nominal Euro/Dollar exchange rate over the period January 2, 1975 to December 29, 2003 (weekends and holidays are neglected)[10]. The derivation of the NATREX series is discussed in detail in Federici and Belloc (2010)[11]. The equation estimated is Eq. (20), written in the form

$$\dddot{r}(t) = a_1\ddot{r}(t) + [a_2 + a_3 r(t)]\dot{r}(t) + a_4 r(t) - a_4 N_n(t) - a_5 \dot{N}_n(t). \quad (26)$$

where

$$\begin{array}{l} a_1 \equiv -\frac{b_1}{b_2} < 0, \\ a_2 \equiv \frac{m-\alpha}{nb_2} > 0, \\ a_3 \equiv -\frac{\beta}{nb_2} < 0, \\ a_4 \equiv \frac{g}{nb_2} \gtrless 0, \\ a_5 \equiv -\frac{m}{nb_2} < 0. \end{array} \quad (27)$$

The expected signs of the a_i coefficients reflect our theoretical hypotheses set out in the previous sections. We note that the "original" parameters are seven $(b_1, b_2, m, \alpha, n, \beta, g)$ while we can estimate only five coefficients. Hence it is impossible to obtain the values of the original parameters. What we can do is to check the agreement between the signs listed in (27) and the coefficient estimates. The estimates are reported in Table 1.

Table 1: Estimation results

Coefficient	Estimate	ASE	Ratio
a_1	-12.405	1.538	8.06
a_2	16.976	2.823	6.01
a_3	-27.421	3.545	7.73
a_4	-0.01064	0.003596	2.96
a_5	-1.226	0.184	6.66
Log-likelihood value	0.3287539E+05		

The last column (Ratio) gives the (absolute value of the) ratio of the parameter estimate to the estimate of its asymptotic standard error (ASE). This ratio does not have a Student's t distribution, but has an asymptotic normal distribution. Thus in a sufficiently large sample it is significantly different from zero at the 5% level if it is greater than1.96 and significantly different from zero at the 1% level (i.e., highly significant) if it it is greater than 2.58.

[10] Source: EUROSTAT.

[11] We have generated daily data over the sample period used in estimation.

The estimation of the model shows a remarkable agreement between estimates and theoretical assumptions. In fact, not only all the coefficients have the expected sign and are highly significant, but, in addition, the observed

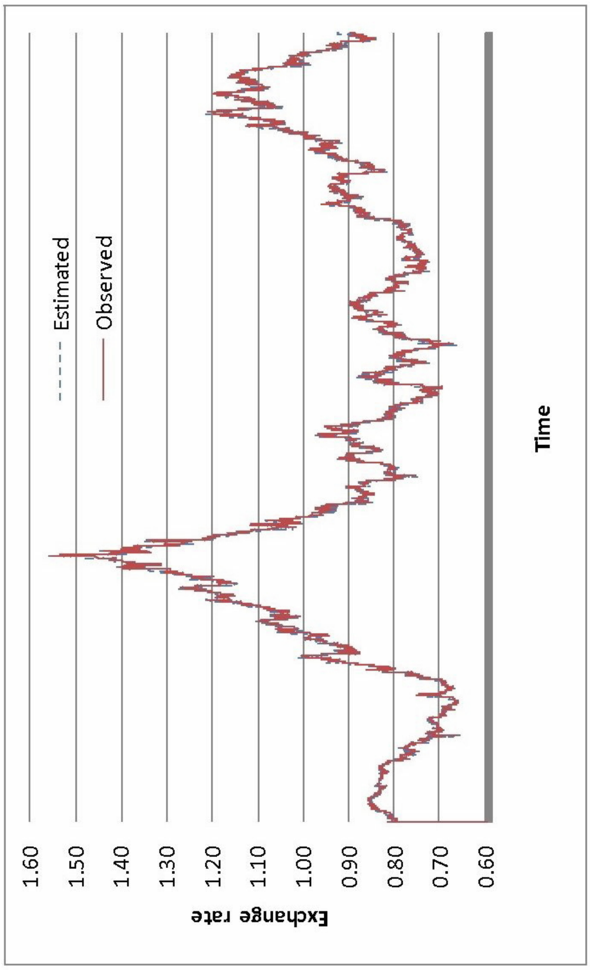

Figure 2: Observed and estimated values

and the estimated values are very close, as shown by Fig. 2 (the correlation coefficient turns out to be 0.9959).

The in-sample root mean square error ($RMSE$) of forecasts[12] of the en-

[12] To obtain these forecasts, the differential equation is re-initialized and solved n times (if one wants forecasts for n periods), each time using the observed value of the endogenous variable in period t as initial value in the solution, which is then employed to obtain the forecast for perod $t+1$.In other words, the re-initialization is at the same frequency as the sample observations.

dogenous variable r turns out to be 0.005475, a very good result.

As regards the out-of-sample, ex post forecasts, we simulated the model over the period January 5, 2004 to June 30, 2006 (weekdays only) and obtained a *RMSE* of 0.091338. This value, although higher than the in-sample value (which is a normal occurrence), is satisfactory.

5 Testing for chaos

Our first step[13] was that of looking for a strange attractor through *phase diagrams*.

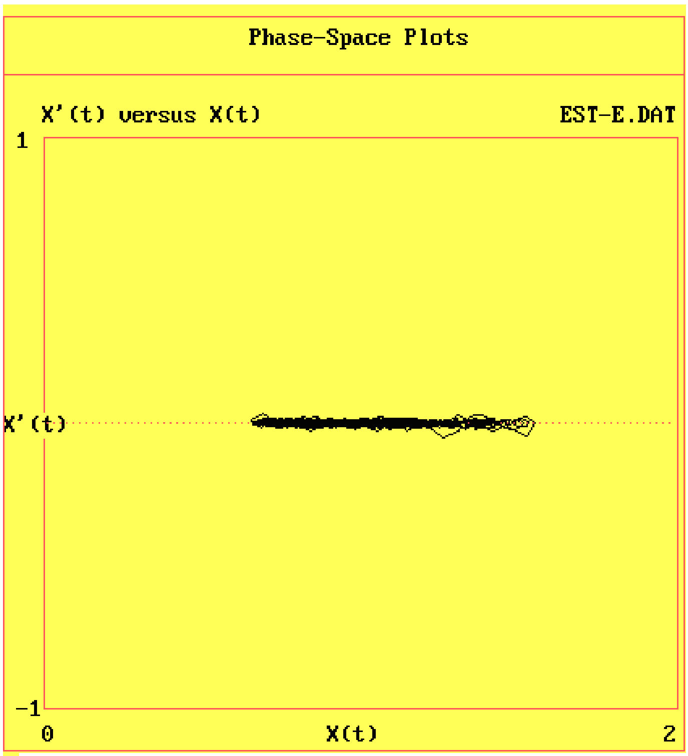

Figure 3: Phase diagram

Figure 3 plots $r'(t)$ against $r(t)$ (these are denoted by $X'(t), X(t)$ in the figure). No discernible structure appears. There does not seem to be a point around which the series evolves, approaching it and going away from it infinite times. On the contrary, the values are very close and no unequivocal

[13]The following tests were carried out using the software Chaos Data Analyzer by Sprott and Rowlands (1992).

closed orbits or periodic motions seem to exist. If we lengthen the time interval for which the phase diagram is built we obtain closed figures, but we cannot clearly classify them as strange attractors because when the data contain such an attractor, this should remain substantially similar as the time interval changes. Such a feature is absent. This test, however, is hardly conclusive, as it relies on impression rather than on quantitative evaluation.

We then computed the *power spectrum* (Fig. 4). Power spectra that are straight lines on a log-linear scale are thought to be good candidates for chaos. This is clearly not the case.

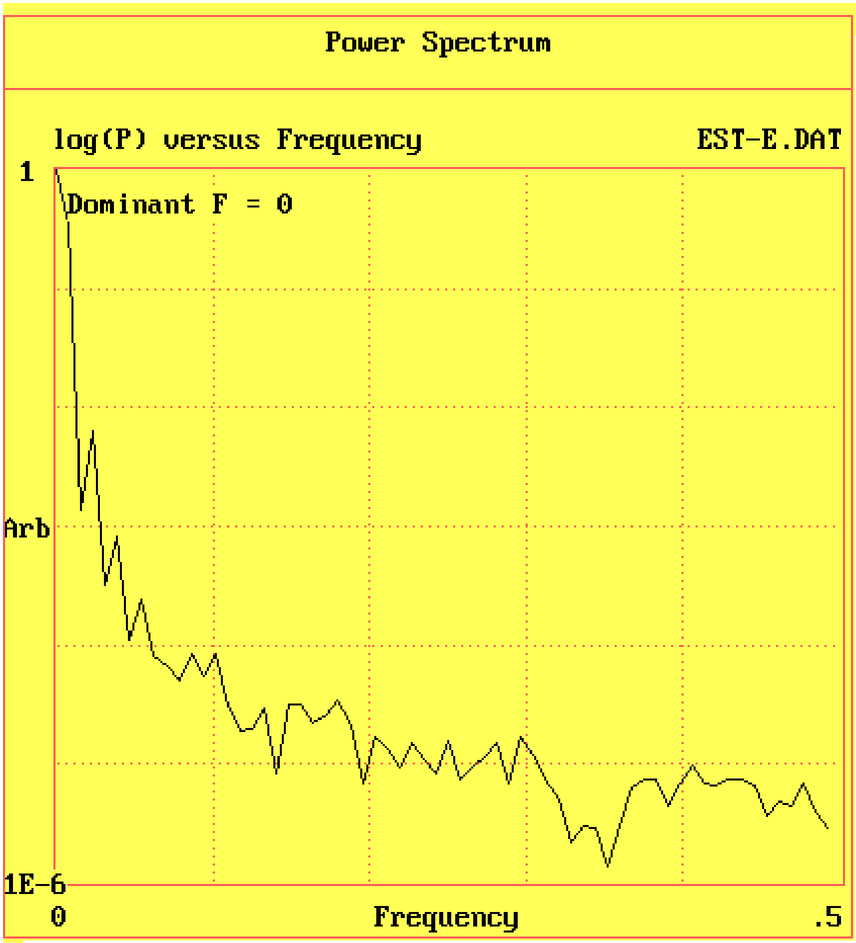

Figure 4: Power spectrum

Quantitative tests are based on the *correlation dimension* and *Liapunov exponents*.

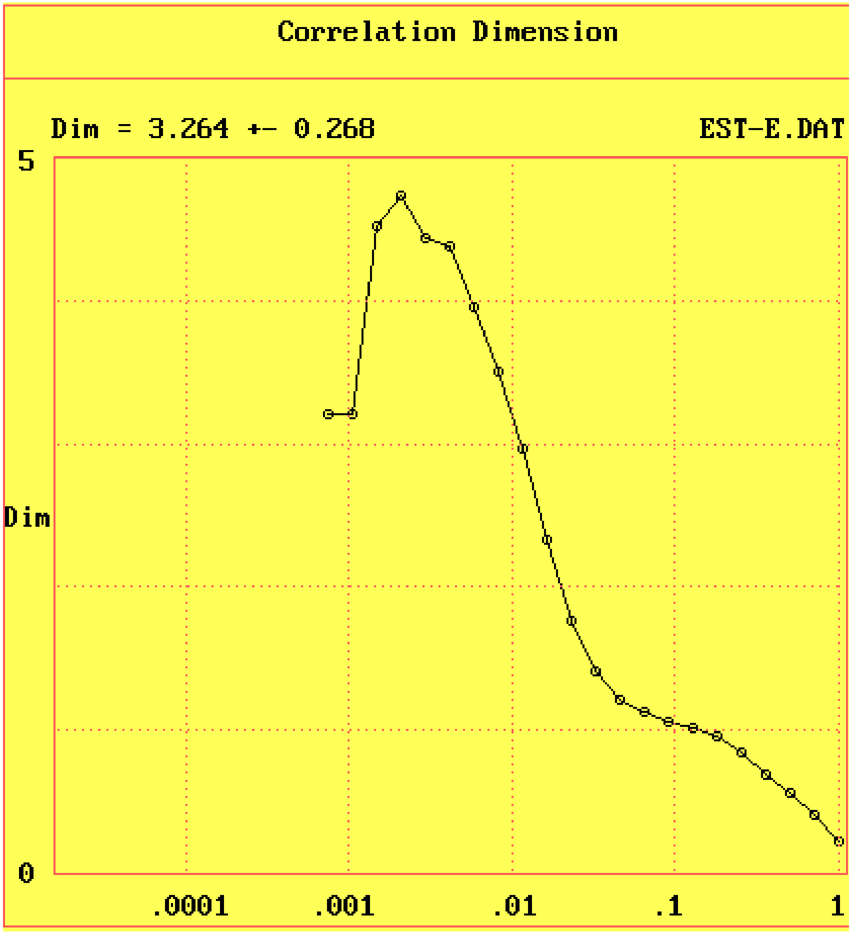

Figure 5: Correlation dimension

The Grassberger-Procaccia algorithm for the computation of the *correlation dimension* requires the presence of a flat plateau in the diagram where the log of the dimension is plotted against the log of the radius.

Since no such plateau exists (see Fig. 5), the computation of the dimension (which turned out to be 3.264 ± 0.268) is not reliable. In any case, it should be noted that saturation of the correlation dimension estimate is just a necessary, but not sufficient, condition for the existence of a chaotic attractor, since also nonlinear nonchaotic stochastic systems are capable of exhibiting this property (Scheinkman and LeBaron, 1989).

Arguably, the only test specific for chaos is provided by *Liapunov exponents*. The Liapunov exponent is a measure of the rate at which nearby trajectories in phase space diverge (SDIC). Chaotic orbits have at least one positive Liapunov exponent.

Inserting the estimated parameters into the original non-linear model and solving the differential equation, we obtained the values (daily data) of the exchange rate generated by the model. Then we applied to this series the Lyapunov exponents test. In this case the greatest Lyapunov exponent is 0.103 ± 0.016. This is evidence for chaos, but the reshuffled (surrogate) data procedure[14] refutes such a result. The basic idea is to produce from the original data a new series with the same distributional properties but with any non-linear dependence removed. The maximum Lyapunov exponent test is then applied to this surrogate series to check whether it gives the same (pro chaos) results as those obtained from the original series. If the results are the same, we should suspect the veracity of our conclusions. We obtained a positive largest Lyapunov exponent of 0.419 ± 0.16. Hence we can conclude that the series generated by the estimated model cannot be considered as chaotic.

The previous results are confirmed by a different procedure, which is the following.

Lyapunov exponents have been calculated from the underlying non-linear model[15] for the estimated parameter values, using the variational matrix equation, and these concentrate information on the nature of the non-linear dynamics. In our case all exponents are negative, and are -0.218691, -0.620225, -0.620229.

On the basis of these results the model is stable dynamically (i.e. for a given set of parameters) and structurally stable (i.e. the results did not change in a substantial way even for large changes in the parameter values[16]).

The stability properties of the model suggest its importance not only to the foreign exchange market but, given the events over the past few years, to financial markets more generally. The Lyapunov exponents of the model show that it is stable at the estimated values and apparently in a wide neigh-

[14] See Scheinkman and LeBaron (1989), Theiler (1991) and Rapp et al. (1993) for a discussion about shuffle diagnostics.

[15] See Wymer (2009) on the advantage of calculating Lyapunov exponents from an estimated model.

[16] These changes were not arbitrary. In fact, we are dealing with *estimated* parameters; it follows that the "true" value of the parameter can lie anywhere in the confidence interval, calculated as

point estimate $\pm$ 1.96σ (95%)

or

point estimate $\pm$ 2.58σ (99%)

where σ is the ASE.

On this point see Gandolfo, 1992.

bourhood of those values. If parameter a_2 is set to zero, however, which means that fundamentalists are not active in the market, the model is unstable. Moreover, in a fairly wide neighbourhood of the other parameters, the model remains unstable. There is a major change in the dynamic structure depending on whether or not fundamentalists are in the market.

6 Conclusion

Our results have important economic implications.

I) The implications for the foreign exchange market, and almost certainly other financial markets, is striking. The stabilizing role of fundamentalists is not surprising given their longer horizons, but the need for fundamentalists to stabilize a market that would otherwise be unstable raises questions about the role of the other players. In recent years, it has been argued that day-traders and other short-term players are important in providing liquidity to the market. If so, they should make the market more stable but they do not. Some (largely anecdotal) evidence suggests that as risk rises these traders disappear from the market. If that is the case their role in providing liquidity is superficial, providing liquidity when it is not needed and not when it is. If that is so, from a macro-economic point of view it is an inefficient use of capital.

II) The second implications is methodological. As stated in the Introduction, after the failure of the standard structural models of exchange rate determination in out-of-sample ex-post forecasts (the most notable empirical rejection was that by Meese and Rogoff, confirmed by subsequent studies), exchange rate forecasting has come to rely on technical analysis and time series procedures, with no place for economic theory. Economic theory can be reintroduced:

a) through a non-linear purely deterministic structural model giving rise to chaos;

b) through a non-linear non-chaotic but stochastic structural model.

The fact that our model fits the data well but does not give evidence for chaos means that non-linear (non-chaotic but stochastic) differential equations econometrically estimated in continuous time are the most promising tool for coping with this phenomenon.

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