A Complete and Strongly Anonymous Leximin Relation on Infinite Streams

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Abstract

Various extensions of the leximin order to the infinite dimensional setting have been suggested. They relax completeness and strong anonymity. Instead, by removing sensitivity to generations at infinite rank this paper defines a complete and strongly anonymous leximin relation on infinite streams. The order is axiomatized, and it is shown to be the limit of extended rank-discounted utilitarianism for any utility function, as the discount factor approaches zero.

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1 Introduction

Maximin — which is often identified with Rawls' (1999) 'difference principle' and which in the intergenerational setting is the principle of giving extreme priority to the generation whose consumption has lowest rank — satisfies procedural equity: it treats generations equally by not being dependent on the sequence in which generations appear. However, it is only sensitive to the interests of the worst off generation.

In order to enhance sensitivity, Sen (1970) suggested maximin in its lexicographic form. This criterion, referred to as *leximin*, gives extreme priority to the generation whose consumption has lowest rank, while taking into account a generation whose consumption has higher rank if and only if there are ties at all lower ranks. In a finite setting, it is a complete and transitive criterion which satisfies both procedural equity (by being indifferent to all permutations) and the strong Pareto principle (by being sensitive to each component of the consumption vector).

However, in a setting where an infinite, but countable, number of generations follow each other in sequence, it is impossible to combine the strong Pareto principle with full procedural equity (referred to as the axiom of strong anonymity), as shown by Van Liedekerke and Lauwers (1997). Moreover, any explicitly described complete and transitive social preferences satisfying the strong Pareto principle cannot even be combined with a weaker form of procedural equity where the social evaluation is indifferent to only finite permutations; this is the Lauwers-Zame impossibility result (Lauwers, 2010; Zame, 2007).

To resolve the dilemma that arises due to the Lauwers-Zame impossibility result, completeness is usually dropped when leximin is extended to an infinite setting, while observing the strong Pareto principle and indifference to finite permutations (see Asheim, 2010, Section 4, for an overview of this literature).

An alternative approach to the problem of extending leximin to an infinite setting is to start with maximin. Recall that maximin is a complete and transitive criterion satisfying the axiom of strong anonymity, even in the infinite setting. One can then ask how much sensitivity for the interests of any one

generation can be introduced without producing a conflict with completeness and the axiom of strong anonymity. The present paper shows that the conflict arises only when trying to take into account generations whose consumption has infinite rank.

In the context of an extreme prioritarian criterion like leximin, one can argue that the interests of generations whose consumption has infinite rank should not matter, as for any such generation there are infinitely many generations that are worse off. Consequently, in the present paper we restrict the domain of the strong Pareto principle to finitely ranked generations, while retaining completeness and the axiom of strong anonymity. This produces a complete and transitive leximin criterion which is fully procedurally equitable by satisfying the axiom of strong anonymity, but which does not take into account generations whose consumption has infinite rank.

The paper is organized as follows. In Section 2 we introduce the framework of our analysis. In Section 3 we define and provide an axiomatic foundation for the new leximin criterion for infinite consumption streams. In Section 4 we show how the proposed leximin criterion is related to the extended rank-dependent utilitarian criterion recently introduced and analyzed by Zuber and Asheim (2011). In Section 5 we show how this new leximin criterion is related other leximin criteria suggested in the infinite setting, while in Section 6 we offer concluding remarks.

2   The framework

Let $\mathbb{N}$ denote as usual the set of natural numbers $\{1, 2, 3, \dots\}$. Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{R}_+$ the set of nonnegative real numbers.

Denote by $\mathbf{x} = (x_1, x_2, \dots, x_t, \dots)$ an infinite stream (or allocation), where $x_t \in \mathbb{R}_+$ is a one-dimensional indicator of the well-being of generation t. We refer to this indicator as the consumption of generation t, restrict attention to allocations consisting of bounded consumption streams, and denote by

$$\mathbf{X} = \left\{ \mathbf{x} = (x_1, \dots, x_t, \dots) \in \mathbb{R}_+^{\mathbb{N}} : \sup_t x_t < +\infty \right\}$$

the set of possible allocations.

For $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, write $\mathbf{x} \geq \mathbf{y}$ whenever $x_t \geq y_t$ for all $t \in \mathbb{N}$; write $\mathbf{x} > \mathbf{y}$ if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$; and write $\mathbf{x} \gg \mathbf{y}$ whenever $x_t > y_t$ for all $t \in \mathbb{N}$. For any $T \in \mathbb{N}$ and $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, denote by $\mathbf{x_T y}$ the consumption stream $\mathbf{z}$ such that $z_t = x_t$ for all $t \leq T$ and $z_t = y_t$ for all $t > T$. For any $z \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbf{X}$, denote by $(z, \mathbf{x})$ the stream $(z, x_1, x_2, \dots)$.

A *permutation* π is a one-to-one map from $\mathbb{N}$ onto $\mathbb{N}$. For any $\mathbf{x} \in \mathbf{X}$ and permutation π, write $\mathbf{x}_\pi = (x_{\pi(1)}, x_{\pi(2)}, \dots) \in \mathbf{X}$. Permutations can be represented by a permutation matrix, $P = (p_{ij})_{i,j \in \mathbb{N}}$, which is an infinite matrix satisfying:

(1) For each $i \in \mathbb{N}$, $p_{ij(i)} = 1$ for some $j(i) \in \mathbb{N}$ and $p_{ij} = 0$ for all $j \neq j(i)$.

(2) For each $j \in \mathbb{N}$, $p_{i(j)j} = 1$ for some $i(j) \in \mathbb{N}$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Given any permutation π, there is a permutation matrix P such that, for any $\mathbf{x} \in \mathbf{X}$, $\mathbf{x}_\pi = (x_{\pi(1)}, x_{\pi(2)}, \dots) = P\mathbf{x}$. Conversely, given any permutation matrix P, there is a permutation π defined by $\pi = P\mathbf{a}$, where $\mathbf{a} = (1, 2, 3, \dots)$. The set of all permutations is denoted by $\mathcal{P}$.

A finite permutation π is a permutation such that there is $T \in \mathbb{N}$ with $\pi(t) = t$ for all $t > T$. Thus, a finite permutation matrix has $p_{tt} = 1$ for all $t > T$ for some $T \in \mathbb{N}$. The set of all finite permutations is denoted by $\mathcal{F}$.

Given a permutation matrix $P \in \mathcal{P}$ and $T \in \mathbb{N}$, we denote the $T \times T$ matrix $(p_{ij})_{i,j \in \{1,\dots,T\}}$ by $P(T)$. Let

$$\mathcal{S} = \{P \in \mathcal{P} : \text{there is some } k \in \mathbb{N} \text{ such that, for each } T \in \mathbb{N},$$
$$P(kT) \text{ is a finite dimensional permutation matrix}\}$$

denote the set of fixed-step permutations.

Two subsets of $\mathbf{X}$ will be of particular interest. First, we introduce the set $\mathbf{X}^+$ of non-decreasing streams in $\mathbf{X}$. This set is defined as follows: $\mathbf{X}^+ = \{\mathbf{x} \in \mathbf{X} : x_t \leq x_{t+1}, \forall t \in \mathbb{N}\}$.

The second subset of $\mathbf{X}$, playing a key role in the remainder of the paper, is the set of allocations, $\bar{\mathbf{X}}$, whose elements can be permuted into non-decreasing streams. This set is defined as follows: $\bar{\mathbf{X}} = \{\mathbf{x} \in \mathbf{X} : \exists P \in \mathcal{P}, \ P\mathbf{x} \in \mathbf{X}^+\}$.

The following inclusions hold: $\mathbf{X}^+ \subset \bar{\mathbf{X}} \subset \mathbf{X}$. In a finite setting, $\bar{\mathbf{X}}$ would be the same as $\mathbf{X}$. To see why this does not hold in an infinite setting, consider the stream $\mathbf{x} = (1, 0, 0, \dots)$. For any permutation π, it must be that $\pi(1) < +\infty$ so that any reordered stream has the form $(0, 0, \dots, 0, 1, 0, \dots)$. Hence, $\mathbf{x}$ cannot be reordered to form a non-decreasing stream.

To characterize the set $\bar{\mathbf{X}}$, let $\ell(\mathbf{x})$ denote $\liminf_{t \to +\infty} x_t$ for any $\mathbf{x} \in \mathbf{X}$. Because streams in $\mathbf{X}$ are bounded, $\ell(\mathbf{x})$ is well-defined for all $\mathbf{x} \in \mathbf{X}$. Write

$$L(\mathbf{x}) = \{t \in \mathbb{N} : x_t < \ell(\mathbf{x})\},$$

and denote by $|L(\mathbf{x})|$ the cardinality of $L(\mathbf{x})$. We make the following observation.

Proposition 1.

(a) If an allocation $\mathbf{x} \in \mathbf{X}$ satisfies $|L(\mathbf{x})| < +\infty$, then $\mathbf{x}$ belongs to $\bar{\mathbf{X}}$ if and only if $x_t \leq \ell(\mathbf{x})$ for all $t \in \mathbb{N}$.

(b) If an allocation $\mathbf{x} \in \mathbf{X}$ satisfies $|L(\mathbf{x})| = +\infty$, then $\mathbf{x}$ belongs to $\bar{\mathbf{X}}$ if and only if $x_t < \ell(\mathbf{x})$ for all $t \in \mathbb{N}$.

Proof. See Zuber and Asheim (2011, Proposition 1). □

For $\mathbf{x} \in \bar{\mathbf{X}}$, denote by $\mathbf{x}_{[\ ]} = (x_{[1]}, x_{[2]}, \dots, x_{[r]}, \dots)$ the non-decreasing allocation which is a permutation of $\mathbf{x}$; i.e., there exists $P \in \mathcal{P}$ such that $\mathbf{x}_{[\ ]} = P\mathbf{x} \in \mathbf{X}_+$. Note that the permutation matrix P need not be unique (if, for instance, $x_t = x_{t'}$ for some $t \neq t'$), but the resulting non-decreasing allocation $\mathbf{x}_{[\ ]}$ is unique. Likewise, for $\mathbf{x} \in \mathbf{X}$, denote by $(x_{[1]}, \dots, x_{[|L(\mathbf{x})|]})$ the non-decreasing allocation which is a permutation of the elements of $\mathbf{x}$ satisfying $t \in L(\mathbf{x})$.

A social welfare relation (SWR) on a set $\mathbf{X}$ is a binary relation $\succsim$, where for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ entails that the consumption stream $\mathbf{x}$ is deemed socially at least as good as $\mathbf{y}$. Let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$. An SWR $\succsim'$ is a *subrelation* to SWR $\succsim''$ if for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, (a) $\mathbf{x} \sim' \mathbf{y} \Rightarrow \mathbf{x} \sim'' \mathbf{y}$ and (b) $\mathbf{x} \succ' \mathbf{y} \Rightarrow \mathbf{x} \succ'' \mathbf{y}$.

3 Axiomatic foundation

The difficulty of combining equal treatment of an infinite number of generations with sensitivity to the interests of each of these generations has been the topic of a prolific literature since the seminal contribution by Diamond (1965). Although complete social preferences over infinite streams that combine equal treatment with Paretian sensitivity exist (Svensson, 1980), they cannot be explicitly described (Basu and Mitra, 2003; Zame, 2007; Lauwers, 2010).

In this section we show how this impossibility is overcome by requiring only sensitivity to the interests of generations whose consumption has finite rank. In subsection 3.1 we first introduce this restricted Pareto axiom together with other axioms used to characterized the leximin criterion we propose. In subsection 3.2 this criterion is then formally defined and fully characterized.

3.1 Axioms

Axiom O (*Order*) The SWR $\succsim$ is complete, reflexive and transitive on $\mathbf{X}$.

An SWR satisfying axiom **O** is named a social welfare order (SWO).

Axiom M (*Monotonicity*) For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succsim \mathbf{y}$.

Axiom **M** is implied by the strong Pareto principle.

We then consider an axiom ensuring some sensitivity for the interests of generation at finite rank.

Axiom RSP (*Restricted Strong Pareto*) For any $\mathbf{x}$, $\mathbf{y} \in \bar{\mathbf{X}}$, if $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

We now turn to an axiom combining Koopmans' (1960) stationarity axiom (Postulate 4) with his separability Postulate 3b (the axiom requiring that the evaluation of two streams with the same present consumption not depend on what that level of consumption is).

Axiom IF (*Independent Future*) For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and $z \in \mathbb{R}_+$, $(z, \mathbf{x}) \succsim (z, \mathbf{y})$ if and only if $\mathbf{x} \succsim \mathbf{y}$.

If we extended our framework to also include comparisons at future times, then

axiom **IF** would imply time consistency provided the SWR $\succsim$ is time invariant.

Next, we state two continuity axioms. Note that their use of the product topology is justified by means of prioritarianism for the worse-off because $\mathbf{x}$ is required to belong to $\mathbf{X}^+$.

Axiom RC1 (*Restricted Continuity 1*) For any $\mathbf{x} \in \mathbf{X}^+$ and $\mathbf{y} \in \mathbf{X}$, if a sequence $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k, \ldots$ of allocations in $\mathbf{X}$ is such that $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |x_t^k - x_t| 2^{-t} = 0$ and, for each $k \in \mathbb{N}$, $x_t^k \in \{x_t, y_t\}$ for all $t \in \mathbb{N}$ and $\mathbf{x}^k \succsim \mathbf{y}$ (resp. $\mathbf{x}^k \precsim \mathbf{y}$), then $\mathbf{x} \succsim \mathbf{y}$ (resp. $\mathbf{x} \precsim \mathbf{y}$).

Axiom RC2 (*Restricted Continuity 2*) For any $\mathbf{x} \in \mathbf{X}^+$ and $\mathbf{y} \in \mathbf{X}$, if a sequence $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k, \ldots$ of allocations in $\mathbf{X}$ is such that $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |x_t^k - x_t| 2^{-t} = 0$ and, for each $k \in \mathbb{N}$, there exists $P_k \in \mathcal{P}$ satisfying $\mathbf{x}^k \geq P_k \mathbf{y}$ (resp. $\mathbf{x}^k \leq P_k \mathbf{y}$), then $\mathbf{x} \succsim \mathbf{y}$ (resp. $\mathbf{x} \precsim \mathbf{y}$).

Imposed continuity axioms must be restricted since, of course, an unrestricted continuity axiom is not compatible with leximin. Axiom **RC1** is clearly weaker than unrestricted continuity (as only binary consumption choices are considered), while axiom **RC2** is weaker than unrestricted continuity if $\succsim$ satisfies axioms **M** and **SA** since then $\mathbf{x}^k \geq P_k \mathbf{y}$ (resp. $\mathbf{x}^k \leq P_k \mathbf{y}$) implies $\mathbf{x}^k \succsim \mathbf{y}$ (resp. $\mathbf{x}^k \precsim \mathbf{y}$) while the converse implication does not hold.

Finally, we state the strong axiom of procedural equity, requiring social indifference with respect to all permutation matrices $P \in \mathcal{P}$.

Axiom SA (*Strong Anonymity*) For any $P \in \mathcal{P}$ and $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \sim P\mathbf{x}$.

3.2 Characterization

In this subsection we completely characterize the class of SWOs satisfying the **O**, **M**, **RSP**, **IF**, **RC1**, **RC2** and **SA**. As a first step, we do so within the restricted domain $\bar{\mathbf{X}}$ of streams that can be reordered into non-decreasing streams.

Definition 1 *Strongly anonymous leximin SWO.* An SWR $\succsim_{\mathcal{P}}^{L}$ on $\bar{\mathbf{X}}$ is a strongly anonymous leximin SWO (SAL SWO) if, for any $\mathbf{x}, \mathbf{y} \in \bar{\mathbf{X}}$, $\mathbf{x} \sim_{\mathcal{P}}^{L} \mathbf{y}$ if and only if $\mathbf{x}_{[]} = \mathbf{y}_{[]}$ and $\mathbf{x} \succ_{\mathcal{P}}^{L} \mathbf{y}$ if and only if there exists $R \in \mathbb{N}$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_{[R]} > y_{[R]}$.

Note that $\succsim_{\mathcal{P}}^L$ satisfies completeness and reflexivity on $\bar{\mathbf{X}}$ as, for any $\mathbf{x}, \mathbf{y} \in \bar{\mathbf{X}}$, $\mathbf{x}_{[]} = \mathbf{y}_{[]}$ or there exists $R \in \mathbb{N}$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_{[R]} \neq y_{[R]}$. Showing that $\succsim_{\mathcal{P}}^L$ satisfies transitivity on $\bar{\mathbf{X}}$ is equally straightforward. Hence, $\succsim_{\mathcal{P}}^L$ is an order on $\bar{\mathbf{X}}$.

Proposition 2. *If an SWR $\succsim$ on $\bar{\mathbf{X}}$ satisfies axioms **O**, **RSP**, **IF**, **RC1** and **SA**, then it is an SAL SWO.*

Proof. Assume that $\mathbf{x}, \mathbf{y} \in \bar{\mathbf{X}}$ and that an SWR $\succsim$ on $\bar{\mathbf{X}}$ satisfies axioms **O**, **RSP**, **IF**, **RC1** and **SA**. Since $\succsim_{\mathcal{P}}^L$ is complete, it is sufficient to show that $\mathbf{x} \sim_{\mathcal{P}}^L \mathbf{y}$ implies $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{x} \succ_{\mathcal{P}}^L \mathbf{y}$ implies $\mathbf{x} \succ \mathbf{y}$.

$\mathbf{x} \sim_{\mathcal{P}}^L \mathbf{y}$ *implies* $\mathbf{x} \sim \mathbf{y}$. Assume $\mathbf{x} \sim_{\mathcal{P}}^L \mathbf{y}$. By Definition 1, $\mathbf{x}_{[]} = \mathbf{y}_{[]}$. By **O** and **SA**, $\mathbf{x} \sim \mathbf{y}$.

$\mathbf{x} \succ_{\mathcal{P}}^L \mathbf{y}$ *implies* $\mathbf{x} \succ \mathbf{y}$. Assume $\mathbf{x} \succ_{\mathcal{P}}^L \mathbf{y}$. By Definition 1, there exists $R \in \mathbb{N}$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_{[R]} > y_{[R]}$. Construct $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}^+$ as follows: (i) $x_{[t]} = \tilde{x}_t = \tilde{y}_t = y_{[t]}$ for $t \in \{1, \ldots, R-1\}$, (ii) $x_{[R]} = \tilde{x}_R > \tilde{y}_R = y_{[R]}$, and (iii) $\tilde{x}_t = \tilde{x} = \inf_{r>R} x_{[r]}$ and $\tilde{y}_t = \tilde{y} = \sup_{r>R} y_{[r]}$ for $t > R$. It suffices to show $\tilde{\mathbf{x}} \succ \tilde{\mathbf{y}}$ as this implies $\mathbf{x} \succ \mathbf{y}$ by axioms **O**, **RSP** and **SA**.

In the case where $\tilde{x} \geq \tilde{y}$, then $\tilde{\mathbf{x}} \succ \tilde{\mathbf{y}}$ follows directly from axiom **RSP**. Hence, assume that $\tilde{x} < \tilde{y}$ and suppose $\tilde{\mathbf{x}} \precsim \tilde{\mathbf{y}}$. Consider the following sequence $\tilde{\mathbf{y}}^1, \tilde{\mathbf{y}}^2, \ldots, \tilde{\mathbf{y}}^k, \ldots$ of allocations in $\mathbf{X}$:

$$
\tilde{y}_t^k = \begin{cases} \tilde{x} & \text{if } t = R+1, \ldots R+k, \\ \tilde{y}_t & \text{otherwise.} \end{cases}
$$

It follows from repeated application of axioms **IF** and **SA** that, for each $k \in \mathbb{N}$, $\tilde{\mathbf{x}} \precsim \tilde{\mathbf{y}}^k$. This in turn, by axiom **RC1**, implies that $\tilde{\mathbf{x}} \precsim \tilde{\mathbf{y}}^\infty$ where

$$
\tilde{y}_t^\infty = \begin{cases} \tilde{x} & \text{if } t > R, \\ \tilde{y}_t & \text{otherwise,} \end{cases}
$$

since $\tilde{\mathbf{y}}^\infty \in \mathbf{X}^+$, $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |\tilde{y}_t^k - \tilde{y}_t^\infty| 2^{-t} = 0$, and for each $k \in \mathbb{N}$, $\tilde{y}_t^k \in \{\tilde{y}_t, \tilde{x}_t\}$ for all $t \in \mathbb{N}$. However, this contradicts that, by axiom **RSP**, $\tilde{\mathbf{x}} \succ \tilde{\mathbf{y}}^\infty$. Hence, $\tilde{\mathbf{x}} \succ \tilde{\mathbf{y}}$ also in the case where $\tilde{x} < \tilde{y}$. $\qquad\square$

A second step is to extend the SAL SWO to the entire domain $\mathbf{X}$ without conflict with any of the axioms $\mathbf{O}$, $\mathbf{M}$, $\mathbf{RSP}$, $\mathbf{IF}$, $\mathbf{RC1}$, $\mathbf{RC2}$ and $\mathbf{SA}$. To do so, define, for any $\mathbf{x} \in \mathbf{X}$, $\bar{\mathbf{x}}$ as follows:

$$\begin{cases} \bar{x}_t = \min\{x_t, \ell(\mathbf{x})\} \text{ for all } t \in \mathbb{N} \text{ if } |L(\mathbf{x})| < +\infty \\ \bar{\mathbf{x}} \text{ is the subsequence of } \mathbf{x} \text{ consisting of all } x_t \text{ with } t \in L(\mathbf{x}) \text{ if } |L(\mathbf{x})| = +\infty \end{cases}$$

Proposition 1 implies that, by construction, $\bar{\mathbf{x}}$ belongs to $\bar{\mathbf{X}}$; therefore $\bar{\mathbf{x}}_{[]}$ is well-defined.

Definition 2 *Extended strongly anonymous leximin SWO.* An SWR $\succsim_{\mathcal{P}}^{L}$ on $\mathbf{X}$ is an extended strongly anonymous leximin SWO (ESAL SWO) if, for any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \sim_{\mathcal{P}}^{L} \mathbf{y}$ if and only if $\bar{\mathbf{x}}_{[]} = \bar{\mathbf{y}}_{[]}$ and $\mathbf{x} \succ_{\mathcal{P}}^{L} \mathbf{y}$ if and only if there exists $R \in \mathbb{N}$ such that $\bar{x}_{[r]} = \bar{y}_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $\bar{x}_{[R]} > \bar{y}_{[R]}$.

Proposition 3. *The ESAL SWO $\succsim_{\mathcal{P}}^{L}$ on $\mathbf{X}$ satisfies axioms $\mathbf{O}$, $\mathbf{M}$, $\mathbf{RSP}$, $\mathbf{IF}$, $\mathbf{RC1}$, $\mathbf{RC2}$ and $\mathbf{SA}$.*

Proof. The ESAL SWO satisfies axiom $\mathbf{O}$. This follows from the facts that SAL SWO is an order and $\bar{\mathbf{x}}_{[]}$ is well-defined for any $\mathbf{x} \in \mathbf{X}$.

The ESAL SWO satisfies axiom $\mathbf{M}$. Assume $\mathbf{x} > \mathbf{y}$ so that $\ell(\mathbf{x}) \geq \ell(\mathbf{y})$. Suppose $\mathbf{x} \prec_{\mathcal{P}}^{L} \mathbf{y}$, implying that there exists $R \in \mathbb{N}$ such that $\bar{x}_{[r]} = \bar{y}_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $\bar{x}_{[R]} < \bar{y}_{[R]} \leq \ell(\mathbf{y}) \leq \ell(\mathbf{x})$. Hence,

$$|\{t \in \mathbb{N} : x_t < \bar{y}_{[R]}\}| \geq R > R - 1 = |\{t \in \mathbb{N} : y_t < \bar{y}_{[R]}\}|.$$

However, as $\mathbf{x} > \mathbf{y}$, $\{t \in \mathbb{N} : x_t < \bar{y}_{[R]}\} \subseteq \{t \in \mathbb{N} : y_t < \bar{y}_{[R]}\}$, leading to a contradiction. Hence, by axiom $\mathbf{O}$, $\mathbf{x} \succsim_{\mathcal{P}}^{L} \mathbf{y}$.

The ESAL SWO satisfies axiom $\mathbf{RSP}$. The ESAL SWO corresponds to the SAL SWO on $\bar{\mathbf{X}}$, and the SAL SWO satisfies axiom $\mathbf{RSP}$.

The ESAL SWO satisfies axiom $\mathbf{IF}$. Consider any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $z \in \mathbb{R}_+$, and write $\mathbf{x}' = (z, \mathbf{x})$ and $\mathbf{y}' = (z, \mathbf{y})$. Suppose $\mathbf{x} \prec_{\mathcal{P}}^{L} \mathbf{y}$, so that there exists $R \in \mathbb{N}$ such that $\bar{x}_{[r]} = \bar{y}_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $\bar{x}_{[R]} < \bar{y}_{[R]}$. If $z \leq \bar{x}_R$, then $\bar{x}'_{[r]} = \bar{y}'_{[r]}$ for all $r \in \{1, \ldots, R\}$ and $\bar{x}'_{[R+1]} < \bar{y}'_{[R+1]}$. If $z > \bar{x}_R$, then $\bar{x}'_{[r]} = \bar{y}'_{[r]}$

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for all $r \in \{1, \ldots, R-1\}$ and $\bar{x}'_{[R]} < \min\{z, \bar{y}_{[R]}\} = \bar{y}'_{[R]}$. In either case, $\mathbf{x}' \prec_\mathcal{P}^L \mathbf{y}'$, establishing by axiom **O** that $(z, \mathbf{x}) \succsim_\mathcal{P}^L (z, \mathbf{y})$ implies $\mathbf{x} \succsim_\mathcal{P}^L \mathbf{y}$. The converse follows likewise by showing that $(z, \mathbf{x}) \prec_\mathcal{P}^L (z, \mathbf{y})$ implies $\mathbf{x} \prec_\mathcal{P}^L \mathbf{y}$.

*The ESAL SWO satisfies axiom **RC1***. Consider any $\mathbf{x} \in \mathbf{X}^+$ and $\mathbf{y} \in \mathbf{X}$. Let the sequence $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k, \ldots$ of allocations in $\mathbf{X}$ be such that, for each $k \in \mathbb{N}$, $x_t^k \in \{x_t, y_t\}$ for all $t \in \mathbb{N}$ and $\mathbf{x}^k \succsim_\mathcal{P}^L \mathbf{y}$. Suppose $\mathbf{x} \prec_\mathcal{P}^L \mathbf{y}$, so that there exists $R \in \mathbb{N}$ such that $x_r = \bar{y}_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_R < \bar{y}_{[R]}$. Since $\bar{y}_{[R]} \leq \ell(\mathbf{y})$, $|\{t \in \mathbb{N} : y_t < \bar{y}_{[R]}\}| = R - 1$, implying since, for each $k \in \mathbb{N}$, $x_t^k \in \{x_t, y_t\}$ for all $t \in \mathbb{N}$ and $\mathbf{x}^k \succsim_\mathcal{P}^L \mathbf{y}$ that $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |x_t^k - x_t| 2^{-t} \geq |\bar{y}_R - x_R| 2^{-R} > 0$. Hence, if $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |x_t^k - x_t| 2^{-t} = 0$, then $\mathbf{x} \succsim_\mathcal{P}^L \mathbf{y}$, by axiom **O**. Likewise if, for each $k \in \mathbb{N}$, $x_t^k \in \{x_t, y_t\}$ for all $t \in \mathbb{N}$ and $\mathbf{x}^k \precsim_\mathcal{P}^L \mathbf{y}$.

*The ESAL SWO satisfies axiom **RC2***. Consider any $\mathbf{x} \in \mathbf{X}^+$ and $\mathbf{y} \in \mathbf{X}$. Let the sequence $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k, \ldots$ of allocations in $\mathbf{X}$ be such that, for each $k \in \mathbb{N}$, there exists $P_k \in \mathcal{P}$ satisfying $\mathbf{x}^k \geq P_k \mathbf{y}$. Suppose $\mathbf{x} \prec_\mathcal{P}^L \mathbf{y}$, so that there exists $R \in \mathbb{N}$ such that $x_r = \bar{y}_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_R < \bar{y}_{[R]}$. Since $\bar{y}_{[R]} \leq \ell(\mathbf{y})$, $|\{t \in \mathbb{N} : y_t < \bar{y}_{[R]}\}| = R - 1$, implying since, for each $k \in \mathbb{N}$, there exists $P_k \in \mathcal{P}$ satisfying $\mathbf{x}^k \geq P_k \mathbf{y}$ that $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |x_t^k - x_t| 2^{-t} \geq |\bar{y}_R - x_R| 2^{-R} > 0$. Hence, if $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |x_t^k - x_t| 2^{-t} = 0$, then $\mathbf{x} \succsim_\mathcal{P}^L \mathbf{y}$, by axiom **O**. Likewise if there exists $P_k \in \mathcal{P}$ satisfying $\mathbf{x}^k \leq P_k \mathbf{y}$.

*The ESAL SWO satisfies axiom **SA***. For any $\mathbf{x} \in \mathbf{X}$, if there exists $P \in \mathcal{P}$ such that $\mathbf{y} = P\mathbf{x}$, then $\ell(\mathbf{y}) = \ell(\mathbf{x})$ (as infinite permutations preserve $\liminf$) so that $\bar{\mathbf{x}}_{[]} = \bar{\mathbf{y}}_{[]}$ and $\mathbf{x} \sim_\mathcal{P}^L \mathbf{y}$. $\qquad\square$

A third step is to show that the ESAL SWO is the only SWR satisfying axioms **O**, **M**, **RSP**, **IF**, **RC1**, **RC2** and **SA** on the full domain $\mathbf{X}$. This follows from the following proposition and is stated as the paper's main theorem.

Proposition 4. *If an SWO $\succsim$ on $\mathbf{X}$ satisfies axioms **O**, **M**, **RC2** and **SA**, then $\bar{\mathbf{x}} \sim \mathbf{x}$ for any $\mathbf{x} \in \mathbf{X}$.*

Proof. Let $s(\mathbf{x})$ denote $\sup_{t \in \mathbb{N}} x_t$ for any $\mathbf{x} \in \mathbf{X}$.

Case 1: $|L(\mathbf{x})| < +\infty$. Since $\bar{\mathbf{x}} \leq \mathbf{x}$, it follows that $\bar{\mathbf{x}} \precsim \mathbf{x}$ by axioms **O** and **M**. To show $\bar{\mathbf{x}} \succsim \mathbf{x}$, construct, for each $k \in \mathbb{N}$, $\bar{\mathbf{x}}^k$ as follows:

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$$
\bar{x}^k_t = \begin{cases} \bar{x}_{[t]} & \text{if } t \le |L(\mathbf{x})| \\ \ell(\mathbf{x}) + \frac{1}{k}\big(s(\mathbf{x}) - \ell(\mathbf{x})\big) & \text{if } |L(\mathbf{x})| + 1 \le t \le |L(\mathbf{x})| + k \\ s(\mathbf{x}) & \text{if } t > |L(\mathbf{x})| + k\,. \end{cases}
$$

By Zuber and Asheim (2011, proof of Lemma 3), there exists, for each $k \in \mathbb{N}$, $P_k \in \mathcal{P}$ such that $\bar{\mathbf{x}}^k \ge P_k \mathbf{x}$. Since

$$
\sum\nolimits_{t \in \mathbb{N}} |\bar{x}^k_t - \bar{x}_{[t]}| 2^{-t} = \left(\tfrac{1}{k}(1 - 2^{-k}) + 2^{-k}\right)\big(s(\mathbf{x}) - \ell(\mathbf{x})\big) 2^{-|L(\mathbf{x})|}\,,
$$

we have that $\bar{\mathbf{x}}_{[]} \succsim \mathbf{x}$ by axiom **RC2**, and $\bar{\mathbf{x}} \succsim \mathbf{x}$ by axioms **O** and **SA**.

Case 2: $|L(\mathbf{x})| = +\infty$. Construct, for each $k \in \mathbb{N}$, $\bar{\mathbf{x}}^k$ as follows:

$$
\bar{x}^k_t = \begin{cases} \bar{x}_{[t]} & \text{if } t \le k \\ s(\mathbf{x}) & \text{if } t > k\,. \end{cases}
$$

Since there exists, for each $k \in \mathbb{N}$, $P_k \in \mathcal{P}$ such that $\bar{\mathbf{x}}^k \ge P_k \mathbf{x}$ and, furthermore, $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |\bar{x}^k_t - \bar{x}_{[t]}| 2^{-t} = 0$, we have that $\bar{\mathbf{x}}_{[]} \succsim \mathbf{x}$ by axiom **RC2**, and $\bar{\mathbf{x}} \succsim \mathbf{x}$ by axioms **O** and **SA**. Construct, for each $k \in \mathbb{N}$, $\mathbf{y}^k$ as follows:

$$
y^k_t = \begin{cases} \bar{x}_{[t]} & \text{if } t \le k \\ \bar{x}_{[k]} & \text{if } t > k\,. \end{cases}
$$

Since there exists, for each $k \in \mathbb{N}$, $P_k \in \mathcal{P}$ such that $\mathbf{y}^k \le P_k \mathbf{x}$ and, furthermore, $\lim_{k \to \infty} \sum_{t \in \mathbb{N}} |y^k_t - \bar{x}_{[t]}| 2^{-t} = 0$, we have that $\bar{\mathbf{x}}_{[]} \precsim \mathbf{x}$ by axiom **RC2**, and $\bar{\mathbf{x}} \precsim \mathbf{x}$ by axioms **O** and **SA**. Hence, $\bar{\mathbf{x}} \sim \mathbf{x}$ also in this case. $\square$

Theorem 1. *Consider an SWR $\succsim$ on $\mathbf{X}$. The following two statements are equivalent.*

*(1) $\succsim$ satisfies axioms **O**, **M**, **RSP**, **IF**, **RC1**, **RC2** and **SA**.*

(2) $\succsim$ is an ESAL SWO.

Proof. (1) implies (2). This follows from Propositions 2 and 4. *(2) implies (1).* This follows from Proposition 3. $\square$

4 Limit of rank-discounted utilitarianism

The extended rank-discounted utilitarian criterion is introduced and characterized by Zuber and Asheim (2011). This criterion coincides with discounted utilitarianism on the set of non-decreasing consumption streams. Utility discounting is then justified as an expression of inequality aversion when future generations are better off. However, and contrary to the discounted utilitarian approach, extended rank-discounted utilitarianism also satisfies procedural equity: two intergenerational consumption streams that are identical up to a permutation are deemed equally good. On streams that are not non-decreasing, discounting becomes the mere expression of intergenerational inequality aversion.

Definition 3 *Extended Rank-Discounted Utilitarian SWO.* An SWR on $\mathbf{X}$ is an Extended Rank-Discounted Utilitarian SWO (ERDU SWO) if it is represented by an SWF $W : \mathbf{X} \to \mathbb{R}$ defined by:

$$W(\mathbf{x}) = u(\ell(\mathbf{x})) + (1 - \beta) \sum\nolimits_{r=1}^{|L(\mathbf{x})|} \beta^{r-1} \left(u(x_{[r]}) - u(\ell(\mathbf{x})) \right), \tag{1}$$

where $0 < \beta < 1$ is a real number and the function u is continuous and increasing.

Write $\succsim_{\beta,u}$ for the ERDU SWO characterized by β and u. The following result establishes that, for any increasing and continuous function u, the ESAL SWO $\succsim_{\mathcal{P}}^{L}$ is the limit of the ERDU SWO $\succsim_{\beta,u}$ as β — the utility discount factor according to rank — approaches 0.

Proposition 5. *For any* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *and any continuous and increasing function* u, $\mathbf{x} \succsim_{\mathcal{P}}^{L} \mathbf{y}$ *if and only if there exists* $\bar{\beta} \in (0,1)$ *such that* $\mathbf{x} \succsim_{\beta,u} \mathbf{y}$ *for all* $\beta \in (0, \bar{\beta})$.

Proof. Assume that $\mathbf{x}, \mathbf{y} \in \bar{\mathbf{X}}$, and that u is a continuous and increasing function. Since $\succsim_{\mathcal{P}}^{L}$ is complete, it is sufficient to show that $\mathbf{x} \sim_{\mathcal{P}}^{L} \mathbf{y}$ implies the existence of $\bar{\beta} \in (0,1)$ such that $\mathbf{x} \sim_{\beta,u} \mathbf{y}$ for all $\beta \in (0, \bar{\beta})$, and that $\mathbf{x} \succ_{\mathcal{P}}^{L} \mathbf{y}$ implies the existence of $\bar{\beta} \in (0,1)$ such that $\mathbf{x} \succ_{\beta,u} \mathbf{y}$ for all $\beta \in (0, \bar{\beta})$.

$\mathbf{x} \sim_{\mathcal{P}}^{L} \mathbf{y}$ *implies the existence of* $\bar{\beta} \in (0,1)$ *such that* $\mathbf{x} \sim_{\beta,u} \mathbf{y}$ *for all* $\beta \in (0, \bar{\beta})$. By Definition 2, $\bar{\mathbf{x}}_{[]} = \bar{\mathbf{y}}_{[]}$. By Definition 3, $\mathbf{x} \sim_{\beta,u} \mathbf{y}$ for all $\beta \in (0,1)$.

$\mathbf{x} \succ^L_{\mathcal{P}} \mathbf{y}$ *implies the existence of* $\bar{\beta} \in (0,1)$ *such that* $\mathbf{x} \succ_{\beta,u} \mathbf{y}$ *for all* $\beta \in$ $(0,\bar{\beta})$. By Definition 2, there exists $R \in \mathbb{N}$ such that $\bar{x}_{[r]} = \bar{y}_{[r]}$ for all $r \in \{1,\ldots,R-1\}$ and $\bar{x}_{[R]} > \bar{y}_{[R]}$. Construct $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}^+$ as follows: (i) $\bar{x}_{[t]} = \tilde{x}_t = \tilde{y}_t = \bar{y}_{[t]}$ for $t \in \{1,\ldots,R-1\}$, (ii) $\bar{x}_{[R]} = \tilde{x}_R > \tilde{y}_R = \bar{y}_{[R]}$, and (iii) $\tilde{x}_t = \tilde{x} = \inf_{r>R} \bar{x}_{[r]}$ and $\tilde{y}_t = \tilde{y} = \sup_{r>T} \bar{y}_{[r]}$ for $t > T$. It suffices to show $\tilde{\mathbf{x}} \succ_{\beta,u} \tilde{\mathbf{y}}$ as this implies $\mathbf{x} \succ \mathbf{y}$ because $\succsim^L_{\mathcal{P}}$ satisfies axioms **O**, **M** and **SA**.

In the case where $\tilde{x} \geq \tilde{y}$, then $\tilde{\mathbf{x}} \succ_{\beta,u} \tilde{\mathbf{y}}$ for all $\beta \in (0,1)$ follows directly because $\succsim^L_{\mathcal{P}}$ satisfies axiom **RSP**. Hence, assume that $\tilde{x} < \tilde{y}$, and define $\bar{\beta}$ by

$$(1-\bar{\beta})u(\tilde{x}_R) + \bar{\beta}u(\tilde{x}) = (1-\bar{\beta})u(\tilde{y}_R) + \bar{\beta}u(\tilde{y}).$$

Then, by (1), $\tilde{\mathbf{x}} \sim_{\bar{\beta},u} \tilde{\mathbf{y}}$, and $\tilde{\mathbf{x}} \succ_{\beta,u} \tilde{\mathbf{y}}$ for all $\beta \in (0,\bar{\beta})$. $\qquad\square$

Proposition 5 shows that the ERDU SWO has a robustness property, as suggested by Basu and Mitra (2007, p. 361).[1]

5　Comparison with other extensions of leximin

In the recent literature on intertemporal social choice, leximin is usually defined in the setting of infinite consumption streams by extending the definition of leximin on the set of finite consumption streams to the infinite setting. To consider this literature and its relation to the present definition of leximin, let us reproduce the definition of leximin in the finite setting.

For this purpose, denote by $\mathbf{x_T}$ the finite stream $(x_1, x_2, \ldots, x_t, \ldots, x_T)$ and denote by $\mathbf{x_{[T]}} = (x_{[1]}, x_{[2]}, \ldots, x_{[r]}, \ldots, x_{[T]})$ the non-decreasing allocation which is a permutation of $\mathbf{x_T}$; i.e., there exists $P(T)$ such that $\mathbf{x_{[T]}} = P(T)\mathbf{x_T}$ is non-decreasing. Then $\succsim^L_T$ is defined by, for any $\mathbf{x_T}, \mathbf{y_T} \in \mathbb{R}^T_+$, $\mathbf{x_T} \sim^L_T \mathbf{y_T}$ if and only if $\mathbf{x_{[T]}} = \mathbf{y_{[T]}}$ and $\mathbf{x_T} \succ^L_T \mathbf{y_T}$ if and only if there exists $R \leq T$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1,\ldots,R-1\}$ and $x_{[R]} > y_{[R]}$.

In the finite setting leximin satisfies both the strong Pareto principle and anonymity. However, in the infinite setting, as pointed out by Van Liedek-erke and Lauwers (1997), the strong Pareto principle is in conflict with strong

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[1]We thank Kohei Kamaga for making this observation.

anonymity, i.e., axiom **SA**. Furthermore, Zame (2007) and Lauwers (2010) demonstrate that SWOs satisfying strong Pareto cannot be explicitly described even when finite anonymity (i.e., anonymity in its weaker form, involving only finite permutations) is imposed. Hence, when extending leximin to infinite streams in a manner that allows explicit description, one cannot keep both axioms **O** and **SA** while insisting on the strong Pareto principle.

In the present paper, we have kept axioms **O** and **SA** and weakened the strong Pareto principle, by making it applicable only on the restricted domain $\bar{\mathbf{X}}$ of streams that can be re-ordered into non-decreasing allocations. In fact, axiom **SA** is even in conflict with the weak Pareto principle whereby one stream is preferred to another stream if the former has higher consumption than the latter at all times.[2] The ESAL SWO does not satisfy the weak Pareto principle (since $\mathbf{x} \sim_{\mathcal{P}}^{L} \mathbf{y}$ whenever $\mathbf{x}$ and $\mathbf{y}$ satisfy $x_t > y_t \geq \ell(\mathbf{y}) = \ell(\mathbf{x})$ for all $t \in \mathbb{N}$), and the above observation entails that this feature is necessary for an SWO satisfying axiom **SA**.

Other explicitly describable extensions of leximin to the infinite setting have all weakened axiom **O**. They can be divided in two parts depending on whether completeness or transitivity is relaxed.

Most of these extensions relax completeness. This is in particular the case of $\succsim^{L}$, under which streams are comparable only if they eventually coincide. The SWR $\succsim^{L}$ is defined as follows (cf. Asheim, d'Aspremont and Banerjee, 2010): For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^{L} \mathbf{y}$ if and only if there exists $T \in \mathbb{N}$ such that $\mathbf{x_T} \succsim_{T}^{L} \mathbf{y_T}$ and $\mathbf{x} = \mathbf{x_T y}$. This criterion does satisfy axiom **SA**, but fails the weak Pareto principle even for constant streams dominating each other. Bossert, Sprumont and Suzumura (2007) suggest $\succsim_{\mathcal{F}}^{L}$, under which streams are comparable only if they eventually coincide or Pareto-dominates each other. The SWR $\succsim_{\mathcal{F}}^{L}$ is

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[2]This is demonstrated by the following adaptation of Fleurbaey and Michel's (2003) proof of their Theorem 1 to a setting where streams are bounded: Consider

$$\mathbf{x} = \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \dots, \tfrac{1}{k+2}, \tfrac{k+1}{k+2}, \dots \right)$$
$$\mathbf{y} = \left(\tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{2}{3}, \dots, \tfrac{1}{k+3}, \tfrac{k}{k+1}, \dots \right),$$

where by axiom **SA** $\mathbf{x}$ is indifferent to $\mathbf{y}$ even though $x_t > y_t$ for all $t \in \mathbb{N}$.

defined as follows: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^{L}_{\mathcal{F}} \mathbf{y}$ if and only if there exists $T \in \mathbb{N}$ such that $\mathbf{x_T} \succsim^{L}_{T} \mathbf{y_T}$ and $\mathbf{x} \geq \mathbf{x_T y}$. This criterion satisfies the strong Pareto principle and hence not axiom **SA**; rather, it satisfies finite anonymity.

Various other contributions show how comparability can be increased further by imposing anonymity involving fixed-step permutations (Lauwers, 1997; Kamaga and Kojima, 2009), different kinds of overtaking or catching-up (Asheim and Tungodden, 2004; Asheim, d'Aspremont and Banerjee, 2010), or both (Asheim and Banerjee, 2010; Kamaga and Kojima, 2010). However, since these are explicitly described SWRs satisfying reflexivity, transitivity, finite anonymity and the strong Pareto principle, it follows from the results of Zame (2007) and Lauwers (2010) that they are not complete.

Given the incompleteness of the leximin criteria of the previous paragraph, one may ask whether they are subrelations to the complete leximin criterion $\succsim^{L}_{\mathcal{P}}$. This is not case since $(1, 0, 0, 0, \dots)$ is deemed socially better than $(0, 0, 0, 0, \dots)$ according to all of these incomplete criteria while they are equally good according to $\succsim^{L}_{\mathcal{P}}$: $(1, 0, 0, 0, \dots) \sim^{L}_{\mathcal{P}} (0, 0, 0, 0, \dots)$.

In fact, there exist $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{x} \succ^{L}_{\mathcal{P}} \mathbf{y}$, while $\mathbf{x} \prec \mathbf{y}$ for any of different overtaking or catching-up leximin criteria $\succsim$ considered by Asheim and Tungodden (2004), Kamaga and Kojima (2010), Asheim, d'Aspremont and Banerjee (2010) and Asheim and Banerjee (2010). Since the time-invariant overtaking leximin criterion $\succsim^{L}_{\mathcal{I}}$ of Asheim, d'Aspremont and Banerjee (2010) is a subrelation to all the other overtaking or catching-up leximin criteria, this result is established through the following proposition.

Proposition 6. *There exist* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *such that* $\mathbf{x} \succ^{L}_{\mathcal{P}} \mathbf{y}$ *and* $\mathbf{x} \prec^{L}_{\mathcal{I}} \mathbf{y}$.

To avoid the additional notation needed to define the time-invariant overtaking leximin criterion $\succsim^{L}_{\mathcal{I}}$, we state the following characterization.

Lemma 1. *The following two statements are equivalent.*

(1) $\mathbf{x} \succsim^{L}_{\mathcal{I}} \mathbf{y}$,

(2) For any $P \in \mathcal{P}$, *there exists* $T \in \mathbb{N}$ *such that* $(P\mathbf{x})_{\mathbf{T}} \succsim^{L}_{T} (P\mathbf{y})_{\mathbf{T}}$,

Proof. (1) implies (2). Assume that (1) is true. As any $P \in \mathcal{P}$ maps a finite subset of $\mathbb{N}$ into another finite subset of $\mathbb{N}$, it follows directly from Asheim, d'Aspremont and Banerjee (2010, Definition 4) that, for any $P \in \mathcal{P}$, there exists $T \in \mathbb{N}$ such that $(P\mathbf{x})_\mathbf{T} \succsim_T^L (P\mathbf{y})_\mathbf{T}$.

(2) implies (1). Suppose that (1) is not true. Then, by Asheim, d'Aspremont and Banerjee (2010, the proof of Proposition 7), there exists $P \in \mathcal{P}$ such that $(P\mathbf{x})_\mathbf{T} \prec_T^L (P\mathbf{y})_\mathbf{T}$ for all $T \in \mathbb{N}$. This contradicts (2). $\quad\square$

Proof of Proposition 6. Let $\mathbf{x} = (0, 1, 1, \dots, 1, \dots)$ and $\mathbf{y} = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$. Then $\bar{x}_{[1]} = 0 = \bar{y}_{[1]}$ and $\bar{x}_{[2]} = 1 > 0 = \bar{y}_{[2]}$ so that $\mathbf{x} \succ_\mathcal{P}^L \mathbf{y}$. We must show that $\mathbf{x} \prec_\mathcal{I}^L \mathbf{y}$; i.e., $\mathbf{x} \precsim_\mathcal{I}^L \mathbf{y}$, but not $\mathbf{x} \succsim_\mathcal{I}^L \mathbf{y}$.

$\mathbf{x} \precsim_\mathcal{I}^L \mathbf{y}$ *is true.* Consider any $P \in \mathcal{P}$. Then there exists $\tau \in \mathbb{N}$ such that $\mathbf{x}_\pi = (x_{\pi(1)}, x_{\pi(2)}, \dots) = P\mathbf{x}$ satisfies $\pi(\tau) = 1$ and $x_{\pi(\tau)} = 0$. Choose $T \geq \tau$. Then $(P\mathbf{x})_\mathbf{T} \precsim_T^L (P\mathbf{y})_\mathbf{T}$. By Lemma 1, this implies $\mathbf{x} \precsim_\mathcal{I}^L \mathbf{y}$.

$\mathbf{x} \succsim_\mathcal{I}^L \mathbf{y}$ *is not true.* By Lemma 1, it suffices to show the existence of $P \in \mathcal{P}$ such that $(P\mathbf{x})_\mathbf{T} \prec_T^L (P\mathbf{y})_\mathbf{T}$ for all $T \in \mathbb{N}$. Since $\mathbf{x}_\mathbf{T} \prec_T^L \mathbf{y}_\mathbf{T}$ for all $T \in \mathbb{N}$, this is obtained by setting P equal to the identity matrix. $\quad\square$

In his Theorem 5, Sakai (2010) considers variants of the leximin criteria satisfying completeness and the strong Pareto principle while failing transitivity. When comparing complete criteria, one SWR is a subrelation to another SWR if and only if they are identical. The leximin variants considered by Sakai (2010) are all different from $\succsim_\mathcal{P}^L$ as they, by the strong Pareto principle, deem $(1, 0, 0, 0, \dots)$ to be socially better than $(0, 0, 0, 0, \dots)$. Furthermore, it cannot be shown that $\mathbf{x} \succ_\mathcal{P}^L \mathbf{y}$ implies $\mathbf{x} \succ \mathbf{y}$ for the whole range of complete but intransitive leximin criteria $\succsim$ considered by Sakai (2010).

6 Concluding remarks

In this paper we have shown that the problem of combining completeness and procedural equity is resolved if one does not insist on sensitivity to the interests of generations whose consumption has infinite rank. In fact, then one can explicitly define a version of leximin over infinite consumption streams that is complete

and transitive, while satisfying the strong form of procedural equity that axiom **SA** entails. Provided that one adopts the extreme form of prioritarianism that leximin represents, perhaps it is reasonable not to care about generations who are better off than an infinite number of other generations.

It is important to note that the use of the product topology in axioms **RC1** and **RC2** does not reflect impatience, but priority for the worse-off, as the stream being approached is restricted to be non-decreasing. This entails that such prioritarianism need not be introduced by invoking the axiom that usually ensures extreme priority of the worse off generation in conflicts between two generations:

Axiom HE (*Hammond Equity*) For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if there exist τ, $\tau' \in \mathbb{N}$ such that $y_\tau < x_\tau < x_{\tau'} < y_{\tau'}$ and $y_t = x_t$ for all $t \neq \tau$, τ', then $\mathbf{x} \succsim \mathbf{y}$.

In the axiomatic foundations for discounted utilitarianism and extended rank-discounted utilitarianism, impatience (in the former case) and priority for the worse-off (in the latter case) are not introduced through the continuity axiom, as the uniform topology is used. Rather, impatience and priority for the worse-off respectively follow when such continuity is combined with versions of Koopmans' stationarity and separability conditions.

We have already shown that the ESAL SWO satisfies stationarity and separable future on the set of all streams (cf. axiom **IF** of Section 3.1), and it is straightforward to show that the criterion satisfies separable present on the set of non-decreasing streams. It is an open question whether introducing axiom **HE**, while weakening axioms **RC1** and **RC1** to versions where the uniform topology is used, is sufficient to characterize the ESAL SWO in view of these separability properties.

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