Optimal Income Taxation with Uncertain Earnings: A Synthesis

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Abstract

We study optimal nonlinear income taxation when earnings can differ because of both ability and luck, so the income tax has both a redistributive role and an insurance role. A substantial literature on optimal redistribution in the absence of uncertainty has evolved since Mirrlees' original contribution. The literature on the income tax as a social insurance device is more limited. It has largely assumed that households are ex ante identical so unequal earnings are due to uncertainty alone. We provide a general treatment of the optimal income tax under uncertainty when households differ in ability. We characterize optimal marginal tax rates and interpret them in terms of redistribution, insurance and incentive effects. The case of ex ante identical households and the no-risk case with heterogeneous abilities come out as special cases.

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1 Introduction

Redistributive income taxation serves to mitigate the social welfare consequences of market-generated inequalities in earnings. These inequalities can be characterized as having two different sources. As emphasized in the traditional optimal income tax literature following from Mirrlees (1971), they can be a result of differences in the endowed ability or productivity of households. The inability of the government to observe each household's ability constrains the benevolent government from achieving a first-best outcome, and limits considerably the progressivity of the tax system. Alternatively, as Varian (1980) and Tuomala (1984) studied, inequality might be a result of uncertainty in the earnings obtained from a given effort. In the absence of market-provided earnings insurance, the income tax system acts as a social insurance device, albeit an imperfect one because of the inability of the government to observe individual effort, a sort of moral hazard. Here, too, progressivity will be compromised by imperfect information. If the government were fully informed and uncertainty was the only source of inequality, the tax system would mimic full insurance and have 100 percent marginal tax rates, which would be highly progressive indeed. The inability to observe individual effort precludes that, and, as in the optimal income tax case, constrains progressivity considerably.[1]

The design of optimal redistributive taxation to address ability differences and to address earnings uncertainty have largely been studied separately. The former literature is vast, and is summarized in Atkinson and Stiglitz (1980), Tuomala (1990), Myles (1995) and Kaplow (2008). Given the complexity of the modeling, simulation techniques are usually relied on to shed light on the optimal income tax structure. They typically result in optimal marginal tax rates that are relatively constant except at the two ends of the ability distribution, where they fall to zero.

The literature on optimal income taxation to deal with earnings uncertainty is much more limited, and has generally assumed away ability differences. Thus, Tuomala (1984) and Low and Maldoom (2004) assume that all households are ex ante identical, so supply the same amount of labor, but differ in earnings because of some innate idiosyncratic uncertainty that is resolved after labor is supplied. Tuomala's (1984) simulation analysis seems to indicate that the optimal degree of progressivity to address earnings uncertainty is qualitatively comparable to that found by Mirrlees (1971) (and confirmed by Tuomala 1990) to address earnings inequality arising from ability differences. Low and Maldoom show how the degree of progressivity reflects a trade-off between an insurance effect, which favors progressivity in the sense of increasing marginal tax rates, and

[1] A third source of earnings inequality we do not consider arises from differences in preferences for work among households. This raises difficult issues with respect to how social preferences should treat persons with different preferences, summarized in Fleurbaey and Maniquet (2006, 2007). Different views on that can change redistribution policy significantly, as illustrated in Boadway, cuff and Marchand (2000) and Cuff (2000).

an incentive effect, which works against such progressivity. There is no general presumption that on balance marginal tax rates will be increasing. Cremer and Gahvari (1999) also assume ex ante identical individuals facing uncertain wages where the government sets redistribution, or social insurance, policy before the uncertainty is resolved. They consider cases in which labor supply and/or commodity purchases can be made either before or after the uncertainty is resolved.

There has been relatively little attention devoted to studying optimal income taxation when both ability differences and earnings uncertainty are present. A well-known exception is Eaton and Rosen (1980a). They considered the choice of a linear progressive income tax in a model with two ability-types and uncertain earnings. Their interest was in learning whether adding uncertainty to the earnings-generation process would increase or decrease the progressivity of the linear tax. Given the difficulty of obtaining analytical results in even this simple setting, they resorted to a series of simulations. The results turned out to be agnostic. Depending on the parameters chosen, such as the degree of risk aversion, adding uncertainty to the standard optimal redistribution problem with two ability-types could either increase or decrease the optimal linear tax rate. Their models were exploratory, and they made no attempt to calibrate them to an actual economy. Diamond, Helms and Mirrlees (1980) also study optimal policy when individuals of different ability face earnings uncertainty. Their policy is also restricted to linear progressive income taxation, and their uncertainty takes a particularly simple form in which persons are either able or not able to work in the second period.

There is another, more recent, literature on the effect of uncertainty on optimal redistribution policy. In the self-labeled new dynamic public finance literature, the emphasis is on uncertainty in an intertemporal setting (Golosov, Tsyvinski and Werning 2007; Kocherlakota 2010). Ability is heterogeneous, but evolves in a stochastic manner period-by-period. In each period, households choose their labor supply and their saving knowing their current skills, but having only expectations of their future skills. Much of the emphasis in this literature is on the implications for the taxation of capital income, with the typical finding that capital income should face positive taxation, especially in the case where there are borrowing constraints (Aiyagari 1995; Conesa, Kitao and Krueger 2009). A lower level of saving makes it more difficult for persons who turn out to be high-skilled to mimic those with lower skills.

In a related context, Cremer and Gahvari (1995) show that with wage uncertainty, a case can be made for giving preferential commodity tax treatment to consumer durables. They assume that ex ante identical households choose their labor supply after the wage has been revealed, and then allocate their disposable income to many goods. Some goods purchases can be chosen after wages are known, while others — durables — must be chosen before wage uncertainty is resolved. Subsidizing the purchase of consumer durables then makes it more difficult for the high-skilled to mimic the low-skilled. Our analysis does not touch directly on these issues, since we focus on uncertainty of earnings that is resolved only after labor is supplied. However, we shall consider the taxation of consumer durables in our model.

In this paper, we revisit the optimal nonlinear income tax problem in a Mirrleesian setting but with earnings uncertainty added. The purpose is partly to synthesis and generalize the existing analysis, and partly to uncover the various influences that bear on the progressivity of the tax. The analysis is inherently more complicated than the standard problem of Mirrlees (1971) and than the pure insurance problem of Tuomala (1984) because there are elements of both settings at work. These include an equity effect familiar from the standard problem, an incentive effect common to both problems, and an insurance effect from the pure insurance problem. Our analysis uncovers how each of these affects the structure of marginal tax rates. The standard approach and the pure insurance approach naturally emerge as special cases.

The analysis differs from the standard approach in a fundamental way. Because of earnings uncertainty, a given amount of earnings will be associated with different amounts of effective labor supply by households of differing ability. This implies that we cannot use the standard mechanism design approach to optimal income taxation because the revelation principle does not apply. The approach we adopt is analogous to the general principal-agent problem of moral hazard set out by Mirrlees (1974). To simplify matters, we assume the first-order approach can be used, which leads to some useful restrictions on certain functions along the line proposed by Jewitt (1988), following Rogerson (1985).

We proceed by setting out the basic model we are using in the next section. This is followed by solving the government's optimal redistribution planning problem. The implementation of this planning outcome using an income tax function is then considered, and the ex ante identical household case is shown as a special case. Next, an alternative formulation of the problem is considered which has the advantage of being directly comparable to the standard approach. The standard approach falls out as a special case of this formulation when risk vanishes. Finally, we confirm that, when there is more than one consumer good, the Atkinson and Stiglitz (1976) Theorem continues to apply with uncertainty, unless some goods must be purchased before uncertainty is resolved.

2 Basic Setting

The economy is populated by a continuum of workers with different earning abilities a, distributed according to $F(a)$ for $\underline{a} \leqslant a \leqslant \overline{a}$, with density $F'(a) = f(a) > 0$ for all a. They are exposed to an exogenous earnings shock ε that is idiosyncratic and occurs after labor supply or effort ℓ is decided. Define effective labor supply for a person of ability a by $z \equiv a\ell$. Actual earnings, denoted by y, are stochastic and are given by the function $y(z, \varepsilon)$, which is increasing in effective labor supply z and the shock ε. This general form accommodates the special cases where the shock is additive, so

$y = z + \varepsilon$, or multiplicative, so $y = \varepsilon z$. By assumption, the distribution of the wage shock does not depend on earning ability a.

It is useful, following Tuomala (1984), to invert the earnings function to yield the shock function $\varepsilon(y, z)$, which is increasing in y and decreasing in z. The shock is drawn from the distribution function $G(\varepsilon)$, which given $\varepsilon(y, z)$ can be written equivalently as $G(y, z)$, where $G_y(y, z) \geqslant 0$ and $G_z(y, z) \leqslant 0$. Thus, given effective labor supply z, $G(y, z)$ is the proportion who earn no more than y, $G_y(y, z)$ is the density of workers with income y, and $G_z(y, z)$ is the change in the proportion of workers who earn no more than y as z increases. We assume that the distribution of earnings is bounded in the sense that for any value of effective labor supply z, there will be an upper bound on earnings, $\overline{y}(z)$, such that $G\big(\overline{y}(z), z\big) = 1$, so $G_y(y, z) = 0$ for all $y > \overline{y}(z)$. Earnings will also be bounded below by $\underline{y}(z) \geqslant 0$. Given that higher effective labor supply leads to higher earnings on average, we assume $G_{zz} > 0$ for $y < \overline{y}(z)$. In the absence of risk, $y = z$ for certain, so $G(\tilde{y}, z) = 0$ for $\tilde{y} < z$ and $G(\tilde{y}, z) = 1$ for $\tilde{y} \geqslant z$.

The government can observe realized income y, and imposes a nonlinear income tax function $T(y)$. It cannot observe a household's type a or either the actual labor ℓ or the effective labor z it supplies. Household disposable income, or consumption, is given by $c(y) = y - T(y)$. For a worker of type a, a given amount of effective labor supplied $z = a\ell$ is associated with a distribution of realized incomes y. This implies that workers of different ability-types will end up earning the same income and will thus be treated alike by the income tax system. There will not be a separating equilibrium as in the standard optimal nonlinear income tax case, and the usual incentive constraints will not be binding.

Individual decisions are made before the shock is revealed. Let expected utility for a type-a worker choosing an effective labor supply $z = a\ell$ be:

$$v(a) = \int_y u\big(c(y)\big) G_y(y, z) dy - z/a \tag{1}$$

where, following Lollivier and Rochet (1983), Weymark (1986) and Boadway, Cuff and Marchand (2000), we assume for simplicity that utility is quasilinear in labor (or leisure). This utility function is useful because it isolates individuals' risk aversion by making it depend only on consumption.

Consider the behavior of a type-a worker. The worker chooses z to maximize $v(a)$ in (1). The first-order condition, assuming an interior solution, is:

$$\frac{1}{a} = \int_y u\big(c(y)\big) G_{yz}(y, z) dy = -\int_y u'\big(c(y)\big) c'(y) G_z(y, z) dy \tag{2}$$

where the second equality is obtained by integrating by parts.[2] The second-order condition is

[2] More precisely, partial integration yields $\int_{\underline{y}}^{\overline{y}} u(c) G_{yz} dy + \int_{\underline{y}}^{\overline{y}} u'(c) c' G_z dy = [u(c) G_z]_{\underline{y}}^{\overline{y}}$. Since $G(\underline{y}, z) = 0$ and $G(\overline{y}, z) = 1$, where $\underline{y}$ and $\overline{y}$ are the lower and upper bounds on income, $G_z(y, z) = 0$ at both ends.

obtained by differentiating either of the two forms of first-order condition in (2) to give:

$$\int_y u\big(c(y)\big)G_{yzz}dy < 0, \quad \text{or} \quad -\int_y u'\big(c(y)\big)c'(y)G_{zz}dy < 0 \tag{3}$$

We assume the second-order conditions are satisfied. Sufficient conditions would be $c'(y) > 0$ and $G_{zz}(y,z) \geqslant 0$, which we have assumed above. In principle, $c'(y)$ need not be positive for all values of y for the second-order conditions to be satisfied. Moreover, given that $G_z < 0$, (2) requires only that on average $c'(y) > 0$ for an interior solution, so does not preclude the possibility that for some values of y, $c'(y) < 0$. Since the budget constraint faced by all households is $c(y) = y - T(y)$, $c'(y) < 0$ implies $T'(y) > 1$, so the marginal tax rate might be greater than 100 percent at some income levels without violating the second-order conditions. We return to this possibility when we investigate the optimal income tax structure.

Note that in the full-information case where the government can observe ability a, the redistributive tax will be based on a and there will be full insurance, or equivalently a 100 percent tax on income. In this case, $c'(y(a)) = 0$. This is not feasible with asymmetric information because of the adverse incentive effect that such a tax would have on effort.

It is useful to define the likelihood ratio of the distribution of y conditional on z as follows:

$$h(y,z) \equiv \frac{G_{yz}(y,z)}{G_y(y,z)} \tag{4}$$

We assume that $h_y \geqslant 0$, with $h_y > 0$ for $y < \overline{y}(z)$. This follows Low and Maldoom, who assume $h_y > 0$ in their setting with ex ante identical individuals. The property $h_y > 0$ is the so-called monotone likelihood ratio property (Milgrom 1981), and is one of the sufficient conditions that Jewitt (1988) shows will validate the first-order approach to the government's problem in the next section. We summarize these assumptions as follows for use later.

Assumption 1: *$h_y > 0$ and $G_{zz}(y,z) \geqslant 0$, with $G_{zz}(y,z) > 0$ for $y < \overline{y}(z)$.*

Note, however, that $h(y,z)$, or equivalently G_{yz}, can be positive or negative. This follows from the fact that $\int_y G_y(y,z)dy = 1$, so $\int_y G_{yz}dy = 0$. This will be important in interpreting our results below.

The solution to (2) yields the supply of effective labor $z(a)$, given the tax system in place. Differentiating (2), we obtain:

$$\dot{z}(a) \equiv \frac{dz(a)}{da} = -\frac{\int_y u\big(c(y)\big)G_{yz}(y,z(a))dy}{a\int_y u\big(c(y)\big)G_{yzz}\big(y,z(a)\big)dy} = -\frac{1}{a^2\int_y u(c(y))G_{yzz}(y,z(a))dy} \tag{5}$$

The second-order condition (3) requires the denominator in (5) to be positive, and the numerator is positive by (2). This implies the following lemma.

Lemma 1. *Assuming the second-order condition for the worker's choice of $z(a)$ is satisfied, effective labor supply $z(a)$ is increasing in ability a.*

Note that since workers of any type a choose the optimal level of effective labor supply $z(a)$, they have no incentive to mimic $z(a')$ for any other type $a' \neq a$. Therefore, the only incentive constraint the policymaker faces is (2), which is like a moral hazard condition.

Suppose for simplicity that the government is purely redistributive so has no net revenue requirements. Then, its budget constraint may be written:

$$\int_y \Big(y - c(y)\Big)\Big(\int_a G_y\big(y, z(a)\big) f(a)da\Big)dy = 0 \tag{6}$$

where $\int_a G_y\big(y, z(a)\big)f(a)da$ is the number of workers (of all ability-types) earning an ex post income of y. We assume that the population is large enough such that the government budget is deterministic.

3 The Government's Optimal Income Tax Problem

We begin by solving the planning solution for the optimal redistribution problem, which involves the government planner choosing optimal quantities. In the following sections, we consider how this planning solution can be implemented using a nonlinear income tax. We assume the government is benevolent and its objective function takes the general form of a weighted utilitarian social welfare function:[3]

$$W = \int_a \beta(a)v(a)f(a)da, \quad \text{with} \quad \dot{\beta}(a) \leqslant 0$$

The assumption of nonincreasing welfare weights $\beta(a)$ ensures that the government has some redistribution motive. The government maximizes social welfare subject to its revenue constraint (6), the definition of $v(a)$ in (1), and an incentive constraint. We assume that the conditions for a first-order approach to this principle-agent problem are satisfied, for example along the lines of Jewitt (1988). In that case, the incentive constraint for a type-a worker is given by the worker's first-order condition (2). This is a straightforward constrained optimization problem with government control variables $v(a)$, $z(a)$ and $c(y)$.

The Lagrangian function can be written:

$$\mathcal{L} = \int_a \beta(a)v(a)f(a)da + \lambda \int_y \Big(y - c(y)\Big)\Big(\int_a G_y\big(y, z(a)\big)f(a)da\Big)dy \tag{7}$$

$$+ \int_a \mu(a)\Big(\int_y u\big(c(y)\big)G_y\big(y, z(a)\big)dy - \frac{z(a)}{a} - v(a)\Big)da + \int_a \gamma(a)\Big(\int_y u\big(c(y)\big)G_{yz}\big(y, z(a)\big)dy - \frac{1}{a}\Big)da$$

[3] The use of a weighted-sum social welfare function is for analytical convenience. A more conventional form would be $W = \int_a w(v(a))f(a)da$, where $w(\cdot)$ is a concave social utility function. This would give qualitatively similar results.

where λ is the shadow value of government revenues, and $\mu(a)$ and $\gamma(a)$ are the type-specific shadow prices of individual utility and incentive constraints. The first-order conditions with respect to $v(a)$, $z(a)$ and $c(y)$ respectively may be written:

$$\beta(a)f(a) - \mu(a) = 0 \tag{8}$$

$$\lambda f(a)\int_y \big(y - c(y)\big)G_{yz}\big(y, z(a)\big)dy + \gamma(a)\int_y u\big(c(y)\big)G_{yzz}\big(y, z(a)\big)dy = 0 \tag{9}$$

$$-\lambda\int_a G_y\big(y, z(a)\big)f(a)da + u'(c(y))\Big(\int_a \mu(a)G_y\big(y, z(a)\big)da + \int_a \gamma(a)G_{yz}\big(y, z(a)\big)da\Big) = 0 \tag{10}$$

To interpret these conditions, first rearrange (9) to obtain:

$$-\lambda\frac{\int_y \big(y - c(y)\big)G_{yz}\big(y, z(a)\big)dy}{\int_y u\big(c(y)\big)G_{yzz}\big(y, z(a)\big)dy} = \frac{\gamma(a)}{f(a)} \equiv \theta(a) \tag{11}$$

The variable $\theta(a)$ is a modified version of the Lagrangian multiplier $\gamma(a)$ on the incentive or moral hazard constraint of type-a persons. Note that the denominator on the lefthand side is positive if the second-order condition for the individual's choice of effective labor supply $z(a)$ is satisfied. Then, using the definition of the likelihood ratio in (4) and the first-order condition (8), condition (10) can be written:

$$\frac{\lambda}{u'\big(c(y)\big)} = \frac{1}{\int_a G_y\big(y, z(a)\big)f(a)da}\int_a \Big(\beta(a) + \theta(a)h\big(y, z(a)\big)\Big)G_y\big(y, z(a)\big)f(a)da \tag{12}$$

Suppose we normalize the social welfare weights such that $\int_a \beta(a)f(a)da = 1$, which is innocuous since social welfare is simply an ordering. From (12), we can deduce the following proposition. The proof is in the Appendix.

Proposition 1. *Assuming* $\int_a \beta(a)f(a)da = 1$, *then*

$$\mathrm{E}\Big[\frac{\lambda}{u'(c(y))}\Big] = 1 \tag{13}$$

The ratio $\lambda/u'(c(y))$ is analogous to the marginal cost of public funds in a standard optimal commodity tax setting (Atkinson and Stern 1974): the value of an increment of revenue to the government relative to an increment of revenue in the hands of an individual. Proposition 1 says that the expected value of the marginal cost of public funds is unity.

Eq. (12) incorporates all three first-order conditions of the government's problem: (8), (9) and (10). It can be simplified further by defining the proportion of workers at income level y who are type-a as follows:

$$\phi(a, y) \equiv \frac{G_y\big(y, z(a)\big)f(a)}{\int_a G_y\big(y, z(\tilde{a})\big)f(\tilde{a})d\tilde{a}} = \frac{G_y\big(y, z(a)\big)f(a)}{\mathrm{E}\big[G_y(y, z(\tilde{a}))|y\big]} \tag{14}$$

For future reference, let $\Phi(a,y) \equiv \int_{\underline{a}}^{a} \phi(\tilde{a},y)d\tilde{a}$ with $\Phi(\overline{a},y) = 1$ and $\Phi(\underline{a},y) = 0$, recalling that $\overline{a}$ and $\underline{a}$ are the upper and lower bounds on skills. Thus, $\Phi(a,y)$ is the proportion of workers at income level y who are of ability-type a or less. Given the presumption that higher income levels draw in higher ability-types, the following assumption is reasonable:

Assumption 2: $\Phi_y(a,y) < 0$ *for a in the interior of the ability distribution.*

Note that $\Phi_y(\underline{a},y) = \Phi_y(\overline{a},y) = 0$ at the boundaries.

Given the definition of $\phi(a)$ in (14), (12) can be rewritten:

$$\frac{\lambda}{u'\big(c(y)\big)} = \int_a \Big(\beta(a) + \theta(a)h\big(y, z(a)\big)\Big)\phi(a,y)da \tag{15}$$

This equation indicates how the marginal utility of consumption varies with income, and takes into account both the redistributive role of government policy, reflected in the social weights $\beta(a)$, and the constraint on its insurance role, reflected in $\theta(a)$, given that redistribution involves moral hazard. It is useful for subsequent comparisons to rewrite (15) in the following way (omitting arguments of functions for simplicity):[4]

$$\frac{\lambda}{u'\big(c(y)\big)} = \underbrace{\int_a \big(1+\theta h\big)f(a)da}_{\text{Insurance}} - \underbrace{\big(1 - \mathrm{E}[\beta|y]\big)}_{\text{Equity}} + \underbrace{\mathrm{Cov}\Big[\theta h, \frac{G_y}{\mathrm{E}\big[G_y|y\big]}\Big]}_{\text{Distribution of } \theta h} \tag{16}$$

The first term is an insurance term, indicating how the incentive constraint favors a deviation of $u'\big(c(y)\big)$ from uniformity. The second term is an equity term that will be positive for low income levels where $\mathrm{E}[\beta|y] > 1$, and vice versa at high incomes. The third term indicates how the incentive effect varies with income.

We next consider how this planning solution can be implemented in a decentralized economy using an income tax function. It is useful for what follows to make the following further assumption.

Assumption 3: $\theta(a) > 0$ *for all a.*

That is, the incentive constraint is binding for all workers. This assumption also implies that an increase in $z(a)$ increases average tax revenue. To see this, note that the numerator of (11) is the increase in tax revenue from an increase in $z(a)$, which is positive if $\theta(a) > 0$.[5]

4 To see this, rewrite (15) using (14) and $\int_a f(a)da = 1$ to give:

$$\frac{\lambda}{u'(c(y))} = \mathrm{E}[\beta|y] + \int_a (\theta h(\phi - f(a)))da + \int_a \theta h f(a)da$$

$$= \int_a (1+\theta h)f(a)da - (1 - \mathrm{E}[\beta|y] + \int_a \bigg(\theta h\Big(\frac{G_y}{\mathrm{E}[G_y|y]} - 1\Big)\bigg)f(a)da$$

which is equivalent to (16).

5 The numerator can be written as $\int_y T(y)G_{yz}dy = -\int_y T'(y)G_z dy$, with $G_z < 0$.

4 Marginal Income Tax Rates

As usual, the first-order conditions for the planning problem can be interpreted in terms of tax wedges, or marginal tax rates. The tax liability of a person with income y is $T(y) = y - c(y)$, so the marginal tax rate is $T'(y) = 1 - c'(y)$, which we assume to be uniquely defined. To determine the pattern of marginal tax rates in the optimum, we can differentiate (15) with respect to y. It is useful to start with a benchmark case in which all workers are ex ante identical, before turning to the more general case.

Workers Ex Ante Identical

Suppose all workers have the same ability level, which we normalize to $a = 1$. This corresponds with the case considered by Low and Maldoom (2004). In this case, assuming $\beta(1) = 1$, (15) or (16) reduce to:

$$\frac{\lambda}{u'\big(c(y)\big)} = 1 + \theta h(y, z) \tag{17}$$

where $\theta = \gamma$, the Lagrange multiplier on the incentive constraint. As mentioned earlier, $h(y, z)$ can be positive or negative, so $\lambda/u'\big(c(y)\big)$, the marginal cost of public funds, can be greater than or less than unity at any given income level. Since Proposition 1 still applies, its average value over all income levels will be unity. Differentiating (17) with respect to y and using $c'(y) = 1 - T'(y)$, we obtain (deleting the arguments of functions for simplicity):[6]

$$T' = 1 - \underbrace{\left(\frac{-u''}{u'}\right)^{-1}}_{A} \cdot \underbrace{\frac{u'}{\lambda}}_{B} \cdot \underbrace{\theta h_y}_{C} \tag{18}$$

The marginal tax rate depends on three effects, labeled A, B and C. Given that $u''(c) < 0$ and $h_y > 0$, all three terms are positive, which implies that $T'(y) < 1$ in this identical-worker case. Equivalently, $c'(y) > 0$ in this case, which guarantees that the second-order conditions for the workers' optimal choice of z are satisfied. Consider each of the three terms in turn.

The first one, $A = -(u''/u')^{-1} > 0$, is the reciprocal coefficient of absolute risk aversion and represents an individual insurance effect. The more risk-averse are individuals, the higher is the marginal tax rate $T'(y)$, that is, the more insurance the tax system provides to them. Suppose further that the coefficient of absolute risk aversion is decreasing in income, as is commonly assumed. This would work in favor of $T'(y)$ falling with income, though the overall effect depends on what happens to B and C as income rises.

The second term, $B = u'/\lambda > 0$, or $B = u'\mathrm{E}\big[1/u'\big]$ by (13), reflects the marginal utility of consumption at a given income level relative to its average. The larger it is, the smaller is the marginal tax rate. It can be thought of as an ex post equity effect, or a social insurance effect, since

[6] This is (8) in Low and Maldoom (2004).

the government puts a higher social value on income of individuals whose marginal utility of income is higher. Given our assumption that $h_y > 0$, u'/λ is decreasing in income by (17). Therefore, term B tends to cause the marginal tax rate to rise with income, possibly working against the risk aversion effect noted above. In effect, B is a measure of the deviation of the outcome from the first best, where $u'\big(c(y)\big) = \lambda$ for all y. At low income levels, $u' > \lambda$, so B works to reduce the marginal tax rate, and vice versa.

The third term, $C = \theta h_y > 0$, positive by Assumptions 1 and 3, is an incentive or efficiency effect, given that θ is the shadow price of the incentive constraint and the likelihood ratio $h(y,z) = G_{yz}/G_y$ reflects the responsiveness of the distribution of outcomes to effort. A higher value of θh_y contributes to a lower marginal tax rate, so less consumption smoothing. Indeed, it is possible that the marginal tax rate is negative at some income levels, for example, if the coefficient of absolute risk aversion is small enough. Note that (18) confirms the fact that in the full-information case where the government can observe ex post wage rates, earnings risk is fully insured. In this case, the incentive constraint is not binding, so $\theta = 0$, leading to a marginal tax rate of 100 percent.[7]

In general, the way in which the marginal tax rate changes with income is ambiguous. Following Low and Maldoom (2004), we can see how $T'(y)$ varies with income by differentiating (18) to obtain:

$$T'' = \Big(\frac{-u''}{u'}\Big)^{-1} \frac{\theta}{1+\theta h} \Big(\frac{\theta h_y^2}{1+\theta h}\big(P - 2 + h_{yy}\big)\Big)$$

where
$$P = -\frac{u'''}{u''} \Big/ \frac{u''}{u'}$$

Thus, if $P(c(y))$ is large enough, the marginal tax rate will be increasing in income. The expression for $P(c(y))$ is the ratio of a precautionary effect, reflecting the desire of a household to supply labor for precautionary purposes given the uncertainty of outcomes, to a risk-aversion effect. A relatively strong precautionary effect leads to increasing marginal tax rates, while strong risk-aversion leads to declining marginal tax rates. If the utility function exhibits constant relative risk aversion, one obtains $P = 1 + 1/\sigma$, where $\sigma = -u''c/u'$ is the coefficient of relative risk aversion. Higher σ reduces P and therefore reduces progressivity.

[7] Cremer and Gahvari (1999) show that if all workers are ex ante identical and choose their labor supply before their wage rate is revealed, the full-information outcome can be achieved if the government can commit to a punitive enough tax rate for individuals who deviate from full-information labor supply. Specifically, suppose that $a = 1$ for all individuals and $z^* = \ell^*$ is the full-information labor supply. Let $\underline{y}^*$ be the minimum income that can be earned when labor supply is ℓ^*. Then, the government imposes a tax of $T(y) = y - E[y]$ for all $y \geqslant \underline{y}^*$, and $T(y) = y$ for $y < \underline{y}^*$. Full insurance is provided for those who choose ℓ^*, while for those with $\ell < \ell^*$, there is a possibility that $c(y) = T(y) - y = 0$. As long as $u(0)$ is sufficiently low, all individuals will choose $\ell = \ell^*$. The same result applies to our model with ex ante identical individuals, though not to our general case with ex ante heterogeneity. We choose not to emphasize it because we view the ex ante identical case as simply a limiting case of use for illustrating the intuition of the more general case.

Workers Differ in Ability

Turn now to the more general case where workers differ in ability a, but face the same distribution of earnings shocks, $G(y,z)$. Differentiating (15) with respect to y and using $c'(y) = 1 - T'(y)$, we obtain:

$$-\frac{u''(c(y))}{u'\big(c(y)\big)}\frac{\lambda}{u'\big(c(y)\big)}\big(1-T'(y)\big) = \int_a \theta(a)\Big(h_y(\cdot)\phi(a,y) + h(\cdot)\phi_y(a,y)\Big)da + \int_a \beta(a)\phi_y(a,y)da \quad (19)$$

which can be written in the following form comparable to (18):

$$T' = 1 - \underbrace{\left(\frac{-u''}{u'}\right)^{-1}}_{A} \cdot \underbrace{\frac{u'}{\lambda}}_{B} \cdot \left(\underbrace{\mathrm{E}\big[\theta h_y|y\big]}_{C} + \underbrace{\int_a \beta\phi_y da}_{D} + \underbrace{\int_a \theta h\phi_y da}_{E}\right) \quad (20)$$

where $1/\lambda = \mathrm{E}[1/u']$ by Proposition 1.

The terms A and B are the analogs of the same terms in (18), where there was only one ability-type. The term C in (20) is simply the expected value of C in (18). These terms have the same interpretation as in the case where workers are ex ante identical. All three are positive. A higher value of absolute risk aversion tends to increase the marginal tax rate, while higher values of deviations from the first-best and of the average incentive effect tend to reduce it. These are intuitive.

The expression for $T'(y)$ in the heterogeneous-ability case differs from the identical-ability case by the addition of the terms D and E. Term D is an enhanced equity effect, reflecting the influence of the social welfare weights $\beta(a)$. By partial integration, the equity effect can be written equivalently as follows:[8]

$$D = \int_a \beta(a)\phi_y(a,y)da = -\int_a \dot{\beta}(a)\Phi_y(a,y)da$$

This implies that $D \leqslant 0$ since $\dot{\beta} \leqslant 0$ and $\Phi_y < 0$ by Assumption 2. Not surprisingly, the more does the equity weight $\beta(a)$ decline with a, the larger is the marginal tax rate.

In the special case of a utilitarian social welfare function, $\dot{\beta}(a) = 0$, so this term disappears. In that case, the government's redistributive objective will be reflected solely in term B: marginal tax rates will be higher for persons with higher incomes (lower values of u'), which will include disproportionately higher-ability persons. The marginal tax rate in (20) then becomes $T = 1 -$

[8] Proof: Term $D = \int_a \beta(a)\phi_y(a,y)da$ can be written:

$$\int_a \beta(a)\frac{d}{da}\Phi_y(a,y)da = [\beta(a)\Phi_y(a,y)]_{\underline{a}}^{\overline{a}} - \int_a \dot{\beta}(a)\Phi_y(a,y)da, \text{ with } [\beta(a)\Phi_y(\cdot)]_{\underline{a}}^{\overline{a}} = 0.$$

$AB(C+E)$, where A is an individual insurance effect, B is a social insurance effect and $C+E$ is an enhanced efficiency or incentive effect.

At the other extreme, in the maximin case, the government objective function is $W = v(\underline{a})$, so $\beta(\underline{a}) = 1$ and $\beta(a) = 0$ for $a > \underline{a}$. The term D then becomes $\phi_y(\underline{a}, y)$. Given that the distribution is truncated so there is a maximum income that households earn given their effort, then $D = 0$ when $\phi(\underline{a}, y) = 0$, that is, at income levels above the maximum that can be attained by type-$\underline{a}$ households. The manner in which D changes as income changes before that maximum is reached depends on how $\phi_y(\underline{a}, y)$ changes with y. As shown in the Appendix, if the density function $G_y(y, z)$ is single-peaked for given z, $\phi_y(\underline{a}, y)$ might be expected to rise starting at the lowest-income level and may eventually fall after the mode of density distribution of the type-$\underline{a}$ worker. In this case, the term D will influence the marginal tax rate to fall starting at the lowest income level, and perhaps rise later on. In the standard case with no uncertainty and under reasonable assumptions about the ability distribution, the marginal tax rate tends to fall throughout the income distribution under a maximin social welfare function (Boadway and Jacquet 2008).

The term $E = \int_a \theta(a)h(y, z(a))\phi_y(a, y)da$ captures the change in the value of the weight on the incentive effect when y changes, given that a higher y contains a higher proportion of high-ability workers. As shown in the Appendix, if the density function $G_y(y, z)$ is single-peaked in y for given z, which is reasonable, $h(y, z(a))$ will be negative for low values of y and positive at high values, while $\phi_y(a, y)$ will be positive for low y and negative for high y. Their product will tend to be negative, implying, since $\theta(a) > 0$, that E will tend to be negative.

Given that A and B are both positive, the sign of the marginal tax rate will depend on the relative magnitudes of C, D and E, where the first two terms represent the classic trade-off between incentives and equity. The equity effect, $D < 0$, tends to increase the marginal tax rate relative to the identical-worker case, and this is reinforced by $E < 0$. There is no guarantee that the sum of the terms C, D and E will be positive at all income levels. That is, $T'(y)$ could be greater than one, so $c'(y) < 0$, for at least some values of y. In an interior solution to the consumer's optimal choice of effective labor supply z effort, the average value of $c'(y)$ must be less than one as we have seen. But neither that nor the second-order condition for the choice of $z(a)$ precludes it from being greater than one for some income levels. For example, if $\beta(a)$ is sufficiently high at low ability-levels and h_y is sufficiently low, D might be high enough relative to C to make $C + D + E < 0$. A simple two-type example is given in the Appendix that shows that the marginal tax rate can be greater than unity.

In the standard model, a well-known result is that as long as the distribution of skills is bounded, the marginal tax rate at the top is zero (Seade 1977). Consider the top of the income distribution in this model. Let $\overline{y}$ be the maximum level of income that can be earned, given the effective labor supply $z(\overline{a})$. Given that the distribution $G(y,z)$ is common for all ability-types and that effective labor supply is increasing in ability by Lemma 1, the income $\overline{y}$ will only be earned by the type-$\overline{a}$ person. Then, $\phi(\overline{a},\overline{y}) = 1$ and $\phi_y(\overline{a},\overline{y}) = 0$, while $\phi(a,\overline{y}) = 0$ for all $a < \overline{a}$. Using (20), the marginal tax rate at the top becomes:

$$T'(\overline{y}) = 1 - \Big(\frac{-u''\big(c(\overline{y})\big)}{u'\big(c(\overline{y})\big)}\Big)^{-1} \cdot \frac{u'(c(\overline{y}))}{\lambda} \cdot \theta(\overline{a})h_y\big(\overline{y}, z(\overline{a})\big) \tag{21}$$

This has the same interpretation as (17) above. Following the same reasoning, $T'(\overline{y}) < 1$, but it can take a positive or negative value.

The same argument can be applied at the bottom. As Seade (1977) has shown, if there is no bunching at the bottom, the marginal tax rate will be zero there as well (unless the social welfare function is maximin). Bunching at the bottom can occur either because a non-negative labor supply constraint is binding for low-ability workers, or because the second-order incentive constraints are binding. In the absence of bunching at the bottom, the lowest income level, $\underline{y}$, will be earned by those of ability $\underline{a}$. An equation similar to (21) will apply for $a = \underline{a}$. Again, the marginal tax rate will be less than 100 percent, but it can be positive or negative.

We can summarize these results for the heterogeneous-household case in the following proposition.

Proposition 2. *When households are ex ante heterogeneous, the marginal tax rate is given by* (20). *Given Assumptions* 1–3, *and assuming the distribution of* $G(y,z)$ *is single-peaked in* y, A, B *and* C *are all positive,* D *is non-positive and* E *is generally negative. Marginal tax rates at lower-income levels will tend to be higher than in the case of ex ante identical households, though not necessarily at higher-income levels. The marginal tax rates at the top and bottom can be positive or negative. The marginal tax rate could exceed 100 percent at some income levels.*

5 An Alternative Formulation

Further insight can be obtained by reformulating the government's problem in way that is closer to the standard deterministic approach following Mirrlees (1971). To do so, we transform the incentive constraint (2) as follows. Differentiate the definition of utility (1) with respect to a to obtain:

$$\dot{v}(a) = \frac{z(a)}{a^2} + \Big(\int_y u\big(c(y)\big)G_{yz}\big(y, z(a)\big)dy - \frac{1}{a}\Big)\dot{z}(a)$$

Since the second term is zero by (2), $v(a)$ satisfies the following equation of motion:

$$\dot{v}(a) = \frac{z(a)}{a^2} \tag{22}$$

We use this as our incentive constraint rather than (2). It is analogous to the first-order incentive constraint in the standard Mirrlees optimal income tax problem The endpoints of this differential equation, $v(\underline{a})$ and $v(\overline{a})$, are constrained by:

$$v(\underline{a}) = \int_y u\big(c(y)\big) G_y\big(y, z(\underline{a})\big) dy - \frac{z(\underline{a})}{\underline{a}} \quad \text{and} \quad v(\underline{a}) = \int_y u\big(c(y)\big) G_y\big(y, z(\overline{a})\big) dy - \frac{z(\overline{a})}{\overline{a}}$$

where $z(\underline{a})$ and $z(\overline{a})$ are chosen by ability-types $\underline{a}$ and $\overline{a}$ to satisfy (2). The Lagrangian function (7) can therefore be rewritten as follows, using the fact that integration by parts implies $-\int_a \pi(a)\dot{v}(a)da = \int_a \dot{\pi}(a)v(a)da + \pi(\underline{a})v(\underline{a}) - \pi(\overline{a})v(\overline{a})$:

$$\begin{aligned}\mathcal{L} = \int_a \beta(a)v(a)f(a)da + \lambda \int_y \big(y - c(y)\big)\Big(\int_a G_y\big(y, z(a)\big)f(a)da\Big)dy \\ + \int_a \hat{\mu}(a)\Big(\int_y u\big(c(y)\big)G_y\big(y, z(a)\big)dy - \frac{z(a)}{a} - v(a)\Big)da \qquad (23) \\ +\underline{\rho}\Big(\int_y u\big(c(y)\big)G_y\big(y, z(\underline{a})\big)dy - \frac{z(\underline{a})}{\underline{a}} - v(\underline{a})\Big) + \overline{\rho}\Big(\int_y u\big(c(y)\big)G_y\big(y, z(\overline{a})\big)dy - \frac{z(\overline{a})}{\overline{a}} - v(\overline{a})\Big) \\ + \int_a \pi(a)\frac{z(a)}{a^2}da + \int_a \dot{\pi}(a)v(a)da + \pi(\underline{a})v(\underline{a}) - \pi(\overline{a})v(\overline{a})\end{aligned}$$

The control variables are now $z(a)$ (including $z(\underline{a})$ and $z(\overline{a})$), $c(y)$, $v(\underline{a})$ and $v(\overline{a})$, while $v(a)$ is a state variable. The first-order conditions on these variables are given by:

$$\lambda f(a) \int_y \big(y - c(y)\big)G_{yz}\big(y, z(a)\big)dy + \frac{\pi(a)}{a^2} = 0 \qquad (24)$$

$$\begin{aligned}-\lambda \int_a G_y\big(y, z(a)\big)f(a)da + u'\big(c(y)\big)\int_a \hat{\mu}(a)G_y\big(y, z(a)\big)da \\ +u'\big(c(y)\big)\Big(\underline{\rho}G_y\big(y, z(\underline{a})\big) + \overline{\rho}G_y\big(y, z(\overline{a})\big)\Big) = 0 \qquad (25)\end{aligned}$$

$$-\underline{\rho} + \pi(\underline{a}) = -\overline{\rho} - \pi(\overline{a}) = 0 \qquad (26)$$

$$\beta(a)f(a) - \hat{\mu}(a) + \dot{\pi}(a) = 0 \qquad (27)$$

The following lemma is proven in the Appendix.

Lemma 2. *The problems characterized by the Lagrangian functions* (7) *and* (23) *are equivalent when* $\pi(a) = -\gamma(a)/\dot{z}(a)$ *and* $\mu(a) = \hat{\mu}(a) - \dot{\pi}(a)$.

To interpret these necessary conditions, it is useful to begin with the case where there is no risk.

The Case with No Risk

This alternative formulation reduces to the standard optimal nonlinear income tax case in the Mirrlees (1971) tradition when risk vanishes. In the absence of risk, effective labor supply and earnings are identical, so $y(a) = z(a)$, and consumption for a type-a person can be written $c(a)$. The utility function (2) simplifies to $v(a) = u\big(c(a)\big) - z(a)/a$, and the budget constraint (5) may be written $\int_a \big(z(a) - c(a)\big) f(a)da = 0$. The incentive constraint (22) still applies.

The first-order condition (24) on $z(a)$ may be written as follows:

$$-\lambda f(a) \int_y T'(y) G_z\big(y, z(a)\big)dy + \frac{\pi(a)}{a^2} = 0$$

In the absence of uncertainty, $y = z(a)$ and $G_z = -1$, so this equation simplifies to $\lambda f(a)T'\big(z(a)\big) + \pi(a)/a^2 = 0$, or:

$$T'\big(y(a)\big) = 1 - \frac{1}{au'\big(c(a)\big)} = -\frac{1}{\lambda}\frac{\pi(a)}{a^2 f(a)} \tag{28}$$

where the middle term follows from the individual's optimal choice of $z(a)$. This is analogous to the marginal tax rate — or the tax wedge — for this quasilinear-in-labor case derived in Boadway, Cuff and Marchand (2000).[9] The marginal tax rate is decreasing in the skill level a and in the value of skills $af(a)$, and is increasing in the shadow price of the incentive constraint $\pi(a)$, which is negative by Lemma 2, given that $\gamma(a) > 0$. The value of $\pi(a)$ is determined by the first-order condition on $v(a)$, (27), which says $-\dot{\pi}(a) = \beta(a)f(a) - \hat{\mu}(a)$. It captures the equity effect of the tax system.

The first-order condition on $c(y)$, (25), can be written as follows for $y(a)$ in the interior, using the fact that $G_y\big(y(a), z(a)\big) = 1$ and $G_y\big(y(a), z(\underline{a})\big) = G_y(y(a), z(\overline{a})) = 0$:

$$-\lambda f(a) + \hat{\mu}(a)u'\big(c(a)\big) = 0 \tag{29}$$

Taking the limit of (29) as $a \to \overline{a}$, we obtain $-\lambda f(\overline{a}) + \hat{\mu}(\overline{a})u'\big(c(\overline{a})\big) = 0$. However, evaluating (25) at $\overline{a}$, using (26), yields $-\lambda f(\overline{a}) + \hat{\mu}(\overline{a})u'(c(\overline{a})) - \pi(\overline{a})u'(c(\overline{a}))f(\overline{a}) = 0$. Therefore, $\pi(\overline{a}) = 0$. By a parallel argument, $\pi(\underline{a}) = 0$. Therefore, by (28), the marginal tax rates at the top and bottom are zero. Thus, not surprisingly, our problem with risk reduces to the no-risk problem when the risk vanishes. For future reference, (29) can be rewritten as follows:

$$T'\big(y(a)\big) = 1 - \frac{1}{au'\big(c(a)\big)} = 1 - \frac{\hat{\mu}(a)}{a\lambda f(a)} \tag{30}$$

The Case with Risk

When earnings are risky, so is the marginal tax rate an individual expects to face. The expected tax rate turns out to be a straightforward generalization of the tax rate when there is no risk. To see this, rewrite the first-order condition (24) on $z(a)$ as follows:

$$\int_y T(y) G_{yz}\big(y, z(a)\big)dy = -\int_y T'(y) G_z\big(y, z(a)\big)dy = -\frac{1}{\lambda}\frac{\pi(a)}{a^2 f(a)}$$

[9] Their equation (12) differs from (28) by having only a in the denominator rather than a^2. That is because they transform their utility function to $V(a) = av(a) = au(c) - z$. That implies that their $\pi(a)$ is equivalent to $\pi(a)/a$ in our formulation.

or, equivalently, (recalling the $G_z < 0$),

$$\mathrm{E}\big[T'(y)|a\big] = -\frac{1}{\lambda}\frac{\pi(a)}{a^2 f(a)} \tag{31}$$

This is a straightforward generalization of (28) to take account of risk. It says the expected marginal tax rate of a type-a person is $-\pi(a)/\big(\lambda a^2 f(a)\big)$, which has the same interpretation as above.[10] The expected marginal tax rates in (31) can follow different patterns as in the no-risk model. In the case of a maximin social welfare function, $\dot{\pi}(a) = \hat{\mu}(a) > 0$ by (27) since $\beta(a) = 0$ for $a > \underline{a}$. Therefore, given that $\pi(a) < 0$, the expected marginal tax rate will be declining in a, as in the standard model (Boadway and Jacquet 2008).

Equation (31) gives the expected marginal tax rate for a person of type a. We can also derive an expected tax wedge for a given income level y, which includes persons of different abilities. The first-order condition (25) on $c(y)$ can be written as follows, using (14) and (26) and the fact that $\int_a \phi(a,y)da = 1$:

$$\frac{1}{u'} = \int_a \frac{\hat{\mu}(a)}{\lambda f(a)}\phi(a,y)da - \frac{\pi(\overline{a})\overline{a}}{\lambda f(\overline{a})}\phi(\overline{a},y) + \frac{\pi(\underline{a})\underline{a}}{\lambda f(\underline{a})}\phi(\underline{a},y)$$

This can be rewritten as follows, after dividing by $\mathrm{E}[a|y]$:

$$1 - \frac{1}{\mathrm{E}[a|y]u'} = 1 - \int_a \frac{1}{\mathrm{E}[a|y]}\frac{\hat{\mu}(a)}{\lambda f(a)}\phi(a,y)da - \frac{\mathit{\Omega}(y)}{\mathrm{E}[a|y]\lambda} \tag{32}$$

where $\mathrm{E}[a|y] = \int_a \tilde{a}\phi(\tilde{a},y)d\tilde{a}$ and $\mathit{\Omega}(y) \equiv \pi(\underline{a})f(\underline{a})\phi(\underline{a},y)/f(\underline{a}) - \pi(\overline{a})\phi(\overline{a},y)/f(\overline{a})$.

As can be seen, (32) is a generalization of the no-risk tax wedge in (30) to the case where a given income can be earned by persons of differing ability. The lefthand side can be interpreted as the expected or average tax wedge at income level y, taking account that the size of the wedge varies with ability. The first term on the righthand side is analogous to the expected value of the righthand side of (30), while the term involving $\mathit{\Omega}(y)$ takes account of the end-point conditions. It applies only when uncertainty is present, since $\pi(\underline{a}) = \pi(\overline{a}) = 0$ in the no-risk case, so $\mathit{\Omega}(y) = 0$. With uncertainty, the sign of $\mathit{\Omega}(y)$ is ambiguous since $\pi(a)$ is non-positive by Lemma 2. For those values of y such that $\phi(\overline{a},y) = 0$, $\mathit{\Omega}(y) \leqslant 0$, which works to raise the marginal tax wedge, while $\mathit{\Omega}(y) \geqslant 0$ for y such that $\phi(\underline{a},y) = 0$.

We can summarize these results in the following proposition.

Proposition 3. *Expected tax wedges under uncertainty are generalizations of the no-risk marginal tax rate as follows:*

[10] The sign of $\pi(a)$ follows from (24), $\int_y T(y)G_{yz}(y,z(a))dy = -\pi(a)/(\lambda a^2 f(a))$. Assuming that increases in $z(a)$ are revenue-enhancing, this implies that $\pi(a) < 0$ in the interior of the ability distribution.

i. The expression for expected marginal tax rate for a type-a person, $\mathrm{E}\big[T'(y)|a\big]$*, is analogous to the no-risk marginal tax rate,* $T'(y(a))$*.*

ii. The expression for the expected tax wedge for earnings y*,* $1 - 1/\mathrm{E}[a|y]u'$*, is analogous to the expected value of the no-risk marginal tax rate, taking account of the possibility of a non-zero expected tax wedge at the endpoints.*

Finally, as shown in the Appendix, further insight into the expected tax wedge at a given income level is found by manipulating (32) to obtain:

$$1 - \frac{1}{\mathrm{E}[a|y]u'\big(c(y)\big)} = \frac{1}{\lambda}\Bigg(\underbrace{\frac{1}{\mathrm{E}[a|y]}\mathrm{E}\Big[\frac{\pi(a)}{f(a)}h\big(y, z(a)\big)\dot{z}(a)\Big|y\Big]}_{I} + \underbrace{\lambda - \frac{\mathrm{E}[\beta(a)|y]}{\mathrm{E}[a|y]}}_{II} - \underbrace{\frac{\mathit{\Omega}(y)}{\mathrm{E}[a|y]}}_{III}\Bigg) \tag{33}$$

The first term, I, is an incentive effect. Given that $\pi(a) < 0$ and $\dot{z}(a) > 0$, it will be negative if $h\big(y, z(a)\big) > 0$, and vice versa. As we argued above, $h(\cdot)$ will tend to be negative where incomes are low, and positive above that. Thus, at low-income levels, $I > 0$, so this incentive effect will tend to increase the tax wedge. At high-income levels, it will lower it. The second term, II, is an equity effect. It will be larger for higher levels of income, so will tend to cause the tax wedge to increase with income. It could take on a negative value at low-income levels. Thus, generally the terms I and II work in opposite directions. As mentioned, the term III takes account of endpoint conditions. It tends to be negative for low-income levels and positive at high incomes.

The Effect of an Increase in Risk

Eaton and Rosen (1980a) studied whether the addition of uncertainty increased or decreased progressivity, and found it to be ambiguous even in their simple setting with two ability-types and linear taxation. Analyzing the effects of an increase in uncertainty in our context would be much more complicated. We can, however, derive the first-order effect of an increase in uncertainty starting with the no-risk optimum.

In the absence of uncertainty, the marginal tax rate at income level $y(a)$, denoted by $T_0'\big(y(a)\big)$, can be written, using (30), as

$$T_0'\big(y(a)\big) = 1 - \frac{\hat{\mu}(a)}{a\lambda f(a)}$$

Suppose we hold the values of all Lagrange multipliers constant at their no-risk levels, and introduce a small amount of risk. Then, since $\mathit{\Omega}(y) = 0$ in the no-risk optimum, (32) may be written

$$1 - \frac{1}{\mathrm{E}[a|y]u'\big(c(y)\big)} = \int_a \frac{\tilde{a}}{\mathrm{E}[a|y]} T_0'(\tilde{a})\phi(\tilde{a}, y)d\tilde{a} \tag{34}$$

Take a first-order Taylor approximation of $T_0'(\tilde{a})$ around $\mathrm{E}[a|y]$ to yield:

$$T_0'(\tilde{a}) \approx T_0'\big(\mathrm{E}[a|y]\big) + T_0''\big(\mathrm{E}[a|y]\big)\big(\tilde{a} - \mathrm{E}[a|y]\big)$$

Substituting this into (34), we obtain:

$$1 - \frac{1}{\mathrm{E}[a|y]u'\big(c(y)\big)} \approx T_0'\big(\mathrm{E}[a|y]\big) + \frac{T_0''\big(\mathrm{E}[a|y]\big)}{\mathrm{E}[a|y]} \int_a \tilde{a}\big(\tilde{a} - \mathrm{E}[a|y]\big)\phi(\tilde{a}, y)d\tilde{a}$$

$$= T_0'\big(\mathrm{E}[a|y]\big) + T_0''\big(\mathrm{E}[a|y]\big)\frac{\mathrm{Var}[a|y]}{\mathrm{E}[a|y]}$$

The following proposition follows immediately.

Proposition 4 *Let $a(y) \equiv \mathrm{E}[a|y]$ where $a(y)$ is the inverse of $y(a)$ in the absence of risk. Introducing a small amount of risk while holding the no-risk Lagrange multipliers fixed raises the expected tax wedge if and only if $T_0''\big(a(y)\big) > 0$.*

Of course, this result is both local and approximate since it is derived holding the Lagrange multipliers constant.

6 The Atkinson-Stiglitz Theorem

Suppose we now allow for two consumer goods in the model. Let $c(y)$ be disposable income, which can be allocated to two goods, x and q, after the state of the world is revealed. Let ex post utility from these two goods be given by $b(x, q)$. The overall utility function is still quasilinear in labor, so the weak separability condition required for the Atkinson and Stiglitz (1976) Theorem in the standard model is satisfied. Producer prices for the two goods are unity, and a tax at the rate τ can be imposed on good x, so its consumer price is $1 + \tau$.[11]

Consider first the ex post problem of a given household. Once the state is revealed, disposable income is $c(y)$, and the household's budget is $c(y) = (1 + \tau)x + q$. The household chooses x to maximize $b\big(x, c(y) - (1 + \tau)x\big)$. The solution to this problem is $x\big(c(y), \tau\big)$, and the indirect ex post utility function is $u\big(c(y), \tau\big)$. The envelope theorem gives:

$$u_c\big(c(y), \tau\big) = b_q(\cdot), \quad u_\tau\big(c(y), \tau\big) = -b_q(\cdot)x = -u_c\big(c(y), \tau\big)x \tag{35}$$

Revising the Lagrangian (7) for the government to take account of the possible tax on x, we have:

$$\mathcal{L} = \int_a \beta(a)v(a)f(a)da + \lambda \int_y \Big(y - c(y) + \tau x\big(c(y), \tau\big)\Big)\Big(\int_a G_y\big(y, z(a)\big)f(a)da\Big)dy$$

$$+ \int_a \mu(a)\Big(\int_y u(c(y), \tau)G_y\big(y, z(a)\big)dy - \frac{z(a)}{a} - v(a)\Big)da + \int_a \gamma(a)\Big(\int_y u(c(y), \tau)G_{yz}\big(y, z(a)\big)dy - \frac{1}{a}\Big)da$$

[11] We can dispense with a tax on good q since proportional taxes on x and q are equivalent to a proportional tax on income so can be absorbed into the income tax. A non-negative value of τ can be interpreted as a differential set of commodity taxes on x and q.

Suppose τ is given, and let $W(\tau)$ be the value function of the solution to this problem for the optimal income tax. Apply the envelope theorem to $W(\tau)$ at $\tau = 0$ and using (35) yields:

$$\frac{dW}{d\tau}\Big|_{\tau=0} = \lambda \int_y x\Big(\int_a G_y f(a)da\Big)dy - \int_a \mu(a)\Big(\int_y u_c x G_y dy\Big)da - \int_a \gamma(a)\Big(\int_y u_c x G_{yz} dy\Big)da \quad (36)$$

The first-order condition on $c(y)$ can be written:

$$\lambda \int_a G_y f(a)da + \int_a \mu(a) u_c G_y da + \int_a \gamma(a) u_c G_{yz} da = 0 \quad (37)$$

Eqs. (36) and (37) imply that

$$\frac{dW}{d\tau}\Big|_{\tau=0} = 0$$

Therefore, the following proposition applies.

Proposition 5. *The Atkinson-Stiglitz Theorem applies if goods purchases are made after earnings are known.*

We have assumed that, although households supply labor before uncertainty is resolved, their goods' purchases are chosen ex post.[12] Goods of a durable nature, such as housing, might have to be chosen before earnings are known. Cremer and Gahvari (1995, 1999) have studied this problem in a setting in which individuals are ex ante identical and subject to wage rate uncertainty, and choose labor after their wage rate is revealed. They show that if durable goods must be purchased before wage rates are revealed, the Atkinson-Stiglitz Theorem does not hold. When the income tax applies optimally to ex post income, taxing durable goods at preferential rates will be welfare-improving.

To investigate this case in our context, suppose x is a durable good and must be purchased before earnings are revealed. Good q continues to be purchased after y is known. Now, the individual must choose both z and x ex ante. Using the same notation as above, the value of transformed expected utility for a type-a person is now given by $v(a) = \int_y b\big(x, c(y) - (1+\tau)x\big)G_y(y,z)dy - z/a$. The choice of z satisfies the analog of (2) as before, while optimal x satisfies the first-order condition from maximizing $v(a)$:

$$\int_y b_x(x, c(y) - (1+\tau)x)G_y(y,z)dy - (1+\tau)\int_y b_q(x, c(y) - (1+\tau)x\big)G_y(y,z)dy = 0 \quad (38)$$

Let $x(a)$ be the solution to (38), and write indirect ex post utility as $u\big(c(y), \tau, x(a)\big)$. Assuming that the second-order conditions for the choice of x are satisfied, the following is shown in the Appendix.

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[12] Cremer and Gahvari (1999) refer to this as the case where households can commit to labor supply, but not to goods purchases.

Lemma 3. *Assuming the second-order condition for the worker's choice of $x(a)$ is satisfied, $\dot{x}(a) > 0$.*

In solving the government's problem, we treat $x(a)$ as a control variable and add as a constraint (38), or equivalently, $\int_y u_x\big(c(y),\tau,x(a)\big)G_y\big(y,z(a)\big)dy = 0$. The Lagrangian expression can be written:

$$\mathcal{L} = \int_a \beta(a)v(a)f(a)da + \lambda\Big(\int_y \big(y-c(y)\big)\Big(\int_a G_y\big(y,z(a)\big)f(a)da\Big)dy + \tau\int_a x(a)f(a)da\Big)$$
$$+\int_a \mu(a)\Big(\int_y u\big(c(y),\tau,x(a)\big)G_y\big(y,z(a)\big)dy - \frac{z(a)}{a} - v(a)\Big)da$$
$$+\int_a \gamma(a)\Big(\int_y u\big(c(y),\tau,x(a)\big)G_{yz}\big(y,z(a)\big)dy - \frac{1}{a}\Big)da + \int_a \delta(a)\Big(\int_y u_x\big(c(y),\tau,x(a)\big)G_y\big(y,z(a)\big)dy\Big)da$$

The first-order conditions with respect to $x(a)$ and $c(y)$are as follows:

$$\lambda\tau f(a) + \gamma(a)\int_y u_x G_{yz}dy + \delta(a)\int_y u_{xx}G_y dy = 0 \tag{39}$$

$$-\lambda\int_a G_y f(a)da + \int_a \mu(a)u_c G_y da + \int_a \gamma(a)u_c G_{yz}da + \int_a \delta(a)u_{xc}G_y da = 0 \tag{40}$$

Using partial integration and the definition of $\theta(a)$ in (11), (39) can be written:[13]

$$\Big(\lambda\tau + \theta(a)\mathrm{Cov}\big[u_x, h(y,z(a))\big]\Big)f(a) + \delta(a)\int_y u_{xx}G_y dy = 0 \tag{41}$$

From this we infer the following lemma.[14]

Lemma 4. *Given $\theta(a) > 0$ by Assumption 3, $\delta(a) > 0$ at $\tau = 0$.*

That is, the incentive constraint on $x(a)$ is binding for all ability-types.

The effect of a change in τ on the value of the Lagrangian expression is:

$$\frac{\partial\mathcal{L}}{\partial\tau} = \lambda\overline{x}(a) + \int_a \mu(a)\Big(\int_y u_\tau G_y dy\Big)da + \int_a \gamma(a)\Big(\int_y u_\tau G_{yz}dy\Big)da + \int_a \delta(a)\Big(\int_y u_{x\tau}G_y dy\Big)da \tag{42}$$

where $\overline{x} = \int_a x(a)f(a)da$. Using the first-order conditions (39) and (40) on $x(a)$ and $c(y)$ and evaluating (42) at $\tau = 0$, we obtain the following proposition, as shown in the Appendix.

Proposition 6. *The welfare effect of introducing a tax on the durable good is given by:*

$$\frac{\partial\mathcal{L}}{\partial\tau}\Big|_{\tau=0} = -\underbrace{\mathrm{Cov}\big[x(a),\beta(a)\mathrm{E}_y[u_c|a]\big]}_{\text{equity}} - \underbrace{\mathrm{Cov}\Big[x(a),\theta(a)\mathrm{E}_y\Big[\frac{du}{dm}h(y,z(a))\Big|a\Big]\Big]}_{\text{incentive}} \gtreqless 0$$

---

[13] That is, $\int_y u_x G_{yz}dy = \int_y u_x h G_y dy = \mathrm{E}[u_x h|a] = \mathrm{Cov}[u_x, h]$ since $\mathrm{E}[u_x|a] = \mathrm{E}[h|a] = 0$.

[14] By Assumption 1, $h(y) > 0$, and $du_x/dy = u_{xq}c'(y) > 0$. Since both h_y and u_x are increasing in y, $\mathrm{Cov}[u_x, h] > 0$, the lemma follows.

where
$$\frac{d}{dm}\Big(u\big(c(y)+m,\tau,x(a)\big)\Big) = u_c + u_x\frac{dx(a)}{dm}$$
represents the change in ex post utility from an increase in income m in all states.

The first term is an equity effect. Since $x(a)$ is increasing in a and $\beta(a)$ is decreasing in a, the equity term will be negative if $\mathrm{E}_y[u_c|a]$ is decreasing in a, thereby tending to make τ negative. The second term involves an incentive effect and can be positive or negative. Thus, the sign of the optimal tax on x is ambiguous.

Assume, following Cremer and Gahvari (1995, 1999), that individuals are ex ante identical with $a = 1$. In this case, $x(a) = \overline{x}$ so the covariance terms in Proposition 6 are both zero. The following corollary is apparent.

Corollary 6.1. *If individuals are ex ante identical, the optimal tax rate on the durable good is zero, $\tau = 0$.*

This result differs from that of Cremer and Gahvari (1995), who find that the durable good should be taxed preferentially compared with goods purchased after the wage rate becomes known. The reason is rather subtle, and is as follows. In Cremer and Gahvari, labor supply is chosen after wage rates are revealed. Persons who plan to mimic low-wage workers in the event that they turn out to be high-wage will demand less of the durable good than persons who do not intend to mimic. In these circumstances, subsidizing purchases of the durable good makes it less attractive to mimic, so the incentive constraint is relaxed. In our model where labor is chosen before uncertainty is resolved, there is no binding incentive constraint that precludes high-wage persons from pretending to be low-wage persons so this argument does not arise. Moreover, there is no redistributive reason for taxing the durable good since all persons are ex ante identical.

7 Concluding Remarks

In this paper, we have provided a fairly general treatment of optimal income taxation when differences in income can be due to both ability differences and uncertainty (luck). We derived a general formula for the marginal income tax rate and disaggregated its determinants into factors involving incentive, equity and insurance effects. The cases of no uncertainty and no ability differences came out as special cases. We also showed that the Atkinson-Stiglitz Theorem continues to be satisfied when earnings are uncertain, as long as goods' purchases can be delayed until after the state is revealed.

Our analysis was facilitated by some simplifying assumptions. Preferences were assumed to be quasilinear in leisure, which eliminates income effects in the demand for consumption. The concept of risk-aversion is transparent in this case since it depends only on consumption and not on labor supply.

We also assumed that labor supply varied only along the intensive margin. Introducing an extensive margin along the lines of Diamond (1980) and Saez (2002) would involve restricting the effective labor supply in each job and focusing on participation and job choice decisions, but it would be a useful extension.

We assumed that the same earnings distribution function applied to the effective labor supply of all workers regardless of their ability. Allowing earnings risk to vary with ability would be an interesting extension, although it is not clear what one would assume about the relation between risk and ability. One could also let risk aversion vary with ability as in Eaton and Rosen (1980a), though again it is not obvious how attitudes to risk would be expected to vary with ability.

Finally, we assumed that ability was exogenous. There is a substantial literature on ability being affected by human capital or education investments, and these naturally raise issues of uncertainty. Seminal papers include Eaton and Rosen (1980b) and Hamilton (1987), and more recent papers include Anderberg and Andersson (2003), Da Costa and Maestri (2007) and Jacobs, Schindler and Yang (2010). For a fuller summary of these, see Schindler and Yang (2010).

Appendix

I. Proofs

Proof of Proposition 1

From (12), integrating over income y yields:

$$\lambda \int_y \frac{1}{u'\big(c(y)\big)} \Big(\int_a G_y(\cdot) f(a) da \Big) dy = \int_a \int_y \big(\beta(a) G_y(\cdot) + \theta(a) G_{yz}(\cdot) \big) f(a) dady$$

or, since $\int_a G_y f(a) da$ is the number of persons earning income y,

$$\lambda \mathrm{E}\Big[\frac{1}{u'\big(c(y)\big)} \Big] = \int_a \beta(a) \Big(\int_y G_y(\cdot) dy \Big) f(a) da + \int_a \theta(a) \Big(\int_y G_{yz}(\cdot) dy \Big) f(a) da$$

$$= \int_a \beta(a) f(a) da \quad \text{since} \quad \int_y G_y dy = 1 \text{ and } \int_y G_{yz} dy = 0.$$

Since $\int_a \beta(a) f(a) da = 1$, (13) follows.

The Value of D in the Maximin Case

With a maximin social welfare function, $D = \phi_y(\underline{a}, y)$. Using (14), we have:

$$\phi(\underline{a}, y) = \frac{G_y(y, z(\underline{a})) f(\underline{a})}{\int_a G_y(y, z(\tilde{a})) f(\tilde{a}) d\tilde{a}} = \frac{G_y f(\underline{a})}{\mathrm{E}\big[G_y | y\big]}$$

Differentiating this with respect to y, we have:

$$\phi_y(\underline{a}, y) = \frac{G_{yy}(y, z(\underline{a})) f(\underline{a})}{\mathrm{E}\big[G_y | y\big]} - \frac{G_y(y, z(\underline{a})) f(\underline{a})}{\big(\mathrm{E}\big[G_y | y\big]\big)^2} \mathrm{E}\big[G_{yy} | y\big]$$

$$= \phi(\underline{a}, y) \left(\frac{G_{yy}(y, z(\underline{a}))}{G_y(y, z(\underline{a}))} - \frac{\mathrm{E}\big[G_{yy} | y\big]}{\mathrm{E}\big[G_y | y\big]} \right)$$

If we assume that $G(y, z(\underline{a}))$ is single-peaked in y for given $z(\underline{a})$, $G_{yy}(y, z(\underline{a}))/G_y(y, z(\underline{a}))$ is positive for low values of y, and becomes negative after the mode. Since $\mathrm{E}\big[\phi(y, a) | y\big] = 1$, we have $\mathrm{E}\big[\phi_y(a, y) | y\big] = 0$. This would suggest that $\phi_y(\underline{a}, y)$ takes a positive and increasing value for low y. It may eventually begin to fall after the mode depending on the value of income at which $\mathrm{E}\big[G_{yy} | y\big] / \mathrm{E}\big[G_y | y\big]$ peaks.

The Sign of E

Recall that $E = \int_a \theta h \phi_y da$. Given that $\theta(a)$ is positive, the sign of E will depend on the signs of $h(y, z(a))$ and $\phi_y(a, y)$, where by (4) and (14):

$$h(y, z) = \frac{G_{yz}(y, z)}{G_y(y, z)}, \qquad \phi(a, y) = \frac{G_y(y, z(a)) f(a)}{\int_a G_y(y, z(\tilde{a})) f(\tilde{a}) d\tilde{a}} = \frac{G_y f(a)}{\mathrm{E}\big[G_y | y\big]}$$

Given $z(a)$, we assume that G_y is single-peaked. An increase in $z(a)$ causes the density function G_y to shift right. Therefore, G_{yz} is negative for low values of y and then becomes positive for higher values, implying that $h(y, z(a))$ is also negative for low y and positive for higher values.

Next, differentiating $\phi(a, y)$ with respect to y, we obtain:

$$\phi_y(a, y) = \frac{G_{yy} f(a)}{\mathrm{E}\big[G_y|y\big]} - \frac{G_y f(a)}{\big(\mathrm{E}\big[G_y|y\big]\big)^2} \mathrm{E}\big[G_{yy}|y\big] = \phi(a, y) \left(\frac{G_{yy}}{G_y} - \frac{\mathrm{E}\big[G_{yy}|y\big]}{\mathrm{E}\big[G_y|y\big]} \right)$$

We know that $\mathrm{E}\big[\phi_y(a, y)|y\big] = 0$. Since we assume that $G(y, a)$ is single-peaked in y for given a, this would suggest that ϕ_y takes a positive value for low y and a negative value for high y. Therefore, E, which involves the product of $h(y, z(a))$ and $\phi_y(a, y)$ will tend to be negative over the range of a, leading to a presumption that E will be negative.

Proof of Lemma 2

Substituting (5) into (9) and using $\pi(a) = -\gamma(a)/\dot{z}(a)$ yields (24).

Substituting $\mu(a) = \hat{\mu}(a) - \dot{\pi}(a)$ into (27) yields (8).

Substituting $\mu(a) = \hat{\mu}(a) - \dot{\pi}(a)$ and (26) into (25) yields:

$$-\lambda \int_a G_y f(a) da + u' \int_a \big(\mu(a) + \dot{\pi}(a)\big) G_y da + u' \Big(\pi(\underline{a}) G_y(\underline{a}) - \pi(\overline{a}) G_y(\overline{a})\Big) = 0$$

The term involving $\dot{\pi}(a)$ may be partially integrated to yield:

$$\int_a \dot{\pi}(a) G_y da = -\int_a \pi(a) G_{yz} \dot{z} da + \big[\pi(a) G_y\big]_{\underline{a}}^{\overline{a}}$$

Using $\pi(a) = -\gamma(a)/\dot{z}(a)$, and substituting this into the above equation yields (10).

Derivation of (33)

Using (27), (32) may be written:

$$\begin{aligned} 1 - \frac{1}{\mathrm{E}[a|y] u'} &= 1 - \int_a \frac{1}{\mathrm{E}[a|y]} \Big(\frac{\beta(a) f(a) + \dot{\pi}(a)}{\lambda f(a)} \Big) \phi(a, y) da - \frac{\Omega(y)}{\mathrm{E}[a|y]\lambda} \\ &= -\int_a \frac{1}{\mathrm{E}[a|y]} \frac{\dot{\pi}(a)}{\lambda f(a)} \phi(a, y) da + 1 - \frac{\mathrm{E}[\beta(a)|y]}{\mathrm{E}[a|y]\lambda} - \frac{\Omega(y)}{\mathrm{E}[a|y]\lambda} \end{aligned}$$

Partially integrating the first term and using (14),

$$1 - \frac{1}{\mathrm{E}[a|y] u'} = \int_a \frac{1}{\mathrm{E}[a|y]\lambda} \frac{\pi(a) G_{yz} \dot{z}(a)}{\mathrm{E}[G_y|y]} da + 1 - \frac{\mathrm{E}[\beta(a)|y]}{\mathrm{E}[a|y]\lambda} - \frac{\Omega(y)}{\mathrm{E}[a|y]\lambda}$$

Using (4), this may be written:

$$1 - \frac{1}{\mathrm{E}[a|y] u'} = \int_a \frac{1}{\mathrm{E}[a|y]\lambda} \frac{\pi(a) h(y, z(a)) G_y \dot{z}(a)}{f(a) \mathrm{E}[G_y|y]} f(a) da + 1 - \frac{\mathrm{E}[\beta(a)|y]}{\mathrm{E}[a|y]\lambda} - \frac{\Omega(y)}{\mathrm{E}[a|y]\lambda}$$

which immediately reduces to (33).

Proof of Lemma 3

Differentiating (38) with respect to a yields:

$$\dot{x}(a)\int_y\Big(\frac{\partial\big(b_x-(1+\tau)b_q\big)}{\partial x}\Big)G_y dy+\dot{z}(a)\int_y\big(b_x-(1+\tau)b_q\big)G_{yz}dy=0$$

The term multiplying $\dot{x}(a)$ is negative by the second-order conditions, and $\dot{z}(a)>0$ by Lemma 1. The term multiplying $\dot{z}(a)$ may be written, using partial integration:

$$\int_y u_x G_{yz}dy=\big[u_x G_z\big]_{\underline{y}}^{\overline{y}}-\int_y\frac{du_x}{dy}G_z dy=-\int_y u_{xc}c'(y)G_z dy$$

Since $G_z<0$, $u_{xc}>0$ and $c'(y)>0$, the result follows.

Proof of Proposition 6

Integrating (40) over y, multiplying by $\overline{x}$ and using $\int_y G_y dy=1$:

$$-\lambda\overline{x}\int_a f(a)da+\int_a\mu(a)\Big(\int_y\overline{x}u_c G_y dy\Big)da+\int_a\gamma(a)\Big(\int_y\overline{x}u_c G_{yz}dy\Big)da$$

$$+\int_a\delta(a)\Big(\int_y\overline{x}u_{xc}G_y dy\Big)da=0$$

Combining this with (42) and using $u_\tau=-xu_c$, $u_{x\tau}=-xu_{xc}$ and $h=G_{yz}/G_y$:

$$\frac{\partial\mathcal{L}}{\partial\tau}=-\int_a\mu(a)(x-\overline{x})\Big(\int_y u_c G_y dy\Big)da-\int_a\gamma(a)(x-\overline{x})\Big(\int_y u_c h G_y dy\Big)da$$

$$-\int_a\delta(a)(x-\overline{x})\Big(\int_y u_{xc}G_y dy\Big)da$$

Since $\gamma(a)=\theta(a)f(a)$ by (11) and $\mu(a)=\beta(a)f(a)$ by the first-order condition with respect to $v(a)$, this may be written:

$$\frac{\partial\mathcal{L}}{\partial\tau}=-\mathrm{Cov}\big[x,\beta(a)\mathrm{E}_y[u_c|a]\big]-\mathrm{Cov}\big[x,\theta(a)\mathrm{E}_y[u_c h|a]\big]-\mathrm{Cov}\Big[x,\frac{\delta(a)}{f(a)}\mathrm{E}_y[u_{xc}|a]\Big]$$

At $\tau=0$, (41) can be written, using (11) and footnote 10,

$$\theta(a)f(a)\mathrm{E}\big[u_x h|a\big]=-\delta(a)\int_y u_{xx}G_y dy=-\delta(a)\mathrm{E}_y\big[u_{xx}|a\big]$$

Therefore, using this for $\delta(a)/f(a)$, we obtain:

$$\left.\frac{\partial\mathcal{L}}{\partial\tau}\right|_{\tau=0}=-\mathrm{Cov}\big[x,\beta(a)\mathrm{E}_y[u_c|a]\big]-\mathrm{Cov}\left[x,\theta(a)\Big(\mathrm{E}_y\big[u_c h|a\big]-\frac{\mathrm{E}\big[u_{xc}|a\big]}{\mathrm{E}_y\big[u_{xx}|a\big]}\mathrm{E}_y[u_x h|a]\Big)\right]$$

Eq. (38) may be written $\int_y u_x\big(c(y)+m,\tau,x\big)G_y dy = 0$. Differentiation yields

$$\frac{dx}{dm} = -\frac{\int u_{xc}G_y dy}{\int u_{xx}G_y dy} = -\frac{\mathrm{E}_y[u_{xc}|a]}{\mathrm{E}_y[u_{xx}|a]}, \qquad \text{so}$$

$$\begin{aligned}\left.\frac{\partial \mathcal{L}}{\partial \tau}\right|_{\tau=0} &= -\mathrm{Cov}\big[x,\beta(a)\mathrm{E}_y[u_c|a]\big] - \mathrm{Cov}\left[x,\theta(a)\Big(\mathrm{E}_y\big[u_c h|a\big] + \frac{dx}{dm}\mathrm{E}_y[u_x h|a]\Big)\right] \\ &= -\mathrm{Cov}\big[x,\beta(a)\mathrm{E}_y[u_c|a]\big] - \mathrm{Cov}\left[x,\theta(a)\Big(\mathrm{E}_y\Big[\frac{d}{dm}(u(c(y),\tau,x)h|a\Big]\Big)\right]\end{aligned}$$

II. Example where the Marginal Tax Rate Exceeds Unity

Assume two ability-types, $a_2 > a_1$. Let $\overline{y}_1$ be type-1's maximum income, so $y > y_1$ implies $G\big(y,z(a_1)\big) = 1$, $\phi(a_1,y) = 0$ and $\phi(a_2,y) = 1$. Eq. (15) can be written as:

$$\frac{\lambda}{u'\big(c(y)\big)} = \begin{cases} \sum_{i=1,2}\Big(\beta(a_i)+\theta(a_i)h\big(y,z(a_i)\big)\Big)\phi(a_i,y) & \text{if } y \leqslant \overline{y}_1 \\ \beta(a_2)+\theta(a_2)h\big(y,z(a_2)\big) & \text{if } y > \overline{y}_1 \end{cases} \tag{15$'$}$$

where $\beta(a_1) > \beta(a_2)$, and $\theta(a_i) > 0$ for $i = 1,2$ by Assumption 3. Assume that $h\big(\overline{y}_1,z(a_2)\big) < 0$ or $G_{yz}\big(\overline{y}_1,z(a_2)\big) < 0$, which implies that $h\big(\overline{y}_1,z(a_1)\big) > 0$ (since $\int h(y,z)G_y dy = 0$ and $h_y > 0$ by Assumption 1). Then,

$$\begin{aligned}\lim_{y\to\overline{y}_{1-}} \frac{\lambda}{u'\big(c(y)\big)} &= \lim_{y\to\overline{y}_{1-}} \sum_{i=1,2}\Big(\beta(a_i)+\theta(a_i)h\big(y,z(a_i)\big)\Big)\phi(a_i,y) \\ &> \beta(a_2)+\theta(a_2)h\big(\overline{y}_1,z(a_2)\big) = \lim_{y\to\overline{y}_{1+}} \frac{\lambda}{u'\big(c(y)\big)}\end{aligned}$$

Therefore,

$$\lim_{y\to\overline{y}_{1-}} c(y) \;>\; \lim_{y\to\overline{y}_{1+}} c(y), \quad \text{or} \quad \lim_{y\to\overline{y}_{1-}} \big(y-T(y)\big) \;>\; \lim_{y\to\overline{y}_{1+}} \big(y-T(y)\big)$$

This implies that $T'(y) > 1$ at $y = \overline{y}_1$. Note than $h\big(\overline{y}_1,z(a_2)\big) < 0$ is a sufficient condition for this demonstration, but it is not necessary. The marginal tax rate will still be greater than 100 percent as long as $\theta(a_1)h\big(\overline{y}_1,z(a_1)\big) > \theta(a_2)h\big(\overline{y}_1,z(a_2)\big)$. Furthermore, $\overline{y}_1$ could be endogenous and based on $z(a_1)$ chosen by type-1's in the optimum.

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