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Yosef Mealem
Shmuel Nitzan

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Abstract

The means of contest design may include differential taxation of the prize. This paper establishes that, given a revenue-maximizing contest designer who faces a balanced-budget constraint, the optimal taxation scheme corresponding to an all-pay auction is appealing in two senses. First, it ensures exceptional equitable final prize valuations. Second, it is effective; it yields total contestants' efforts that are larger than those obtained under almost any Tullock-type lottery. Furthermore, when a budget surplus is allowed, the superiority of optimal taxation under the APA is preserved in terms of equity and effectiveness relative to optimal taxation under any contest success function.

JEL-Code: D700, D720, D740, D780.

Keywords: contest design, revenue maximization, balanced-budget constraint, budget surplus, optimal differential taxation, endogenous stakes, all-pay auction, lottery.

Yosef Mealem
The School of Banking & Finance
Netanya Academic College
Netanya / Israel, 42365
yossime@bezeqint.net

Shmuel Nitzan
Department of Economics
Bar-Ilan University
Ramat Gan / Israel, 52900
nitzans@biu.ac.il

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1. Introduction

Applications of contest theory include promotional competitions, litigation, internal labor market tournaments, rent-seeking, R&D races, political and public policy competitions and sports, Konrad (2009), Congleton et al. (2008). Contest design may involve the endogenous determination of relevant institutional characteristics by contest designers; economic and political entrepreneurs who wish to maximize the total efforts made by the contestants. These characteristics may include various forms of discrimination between the contestants. In particular, discrimination can take the form of differential taxation of the contested prize.[1] Such taxation determines the actual stakes of the contestants from the prize, given their initial prize valuations. If an effort (revenue)-maximizing contest designer faces a balanced-budget constraint, then, by definition, if one contestant's possible winning of the prize is subjected to a tax, then the possible winning of the prize by another contestant must be associated with a negative tax, viz., the granting of a subsidy. In a two-contestant setting, optimal differential taxation (discrimination) may therefore result in an increase or a decrease in the gap between the contestants' stakes from the prize. The effect of the designer's preferred tax-subsidy scheme on the contestants' actual stakes from the prize hinges on the contest-success function (CSF) – the rule that determines the contestants' chances of winning the prize, given their exerted efforts in trying to win the prize.[2] In light of the existing results in the contest literature, see Konrad (2009), one might intuitively expect that equalization of stakes is always the optimal strategy for a revenue-maximizing contest designer. Such expectation is plausible because equal stakes imply maximal competition that apparently induces the largest contestants' efforts, as in Gradstein (1995).

The first objective of this paper is to show that this expectation is indeed realized when the CSF is the widely used all-pay auction (APA). However, as will be shown, the fulfillment of this expectation under an APA is the exception rather than the rule. That is, in general, this seemingly plausible expectation is not fulfilled. This is the case when the CSF is the most commonly assumed lottery, Tullock (1980), and, in particular, the simple lottery that will be used to diagrammatically illustrate our

[1] Alternative forms of discrimination via the control of the contest success function are examined in Clark and Riis (2000), Epstein et al. (2011a), (2011b), Franke (2007) , Franke et al. (2011) and Lien (1990).

[2] For a recent study on the meaning and rationalization of CSFs, see Corchon and Dahm (2010).

claim. Whereas stake equalization is optimal in the APA case, it is not optimal in the lottery case, although the optimal taxation scheme reduces the gap between the contestants' stakes from the prize, but does not eliminate it. The proof of the extreme equalitarian nature of optimal differential taxation under an APA has to deal with two possibilities. In the first possibility, the difference between the contestants' stakes is mild and there is an inverse relationship between their taxes. Consequently, the use of the well known properties of the equilibrium strategies in an APA enables a straightforward proof. In contrast, in the second possibility, the asymmetry between the contestants' prize valuations is large, the balanced budget constraint allows direct (not inverse) relationship between the taxes imposed on the contestants and, in turn, an increase in the sum of their prize valuations. This complicates matters and requires a different more subtle proof strategy. The challenge of clarifying the economic intuition behind the second part of the proof can nevertheless be met by applying standard microeconomic arguments. As we will show, the different more equalitarian nature of optimal taxation under the APA is due to the different relationship between the balanced-budget curve and the equi-effort curves under an APA and under a lottery. We also supplement the intuitive justification of the optimal taxation scheme under an APA and under a lottery with an economic interpretation. The proposed interpretation stresses the different role of leveling the playing field in attaining the maximal revenue via optimal taxation under these CSFs.

The second objective of the paper is to compare the appeal of the optimal differential taxation of the prize under the APA and any Tullock-type lottery in terms of their effectiveness as a means of revenue maximization for a contest designer who determines his preferred taxation scheme subject to a balanced-budget constraint. It turns out that optimal taxation yields larger revenue (total efforts) under the APA than under almost any lottery and, in particular, the simple lottery[3]. Finally, we establish that optimal differential taxation under the APA is the most effective means of generating revenue when a budget surplus is allowed.

The remainder of the paper is laid out as follows. In the next section we present the contest designer's problem under the balanced-budget constraint. The

[3] When the contestants stakes are given, the APA does not necessarily yield larger efforts than the simple lottery, as shown by Fang (2002), Epstein et al. (2011b). In our setting where the stakes can be controlled, the efforts under the APA are always larger than or equal to those obtained under any lottery.

optimal taxation scheme under the APA is presented in Section 3 We clarify the economic intuition behind the proof of the first result establishing the superiority of the APA in terms of equity, by applying standard microeconomic arguments. The exceptional nature of the first result is clarified by contrasting it with the non-extreme equalitarian nature of optimal taxation under almost any Tullock-type lottery. The superior effectiveness of optimal taxation under the APA as a means of generating efforts is established in Section 4. Concluding remarks are presented in Section 5. The proofs of the three main results that deal with optimal taxation under a balanced-budget constraint are relegated to an Appendix.

2. The problem of the contest designer under a balanced-budget constraint

In our contest there are two risk-neutral contestants, the high and low benefit contestants, 1 and 2. The prize valuations of the contestants are denoted by n_1 and n_2 and, with no loss of generality, we assume that $n_1 \geq n_2$ or $k = \frac{n_1}{n_2} \geq 1$ and that the contest designer has complete information on the contestants' prize valuations. Given the contestants' fixed prize valuations and the CSF, the function $p_i(x_1, x_2)$ that specifies the contestants' winning probability given their efforts x_1 and x_2, the expected net payoff (surplus) of contestant *i* is:

$$E(u_i) = p_i(x_1, x_2)n_i - x_i, \quad (i=1,2) \tag{1}$$

Direct discrimination via differential taxation of the contested prize that affects the contestants' actual prize valuations, n_1 and n_2, is a pair of (positive or negative) amounts, ε_1 and ε_2 that changes the prize valuations to $(n_1 + \varepsilon_1)$ and $(n_2 + \varepsilon_2)$. A contest designer who applies such a taxation scheme must ensure that the transformed prize valuations are positive. Otherwise the contestants will not voluntarily take part in the contest and the designer's revenue will be equal to zero. We also assume that the contest designer faces a balanced-budget constraint, that is, ε_1 and ε_2 must also satisfy the requirement that the designer's expected expenditures are equal to zero, that is, $p_1\varepsilon_1 + p_2\varepsilon_2 = 0$.[4] This ex-ante balanced-budget constraint is reasonable when the designer is "risk neutral" in the sense that he does not mind to face an ex-post deficit situation after the outcome of the contest has been revealed. The balanced-budget constraint is more plausible when the designer controls a series of identical contests that are held during a fixed period (typically weekly, monthly or quarterly contests that are held during the budget year). In such a case, the designer actually tries to ensure that during the relevant period the net transfers between the contestants are cancelled out such that his budget is balanced.

In the optimal contest design setting, the objective function of the contest designer is:

(2) $$G = x_1 + x_2$$

The designer maximizes his objective function (2) subject to the relevant constraints by selecting ε_1 and ε_2, given the anticipated Nash equilibrium efforts of the contestants. The particular choice of the taxation scheme together with its corresponding efforts of the contestants, constitute the equilibrium of the game. The contest game that we study has therefore a two-stage structure. In the first stage the designer determines the taxation scheme. In the second stage the contestants make decisions on their exerted efforts taking as given the (positive and negative) taxes levied on the prize.

3. The superiority of the APA in terms of equity

Under the APA, the certain winner is the contestant who makes the largest effort. That is, the APA is given by:

(3) $$p_1(x_1, x_2) = \begin{cases} 1 & \text{if} \quad x_1 > x_2 \\ 0.5 & \text{if} \quad x_1 = x_2 \\ 0 & \text{if} \quad x_1 < x_2 \end{cases}$$

[4] The possibility of a balanced-budget constraint faced by the contest designer has not been dealt with in the contest literature. The possibility of caps on the contestants' efforts has been examined, for example, by Che and Gale (1998), Ujhelyi (2009).

Given a tax scheme implemented by ε_1 and ε_2, the two contestants maximize their expected payoffs:

(4) $$E(u_1) = p_1(x_1, x_2)(n_1 + \varepsilon_1) - x_1 \text{ and } E(u_2) = [1 - p_1(x_1, x_2)](n_2 + \varepsilon_2) - x_2$$

If the stake of contestant 1 is larger than or equal to the stake of contestant 2, that is, $n_1 + \varepsilon_1 \geq n_2 + \varepsilon_2 > 0$, then the optimal efforts of the contestants and their corresponding equilibrium winning probabilities are given, as is well known (see Konrad (2009)), by:

(5) $$x_1^* = 0.5(n_2 + \varepsilon_2),\ x_2^* = \frac{(n_2 + \varepsilon_2)^2}{2(n_1 + \varepsilon_1)},\ p_1^* = 1 - \frac{n_2 + \varepsilon_2}{2(n_1 + \varepsilon_1)} \text{ and } p_2^* = \frac{n_2 + \varepsilon_2}{2(n_1 + \varepsilon_1)}$$

In turn, the objective function of the contest designer is:

(6) $$G_A = x_1^* + x_2^* = \frac{n_2 + \varepsilon_2}{2} + \frac{(n_2 + \varepsilon_2)^2}{2(n_1 + \varepsilon_1)} = \frac{(n_2 + \varepsilon_2)(n_1 + \varepsilon_1 + n_2 + \varepsilon_2)}{2(n_1 + \varepsilon_1)}$$

The designer selects a taxation scheme $(\varepsilon_1, \varepsilon_2)$ such that he maximizes the contestants' equilibrium efforts (6), subject to the balanced-budget constraint, $p_1\varepsilon_1 + p_2\varepsilon_2 = 0$. Since $p_i > 0$, $i = 1,2$, the budget constraint implies that $\varepsilon_1\varepsilon_2 \leq 0$. Taking into account the equilibrium efforts of the contestants and assuming that $n_1 + \varepsilon_1 \geq n_2 + \varepsilon_2 > 0$ [5], the balanced-budget constraint takes the form:

(7) $$p_1\varepsilon_1 + p_2\varepsilon_2 = \left[1 - \frac{n_2 + \varepsilon_2}{2(n_1 + \varepsilon_1)}\right]\varepsilon_1 + \frac{n_2 + \varepsilon_2}{2(n_1 + \varepsilon_1)}\varepsilon_2 = 0$$

or, equivalently,

(8) $$[2(n_1 + \varepsilon_1) - (n_2 + \varepsilon_2)]\varepsilon_1 + (n_2 + \varepsilon_2)\varepsilon_2 = 0$$

[5] Later on we discuss the other possible case where $n_2 + \varepsilon_2 > n_1 + \varepsilon_1 > 0$.

The designer's problem is, therefore, given by:

$$\underset{\varepsilon_1,\varepsilon_2}{Max}\left(x_1^* + x_2^*\right) = \frac{(n_2+\varepsilon_2)(n_1+\varepsilon_1+n_2+\varepsilon_2)}{2(n_1+\varepsilon_1)}$$

$$s.t.$$

$$p_1\varepsilon_1 + p_2\varepsilon_2 = 0 \rightarrow [2(n_1+\varepsilon_1)-(n_2+\varepsilon_2)]\varepsilon_1 + (n_2+\varepsilon_2)\varepsilon_2 = 0$$

$$n_1+\varepsilon_1 \geq n_2+\varepsilon_2 > 0$$

Our first result specifies the optimal taxation scheme under the APA.

Proposition 1: *The optimal taxation scheme under the APA equalizes the contestants final stakes, that is,* $(\varepsilon_1,\varepsilon_2) = (-0.5(n_1-n_2), 0.5(n_1-n_2))$.

Proof: See Appendix.

Let us clarify the idea of the proof by using Figures 1 and 2 and the well known properties of the equilibrium strategies under the APA. First, notice that the balanced-budget constraint requires that a move from the initial situation involves opposite-sign changes in the contestants' stakes and that $\varepsilon_1 \leq 0 \leq \varepsilon_2$ (the designer does not increase the stake of contestant 1 and reduce the stake of contestant 2 because such a strategy increases the gap between the contestants' stakes, so the intensity of the competition and, in turn, the contestants' efforts are reduced).

Two feasible taxation schemes are represented by points D and E. Obviously, the scheme $(\varepsilon_1,\varepsilon_2)$ represented by E that equalizes the contestants' final stakes is feasible (satisfies the balanced-budget constraint). Applying this strategy the designer imposes a tax (grants a subsidy) equal to half of the gap between the initial stakes on contestants 1 (to contestant 2) and this equalizes the winning probabilities of the contestants, so the balanced-budget constraint is indeed satisfied. This scheme generates larger efforts than the scheme $(\varepsilon_1,\varepsilon_2) = (0,0)$ represented by D, because it increases the intensity of competition and, in turn, the contestants' efforts. The feasible (potentially optimal) schemes for the designer are those represented by points on the curve connecting points D and E where $-0.5(n_1-n_2) \leq \varepsilon_1 < 0$.

The crucial issue to which we now turn is the reason that the optimal taxation scheme is the equalitarian scheme represented by point E. To answer this question, we proceed by clarifying the properties of the equi-effort and balanced-budget curves.

A typical equi-effort $\overline{G}_A$ curve is given by $\overline{G}_A = \frac{n_2+\varepsilon_2}{2} + \frac{(n_2+\varepsilon_2)^2}{2(n_1+\varepsilon_1)}$. In the $(\varepsilon_1, \varepsilon_2)$ plane, this curve is positively-sloped since an increase in ε_1 reduces the intensity of competition and, in turn, the contestants' efforts. To bring total effort back to its original level, ε_2 must be increased. Similarly, a reduction in the stake of contestant 1 or an increase in the stake of contestant 2 result in an increase in total effort. Four typical equi-effort curves G_A^A, G_A^B, G_A^C and G_A^E, $G_A^E > G_A^A > G_A^B > G_A^C$, are presented in Figures 1 and 2. These curves are positively-sloped and concave.

As already noted, in the neighborhood of point D, the balanced-budget curve is negatively sloped. In addition, the curve is also concave. These properties imply that, in the relevant range where $n_1+\varepsilon_1 \geq n_2+\varepsilon_2 > 0$, the balanced-budget curve can be always negatively sloped, as in Figure 1. In such a case, the optimality of the equalitarian tax scheme could be directly inferred from the equilibrium properties of the APA (see, for example, Baye et al. (1993), equation (10)) that imply that an increase of the lower stake or a decrease in the higher stake, viz., stake equalization, positively affect the total efforts. Alternatively, one could argue that in an all-pay auction equilibrium, the sum of the players' expenditures is weakly less than the lower prize valuation n_2. This follows from the players' equilibrium mixed strategies, which are uniform on [0, n_2] . Thus, the optimal tax scheme maximizes the lower of the two players' valuations, i.e., equates these valuations. In fact, when the balanced-budget curve is always negatively sloped, one can immediately realize that the optimal point cannot be interior and lie on the negatively-sloped part of the balanced-budget curve because at such a point the positive slope of the equi-effort curve is larger than the negative slope of the balanced-budget curve, as at points A, B, C in Figure 1. Hence, the optimal point in Figure 1 is E.

What complicates the proof, however, is the fact that, with sufficiently asymmetric players, the balanced-budget curve can also be positively sloped and have the typical shape depicted in Figure 2.[6] Note that in such a case the balanced-budget constraint enables a simultaneous increase in the contestants' prize valuations and, in turn, an invrease in the sum of these valuations beyond the initial (n_1+n_2). So the

[6] As shown in the proof, the two possible shapes of the balanced-budget curve presented in Figure 1 and in Figure 2 are, respectively, obtained when $1 < k \leq 3$ and $k > 3$.

two alternative straightforward proofs based on the equilibrium properties of the APA or on the comparison of the positive and negative slopes of the equi-effort and balanced-budget curves can no longer be used. To face the challenge of this possibility, we prove that the optimal taxation scheme is represented by the extreme point E, and not by an interior tax scheme represented by a point between A and E in Figure 2, by establishing that, at any point in the positively-sloped part of the balanced-budget curve, the positive slope of the equi-effort curve is larger than the positive slope of the balanced-budget curve. In economic terms, at any point in the positively-sloped part of the balanced-budget curve, any move towards E has two contrasting effects on the intensity of competition; the reduction in the stake of contestant 1 increases the intensity of competition and, in turn, the contestants' efforts, whereas the reduction in the stake of contestant 2 decreases the intensity of competition and, in turn, the contestants' efforts. Since the former effect is dominant, any move toward E increases the contestants' effort. That is, point E represents the optimal equalitarian taxation scheme.

Figure 1: The APA case where $1 < k \leq 3$

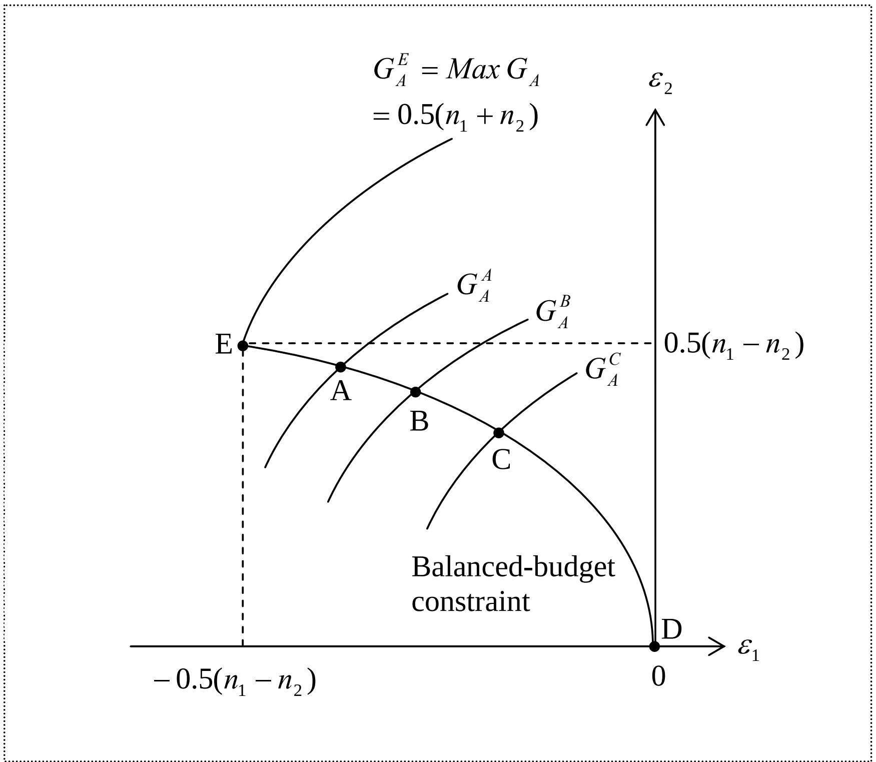
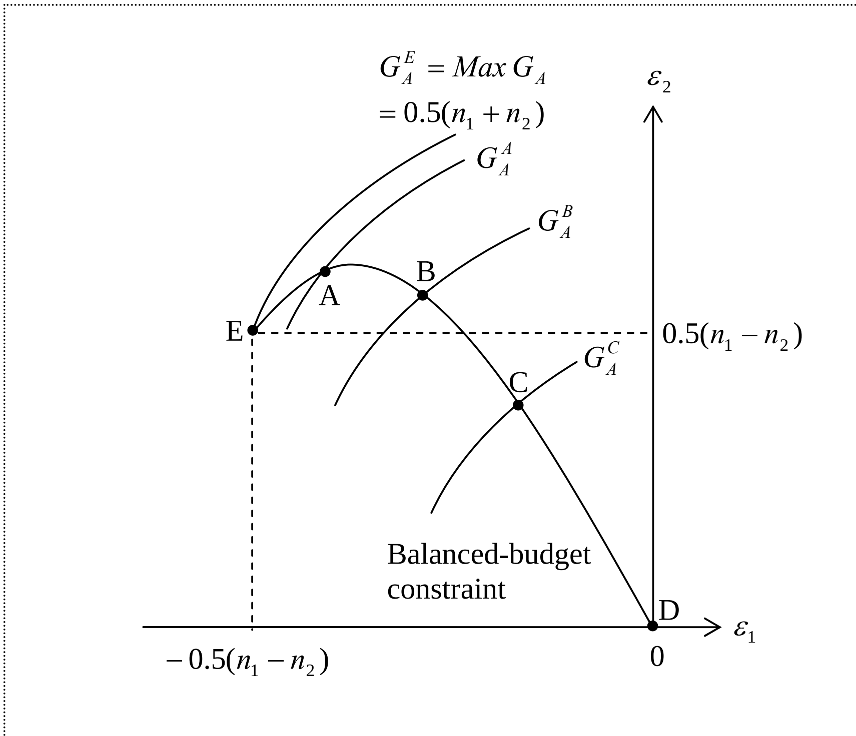


Figure 2: The APA case where $k > 3$



In light of the existing results in the contest literature, see Konrad (2009), one may intuitively expect that equalization of stakes is *always* the optimal strategy for a revenue-maximizing contest designer. At first glance, such expectation is plausible because equal stakes imply maximal competition that apparently induces the largest efforts. Our next objective is to explain why the fulfillment of this expectation under an APA is the exception rather than the rule. The extreme nature of the APA results in an extreme optimal taxation scheme (the optimal point E is not interior). In contrast, optimal taxation under the widely studied lottery CSFs proposed by Tullock (1980), is not extreme; it reduces the gap between the contestants' stakes, but does not eliminate it. A Tullock-type lottery is given by:

(9) $$p_1(x_1, x_2) = \frac{x_1^\alpha}{x_1^\alpha + x_2^\alpha}$$

where $\alpha > 0$. The CSF is a simple lottery, apparently the most commonly assumed CSF in the rent-seeking literature, when $\alpha = 1$. In this case, a contestant's probability of winning the contest is equal to his relative effort.

Given a taxation scheme represented by ε_1 and ε_2, the two contestants maximize their expected payoffs:

(10) $$E(u_1) = \frac{x_1^\alpha}{x_1^\alpha + x_2^\alpha}(n_1 + \varepsilon_1) - x_1 \text{ and } E(u_2) = \frac{x_2^\alpha}{x_1^\alpha + x_2^\alpha}(n_2 + \varepsilon_2) - x_2$$

Let $a = \frac{n_1 + \varepsilon_1}{n_2 + \varepsilon_2}$. By the first order conditions,

(11) $$x_1^* = \frac{\alpha a^\alpha (n_1 + \varepsilon_1)}{(a^\alpha + 1)^2} \text{ and } x_2^* = \frac{\alpha a^\alpha (n_2 + \varepsilon_2)}{(a^\alpha + 1)^2}$$

and, therefore,

(12) $$G_L = x_1^* + x_2^* = \frac{\alpha a^\alpha (n_1 + \varepsilon_1 + n_2 + \varepsilon_2)}{(a^\alpha + 1)^2}$$

(13) $$p_1 = \frac{a^\alpha}{a^\alpha + 1} \text{ and } p_2 = \frac{1}{a^\alpha + 1}$$

and the balanced-budget constraint takes the form

(14) $$p_1\varepsilon_1 + p_2\varepsilon_2 = \frac{a^\alpha}{a^\alpha + 1}\varepsilon_1 + \frac{1}{a^\alpha + 1}\varepsilon_2 = 0$$

or

(15) $$a^\alpha \varepsilon_1 + \varepsilon_2 = 0$$

The designer's problem is therefore:

$$\underset{\varepsilon_1,\varepsilon_2}{Max}\left(x_1^* + x_2^*\right) = \frac{\alpha a^{\alpha}\left(n_1+\varepsilon_1+n_2+\varepsilon_2\right)}{\left(a^{\alpha}+1\right)^2}$$

(16)

$$s.t.$$

1. $p_1\varepsilon_1 + p_2\varepsilon_2 = 0 \rightarrow a^{\alpha}\varepsilon_1 + \varepsilon_2 = 0$
2. $1-\alpha + a^{\alpha} \geq 0$
3. $(1-\alpha)a^{\alpha} + 1 \geq 0$
4. $n_1 + \varepsilon_1 > 0$
5. $n_2 + \varepsilon_2 > 0$

Note that constraints 2 and 3 guarantee that the contestants' utilities are not negative as well as the fulfillment of the second-order conditions in the contestants' maximization problems.[7] The solution of this problem yields our second result.

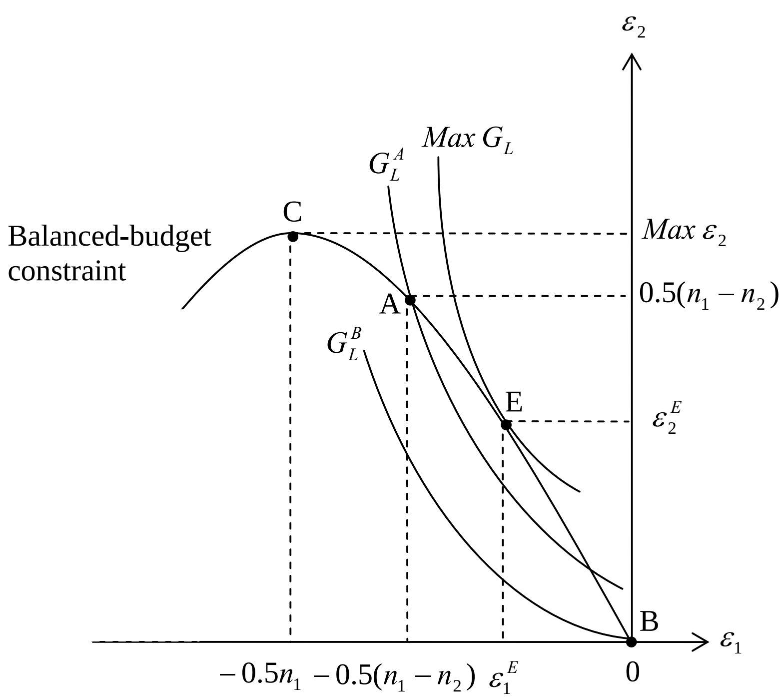
Proposition 2: When $k > 1$, *the optimal taxation scheme under any Tullock-type lottery with* $0 < \alpha < 2$ *does not equalize the contestants' final stakes, but preserves their relative magnitude. That is,* $(\varepsilon_1, \varepsilon_2) \neq (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$ *and* $\varepsilon_1 + \varepsilon_2 > 0$.

Proof: See Appendix.

That is, if the contestants' initial stakes are different, then a designer who chooses a taxation scheme subject to a balanced-budget constraint does not have an incentive to eliminate the gap between the contestants' prize valuations and the reduced initially higher stake is still larger than the increased initially lower stake. To illustrate this result in a tractable geometric way, consider the special case of a simple lottery ($\alpha = 1$) where the designer has an incentive to reduce the gap between the contestants' stakes but not to eliminate it. The diagrammatic illustration of the typical interior equilibrium in this case appears in Figure 3. Whereas the typical shape of the balanced-budget curve is unchanged, the equi-effort curves are now negatively sloped and convex. Furthermore, at point A, which represents the equalitarian taxation scheme, the slope of the balanced-budget curve is larger than the slope of the equi-effort curve. This implies that the point E, which represents the interior equilibrium taxation scheme, must be to the right of A. That is, the reduced stake of contestant 1 is

[S] See Epstein et al. (2011b).

Figure 3: The simple lottery case



still larger than the increased stake of contestant 2. The equalitarian taxation scheme represented by point A enables the designer to neutralize the initial difference in the contestants' stakes and thus increase the intensity of competition and, in turn, the contestants' efforts relative to the initial situation represented by point B. The move from point A to point E enables the designer to further increase the contestants' efforts by fully taking advantage of the potential "income effect" associated with a scheme that increases the sum of the final stakes from $(n_1 + n_2)$ at A to $(n_1 + \varepsilon_1 + n_2 + \varepsilon_2)$ at E (by Proposition 2, the optimal taxation scheme satisfies $\varepsilon_1 + \varepsilon_2 > 0$). This positive income effect dominates the negative effect on total efforts due to the reduced competition associated with the creation of a gap between the contestants' final stakes.

It seems to us that Proposition 2 has significant implications in public economics.[8] In particular, it can be used to explain why contingent taxation of the prize won in a lottery contest between two lobbyists representing two interest groups, such as, the "rich" and the "poor" or "consumers" and a "monopoly", tends to preserve the initial ex-ante inequality between the interest groups represented by the lobbyists. Such applications certainly deserve further examination, which is beyond the scope of the current work.

4. The superiority of the APA in terms of revenue maximization

If the designer faces a balanced-budget constraint, then optimal taxation under the APA yields total efforts that are equal to the average of the initial stakes. These efforts are larger than those obtained under almost any Tullock-type lottery and, in any event, they are always larger than or equal to those obtained under any lottery.

P**roposition 3:** *The total efforts of the contestants corresponding to the optimal taxation scheme und er the APA are equa l to th e average prize valuation,* $G_A = 0.5(n_1 + n_2)$. *These total efforts are larger than or equal to those obtained under any Tullock-type lottery with* $0 < \alpha \leq 2$.

Proof: See Appendix.

As shown in the proof, the maximal efforts under an APA can also be secured under a Tullock-type lottery with the exponent α being equal to 2. In other words, maximal performance of optimal differential taxation can be attained in the mixed-strategy equilibrium of the extreme logit CSF where $\alpha = \infty$ (the APA) or in the pure-strategy equilibrium of the extreme logit CSF where $\alpha = 2$. Note that such equivalence has the flavor of the neutrality result obtained in Alcalde and Dahm (2010). However, in our setting of contest design, the contestants' maximal efforts are larger than those obtained in the setting of Alcalde and Dahm (2010) because we allow discrimination between the contestants via the optimal scheme of differential taxation of the prize.

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[8] Applications in other disciplines, e.g., evolutionary biology, also seem natural because the assumption of contest resolution based on a lottery and the assumption of effort maximization (by nature) seem plausible.

Let us explain the intuition behind Proposition 3. For $k > 1$ and $\alpha = 2$, it can be verified that, at the point representing the optimal taxation scheme, the slope of the balanced-budget curve is still larger than the slope of the equi-effort curve. This means that the designer is aware of the kind of "income effect" described in the last paragraph of the preceding section. So why does not he take advantage of this effect and increase the contestants' efforts by creating a gap between their final stakes (see Proposition 2). The reason is that, when the designer modifies the contestants' prize valuations, he must be certain that his intervention preserves the contestants' incentives to take part in the contest. When $\alpha = 2$, it is known that the existence of a pure-strategy equilibrium requires that the contestants' stakes are equal. The designer must therefore equalize the stakes because otherwise the contestant with the lower prize valuation attains a negative utility, which prevents his participation in the contest. Hence, when the contestants' stakes are equalized, despite the existence of the "income effect", its application is not feasible; the utility of each contestant is equal to zero and any modification of the stakes by resorting to taxation will result in the withdraw of the lower-stake player from the contest and, in turn, in the reduction of the total efforts to zero. The above explanation implies that a change in the exponent of the lottery from $\alpha = 2$ to $\alpha < 2$ enables the designer to increase the contestants' efforts by taking advantage of the "income effect" (note that for $\alpha < 2$, taxation that equalizes the stakes results in positive utility for both of the players). So how can $\alpha = 2$ yield the maximal efforts $G_L = 0.5(n_1 + n_2)$. The answer to this question is that the move from equal stakes to non-equal stakes involves two negative effects that reduce the contestants' efforts. First, the move implies reduced competition that reduces the contestants' incentive to exert effort. Second, the reduction in α means that the impact of effort on the winning probability is reduced, and this effect also lowers the contestants' incentive to exert effort. The combined negative effect more than counterbalances the positive "income effect" and this explains why the maximal efforts are attained at $\alpha = 2$.

Finally, suppose that the contestants' participation in the contest is voluntary and the designer does not face a balanced-budget constraint and any surplus is allowed. The utility of the contest designer is now given by the contestants' efforts and the net expected surplus in the budget used for the differential taxation of the prize. That is, his objective function is given by

$\left[\frac{(n_2+\varepsilon_2)(n_1+\varepsilon_1+n_2+\varepsilon_2)}{2(n_1+\varepsilon_1)}-(p_1\varepsilon_1+p_2\varepsilon_2)\right]$. When budget surplus is allowed, the sum of the contestants' efforts and the surplus in the budget of the contest designer can be equal to the stake of contestant 1, n_1. The proof is straightforward. The designer can now reduce contestant 2's stake to zero, $\varepsilon_2=-n_2$, and contestant 1's stake almost to zero, $\varepsilon_1\to -n_1^+$. Such a taxation scheme induces contestant 2 not to take part in the contest and contestant 1 to make a negligible effort, which guarantees his winning. The contestants' efforts therefore converge to zero, the designer expropriates almost all the stake of contestant 1 so his benefit is equal to n_1. This 'take it or leave it' - type result has been obtained in the literature by applying different mechanisms. In the current study, it is obtained by resorting to optimal differential taxation of the prize. A similar result has been derived in Nti (2004) by applying a Tullock-type lottery and a transformation of the contestants' efforts that is equivalent to the setting of a reservation effort n_1 for contestant 1. If contestant 1's effort is smaller than the reservation effort, then contestant 2 wins the contest. Alternatively, we could use a first-price APA with a reservation price of n_1 (see Hillman and Riley, 1989).

5. Conclusion

As in a standard public finance context, taxation in a contest setting has efficiency and distributional implications. In this study efficiency (inefficiency) is measured in terms of the total efforts exerted by the contestants and the distributional effect is measured by the gap between the contestants' relevant final prize valuations.

Optimal contest design can be implemented by applying direct discrimination that affects the contestants' prize valuations via differential taxation of the prize. Interestingly, when the contest designer faces a balanced-budget constraint, differential taxation of the prize under the APA is sufficient to secure the exertion of the largest efforts by the contestants, relative to optimal taxation under any Tullock-type lottery. Such superiority is attained without resorting to structural discrimination that affects the parameters of the contest success function, as in Clark and Riis (2000), Epstein et al. (2011a), (2011b), Franke (2007), Franke et al. (2011) and Lien (1990), which may be difficult to control or even illegal. Furthermore, allowing taxation that result in a budget surplus, the optimal differential taxation scheme under the APA generates the maximal possible total efforts, which are equal to the highest contestant's value of the prize.

Optimal taxation under the APA is also superior in terms of equity: equality of the contestants' final prize valuations. When the contest designer faces a balanced-budget constraint, optimal taxation under the APA eliminates the gap between the contestants' initial prize valuations. Such equalization of the contestants' stakes is an exception and not the rule. In fact, it is not obtained under almost any Tullock-type lottery and we conjecture that this finding is more general. That is, optimal taxation under any regular lottery that satisfies some standard properties closes the gap between the contestants' stakes from the prize, but does not eliminate it. The economic rationale of this finding is due to the dominance of the positive "income effect" on total efforts, which is attained by taxation that increases the sum of the final stakes, over the negative "inequality effect"; the negative effect on total efforts of the preserved stake inequality, which implies giving up some potential extra competition between the contestants that could enhance the exertion of efforts. Note that the existence of the "income effect" in our strategic contest setting crucially depends on two assumptions: the initial difference between the contestants' prize valuations and the balanced-budget constraint that enables taxation that can increase the initial sum of the contestants' stakes. But, under the APA, these necessary assumptions are not sufficient to ensure the existence of a positive "income effect". The reason is that in the case of an APA, the equi-effort curves are positively sloped and not negatively sloped as in the case of a lottery when the stakes are equalized. This means that an interior optimal taxation can only be obtained along the positively-sloped part of the balanced-budget constraint (see Figure 2). But in this range the optimal taxation scheme is not interior; it yields equal final stakes because any feasible alternative taxation scheme that involves a simultaneous increase of the equal stakes of the contestants negatively affects their total efforts. The increase in the stake of contestant 1 reduces the intensity of competition and, in turn, the exerted efforts. This decline in the exerted efforts is moderated, but not neutralized or more than counter balanced, by the required increase in the stake of contestant 2 while moving along the positively-sloped part of the balanced-budget curve. In other words, in the case of an APA, when the contestants' stakes are equalized, the "income effect" (the effect on total efforts of the increase in the sum of the contestants' stakes) is negative. So the designer prefers the equalitarian corner solution. Finally, we have shown that when a surplus is allowed in the contest designer's budget, again, optimal taxation under the APA almost eliminates the gap between the contestants' initial prize valuations; the prize of one contestant is reduced to zero and the prize of the other contestant is reduced to a positive value slightly higher than zero.

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Appendix

Proposition 1: The optimal taxation scheme under the APA equalizes the contestants final stakes, that is, $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$.

Proof: The proof includes three parts. We first clarify the properties of the feasible (potentially equilibrium) strategies $(\varepsilon_1, \varepsilon_2)$. For these strategies, we describe in Part 2 the properties of the balanced-budget constraint (*bbc*). We then present in the third part the properties of an equi-effort curve and by comparing its slope to that of the balanced-budget curve complete the proof.

Part 1: Let us show that in equilibrium, if $k = 1$, then $(\varepsilon_1, \varepsilon_2) = (0,0)$ and if $k > 1$, then $-0.5(n_1 - n_2) \leq \varepsilon_1 < 0 < \varepsilon_2$.

Since $\frac{\partial G_A}{\partial \varepsilon_1} = -0.5\left(\frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}\right)^2 < 0$, $\frac{\partial G_A}{\partial \varepsilon_2} = 0.5 + \frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1} > 0$ and, by the bbc, $\varepsilon_1 \varepsilon_2 \leq 0$, considering a deviation from $(\varepsilon_1, \varepsilon_2) = (0,0)$, the designer does not have an incentive to reduce ε_2 (so $\varepsilon_2 \geq 0$) or increase ε_1 (so $\varepsilon_1 \leq 0$). That is, in any equilibrium, $\varepsilon_1 \leq 0 \leq \varepsilon_2$ must be satisfied.

When $k = 1$ ($n = n_1 = n_2$), by (6), for $(\varepsilon_1, \varepsilon_2) = (0,0)$, the total efforts are equal to $\frac{n}{2} + \frac{n^2}{2n} = n$. It can be easily verified, by (6), that any alternative feasible taxation scheme attains smaller efforts. Henceforth we therefore assume that $k > 1$.

When $k > 1$, by (6), for $(\varepsilon_1, \varepsilon_2) = (0,0)$, the total efforts are equal to $\frac{n_1 + n_2}{2k}$ and for the feasible stake-equalizing scheme $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$, the total efforts are equal to $0.5(n_1 + n_2)$. Since, for $k > 1$, $0.5(n_1 + n_2) > \frac{n_1 + n_2}{2k}$, we can conclude that $(\varepsilon_1, \varepsilon_2) = (0,0)$ does not maximize the contestants' efforts. Hence, for $k > 1$, in equilibrium, $\varepsilon_1 < 0 < \varepsilon_2$.

Let us complete the proof of Part 1 (establish that, in equilibrium, $-0.5(n_1 - n_2) \leq \varepsilon_1$) by showing that when $-n_1 < \varepsilon_1 < -0.5(n_1 - n_2)$, the corresponding efforts cannot be maximal.

Let $(\varepsilon_1^E, \varepsilon_2^E) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$ be our benchmark taxation scheme, where $n_1 + \varepsilon_1^E = n_2 + \varepsilon_2^E = 0.5(n_1 + n_2)$, efforts are equal to $G_A^E = \frac{n_2 + \varepsilon_2^E}{2} + \left(\frac{n_2 + \varepsilon_2^E}{2}\right)\left(\frac{n_2 + \varepsilon_2^E}{n_1 + \varepsilon_1^E}\right) = 0.5(n_1 + n_2)$ and $\frac{n_2 + \varepsilon_2^E}{n_1 + \varepsilon_1^E} = \frac{n_1 + \varepsilon_1^E}{n_2 + \varepsilon_2^E} = 1$.

Starting from this scheme, let us reduce ε_1 below $-0.5(n_1 - n_2)$ and show that such a change reduces the efforts, independent of the balanced-budget constraint:[9]

(i) If, after the reduction in ε_1, $n_1 + \varepsilon_1 \geq n_2 + \varepsilon_2$, then efforts are still given by (6), $G_A = \frac{n_2 + \varepsilon_2}{2} + \left(\frac{n_2 + \varepsilon_2}{2}\right)\left(\frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}\right)$, and $n_2 + \varepsilon_2^E = n_1 + \varepsilon_1^E > n_1 + \varepsilon_1 \geq n_2 + \varepsilon_2$.

Therefore, $\frac{n_2 + \varepsilon_2^E}{2} > \frac{n_2 + \varepsilon_2}{2}$ and $\frac{n_2 + \varepsilon_2^E}{n_1 + \varepsilon_1^E} = 1 \geq \frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}$ and, consequently, the move from G_A^E to the new G_A reduces the efforts.

(ii) If, after the reduction in ε_1, $n_1 + \varepsilon_1 < n_2 + \varepsilon_2$, then efforts are given by $G_A = \frac{n_1 + \varepsilon_1}{2} + \left(\frac{n_1 + \varepsilon_1}{2}\right)\left(\frac{n_1 + \varepsilon_1}{n_2 + \varepsilon_2}\right)$ and $n_2 + \varepsilon_2^E = n_1 + \varepsilon_1^E > n_1 + \varepsilon_1$. Therefore, $\frac{n_2 + \varepsilon_2^E}{2} > \frac{n_1 + \varepsilon_1}{2}$ and $\frac{n_2 + \varepsilon_2^E}{n_1 + \varepsilon_1^E} = 1 > \frac{n_1 + \varepsilon_1}{n_2 + \varepsilon_2}$ and consequently the move from G_A^E to the new G_A reduces the total efforts.

Part 2: In this part we examine the properties of the *bbc* for the relevant schemes satisfying $-0.5(n_1 - n_2) \leq \varepsilon_1 < 0 < \varepsilon_2$. The *bbc* can be written as $\varepsilon_2^2 + \varepsilon_2(n_2 - \varepsilon_1) + \varepsilon_1[2(n_1 + \varepsilon_1) - n_2] = 0$. Since $\varepsilon_1 \leq 0 \leq \varepsilon_2$, the solution of this equation must be the positive root. That is:

(A.1) $$\varepsilon_2(\varepsilon_1) = \frac{\varepsilon_1 - n_2 + \left\{(n_2 - \varepsilon_1)^2 - 4\varepsilon_1[2(n_1 + \varepsilon_1) - n_2]\right\}^{0.5}}{2}.$$

A taxation scheme $(\varepsilon_1, \varepsilon_2)$ that satisfies the *bbc* has the following properties:

a. $n_1 + \varepsilon_1 = n_2 + \varepsilon_2 (> 0)$ iff $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$.

b. If $-0.5(n_1 - n_2) < \varepsilon_1 < 0$, then $n_1 + \varepsilon_1 > n_2 + \varepsilon_2 (> 0)$.

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[9] In this range, it is possible that the contestant who initially has the higher stake becomes the one with the lower stake and then, under the balanced-budget constraint (8), the roles of contestants 1 and 2 are reversed. Therefore, the constraint (8) is no longer applicable.

c. For $n_1 + \varepsilon_1 \geq n_2 + \varepsilon_2 > 0$, the function (A.1) that defines the *bb*-curve is (i) concave, $\frac{\partial^2 \varepsilon_2}{\partial \varepsilon_1^2} < 0$ and (ii) at $(\varepsilon_1, \varepsilon_2) = (0,0)$, $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} < 0$.

The proof of property **a** is straightforward and therefore omitted.

To prove property **b,** note that in the range $-0.5(n_1 - n_2) \leq \varepsilon_1 \leq 0$ on the bb-curve, only in the extreme point where $\varepsilon_1 = -0.5(n_1 - n_2)$, the contestants' stakes are equal, that is, $n_1 + \varepsilon_1 = n_2 + \varepsilon_2$. In the other extreme point where $\varepsilon_1 = 0$, $n_1 + \varepsilon_1 > n_2 + \varepsilon_2$. Hence, by the continuity of the bbc (A.1), for every ε_1, $-0.5(n_1 - n_2) < \varepsilon_1 \leq 0$, we get that $n_1 + \varepsilon_1 > n_2 + \varepsilon_2$.

To prove property **c,** notice that according to (A.1), the slope of the *bb*-curve is:

(A.2) $$\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = 0.5 + 0.5(n_2 - 4n_1 - 7\varepsilon_1)\left\{(n_2 - \varepsilon_1)^2 - 4\varepsilon_1[2(n_1 + \varepsilon_1) - n_2]\right\}^{-0.5}$$

and, therefore,

(A.3) $$\frac{\partial^2 \varepsilon_2}{\partial \varepsilon_1^2} = -0.5(n_2 - 4n_1 - 7\varepsilon_1)^2\left\{(n_2 - \varepsilon_1)^2 - 4\varepsilon_1[2(n_1 + \varepsilon_1) - n_2]\right\}^{-1.5}$$
$$-3.5\left\{(n_2 - \varepsilon_1)^2 - 4\varepsilon_1[2(n_1 + \varepsilon_1) - n_2]\right\}^{-0.5} < 0$$

Substituting $\varepsilon_1 = 0$ in (A.2), we get that $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = 1 - 2k < 0$.

Let us, finally, find out what is the value of ε_1 that yields the maximal value of $\varepsilon_2(\varepsilon_1)$ on the *bb*-curve. By (A.2), for $(\varepsilon_1, \varepsilon_2) = (0,0)$, we get that $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} < 0$ and for $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$, we get that $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = \frac{k-3}{3k-1}$. By (A.3), the function $\varepsilon_2(\varepsilon_1)$ defining the *bb-curve* is concave in the relevant range $-0.5(n_1 - n_2) \leq \varepsilon_1 < 0$. Therefore, for $k > 3$, $\varepsilon_2(\varepsilon_1)$ has a negative slope at $\varepsilon_1 = 0$, a positive slope at $\varepsilon_1 = -0.5(n_1 - n_2)$ and a zero slope at some intermediate value ε_1, $-0.5(n_1 - n_2) < \varepsilon_1 < 0$, that yields the maximal value of ε_2. Notice that when $k \leq 3$, $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} < 0$ for any ε_1, $-0.5(n_1 - n_2) \leq \varepsilon_1 < 0$. That is, the *bb*-curve is declining in the relevant domain and the maximal value of $\varepsilon_2 = 0.5(n_1 - n_2)$ is obtained at $\varepsilon_1 = 0.5(n_1 - n_2)$.

For $k \le 3$ and $-0.5(n_1 - n_2) \le \varepsilon_1 < 0$, an increase in ε_1 is always associated with a decrease in ε_2. Since $\frac{\partial G_A}{\partial \varepsilon_1} < 0$ and $\frac{\partial G_A}{\partial \varepsilon_2} > 0$, we directly get that the maximal efforts are obtained in the equalitarian scheme $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$. But when $k > 3$ and $-0.5(n_1 - n_2) \le \varepsilon_1 < 0$, an increase in ε_1 can be associated with an increase in ε_2. The optimality of the equalitarian tax scheme needs therefore to be proved taking into account also this possibility (an increasing *bb*-curve). The third part of the proof establishes the optimality of $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$ also in this case.

Part 3: An equi-effort curve (*ee*-curve) $\overline{G}_A$ is defined for $n_1 + \varepsilon_1 \ge n_2 + \varepsilon_2 > 0$ as follows $\overline{G}_A = \frac{n_2 + \varepsilon_2}{2} + \frac{(n_2 + \varepsilon_2)^2}{2(n_1 + \varepsilon_1)}$. Let us show that the function $\varepsilon_2(\varepsilon_1)$ that defines $\overline{G}_A$ is positively sloped, $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} > 0$, and concave, $\frac{\partial^2 \varepsilon_2}{\partial \varepsilon_1^2} < 0$. Differentiating the function $\varepsilon_2(\varepsilon_1)$ we get $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = -\frac{-0.5\left(\frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}\right)^2}{0.5 + \frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}} = \frac{\left(\frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}\right)^2}{1 + 2\left(\frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}\right)} > 0$. Letting $b = \frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}$, we get that $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = \frac{b^2}{1 + 2b} > 0$ and $\frac{\partial^2 \varepsilon_2}{\partial \varepsilon_1^2} = \frac{2b\left(\frac{\partial \varepsilon_2}{\partial \varepsilon_1} - b\right)(b + 1)}{(n_1 + \varepsilon_1)(1 + 2b)^2}$. Substituting $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = \frac{b^2}{1 + 2b}$, we get that $\frac{\partial^2 \varepsilon_2}{\partial \varepsilon_1^2} = -\frac{2b^2(1 + b)^2}{(n_1 + \varepsilon_1)(1 + 2b)^3} < 0$. Given the properties of the *bb*-curve and the *ee*-curve, we will complete the proof by showing that, for $-0.5(n_1 - n_2) \le \varepsilon_1 < 0$, where, $n_1 + \varepsilon_1 \ge n_2 + \varepsilon_2$ (see Part 2), at every point on the bb-curve, the slope of the ee-curve is larger than the slope of the bb-curve.

Substituting $b = \frac{n_2 + \varepsilon_2}{n_1 + \varepsilon_1}$ in the *bbc*, we get that

$$\left[1-\frac{n_2+\varepsilon_2}{2(n_1+\varepsilon_1)}\right]\varepsilon_1+\frac{n_2+\varepsilon_2}{2(n_1+\varepsilon_1)}\varepsilon_2=0, \text{ or } (1-0.5b)\varepsilon_1+0.5b\varepsilon_2=0 \text{ or } b=\frac{2\varepsilon_1}{\varepsilon_1-\varepsilon_2}.$$ [10]

We have to show that $\frac{b^2}{1+2b}>\frac{2(n_1+2\varepsilon_1)-(n_2+\varepsilon_2)}{\varepsilon_1-(n_2+2\varepsilon_2)}$, where the LHS (RHS) expression is the slope of the *ee*-curve (*bb*-curve). This inequality can be equivalently written as $\frac{b^2}{1+2b}>\frac{2(n_1+\varepsilon_1)-(n_2+\varepsilon_2)+2\varepsilon_1}{\varepsilon_1-\varepsilon_2-(n_2+\varepsilon_2)}$. Dividing the nominator and denominator of the RHS by $(n_1+\varepsilon_1)$ we get $\frac{b^2}{1+2b}>\frac{2-\frac{n_2+\varepsilon_2}{n_1+\varepsilon_1}+\frac{2\varepsilon_1}{n_1+\varepsilon_1}}{\frac{\varepsilon_1-\varepsilon_2}{n_1+\varepsilon_1}-\frac{n_2+\varepsilon_2}{n_1+\varepsilon_1}}$ and, in terms of *b*, we get $\frac{b^2}{1+2b}>\frac{2-b+\frac{2\varepsilon_1}{n_1+\varepsilon_1}}{\frac{\varepsilon_1-\varepsilon_2}{n_1+\varepsilon_1}-b}$. Since the denominator of the RHS expression is negative (because $\varepsilon_1<0<\varepsilon_2$ and $b>0$), $b^2\left(\frac{\varepsilon_1-\varepsilon_2}{n_1+\varepsilon_1}-b\right)<\left(2-b+\frac{2\varepsilon_1}{n_1+\varepsilon_1}\right)(1+2b)$. After some algebraic manipulations, this inequality takes the form:

$$b(1-b)^2+2(b+1)+\frac{1}{n_1+\varepsilon_1}\left[2\varepsilon_1(1+2b)-b^2(\varepsilon_1-\varepsilon_2)\right]>0$$

Substituting $b=\frac{2\varepsilon_1}{\varepsilon_1-\varepsilon_2}$ (which has been calculated above) in all the terms in the above inequality with the exception of $b(1-b)^2$, we get after some algebraic manipulations that the above inequality is equivalent to $b(1-b)^2+2\left(\frac{2\varepsilon_1}{\varepsilon_1-\varepsilon_2}+1\right)+\frac{1}{n_1+\varepsilon_1}\left[2\varepsilon_1\left(1+2\frac{2\varepsilon_1}{\varepsilon_1-\varepsilon_2}\right)-\left(\frac{2\varepsilon_1}{\varepsilon_1-\varepsilon_2}\right)^2(\varepsilon_1-\varepsilon_2)\right]>0$ or $b(1-b)^2+2\left(\frac{3\varepsilon_1-\varepsilon_2}{\varepsilon_1-\varepsilon_2}\right)\left(\frac{n_1+2\varepsilon_1}{n_1+\varepsilon_1}\right)>0$. Since $-0.5(n_1-n_2)\le\varepsilon_1<0$, $n_1+2\varepsilon_1\ge n_1+2[-0.5(n_1-n_2)]=n_2>0$ and, therefore, $\frac{n_1+2\varepsilon_1}{n_1+\varepsilon_1}>0$. Since

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[10] Note that the condition $-0.5(n_1-n_2)\le\varepsilon_1<0$ requires that $\varepsilon_1<0<\varepsilon_2$. Therefore, $\varepsilon_1-\varepsilon_2<0$.

$\frac{3\varepsilon_1 - \varepsilon_2}{\varepsilon_1 - \varepsilon_2} > 0$ (because $\varepsilon_1 < 0 < \varepsilon_2$) and $b > 0$, we get that, for $-0.5(n_1 - n_2) \le \varepsilon_1 < 0$, $b(1-b)^2 + 2\left(\frac{3\varepsilon_1 - \varepsilon_2}{\varepsilon_1 - \varepsilon_2}\right)\left(\frac{n_1 + 2\varepsilon_1}{n_1 + \varepsilon_1}\right) > 0$. **Q.E.D**

Proposition 2: When $k > 1$ *the optimal taxation scheme under any Tullo ck-type lottery with* $0 < \alpha < 2$ *does not equalize th e contestants' final s takes, but preserves their relative magnitude. That is,* $(\varepsilon_1, \varepsilon_2) \ne (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$ *and* $\varepsilon_1 + \varepsilon_2 > 0$.[11]

Proof: The proof is based on three lemmas.

Lemma 1: For $0 < \alpha \le 2$, the total efforts obtained in the designer's problem, (16), satisfy:

a. For any $(\varepsilon_1, \varepsilon_2)$, $\frac{\alpha a^\alpha (n_1 + \varepsilon_1 + n_2 + \varepsilon_2)}{(a^\alpha + 1)^2} \le 0.25\alpha(n_1 + \varepsilon_1 + n_2 + \varepsilon_2)$.

b. For the equilibrium taxation scheme $(\varepsilon_1^E, \varepsilon_2^E)$, $0.25\alpha(n_1 + n_2) \le \frac{\alpha a^\alpha (n_1 + \varepsilon_1^E + n_2 + \varepsilon_2^E)}{(a^\alpha + 1)^2}$.

Proof of Lemma 1:

a. Consider the expression $c = \frac{a^\alpha}{(a^\alpha + 1)^2}$. Since $\frac{\partial c}{\partial a^\alpha} = \frac{1 - a^\alpha}{(a^\alpha + 1)^3}$, the maximal value of c, which is equal to 0.25, is reached at $a = 1$ (notice that the second order condition is satisfied at $a = 1$, $\frac{\partial^2 c}{\partial (a^\alpha)^2} < 0$). Hence, for any $(\varepsilon_1, \varepsilon_2)$:

$$\frac{\alpha a^\alpha (n_1 + \varepsilon_1 + n_2 + \varepsilon_2)}{(a^\alpha + 1)^2} \le 0.25\alpha(n_1 + \varepsilon_1 + n_2 + \varepsilon_2).$$

b. The selection of $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$, which satisfied all the constraints in the designer's problem (16), yields total efforts that are equal to $0.25\alpha(n_1 + n_2)$. Hence, $0.25\alpha(n_1 + n_2) \le \frac{\alpha a^\alpha (n_1 + \varepsilon_1^E + n_2 + \varepsilon_2^E)}{(a^\alpha + 1)^2}$. **Q.E.D**

[11] For $k = 1$, it can be shown that in equilibrium $(\varepsilon_1, \varepsilon_2) = (0,0)$.

Lemma 2: In equilibrium, for $0<\alpha<2$ and $k>1$, $\varepsilon_1<0<\varepsilon_2$.

Proof of Lemma 2: The proof consists of two steps.

Step 1 - Since $p_i>0$, $i=1,2$, the balanced-budget constraint (14) implies that $\varepsilon_1\varepsilon_2\le 0$. Let us first show that $\varepsilon_1\le 0\le\varepsilon_2$. Suppose to the contrary that, in equilibrium, the inequalities $\varepsilon_1\le 0\le\varepsilon_2$ are not satisfied. Since $\varepsilon_1\varepsilon_2\le 0$, this implies that $\varepsilon_2<0<\varepsilon_1$ and, therefore, after the change in the contestants' stakes, the stake of contestant 1 (2) is increased (decreased), $a=\frac{n_1+\varepsilon_1}{n_2+\varepsilon_2}>1$. By the balanced-budget constraint (15), $\varepsilon_2=-a^{\alpha}\varepsilon_1<-\varepsilon_1$ and using the result $a>1$, we get that $\varepsilon_2=-a^{\alpha}\varepsilon_1<-\varepsilon_1$ or $\varepsilon_1+\varepsilon_2<0$. The total efforts, even if constraints 2 and 3 in the designer's problem (16) are disregarded, are not larger than $0.25\alpha(n_1+\varepsilon_1+n_2+\varepsilon_2)$. Clearly, under the constraints 2 and 3, the total contestants' efforts cannot be larger than this amount. Since $\varepsilon_1+\varepsilon_2<0$, the equilibrium total efforts are smaller than $0.25\alpha(n_1+n_2)$. But this contradicts part (b) of Lemma 1, which implies that the assumption $\varepsilon_2<0<\varepsilon_1$ cannot be true. Hence, $\varepsilon_1\le 0\le\varepsilon_2$.

Step 2 - Let us prove that for $k>1$, $(\varepsilon_1,\varepsilon_2)=(0,0)$ is not optimal. Together with the conditions established in step 1, $\varepsilon_1\le 0\le\varepsilon_2$, this will complete the proof establishing that, in equilibrium, $\varepsilon_1<0<\varepsilon_2$. Let us then show that the selection of $(\varepsilon_1,\varepsilon_2)=(-0.5(n_1-n_2),0.5(n_1-n_2))$ is superior to the selection of $(\varepsilon_1,\varepsilon_2)=(0,0)$, that is, $\frac{\alpha k^{\alpha}(n_1+n_2)}{(k^{\alpha}+1)^2}<0.25\alpha(n_1+n_2)$. This latter inequality is equivalent to $0<(k^{\alpha}-1)^2$, which is always satisfied since $k>1$. **Q.E.D**

Lemma 3: An equi-effort curve $\overline{G}_L$ is defined by $\overline{G}_L=\frac{\alpha[(n_1+\varepsilon_1)(n_2+\varepsilon_2)]^{\alpha}(n_1+\varepsilon_1+n_2+\varepsilon_2)}{[(n_1+\varepsilon_1)^{\alpha}+(n_2+\varepsilon_2)^{\alpha}]^2}$.

a. If $k>1$, then at the point representing $(\varepsilon_1,\varepsilon_2)=(-0.5(n_1-n_2),0.5(n_1-n_2))$,

(1) The slope of the equi-effort curve $\frac{\partial\varepsilon_2}{\partial\varepsilon_1}$ is equal to -1.

(2) In the margin, an increase in ε_i, $i = (1,2)$, increases total efforts.

(3) The slope of the balanced-budget curve $\frac{\partial \varepsilon_2}{\partial \varepsilon_1}$ is larger than -1.

b. For $0 < \alpha < 2$ and $k > 1$, in equilibrium $a = \frac{n_1 + \varepsilon_1}{n_2 + \varepsilon_2} > 1$ (the stake of contestant 1 (2) is reduced (increased), but the final stake of 1 is still larger than that of 2) and therefore $\varepsilon_1 + \varepsilon_2 > 0$.

Proof of Lemma 3:

a.(1) and a.(2). Given an equi-effort curve $\overline{G}_L$.

$$\frac{\partial \overline{G}_L}{\partial \varepsilon_1} = \alpha(n_2 + \varepsilon_2)^\alpha \frac{\left\{ \begin{array}{l} \left[\alpha(n_1+\varepsilon_1)^{\alpha-1}(n_1+\varepsilon_1+n_2+\varepsilon_2)+(n_1+\varepsilon_1)^\alpha\right]\left[(n_1+\varepsilon_1)^\alpha+(n_2+\varepsilon_2)^\alpha\right]^2 \\ -2\left[(n_1+\varepsilon_1)^\alpha+(n_2+\varepsilon_2)^\alpha\right]\alpha(n_1+\varepsilon_1)^{\alpha-1}(n_1+\varepsilon_1)^\alpha(n_1+\varepsilon_1+n_2+\varepsilon_2) \end{array} \right\}}{\left[(n_1+\varepsilon_1)^\alpha+(n_2+\varepsilon_2)^\alpha\right]^4}$$

or, after some simplification,

$$\frac{\partial \overline{G}_L}{\partial \varepsilon_1} = \frac{\alpha a^{\alpha-1}\left\{\alpha(a+1)+a+a^\alpha\left[a-\alpha(a+1)\right]\right\}}{\left(a^\alpha+1\right)^3}$$

In a similar way we get that

$$\frac{\partial \overline{G}_L}{\partial \varepsilon_2} = \frac{\alpha a^\alpha\left\{a^\alpha\left[\alpha(a+1)+1\right]+1-\alpha(a+1)\right\}}{\left(a^\alpha+1\right)^3}$$

The slope of an equi-effort curve is therefore equal to

$$\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = -\frac{\frac{\partial \overline{G}_L}{\partial \varepsilon_2}}{\frac{\partial \overline{G}_L}{\partial \varepsilon_1}} = -\frac{\left\{\alpha(a+1)+a+a^\alpha\left[a-\alpha(a+1)\right]\right\}}{a\left\{a^\alpha\left[\alpha(a+1)+1\right]+1-\alpha(a+1)\right\}}$$

For $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$, $a = 1$. For $a = 1$ we get that $\frac{\partial \overline{G}_L}{\partial \varepsilon_1} = \frac{\partial \overline{G}_L}{\partial \varepsilon_2} = 0.25\alpha$ and $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = -1$. This means that in the neighborhood of $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$, an equi-effort curve is negatively sloped and an increase in ε_1 or in ε_2 increases the total efforts (an increase in total efforts shifts an equi-effort curve upward in the $(\varepsilon_1, \varepsilon_2)$ plane).

a.(3) From the implicit form of the balanced-budget constraint (15) we get that $(n_1+\varepsilon_1)^\alpha \varepsilon_1 + (n_2+\varepsilon_2)^\alpha \varepsilon_2 = 0$. Therefore, the slope of the balanced-budget curve is equal to $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = -\frac{\alpha(n_1+\varepsilon_1)^{\alpha-1}\varepsilon_1 + (n_1+\varepsilon_1)^\alpha}{\alpha(n_2+\varepsilon_2)^{\alpha-1}\varepsilon_2 + (n_2+\varepsilon_2)^\alpha}$ or $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = -\frac{a^{\alpha-1}[n_1+\varepsilon_1(1+\alpha)]}{n_2+\varepsilon_2(1+\alpha)}$ and, therefore, the slope at $(\varepsilon_1,\varepsilon_2) = (-0.5(n_1-n_2), 0.5(n_1-n_2))$ is $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = -\frac{n_1+n_2-\alpha(n_1-n_2)}{n_1+n_2+\alpha(n_1-n_2)}$. Since $n_1+n_2-\alpha(n_1-n_2) < n_1+n_2+\alpha(n_1-n_2)$, the slope of the balanced-budget curve at $(\varepsilon_1,\varepsilon_2) = (-0.5(n_1-n_2), 0.5(n_1-n_2))$ satisfies $\frac{\partial \varepsilon_2}{\partial \varepsilon_1} > -1$.[12]

b. Let us show that a move from $(\varepsilon_1,\varepsilon_2) = (-0.5(n_1-n_2), 0.5(n_1-n_2))$, where $a=1$, that involves a marginal increase in ε_1, which preserves the balanced-budget constraint, increases total efforts. This will prove that, in equilibrium**,** $a \neq 1$.

A marginal change in $a=1$, which is due to a marginal increase in ε_1 that preserves the balanced-budget constraint, still satisfies constraints 2 and 3 in the designer's problem (16), because at $a=1$ these constraints are satisfied as strict inequalities ($2-\alpha>0$). By Lemma 3 part (a), at the point which represents $(\varepsilon_1,\varepsilon_2) = (-0.5(n_1-n_2), 0.5(n_1-n_2))$, the slope of the balanced-budget curve ($-1 < \frac{\partial \varepsilon_2}{\partial \varepsilon_1}$) is larger than the slope of the equi-effort curve ($\frac{\partial \varepsilon_2}{\partial \varepsilon_1} = -1$). Therefore, starting from this point, a marginal increase in ε_1 accompanied by the required change in ε_2, such that the balanced-budget curve is still satisfied, increases the total efforts. Note that if the slope of the balanced budget curve is positive (not positive) an increase (a decrease) in ε_2 is required. We have shown then that $a \neq 1$. Let us show that, in equilibrium, $(0<)a<1$ is impossible. Suppose to the contrary that $a<1$. By Lemma 2, $\varepsilon_1 < 0 < \varepsilon_2$, the balanced-budget constraint, $a^\alpha \varepsilon_1 + \varepsilon_2 = 0$ and the assumption $a<1$, we get that $\varepsilon_2 = -a^\alpha \varepsilon_1 < -\varepsilon_1$ or $\varepsilon_1 + \varepsilon_2 < 0$. By Lemma 1 part (a),

[12] For $0<\alpha\le 1$, $-1<\frac{\partial \varepsilon_2}{\partial \varepsilon_1}<0$. For $1<\alpha<2$, there are three possibilities. (1) If $k<\frac{\alpha+1}{\alpha-1}$, then $-1<\frac{\partial \varepsilon_2}{\partial \varepsilon_1}<0$. (2) If $k=\frac{\alpha+1}{\alpha-1}$, then $\frac{\partial \varepsilon_2}{\partial \varepsilon_1}=0$. (3) If $k>\frac{\alpha+1}{\alpha-1}$, then $0<\frac{\partial \varepsilon_2}{\partial \varepsilon_1}$.

$\frac{\alpha a^{\alpha}(n_1+\varepsilon_1+n_2+\varepsilon_2)}{(a^{\alpha}+1)^2} \le 0.25\alpha(n_1+\varepsilon_1+n_2+\varepsilon_2)$. Since $\varepsilon_1+\varepsilon_2<0$, $\frac{\alpha a^{\alpha}(n_1+\varepsilon_1+n_2+\varepsilon_2)}{(a^{\alpha}+1)^2} < 0.25\alpha(n_1+n_2)$. But this contradicts the second part of Lemma 1. We therefore obtain that the assumption $a<1$ cannot be satisfied. In equilibrium, then $a=1$ and $a>1$ cannot be satisfied. That is, $a>1$. By the balanced-budget constraint we get that $\varepsilon_2=-a^{\alpha}\varepsilon_1>-\varepsilon_1$ and this proves the second part of Proposition 2, that is, $\varepsilon_1+\varepsilon_2>0$. **Q.E.D**

P**roposition 3:** *The total efforts of the contestants corresponding to the optimal taxation scheme und er the APA are equa l to th e average prize valuation,* $G_A=0.5(n_1+n_2)$. *These total efforts are larger than or equal to those obtained under any Tullock-type lottery with* $0<\alpha\le 2$.

Proof: The proof will use the following lemma and its consequences.

Lemma 4: In equilibrium, for $0<\alpha\le 2$,

- **a.** $\varepsilon_1 \le 0 \le \varepsilon_2$.
- **b.** $a\ge 1$ and, therefore, $\varepsilon_1+\varepsilon_2\ge 0$.

Proof of Lemma 4:

a. See step 1 in the proof of Lemma 2.

b. Let us show that, in equilibrium, $(0<)a<1$ is impossible. Suppose to the contrary that $a<1$. This means that $\varepsilon_1<0<\varepsilon_2$. That is, the weak inequalities obtained in the first part of the lemma cannot be satisfied as equalities because such equalities mean that $\varepsilon_1=\varepsilon_2=0$ and so $a=\frac{n_1}{n_2}=k>1$, which contradicts the assumption $a<1$. By the balanced-budget constraint, $a^{\alpha}\varepsilon_1+\varepsilon_2=0$, the assumption $a<1$ and the conclusion that $\varepsilon_1<0<\varepsilon_2$, we get that $\varepsilon_2=-a^{\alpha}\varepsilon_1<-\varepsilon_1$ or $\varepsilon_1+\varepsilon_2<0$. This latter inequality means that the equilibrium efforts are smaller than $0.25\alpha(n_1+n_2)$, which contradicts part (b) of Lemma 1. **Q.E.D**

Recall that by Lemma 1 part (a), for $0<\alpha\le 2$, (1.A) $$\frac{\alpha a^{\alpha}\left(n_1+\varepsilon_1+n_2+\varepsilon_2\right)}{\left(a^{\alpha}+1\right)^2} \le 0.25\alpha\left(n_1+\varepsilon_1+n_2+\varepsilon_2\right)$$

Suppose now that the designer wishes to maximize the total efforts $0.25\alpha(n_1+\varepsilon_1+n_2+\varepsilon_2)$. That is, he faces the problem:

(2.A)
$$\begin{aligned}
&\underset{\varepsilon_1,\varepsilon_2,\alpha}{Max}\ \left(x_1^*+x_2^*\right)=0.25\alpha\left(n_1+\varepsilon_1+n_2+\varepsilon_2\right)\\
&s.t.\\
&1.\ a^{\alpha}\varepsilon_1+\varepsilon_2=0\\
&2.\ \alpha-1-a^{\alpha}\le 0\\
&3.\ (\alpha-1)a^{\alpha}-1\le 0\\
&4.\ \alpha-2\le 0\\
&5.\ n_1+\varepsilon_1>0\\
&6.\ n_2+\varepsilon_2>0
\end{aligned}$$

Let us show that the maximal efforts for this problem are equal to $0.5(n_1+n_2)$. Since this amount can be attained by a Tullock-type lottery with $\alpha=2$ (see Lemma 1 part (b)), inequality (1.A) implies that the maximal efforts under a Tullock-type lottery is also $0.5(n_1+n_2)$, which will complete the proof of Proposition 3.

Consider problem (2.A) and let $k>1$.[13] Since, by Lemma 4 part (b), in equilibrium, $a\ge 1$. The fulfillment of the constraint $a\ge 1$ and constraint 4 imply that constraint 2 is also satisfied. We can therefore omit constraint 2 and add the constraint $1-a\le 0$ to obtain the following equivalent designer's problem:

(3.A)
$$\begin{aligned}
&\underset{\varepsilon_1,\varepsilon_2,\alpha}{Max}\ \left(x_1^*+x_2^*\right)=0.25\alpha\left(n_1+\varepsilon_1+n_2+\varepsilon_2\right)\\
&s.t.\\
&1.\ a^{\alpha}\varepsilon_1+\varepsilon_2=0\\
&2.\ (\alpha-1)a^{\alpha}-1\le 0\\
&3.\ \alpha-2\le 0\\
&4.\ 1-a\le 0\\
&5.\ n_1+\varepsilon_1>0\\
&6.\ n_2+\varepsilon_2>0
\end{aligned}$$

The Lagrangian function is:

[13] For $k=1$, it can be shown that for every α satisfying $0<\alpha\le 2$, in equilibrium, $(\varepsilon_1,\varepsilon_2)=(0,0)$. This implies that in this case the designer will choose $\alpha=2$ and that total efforts will be equal to those obtained under the APA, that is , $n_1=n_2=n$. This also implies that for $0<\alpha<2$, total efforts do not exceed $0.25\alpha(n_1+n_2)=0.5\alpha n$, which is smaller than n.

$$L = 0.25\alpha(n_1 + \varepsilon_1 + n_2 + \varepsilon_2) + \nu_1(a^\alpha \varepsilon_1 + \varepsilon_2) + \mu_1[(1-\alpha)a^\alpha + 1]$$
$$+ \mu_2(2-\alpha) + \mu_3(a-1)$$

and, in addition, by Lemma 4 part (a), in equilibrium, $\varepsilon_1 \le 0 \le \varepsilon_2$ and $\varepsilon_1 + \varepsilon_2 \ge 0$. The following Kuhn-Tucker conditions must therefore be satisfied:

$$a^\alpha \varepsilon_1 + \varepsilon_2 = 0$$
$$\mu_1[(1-\alpha)a^\alpha + 1] = 0 \quad (1-\alpha)a^\alpha + 1 \ge 0 \quad \mu_1 \ge 0$$
$$\mu_2(2-\alpha) = 0 \quad 2-\alpha \ge 0 \quad \mu_2 \ge 0$$
$$\mu_3(a-1) = 0 \quad a-1 \ge 0 \quad \mu_3 \ge 0$$

Hence,

$$\frac{\partial L}{\partial \varepsilon_1} = 0.25\alpha + \nu_1\left(\frac{\alpha a^{\alpha-1}}{n_2+\varepsilon_2}\varepsilon_1 + a^\alpha\right) + \mu_1\frac{\alpha a^{\alpha-1}(1-\alpha)}{n_2+\varepsilon_2} + \frac{\mu_3}{n_2+\varepsilon_2} = 0$$

$$\frac{\partial L}{\partial \varepsilon_2} = 0.25\alpha + \nu_1\left(-\frac{\alpha a^\alpha}{n_2+\varepsilon_2}\varepsilon_1 + 1\right) + \mu_1\frac{\alpha a^\alpha(\alpha-1)}{n_2+\varepsilon_2} - \mu_3\frac{a}{n_2+\varepsilon_2} = 0$$

$$\frac{\partial L}{\partial \alpha} = 0.25(n_1 + \varepsilon_1 + n_2 + \varepsilon_2) + \nu_1 a^\alpha \varepsilon_1 \ln a + \mu_1[(1-\alpha)a^\alpha \ln a - a^\alpha] - \mu_2 = 0$$

$$\frac{\partial L}{\partial \nu_1} = a^\alpha \varepsilon_1 + \varepsilon_2 = 0$$

Suppose that, in equilibrium, $\alpha < 2$, so $\mu_2 = 0$. In this case, in equilibrium, $a > 1$ (see Lemma 3 part (b)) and, therefore, $\mu_3 = 0$. Given these requirements, let us consider the following two possibilities:

Possibility 1: $(1-\alpha)a^\alpha + 1 > 0$. In this case, $\mu_1 = 0$ and we therefore get that:

(4.A) $$\frac{\partial L}{\partial \varepsilon_2} = 0.25\alpha + \nu_1\left(-\frac{\alpha a^\alpha}{n_2+\varepsilon_2}\varepsilon_1 + 1\right) = 0$$

(5.A) $$\frac{\partial L}{\partial \alpha} = 0.25(n_1 + \varepsilon_1 + n_2 + \varepsilon_2) + \nu_1 a^\alpha \varepsilon_1 \ln a = 0$$

By (4.A), since $\varepsilon_1 \le 0 \le \varepsilon_2$, we get that:

(6.A) $$\nu_1 = \frac{0.25\alpha}{\frac{\alpha a^\alpha}{n_2+\varepsilon_2}\varepsilon_1 - 1} < 0$$

and, by (5.A), we get that:

(7.A) $$0.25(n_1 + \varepsilon_1 + n_2 + \varepsilon_2) = -\nu_1 a^\alpha \varepsilon_1 \ln a$$

Since, in equilibrium, $\varepsilon_1 + \varepsilon_2 \geq 0$ (Lemma 4 part (b)), the LHS expression in (7.A) is positive, so the RHS expression must be positive. Since $\varepsilon_1 \leq 0 \leq \varepsilon_2$ and $a > 1$ (Lemma 4 part (a) and lemma 3 part (b)), $\varepsilon_1 < 0$, because otherwise the RHS in (7.A) equals zero. But this implies that if $\left(-\nu_1 a^{\alpha} \varepsilon_1 \ln a\right)$ is positive, then $\nu_1 > 0$ (since $\varepsilon_1 < 0$ and $\ln a > 0$), which contradicts inequality (6.A). We have thus obtained that, in equilibrium, $\alpha < 2$ and $(1-\alpha)a^{\alpha} + 1 > 0$ cannot hold.

Possibility 2: $(1-\alpha)a^{\alpha} + 1 = 0$ (recall that we have assumed that $\alpha < 2$ and, therefore, $a > 1$). Note that this possibility requires that $\alpha > 1$ and that

(8.A) $$\frac{n_1 + \varepsilon_1}{n_2 + \varepsilon_2} = (\alpha - 1)^{-\frac{1}{\alpha}}$$

Since, by the balanced-budget constraint, $\left(a^{\alpha} = \left(\frac{n_1 + \varepsilon_1}{n_2 + \varepsilon_2}\right)^{\alpha} = -\frac{\varepsilon_2}{\varepsilon_1}\right.$, in equilibrium,

$-\frac{\varepsilon_2}{\varepsilon_1} = (\alpha - 1)^{-1}$ or:

(9.A) $$\varepsilon_2 = -\varepsilon_1(\alpha - 1)^{-1}$$

By (8.A) and (9.A), we get the taxation scheme $(\varepsilon_1, \varepsilon_2)$:

(10.A) $$\varepsilon_1 = \frac{n_2(\alpha-1) - n_1(\alpha-1)^{\frac{1}{\alpha}+1}}{(\alpha-1)^{\frac{1}{\alpha}+1} + 1}, \quad \varepsilon_2 = \frac{n_1(\alpha-1)^{\frac{1}{\alpha}} - n_2}{(\alpha-1)^{\frac{1}{\alpha}+1} + 1}$$

Let us show that, in equilibrium, this taxation scheme is impossible. From (9.A) we get that $\varepsilon_2(\alpha - 1) + \varepsilon_1 = 0$ and, therefore, $\varepsilon_2(\alpha - 2) + \varepsilon_1 + \varepsilon_2 = 0$ or $\varepsilon_1 + \varepsilon_2 = \varepsilon_2(2 - \alpha)$. We therefore get that:

(11.A) $$0.25\alpha(n_1 + \varepsilon_1 + n_2 + \varepsilon_2) = 0.25\alpha[n_1 + n_2 + \varepsilon_2(2-\alpha)]$$

Notice that the selection of $(\varepsilon_1, \varepsilon_2) = (-0.5(n_1 - n_2), 0.5(n_1 - n_2))$ and $\alpha = 2$ satisfies all the constraints in (3.A) and yield total efforts that are equal to $0.5(n_1 + n_2)$. Let us show that the total efforts in our case are smaller than this amount and this contradiction would imply that the assumption that, in equilibrium, $(1-\alpha)a^{\alpha} + 1 = 0$, together with $\alpha < 2$, is impossible. Given (11.A), we have to show that $0.25\alpha[n_1 + n_2 + \varepsilon_2(2-\alpha)] < 0.5(n_1 + n_2)$. This latter inequality can be written as $\alpha\varepsilon_2(2-\alpha) < (n_1 + n_2)(2-\alpha)$ and since $1 < \alpha < 2$, we have to prove that, in equilibrium, $\alpha\varepsilon_2 < n_1 + n_2$.

By substituting ε_2, see (10.A), in the last inequality we get that we have to prove the following inequality:

$$\frac{\alpha\left[n_1(\alpha-1)^{\frac{1}{\alpha}}-n_2\right]}{(\alpha-1)^{\frac{1}{\alpha}+1}+1}<n_1+n_2$$

or

(12.A) $$n_1\frac{\alpha(\alpha-1)^{\frac{1}{\alpha}}}{(\alpha-1)^{\frac{1}{\alpha}+1}+1}-n_2\frac{\alpha}{(\alpha-1)^{\frac{1}{\alpha}+1}+1}<n_1+n_2$$

Finally, let us prove inequality (12.A) by showing that the coefficient of n_1 in the LHS of (12.A) is smaller than 1. We have to show then that $\frac{\alpha(\alpha-1)^{\frac{1}{\alpha}}}{(\alpha-1)^{\frac{1}{\alpha}+1}+1}<1$ or $\alpha(\alpha-1)^{\frac{1}{\alpha}}<(\alpha-1)^{\frac{1}{\alpha}+1}+1$ or $\alpha(\alpha-1)^{\frac{1}{\alpha}}<(\alpha-1)(\alpha-1)^{\frac{1}{\alpha}}+1$ or $(\alpha-1)^{\frac{1}{\alpha}}<1$. Since $1<\alpha<2$,the last inequality is satisfied and, therefore, inequality (12.A) is also satisfied.

The conclusion from the two possibilities is that, in equilibrium, $\alpha=2$. By constraints 2 and 4 in problem (3.A), this implies that $a=1$. Therefore, by the balanced-budget constraint, we get that $\varepsilon_1+\varepsilon_2=0$ and, since $a=1$, we get that, in equilibrium, $(\varepsilon_1,\varepsilon_2)=(-0.5(n_1-n_2),0.5(n_1-n_2))$. The maximal total efforts are therefore equal to $0.5(n_1+n_2)$. That is, for any α, $1<\alpha<2$, the maximal efforts are smaller than $0.5(n_1+n_2)$. **Q.E.D**