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Abstract

Evaluation of climate policies and other issues requires a variable population setting where population is endogenously determined. We propose and axiomatize the rank-discounted critical-level utilitarian social welfare order. It is shown to fill out the space between critical-level utilitarianism and (a version of) critical-level leximin. Moreover, it satisfies many conditions and principles used to evaluate variable population criteria. In particular, it avoids the repugnant conclusion even when the critical level is zero.

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1 Introduction

Evaluation of climate policies, and indeed of a number of other policy issues, requires a variable population setting where population is endogenously determined. How can allocations of wellbeing be evaluated when population varies between the alternatives? Since Parfit's (1976; 1982; 1984) critique of total utilitarianism — showing that it leads to the repugnant conclusion of favoring large populations with low wellbeing — this has been the topic of axiomatic population ethics.

Even though the repugnant conclusion as such can easily be avoided, by e.g. average utilitarianism and critical-level utilitarianism, such alternatives criteria have their own short-comings, as discussed extensively by Arrhenius (2012). The current paper contributes to the field of population ethics by proposing a criterion that overcomes some difficulties of previous criteria.

In Zuber and Asheim (2012) we have introduced the concept of rank-discounted utilitarianism, motivated in particular by the question of how to evaluate climate policies from an ethical, impartial perspective. In that paper we introduce rank-discounted utilitarianism in a deterministic setting where population growth is not explicitly discussed. Rank-discounted utilitarianism can, however, in a straightforward manner be generalized to a situation where population size changes exogenously over time, by letting individuals rather than generations be the object of analysis.

A much more challenging situation to analyze is where population changes endogenously, e.g., as a consequence of climate change. Climate change may prevent the existence of a great many people who would otherwise have existed (Broome, 2010). How can we take into account in our evaluation the loss of such potential lives? Broome's (2004) argument for the position that one cannot simply ignore this effect when evaluating climate change is convincing. Rather, it seems natural to assume that there exists a critical level of well-being which, if experienced by an added individual without changing the wellbeing levels of the existing population, leads to an alternative which is as good as the original (Blackorby, Bossert and Donaldson, 2005).

In this paper we provide a solution to the problem of how to merge critical-level population ethics with rank-discounted utilitarianism, leading to the notion of *rank-discounted critical level utilitarianism* (RDCLU). As is the case for Ng's (1989) theory X and Sider's (1991) principle GV, RDCLU is a *variable value principle*, in the sense that the value of an egalitarian population does not change affinely with population size, and a *context sensitive theory*, in the sense that the contributive value of a life depends on the wellbeing of the rest of the population. However, in contrast to Ng's theory X the value of each individual depends on its rank, and in contrast to Sider's principle GV, the values are assigned in a prioritarian manner.

After presenting our formal framework (Section 2), we provide an axiomatic foundation (Section 3) and show how the RDCLU social welfare order fills out the space between critical-level utilitarianism and (a version of) critical-level leximin (Section 4). Moreover, we illustrate how it satisfies many conditions and principles used to evaluate variable population criteria (Section 5). Finally, we illustrate in a context of a simple model of optimal population size how RDCLU escapes the repugnant conclusion (Section 6) and note that the problem of accommodating uncertainty still remains (Section 7). Proofs are contained in an appendix.

2 Framework

Let Λ be a non-empty and countable set of locations, which can be interpreted as locations in space and time. We allow Λ to be infinite. At each location, $\lambda \in \Lambda$, there is either one individual, with $\mathbf{x}(\lambda) \in Y$ indicating the lifetime wellbeing of the individual, or no individual, in which case $\mathbf{x}(\lambda) = \emptyset$. Refer to the mapping $\mathbf{x} : \Lambda \to \{\emptyset\} \cup Y$ as an *allocation*, and say that $\mathbf{x}$ is *finite* if the set $\Lambda(\mathbf{x}) := \{\lambda \in \Lambda : \mathbf{x}(\lambda) \neq \emptyset\}$ of inhabited locations is non-empty and finite. Let $\mathbf{X}$ be the set of finite allocations, and denote, for given finite allocation $\mathbf{x} \in \mathbf{X}$, the finite number $|\Lambda(\mathbf{x})|$ of individuals by $n(\mathbf{x})$.

We assume that $[a, b] \subseteq Y \subseteq \mathbb{R}$, where $a < 0 < b$ and, following the usual convention in population ethics, $\mathbf{x}(\lambda) = 0$ represents *neutrality*. Hence, lifetime wellbeing is normalized so that above neutrality, a life, as a whole, is worth living; below neutrality, it is not. Note that we assume that 0 is an interior point in Y so that the analysis allows from lives subjectively not worth living, with negative wellbeing.

A social welfare relation (SWR) on a set $\mathbf{X}$ is a binary relation $\succsim$, where for any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ implies that the allocation $\mathbf{x}$ is deemed socially at least as good as $\mathbf{y}$. Let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$. A social welfare function (SWF) representing $\succsim$ is a mapping $W : \mathbf{X} \to \mathbb{R}$ with the property that for any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $W(\mathbf{x}) \geq W(\mathbf{y})$ if and only if $\mathbf{x} \succsim \mathbf{y}$.

In our analysis different locations correspond to different individuals. If the same individual lives at different locations (in particular, extends through time), then the analysis need to be appropriately adjusted. The question of how to make such an adjustment is not addressed here.

Let $\pi : \{1, \ldots, n(\mathbf{x})\} \to \Lambda(\mathbf{x})$ be a bijection that assigns to the individual with rank r the location it inhabits:

$$\mathbf{x}(\pi(r)) \leq \mathbf{x}(\pi(r+1)) \quad \text{for all } r \in \{1, \ldots, n(\mathbf{x}) - 1\} .$$

Write $x_{[r]} = \mathbf{x}(\pi(r))$ for all $r \in \{1, \ldots, n(\mathbf{x})\}$. Even though π need not be uniquely determined if different individuals have the same wellbeing level, the resulting rank-ordered allocation, $\mathbf{x}_{[]} = (x_{[1]}, \ldots, x_{[r]}, \ldots, x_{[n(\mathbf{x})]})$, is uniquely determined.

For every $n \in \mathbb{N}$, write $\mathbf{X}_n = \{\mathbf{x} \in \mathbf{X} : n(\mathbf{x}) = n\}$ for the set of finite allocations with population size equal to n. Let $(z)_n$ denote $\mathbf{x} \in \mathbf{X}_n$ with $x_{[r]} = z$ for all $r \in \{1, \ldots, n\}$. Let $(\mathbf{x}, z)$ denote $\mathbf{x} \in \mathbf{X}$ with $z \in Y$ added at an uninhabited location. Let $\big(\mathbf{x}, (z)_n\big)$ denote $\mathbf{x} \in \mathbf{X}$ with $z \in Y$ added at n uninhabited location.

3 Axioms and representation result

Rank-discounted critical-level utilitarianism can be characterized by the following seven axioms. The first three axioms are sufficient to ensure numerical representation for any fixed population size, and entails that individuals are treated anonymously and with sensitivity for their well-being.

Axiom 1 (Order) *The relation* $\succsim$ *is complete, reflexive and transitive on* $\mathbf{X}$.

An SWR satisfying Axiom 1 is called a social welfare order (SWO).

Axiom 2 (Continuity) *For all* $n \in \mathbb{N}$ *and* $\mathbf{x} \in \mathbf{X}_n$, *the sets* $\{\mathbf{y} \in \mathbf{X}_n : \mathbf{y} \succsim \mathbf{x}\}$ *and* $\{\mathbf{y} \in \mathbf{X}_n : \mathbf{x} \succsim \mathbf{y}\}$ *are closed.*

Axiom 3 (Suppes-Sen) *For all* $n \in \mathbb{N}$ *and* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n$, *if* $\mathbf{x}_{[]} > \mathbf{y}_{[]}$, *then* $\mathbf{x} \succ \mathbf{y}$.

While ordinary critical-level utilitarianism allows for unrestricted independence to adding an individual (see Blackorby, Bossert and Donaldson, 2005), our axioms impose such independence only if the added individual is best-off (relative to two allocations with the same population size) or worst-off.

Axiom 4 (Existence independence of the best-off) *For all* $n \in \mathbb{N}$, $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n$ *and* $z \in Y$ *satisfying* $z \geq \max\{x_{[n]}, y_{[n]}\}$, $(\mathbf{x}, z) \succsim (\mathbf{y}, z)$ *if and only if* $\mathbf{x} \succsim \mathbf{y}$.

Axiom 5 (Existence independence of the worst-off) *For all* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *and* $z \in Y$ *satisfying* $z \leq \min\{x_{[1]}, y_{[1]}\}$, $(\mathbf{x}, z) \succsim (\mathbf{y}, z)$ *if and only if* $\mathbf{x} \succsim \mathbf{y}$.

In the spirit of critical-level utilitarianism, we introduce a critical wellbeing level $c \in Y \cap \mathbb{R}_+$, which if experienced by an added individual without changing the utilities of the existing population, leads to an alternative which is as good as the original. However, the following axiom imposes this if $x_{n(\mathbf{x})} \leq c$, not otherwise. Since $a < 0 \leq c$ and $a \in Y$, c is as large as the neutral wellbeing level and strictly larger than the greatest lower bound for the set of wellbeing levels.

Axiom 6 (Existence of a critical level) *There exist* $c \in Y \cap \mathbb{R}_+$ *and* $n \in \mathbb{N}$ *such that, for all* $\mathbf{x} \in \mathbf{X}_n$ *satisfying* $x_{[n]} \leq c$, $(\mathbf{x}, c) \sim \mathbf{x}$.

All axioms above are satisfied also by ordinary critical-level utilitarianism. However, as discussed by Arrhenius (2012, Sect. 5.1), critical-level utilitarianism has the properties that adding sufficiently many individuals with wellbeing just above c makes the allocation better than any fixed alternative, and adding sufficiently many individuals with wellbeing just below c makes the allocation worse than any fixed alternative. These properties might be considered extreme. The following axiom ensures that adding individuals at a given wellbeing level has bounded importance.

Axiom 7 (Constant-equivalence) *For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if $\mathbf{x} \succ \mathbf{y}$, then there exist $z \in Y$ and $N \in \mathbb{N}$ such that $\mathbf{x} \succ (z)_n \succ \mathbf{y}$ for all $n \geq N$.*

We will now state our main result (proven in the appendix), namely that these seven axioms characterize the rank-discounted critical-level utilitarian SWO.

Definition 1 *An SWR on $\mathbf{X}$ is a rank-discounted critical-level utilitarian SWO (RDCLU SWO) if it is represented by an SWF $W : \mathbf{X} \to \mathbb{R}$ defined by:*

$$W(\mathbf{x}) = \sum\nolimits_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c) \right) \,,$$

where $\beta \in (0, 1)$ is a rank utility discount factor and $u : Y \to \mathbb{R}$ is a continuous and increasing utility function.

Theorem 1 *Consider an SWR $\succsim$ on $\mathbf{X}$. The following two statements are equivalent.*

(1) $\succsim$ satisfies Axioms 1–7.

(2) $\succsim$ is an RDCLU SWO.

It follows from the RDCLU SWO that c is the wellbeing level which, if experienced by an added individual without changing the utilities of the existing population, leads to an alternative which is as good as the original *only if* $x_{[n(\mathbf{x})]} \leq c$. If $x_{[n(\mathbf{x})]} > c$, then there is a context-dependent critical level in the open interval $(c, x_{[n(\mathbf{x})]})$ which depends on the wellbeing levels that exceed c. This follows from Definition 1, since adding an individual at wellbeing level $x_{[n(\mathbf{x})]}$ increases welfare, while adding an individual at wellbeing level c lowers the weights assigned to individuals at wellbeing levels that exceed c and thereby reduces welfare.

4 Limits of rank-discounted critical-level utilitarianism

In this section we show that critical-level rank-discounted utilitarianism fills out the space between critical-level utilitarianism and (a version of) critical-level leximin. Before stating and (in the appendix) proving this result, we first define these SWOs.

Definition 2 An SWR $\succsim_{u,c}$ on $\mathbf{X}$ is a *critical-level utilitarian* SWO (CLU SWO) if $\succsim_{u,c}$ is represented by an SWF $w : \mathbf{X} \to \mathbb{R}$ defined by:

$$w(\mathbf{x}) = \sum_{r=1}^{n(\mathbf{x})} \big(u(x_{[r]}) - u(c)\big) \ ,$$

where $u : Y \to \mathbb{R}$ is a continuous and increasing utility function and $c \geq 0$.

What we call CLU coincides with Blackorby, Bossert and Donaldson's *generalized critical-level utilitarianism*, which they have discussed and axiomatized in Blackorby and Donaldson (1984) and Blackorby, Bossert and Donaldson (1995, 2005).

Definition 3 An SWR $\succsim_c^L$ on $\mathbf{X}$ is a *critical-level leximin* SWO (CLL SWO) if, for any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $n(\mathbf{x}) \geq n(\mathbf{y})$,

(a) $\mathbf{x} \sim_c^L \mathbf{y}$ if and only if $(x_{[1]}, \dots, x_{[n(\mathbf{y})]}) = \mathbf{y}_{[\,]}$ and $(x_{[n(\mathbf{y})+1]}, \dots, x_{[n(\mathbf{x})]}) = (c)_{n(\mathbf{x})-n(\mathbf{y})}$,

(b) $\mathbf{x} \succ_c^L \mathbf{y}$ if and only if (i) there exists $R \in \{1, \dots, n(\mathbf{y})\}$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1, \dots, R-1\}$ and $x_{[R]} > y_{[R]}$ or (ii) $(x_{[1]}, \dots, x_{[n(\mathbf{y})]}) = \mathbf{y}_{[\,]}$ and $(x_{[n(\mathbf{y})+1]}, \dots, x_{[n(\mathbf{x})]}) > (c)_{n(\mathbf{x})-n(\mathbf{y})}$, and

(c) $\mathbf{x} \prec_c^L \mathbf{y}$ if and only if (i) there exists $R \in \{1, \dots, n(\mathbf{y})\}$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1, \dots, R-1\}$ and $x_{[R]} < y_{[R]}$ or (ii) $(x_{[1]}, \dots, x_{[n(\mathbf{y})]}) = \mathbf{y}_{[\,]}$ and $x_{[n(\mathbf{y})+1]} < c$.

Note that this SWO differs from the critical-level leximin SWO proposed and axiomatized by Blackorby, Bossert and Donaldson (1996) when comparing allocations with different numbers of individuals. Their SWO is defined as follows: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $n(\mathbf{x}) \geq n(\mathbf{y})$ (where $\tilde{\mathbf{y}}$ denotes $\big(\mathbf{y}, (c)_{n(\mathbf{x})-n(\mathbf{y})}\big)$), (a) $\mathbf{x} \sim \mathbf{y}$ if and only if

$\mathbf{x}_{[]} = \tilde{\mathbf{y}}_{[]}$, (b) $\mathbf{x} \succ \mathbf{y}$ if and only if there exists $R \in \{1, \ldots, n(\mathbf{x})\}$ such that $x_{[r]} = \tilde{y}_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_{[R]} > \tilde{y}_{[R]}$, and (c) $\mathbf{x} \prec \mathbf{y}$ if and only if there exists $R \in \{1, \ldots, n(\mathbf{x})\}$ such that $x_{[r]} = \tilde{y}_{[r]}$ for all $r \in \{1, \ldots, R-1\}$ and $x_{[R]} < \tilde{y}_{[R]}$.[1]

Write $\succsim_{\beta,u,c}$ for the RDCLU SWO characterized by β, u and c. The following result establishes that, for any increasing and continuous function u, the CLU SWO $\succsim_{u,c}$ is the limit of the RDCLU SWO $\succsim_{\beta,u,c}$ as β approaches 1 and the CLL SWO $\succsim_c^L$ is the limit of the RDCLU SWO $\succsim_{\beta,u,c}$ as β approaches 0. Note that in the case of $\succsim_c^L$, the weak preference of $\succsim_{\beta,u,c}$ for small β is both sufficient and necessary.

Proposition 1 *For any* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *and any continuous and increasing function* u*,*

(i) $\mathbf{x} \succsim_{u,c} \mathbf{y}$ *if there exists* $\bar{\beta} \in (0,1)$ *such that, for all* $\beta \in (\bar{\beta}, 1)$*,* $\mathbf{x} \succsim_{\beta,u,c} \mathbf{y}$*.*

(ii) $\mathbf{x} \succsim_c^L \mathbf{y}$ *if and only if there exists* $\underline{\beta} \in (0,1)$ *such that, for all* $\beta \in (0, \underline{\beta})$*,* $\mathbf{x} \succsim_{\beta,u,c} \mathbf{y}$*.*

The case where the rank-discount factor β approaches 0, and thereby the RDCLU SWO approaches the CLL SWO, is related to the case where a given allocation is replicated in the following sense: For any $\mathbf{x} \in \mathbf{X}$ and any $k \in \mathbb{N}$, the *k-replica* of $\mathbf{x}$ is an allocation $\mathbf{x}^k$ with $kn(\mathbf{x})$ individuals having the property that $x_{[r]} = x^k_{[\rho]}$ for all $r \in \{1, \ldots, n(\mathbf{x})\}$ and $\rho \in \{k(r-1)+1, \ldots, kr\}$. Proposition 1 implies that, for fixed $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and $\beta \in (0,1)$, there exists $K \in \mathbb{N}$ such that, for all $k > K$, $\mathbf{x}^k \succsim_{\beta,u,c} \mathbf{y}^k$ if and only if $\mathbf{x} \succsim_c^L \mathbf{y}$. Hence, as a given allocation is replicated, utility weight is redistributed towards the individuals with lowest wellbeing.

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[1] Both definitions have drawbacks. Under our definition — which is also suggested by Arrhenius (2012, Sect. 6.8) — any finite allocation $\mathbf{x}$ is worse than a one-individual allocation with wellbeing above $x_{[1]}$. This leads to an extreme version of Arrhenius' (2012) Reversed Repugnant Conclusion. Under Blackorby, Bossert and Donaldson's (1996) definition, a finite egalitarian allocation with a wellbeing level much above c is worse than an egalitarian allocation with wellbeing level barely above c if the latter contains an additional individual. This leads to an extreme version of Arrhenius' (2012) Weak Repugnant Conclusion. See Section 5 for more on different kinds of repugnant conclusions.

5 Evaluating rank-discounted critical-level utilitarianism

We now evaluate the RDCLU SWO by means of conditions and principles suggested in the literature on population ethics. The conditions referred to below are reproduced in the form they are presented in Arrhenius (2012), except that terminology is adjusted to reflect terms used in this paper. As before we assume that 0 is an interior point in Y so that the analysis allows for lives subjectively not worth living, with negative wellbeing.

For the sake of completeness let us first introduce the average utilitarian social welfare ordering which has been much discussed in the literature.

Definition 4 An SWR $\succsim_u^A$ on $\mathbf{X}$ is an *average utilitarian* SWO (AU SWO) if $\succsim_u^A$ is represented by an SWF $\bar{w} : \mathbf{X} \to \mathbb{R}$ defined by:

$$\bar{w}(\mathbf{x}) = \tfrac{1}{n(\mathbf{x})} \sum\nolimits_{r=1}^{n(\mathbf{x})} u(x_{[r]}) \,,$$

where $u : Y \to \mathbb{R}$ is a continuous and increasing utility function.

Note that the average utilitarian social welfare ordering is not affected by how $u(0)$ is normalized, and thus not affected by the introduction of a critical level c either.

We start by a conclusion argued to be repugnant by Parfit (1976, 1982, 1984), and which has been used to criticize total utilitarianism:

> *The Repugnant Conclusion:* For any egalitarian allocation with very high positive wellbeing, there is an allocation with very low positive wellbeing which is better.

If $c > 0$, then the RDCLU SWO clearly escapes the repugnant conclusion, and so does critical-level utilitarianism. However, in contrast to critical-level utilitarianism, the RDCLU SWO avoids the repugnant conclusion even if $c = 0$, as shown below.

Also if $c > 0$, critical-level utilitarianism is subject to the following weaker form of the repugnant conclusion:

The Weak Repugnant Conclusion: For any egalitarian allocation with very high positive wellbeing, there is an allocation with wellbeing just above the critical level which is better.

In contrast, due to rank discounting, the weak repugnant conclusion does *not* follow from the RDCLU SWO.[2] The formal argument is as follows: If $x > y > c$, then there exists $n \in \mathbb{N}$ determined by the requirement that

$$\tfrac{1-\beta^n}{1-\beta}\left(u(x) - u(c)\right) \geq \tfrac{1}{1-\beta}\left(u(y) - u(c)\right) ,$$

such that if $n(\mathbf{x}) = n$ and and $x_{[r]} = x$ for all $r \in \{1, \ldots, n\}$, then $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ for any $\mathbf{y} \in \mathbf{X}$ with $y_{[r]} = y$ for all $r \in \{1, \ldots, n(\mathbf{y})\}$, independently of large the population size $n(\mathbf{y})$ is. By Definition 1, $\mathbf{x} \in \mathbf{X}$ is as good as any allocation with wellbeing just above the critical level. Moreover, the conclusion that the RDCLU SWO avoids the repugnant conclusion even if $c = 0$ follows since this argument holds also if $c = 0$.

Average utilitarianism leads to following problematic conclusion:

The Reverse Repugnant Conclusion: For any egalitarian allocation with very high positive wellbeing, there is a better one-individual allocation with slightly higher wellbeing.

This conclusion does not follow from the RDCLU SWO as

$$\left(u(x) - u(c)\right) < \tfrac{1-\beta^n}{1-\beta}\left(u(y) - u(c)\right)$$

is clearly consistent with $x > y > c$ — provided that y is sufficiently close to x and n is sufficiently large — where $\mathbf{x}$ with $n(\mathbf{x}) = 1$ and $x_{[1]} = x$ is the one-individual allocation and $\mathbf{y}$ with $n(\mathbf{y}) = n$ and $y_{[r]} = x$ for all $r \in \{1, \ldots, n\}$ is the egalitarian

[2] In our discussion of the Weak Repugnant Conclusion, we associate its mentioning of "the critical level" with the parameter c. As pointed out at the end of Section 3, in the case of the RDCLU SWO, c is the critical level — in the sense of the wellbeing level which, if experienced by an added individual without changing the utilities of the existing population, leads to an alternative which is as good as the original — only if no individual has a wellbeing level that exceeds c.

allocation with very high positive wellbeing. The conclusion does not follow from the CLU SWO either.

Other conditions and principles have been discussed in the literature on variable population ethics. As analyzed in Arrhenius (2012, Sect. 3.8), the AU SWO fails the following condition:

> *The Strong Quality Addition Principle:* There is an egalitarian allocation with very high wellbeing such that its addition to any allocation $\mathbf{x}$ is at least as good as an addition of any allocation with very low positive wellbeing to $\mathbf{x}$.

In contrast, both the RDCLU SWO and the CLU SWO with $c \geq 0$ satisfy this condition (just let the egalitarian allocation be a one-person population with wellbeing above c).

Moreover, the RDCLU SWO and the CLU SWO violate the first (if $c > 0$), but satisfy the second (even if $c = 0$), of the following two conditions, both of which are violated by average utilitarianism:

> *The Mere Addition Principle:* An addition of individuals with positive wellbeing does not make an allocation worse.

> *The Negative Mere Addition Principle:* An addition of individuals with negative wellbeing makes an allocation worse.

While the CLU SWO with $c = 0$ satisfies also the Mere Addition Principle, the RDCLU SWO with $c = 0$ does not. The reason is that adding an individual with low positive wellbeing will decrease the utility weights on individuals with higher wellbeing and might thereby worsen the allocation.

Arrhenius (2012) suggests the following weak version of the Pigou-Dalton transfer principle:

> *The Non-Anti Egalitarianism Principle:* An egalitarian allocation is better than an allocation with the same number of individuals, inequality, and lower average (and thus lower total) wellbeing.

Table 1: Population principles and social welfare orderings

	RDCLU $c \geq 0$	CLU $c = 0$	CLU $c > 0$	AU
Avoiding Repugnant Conclusion	$+$	$-$	$+$	$+$
Avoiding Weak Repugnant Conclusion	$+$	$-$	$-$	$+$
Avoiding Reverse Repugnant Conclusion	$+$	$+$	$+$	$-$
Strong Quality Addition Principle	$+$	$+$	$+$	$-$
Mere Addition Principle	$-$	$+$	$-$	$-$
Negative Mere Addition Principle	$+$	$+$	$+$	$-$
Non-Anti Egalitarianism Principle	$+^{a}$	$+^{b}$	$+^{b}$	$+^{b}$
Weak Non-Sadism Conclusion	$+^{c}$	$+$	$-$	$-$

(a) If $\beta \times \mathcal{C}_u \leq 1$. ($b$) If u is concave. (c) If Y is bounded above.

This is compatible with the RDCLU SWO: Let u satisfy $\beta \times \mathcal{C}_u \leq 1$, where

$$\mathcal{C}_u = \sup_{0<\varepsilon\leq x\leq x'} \frac{u\big(x' + \varepsilon\big) - u\big(x'\big)}{u\big(x\big) - u\big(x - \varepsilon\big)}$$

is an index of non-concavity of the function u. The Non-Anti Egalitarianism Principle is also compatible with the CLU SWO and the AU SWO if u is concave.

With the possible exception of the Mere Addition Principle, this shows that the RDCLU SWO has desirable properties when evaluated by these conditions and principles. We end by noting that the RDCLU SWO with $c \geq 0$ satisfies even the following non-sadism condition, provided that Y is bounded above.

> *The Weak Non-Sadism Condition:* There is a negative wellbeing level and a number of individuals at this level such that an addition of any number of individuals with positive wellbeing is at least as good as an addition of the individuals with negative wellbeing, for any initial allocation.

To see this, let m denote $\sup Y$ and choose $y \in Y \cap \mathbb{R}_{--}$ and $n \in \mathbb{N}$ such that

$$\tfrac{1-\beta^n}{1-\beta}\left(u(y) - u(0)\right) + \tfrac{\beta^n}{1-\beta}\left(u(m) - u(0)\right) \leq 0\,. \tag{1}$$

Then it follows from the fact that $u(m) \geq u(c) \geq u(0)$ that, for any initial allocation $\mathbf{x} \in \mathbf{X}$, adding n individuals with wellbeing y to $\mathbf{x}$ is worse than adding any number of individuals with positive wellbeing to $\mathbf{x}$. This obtains since eq. (1) implies

$$\tfrac{1-\beta^n}{1-\beta}\left(u(y) - u(c)\right) + \tfrac{\beta^n}{1-\beta}\left(u(m) - u(c)\right) \leq \tfrac{1}{1-\beta}\left(u(0) - u(c)\right) ,$$

where the r.h.s. is smaller than the welfare of any allocation where all individuals have positive wellbeing. These positive conclusions are of interest in view of Arrhenius (2012, Sect. 11.14), where it is shown that the Weak Non-Sadism Condition cannot be satisfied unless other desirable properties are given up.[3]

Table 1 summarizes the different principles and conclusions satisfied (or not) by the different social welfare orderings discussed in this section, where the sign '+' denotes that the principle is satisfied (or the conclusion follows or, in the first three lines, the conclusion is avoided) and the sign '−' denotes the opposite.

6 Optimal population size

Following Dasgupta (1988, pp. 123–125), let m be the total available amount of a consumption good, and let, as before, n denote the number of individuals. Let the wellbeing of each individual be equal to allocated consumption minus s, implying that s is the level of consumption needed to attain neutrality. Hence, a life is worth living if consumption exceeds s, while it is not if consumption falls below s.

Under AU, the optimal population size n is equal to 1, as this maximizes average utility. Turn now to CLU and RDCLU. Under the assumption that u is concave, it is optimal to divide the available amount m equally among the n individuals. Hence, the so-called *genesis problem* is to optimize n given that each individual's wellbeing $x(n)$ equals $\frac{m}{n} - s$, with n treated as a continuous variable, for tractability. Dasgupta (2005) argues that the genesis problem might not be the most interesting

[3]These properties of the RDCLU SWO do not, however, contradict Arrhenius (2012, Sixth Impossibility Theorem) as the General Non-Extreme Priority Condition – not discussed here – is not satisfied by the RDCLU SWO.

problem for population ethics. It is also different from the problem studied by Palivos and Yip (1993) and Razin and Yuen (1995), where the development of per capita wellbeing and population size is optimized within models of economic growth. Still, it is illustrative and leads to generalizable insights (cf. Dasgupta, 1988, fn. 16).

Under CLU, the genesis problem becomes

$$\max_n n\left(u\left(\tfrac{m}{n}-s\right)-u(c)\right) ,$$

leading to the first-order condition

$$u(x(n))-u(c)=(x(n)+s)u'(x(n))\,. \tag{2}$$

If $u(x)=\frac{1}{1-\eta}(x+s)^{1-\eta}$ with $\eta>1$, then (2) can be transformed to

$$\frac{x(n)+s}{c+s}=\eta^{\frac{1}{\eta-1}}\,.$$

As the elasticity of marginal utility η goes to infinity, $\eta^{\frac{1}{\eta-1}}$ goes to unity, illustrating how CLU leads to the repugnant conclusion if $c=0$ and to the weak repugnant conclusion otherwise.

Under RDCLU, the genesis problem becomes

$$\max_n \tfrac{1-\beta^n}{1-\beta}\left(u\left(\tfrac{m}{n}-s\right)-u(c)\right) ,$$

leading to the first-order condition

$$\gamma(n,\beta)\left(u(x(n))-u(c)\right)=(x(n)+s)u'(x(n))\,, \tag{3}$$

where

$$\gamma(n,\beta):=\frac{\beta^n(-\ln\beta^n)}{1-\beta^n}$$

can be shown to satisfy $0<\gamma(n,\beta)<1$, $\frac{\partial\gamma}{\partial n}<0$, $\frac{\partial\gamma}{\partial\beta}>0$ and $\lim_{\beta\to1}\gamma(n,\beta)=1$. If $u(x)=\frac{1}{1-\eta}(x+s)^{1-\eta}$ with $\eta>1$, then (3) can be transformed to

$$\frac{x(n)+s}{c+s}=\left(\frac{\gamma(n,\beta)+\eta-1}{\gamma(n,\beta)}\right)^{\frac{1}{\eta-1}}\,.$$

The l.h.s. is a decreasing function of n which equals 1 for $n=\frac{m}{c+s}$ and approaches ∞ as $n\downarrow 0$. The r.h.s. is greater than $\eta^{\frac{1}{\eta-1}}>1$ and an increasing function of n —

implying that the first-order condition determines a unique optimal value of n — and a decreasing function of β which approaches $\eta^{\frac{1}{\eta-1}}$ as $\beta \to 1$ — implying that this optimal population size is lower under RDCLU than under CLU. Thus, this analysis illustrates how RDCLU leads to an escape from the repugnant conclusion.

7 Concluding remarks

We have contributed to population ethics by proposing and axiomatizing the rank-discounted critical-level utilitarian (RDCLU) SWO. By doing so we have taken one step towards preparing the rank-discounted utilitarian criterion (see Zuber and Asheim, 2012) for practical use.

First of all, we have generalized rank-discounted utilitarianism by letting individuals rather than generations be the object of analysis. This generalization has several implications, one of which is particularly interesting to point out: If there is no intragenerational inequality and per capita wellbeing increases over time, then the aggregate marginal utility of a generation increases with the number of individuals belonging to this generation. On the other hand, the average rank-dependent discount rate with which this aggregate marginal utility is discounted between this generation and its immediate predecessor increases with its size.

Secondly, we have allowed for analysis of a situation where population changes endogenously, e.g., as a consequence of climate change. By introducing a critical level which if experienced by an added individual without changing the utilities of the existing population leads to an alternative which is as good as the original *only if* the wellbeing levels of the existing population does not exceed the critical level, we have been able to combine critical-level population ethics with rank-discounted utilitarianism in an appealing manner (when evaluated by a class of conditions and principles).

However, practical application of the rank-discounted utilitarian criterion also requires explicit treatment of uncertainty. This is a topic for future research.

Appendix: Proofs

We first prove the representation result by showing that statements (1) and (2) of Theorem 1 are equivalent. It is straightforward to show that an RDCLU SWO satisfies Axioms 1–7, so that statement (2) implies statement (1). Hence, to prove Theorem 1 we need to show that statement (1) implies statement (2); that is, that an SWR $\succsim$ on $\mathbf{X}$ satisfying Axioms 1–7 is an RDCLU SWO. This is shown by means of Propositions 2–5, which are established in this appendix.

We define the restriction $\succsim_n$ of $\succsim$ to $\mathbf{X}_n$ in the following way: for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n$, $\mathbf{x} \succsim_n \mathbf{y}$ if and only if $\mathbf{x} \succsim \mathbf{y}$. We begin by establishing a representation result for sets of allocations with the same finite population size.

Proposition 2 *If the SWR $\succsim$ satisfies Axioms 1–5, then there exist $\beta \in \mathbb{R}_{++}$ and a continuous increasing function $u : Y \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$ and $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n$,*

$$\mathbf{x} \succsim_n \mathbf{y} \iff \sum\nolimits_{r=1}^{n} \beta^r u(x_{[r]}) \geq \sum\nolimits_{r=1}^{n} \beta^r u(y_{[r]}) .$$

Proof. For any $n \in \mathbb{N}$, we show that the relation $\succsim_n$ satisfies the following properties proposed by Ebert (1988): Continuous Ordering, Monotonicity, Symmetry and Independence with Respect to Ordered Vectors. By Ebert (1988, Theorem 1), this implies that there exist continuous increasing functions u_r^n such that for all $n \in \mathbb{N}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}_n$,

$$\mathbf{x} \succsim_n \mathbf{y} \iff \sum\nolimits_{r=1}^{n} u_r^n(x_{[r]}) \quad \geq \quad \sum\nolimits_{r=1}^{n} u_r^n(y_{[r]}) . \tag{A.1}$$

The fact that $\succsim_n$ is a continuous ordering follows from Axioms 1 and 2. The fact that it satisfies Monotonicity and Symmetry follows from Axiom 3. For the independence condition, we can apply Gorman's (1968) theorem on the ordered set $\mathbf{X}_n^+ = \{\mathbf{x} \in \mathbf{X}_n : x(1) \leq \ldots \leq x(n)\}$. By Axiom 4, we know that all sets $\{1, 2, \ldots, t\}$ for $1 < t < n$ are separable. By Axiom 5, we know that all sets $\{t, t+1, \ldots, n\}$ for $1 < t < n$ are separable. By intersections of such separable subsets, we can obtain any subsets $\{t, t+1\}$, $1 \leq t < n$, which are therefore separable by Gorman's (1968) theorem. By unions of such subsets we can obtain any subset of indices in $\{1, \ldots, n\}$ so that they are also separable by Gorman's (1968) theorem. This corresponds to Ebert's (1988) Independence with Respect to Ordered Vectors. By cardinality, we may set $u_r^n(0) = 0$ for all $r \leq n$ (normalization condition).

Now, representation (A.1) exists for $\succeq_n$ whatever $n \in \mathbb{N}$. Furthermore, by Axiom 4 we have the following equivalences (for $z \geq \{x_{[n]}, y_{[n]}\}$):

$$\sum\nolimits_{r=1}^{n} u_r^n(x_{[r]}) \geq \sum\nolimits_{r=1}^{n} u_r^n(y_{[r]}) \iff \mathbf{x} \succsim_n \mathbf{y} \iff (\mathbf{x}, z) \succsim_{n+1} (\mathbf{y}, z)$$
$$\iff \sum\nolimits_{r=1}^{n} u_r^{n+1}(x_{[r]}) + u_{n+1}^{n+1}(z) \geq \sum\nolimits_{r=1}^{n} u_r^{n+1}(y_{[r]}) + u_{n+1}^{n+1}(z)\,.$$

By standard uniqueness results for additive functions on rank-ordered sets, we can take (after the appropriate normalization) $u_r^n \equiv u_r^{n+1}$. We can henceforth drop the superscript n in functions u_r^n.

By by Axiom 5 we have the following equivalences (for $z \leq \min\{x_{[n]}, y_{[n]}\}$):

$$\sum\nolimits_{r=1}^{n} u_r(x_{[r]}) \geq \sum\nolimits_{r=1}^{n} u_r(y_{[r]}) \iff \mathbf{x} \succsim_n \mathbf{y} \iff (\mathbf{x}, z) \succsim_{n+1} (\mathbf{y}, z)$$
$$\iff \sum\nolimits_{r=1}^{n} u_{r+1}(x_{[r]}) + u_1(z) \geq \sum\nolimits_{r=1}^{n} u_{r+1}(y_{[r]}) + u_1(z)\,.$$

By the cardinality of the additive representation and the normalization condition, there must exists a $\beta > 0$ such that $u_{r+1}(y) = \beta u_r(y)$ for any $y \in Y$. Note that β does not depend on r. We obtain the following representation of $\succeq_n$:

$$W_n(\mathbf{x}) = \sum\nolimits_{r=1}^{n} \beta^r u(x_{[r]}),\ \forall \mathbf{x} \in \mathbf{X}_n\,,$$

where $u \equiv u_1$ is a continuous increasing function from Y to $\mathbb{R}$. ■

Let $c \in Y \cap \mathbb{R}_+$ be the critical level defined in Axiom 6 and define $Y_c = \{y \in Y : y \leq c\}$. Note that Y_c is non-empty since $a < 0 \leq c$ and $[a, c] \subseteq Y$. Define $\mathbf{X}_c$ the non-empty set of finite allocations $\mathbf{x} : \Lambda \to \{\emptyset\} \cup Y_c$ where wellbeing does not exceed c. We obtain a representation result for finite allocations (with variable population) where wellbeing does not exceed c.

Proposition 3 *If the SWR* $\succsim$ *satisfies Axioms 1–6 and* $c \in Y \cap \mathbb{R}_+$ *is the critical level of Axiom 6, then there exist* $\beta \in \mathbb{R}_{++}$ *and a continuous increasing function* $u : Y \to \mathbb{R}$ *such that, for all* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_c$,

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right) .$$

Proof. Assume that $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_c$ and $n(\mathbf{x}) \leq n(\mathbf{y})$ and let $k = n(\mathbf{y}) - n(\mathbf{x})$. Then:

$$\begin{aligned}
\mathbf{x} \succsim \mathbf{y} &\iff \left(\mathbf{x}, (c)_k\right) \succsim_{n(\mathbf{y})} \mathbf{y} \\
&\iff \sum_{r=1}^{n(\mathbf{x})} \beta^r u(x_{[r]}) + \sum_{r=n(\mathbf{x})+1}^{n(\mathbf{y})} \beta^r u(c) \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r u(y_{[r]}) \\
&\iff \sum_{r=1}^{n(\mathbf{x})} \beta^r u(x_{[r]}) + \sum_{r=n(\mathbf{x})+1}^{n(\mathbf{y})} \beta^r u(c) - \sum_{r=1}^{n(\mathbf{y})} \beta^r u(c) \\
&\qquad\qquad \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r u(y_{[r]}) - \sum_{r=1}^{n(\mathbf{y})} \beta^r u(c) \\
&\iff \sum_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right)
\end{aligned}$$

since $\mathbf{x} \sim \left(\mathbf{x}, (c)_k\right)$ by k applications of Axiom 6. ■

The following proposition shows that adding Axiom 7 implies that the rank utility discount factor, β, is smaller than 1.

Proposition 4 *If the SWR* $\succsim$ *satisfies Axioms 1–7 and* $c \in Y \cap \mathbb{R}_+$ *is the critical level of Axiom 6, then there exist* $0 < \beta < 1$ *and a continuous increasing function* $u : Y \to \mathbb{R}$ *such that, for all* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_c$,

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \geq \sum_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right) .$$

Proof. Since $a < 0 \leq c$ and $[a, c] \in Y_c$ is non-empty, by Axiom 3, there exist x, $y \in Y_c$ and $n \in \mathbb{N}$ such that $(x)_n \succ (y)_n$. Assume that there exist $z \in Y$ and $k \in \mathbb{N}$ such that $(x)_n \succ (z)_k \succ (y)_n$ (note that $z < c$, because otherwise, by Axioms 1, 3 and 6, $(z)_k \succsim (c)_k \sim (c)_n \succsim (x)_n$, a contradiction). By Proposition 3, this means that (for $\beta \neq 1$; the case $\beta = 1$ can be treated similarly):

$$\tfrac{\beta^n - 1}{\beta - 1}\left(u(x) - u(c)\right) > \tfrac{\beta^k - 1}{\beta - 1}\left(u(z) - u(c)\right) > \tfrac{\beta^n - 1}{\beta - 1}\left(u(y) - u(c)\right) .$$

When $\beta > 1$, $\lim_{k\to\infty} \frac{\beta^k-1}{\beta-1} = \infty$ so that $\lim_{k\to\infty} \frac{\beta^k-1}{\beta-1}\big(u(z)-u(c)\big) = -\infty$. Hence there exists $K > k$ such that $\frac{\beta^K-1}{\beta-1}\big(u(z)-u(c)\big) < \frac{\beta^{n(\mathbf{x})}-1}{\beta-1}\big(u(y)-u(c)\big)$, a contradiction of Axiom 7. ■

Finally, we extend the representation to the entire domain $\mathbf{X}$ of all finite allocations by showing that any finite allocation $\mathbf{x}$ can be made as bad as an allocation where all individuals are at the critical c by adding sufficiently many people at a low wellbeing level z, and thus indifferent to an egalitarian allocation where each individual's wellbeing equals $x \le c$. We can thus combine the representations of Propositions 2 and 3, showing that statement (1) of Theorem 1 implies statement (2).

Proposition 5 *If the SWR $\succsim$ satisfies Axioms 1–7, then there exist $0 < \beta < 1$ and a continuous increasing function $u : Y \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,*

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{r=1}^{n(\mathbf{x})} \beta^r \left(u(x_{[r]}) - u(c)\right) \ge \sum_{r=1}^{n(\mathbf{y})} \beta^r \left(u(y_{[r]}) - u(c)\right) .$$

Proof. *Step 1: For any $n \in \mathbb{N}$, $\mathbf{x} \in \mathbf{X}_n$ and $z \in Y$ satisfying $z \le x_{[1]}$ and $z < c$, there exists $k(\mathbf{x})$ such that for all $k \ge k(\mathbf{x})$, $(\mathbf{x}, (z)_k) \prec (c)_{n+k}$.*

By Proposition 4, we know that $\beta < 1$ in the representation on $\mathbf{X}_c$. By Proposition 2, the property extends to $\mathbf{X}_n$ for any $n \in \mathbb{N}$.

For any $\mathbf{x} \in \mathbf{X}_n$, the n-equally distributed equivalent of $\mathbf{x}$ denoted $e_n(\mathbf{x})$ is the real number $x \in Y$ such that $(x)_n \sim_n \mathbf{x}$. Axioms 1–3 imply that $e_n : \mathbf{X}_n \to Y$ is well-defined. By Proposition 2 and since Axioms 1–5 hold, it is defined as follows:

$$e_n(\mathbf{x}) = u^{-1}\left(\tfrac{1-\beta}{1-\beta^n}\sum_{r=1}^{n} \beta^{r-1} u(x_{[r]})\right) .$$

Now let $z \le x_{[1]}$ and $z < c$. We obtain the following expression:

$$\begin{aligned} e_{n+k}\big(\mathbf{x}, (z)_k\big) &= u^{-1}\left(\tfrac{1-\beta}{1-\beta^{n+k}}\left(\sum_{r=1}^{k} \beta^{r-1} u(z) + \sum_{r=k+1}^{n+k} \beta^{r-1} u(x_{[r]})\right)\right) \\ &= u^{-1}\left(\tfrac{1-\beta^k}{1-\beta^{n+k}} u(z) + \tfrac{\beta^k-\beta^{n+k}}{1-\beta^{n+k}} \cdot \tfrac{1-\beta}{1-\beta^n} \sum_{r=1}^{n} \beta^{r-1} u(x_{[r]})\right) . \end{aligned}$$

If $x_{[n]} \le c$, then $e_{n+k}\big(\mathbf{x}, (z)_k\big) \le c$ for all $k \in \mathbb{N}$ and Step 1 is completed. Therefore, assume $x_{[n]} > c$ which, since $z \le x_{[1]}$ and $z < c$, implies that $z < e_n(\mathbf{x})$.

Write $a_k := (1-\beta^k)/(1-\beta^{n+k})$; because $0<\beta<1$, $(a_k)_{k\in\mathbb{N}}$ is an increasing sequence converging to 1. Since $z<e_n(\mathbf{x})$ and

$$e_{n+k}\big(\mathbf{x},(z)_k\big)=u^{-1}\big(a_k u(z)+(1-a_k)u\big(e_n(\mathbf{x})\big)\big)\,,$$

it follows that $e_{n+k+1}\big(\mathbf{x},(z)_{k+1}\big)<e_{n+k}\big(\mathbf{x},(z)_k\big)$ and $e_{n+k}\big(\mathbf{x},(z)_k\big)$ tends to z when k tends to infinity. As $z<c$, we deduce that, for any $n\in\mathbb{N}$ and $\mathbf{x}\in\mathbf{X}_n$, there exists $k(\mathbf{x})\in\mathbb{N}$ such that, for any $k\geq k(\mathbf{x})$, $e_{n+k}\big(\mathbf{x},(z)_k\big)<c$.

Step 2: For any $\mathbf{x}$, $\mathbf{y}\in\mathbf{X}$, *choose* z *with* $z\leq\min\{x_{[1]},y_{[1]}\}$ *and* $z<c$, $\ell=\max\{k(\mathbf{x}),k(\mathbf{y})\}$, $x=e_{n(\mathbf{x})+\ell}\big(\mathbf{x},(z)_\ell\big)$ *and* $y=e_{n(\mathbf{y})+\ell}\big(\mathbf{y},(z)_\ell\big)$, *and use* $(\mathbf{x},(z)_\ell)\sim(x)_{n(\mathbf{x})+\ell}$, $(\mathbf{y},(z)_\ell)\sim(y)_{n(\mathbf{y})+\ell}$ *and* $(x)_{n(\mathbf{x})+\ell}$, $(y)_{n(\mathbf{y})+\ell}\in\mathbf{X}_c$ *to establish the result.*

Using the above definitions of x and y, we obtain the following equivalences by repeated applications of Axiom 5 and Proposition 3:

$$\begin{aligned}\mathbf{x}\succsim\mathbf{y}\iff&(x)_{n(\mathbf{x})+\ell}\sim(\mathbf{x},(z)_\ell)\succsim(\mathbf{y},(z)_\ell)\sim(y)_{n(\mathbf{y})+\ell}\\ \iff&\sum\nolimits_{r=1}^{n(\mathbf{x})+\ell}\beta^r\left(u(x)-u(c)\right)\geq\sum\nolimits_{r=1}^{n(\mathbf{y})+\ell}\beta^r\left(u(y)-u(c)\right)\,.\end{aligned}$$

However, by Proposition 2,

$$\begin{aligned}\sum\nolimits_{r=1}^{n(\mathbf{x})+\ell}\beta^r u(x)&=\sum\nolimits_{r=1}^{\ell}\beta^r u(z)+\beta^\ell\sum\nolimits_{r=1}^{n(\mathbf{x})}\beta^r u(x_{[r]})\,,\\ \sum\nolimits_{r=1}^{n(\mathbf{y})+\ell}\beta^r u(y)&=\sum\nolimits_{r=1}^{\ell}\beta^r u(z)+\beta^\ell\sum\nolimits_{r=1}^{n(\mathbf{y})}\beta^r u(y_{[r]})\,,\end{aligned}$$

using the fact $(x)_{n(\mathbf{x})+\ell}\sim(\mathbf{x},(z)_\ell)$ and $(y)_{n(\mathbf{y})+\ell}\sim(\mathbf{y},(z)_\ell)$. We obtain that

$$\sum\nolimits_{r=1}^{n(\mathbf{x})}\beta^r\left(u(x_{[r]})-u(c)\right)\geq\sum\nolimits_{r=1}^{n(\mathbf{y})}\beta^r\left(u(y_{[r]})-u(c)\right)$$

if and only if $\mathbf{x}\succsim\mathbf{y}$ by combining these result and rearranging terms. ■

We then provide a proof of the result (Proposition 1) on the limits of RDCLU.

Proof of Proposition 1. Assume that $\mathbf{x}$, $\mathbf{y}\in\mathbf{X}$ and u is a continuous and increasing function.

Part (i): This part follows directly from the observation that, on the domain $(0,1]$, $\sum_{r=1}^{n(\mathbf{x})}\beta^r\left(u(x_{[r]})-u(c)\right)$ is a continuous function of β.

Part (ii): Since $\succsim_c^L$ is complete, it is sufficient to show that $\mathbf{x} \sim_c^L \mathbf{y}$ implies the existence of $\underline{\beta} \in (0,1)$ such that $\mathbf{x} \sim_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0,\underline{\beta})$, and that $\mathbf{x} \succ_c^L \mathbf{y}$ implies the existence of $\underline{\beta} \in (0,1)$ such that $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0,\underline{\beta})$.

$\mathbf{x} \sim_c^L \mathbf{y}$ *implies the existence of* $\underline{\beta} \in (0,1)$ *such that* $\mathbf{x} \sim_{\beta,u,c} \mathbf{y}$ *for all* $\beta \in (0,\underline{\beta})$. Let $n(\mathbf{x}) \geq n(\mathbf{y})$. By Definition 3, $(x_{[1]},\ldots,x_{[n(\mathbf{y})]}) = \mathbf{y}_{[]}$ and $(x_{[n(\mathbf{x})+1]},\ldots,x_{[n(\mathbf{y})]}) = (c)_{n(\mathbf{x})-n(\mathbf{y})}$. By Definition 1, $\mathbf{x} \sim_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0,1)$.

$\mathbf{x} \succ_c^L \mathbf{y}$ *implies the existence of* $\underline{\beta} \in (0,1)$ *such that* $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ *for all* $\beta \in (0,\underline{\beta})$. Let $x = \min\{x_{[1]}, c\}$ and $y = \max\{y_{[n(\mathbf{y})]}, c\}$. Note that $x \leq y$. If $x = y$ and $\mathbf{x} \succ_c^L \mathbf{y}$, then by Definitions 1 and 3, $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0,1)$. Hence, only the case where $x < y$ remains. By Definition 3, there are three cases. Case 1: there exists $R \in \{1,\ldots,n\}$ such that $x_{[r]} = y_{[r]}$ for all $r \in \{1,\ldots,R-1\}$ and $x_{[R]} > y_{[R]}$, where $n := \min\{n(\mathbf{x}), n(\mathbf{y})\}$. In this case, let $x' = x_{[R]}$ and $y' = y_{[R]}$. Case 2: $n(\mathbf{x}) > n(\mathbf{y})$, $(x_{[1]},\ldots,x_{[n(\mathbf{y})]}) = \mathbf{y}_{[]}$ and $(x_{[n(\mathbf{y})+1]},\ldots,x_{[n(\mathbf{x})]}) > (c)_{n(\mathbf{x})-n(\mathbf{y})}$. In this case, let $x' = x_{[R]}$ and $y' = c$, where $R := \min\{r > n(\mathbf{y}) : x_{[r]} > c\}$. Case 3: $n(\mathbf{x}) < n(\mathbf{y})$, $\mathbf{x}_{[]} = (y_{[1]},\ldots,y_{[n(\mathbf{x})]})$ and $c > y_{[n(\mathbf{x})+1]}$. In this case, let $x' = c$ and $y' = y_{[n(\mathbf{x})+1]}$. Note that in all three cases, $x' > y'$. Define $\underline{\beta}$ by

$$(1-\underline{\beta})u(x') + \underline{\beta}u(x) = (1-\underline{\beta})u(y') + \underline{\beta}u(y)\,.$$

Then, by applying the following affine transformation of W:

$$\tfrac{1-\beta}{\beta}W(\mathbf{x}) + u(c) = (1-\beta)\sum\nolimits_{r=1}^{n(\mathbf{x})}\beta^{r-1}u(x_{[r]}) + \beta^{n(\mathbf{x})}u(c)\,,$$

it follows from Definition 1 that $\mathbf{x} \succ_{\beta,u,c} \mathbf{y}$ for all $\beta \in (0,\underline{\beta})$. ■

References

Arrhenius, G. (2012). *Population Ethics – The Challenge of Future Generations.* Forthcoming.

Blackorby, C., Donaldson, D. (1984). "Social criteria for evaluating population change", *Journal of Public Economics* **25**, 13–33.

Blackorby, C., Bossert, W., Donaldson, D. (1995). "Intertemporal population ethics: Critical-level utilitarian principles", *Econometrica* **63**, 1303–1320.

Blackorby, C., Bossert, W., Donaldson, D. (1996). "Leximin population ethics", *Mathematical Social Sciences* **31**, 115–131.

Blackorby, C., Bossert, W., Donaldson, D. (2005). *Population Issues in Social Choice Theory, Welfare Economics, and Ethics*. Cambridge, UK: Cambridge University Press.

Broome, J. (2004). *Weighing Lives*. Oxford, UK: Oxford University Press.

Broome, J. (2010). "The most important thing about climate change". In *Public Policy: Why ethics matters*, ed. J. Boston, A. Bradstock, D. Eng, pp. 101–116. Canberra: ANU E Press.

Dasgupta, P.S. (1988). "Lives and well-being", *Social Choice and Welfare*, 5, 103–126.

Dasgupta, P.S. (2005). "Regarding optimum population", *Journal of Political Philosophy*, 13, 414–442.

Ebert, U. (1988). "Measurement of inequality: An attempt at unification and generalization", *Social Choice and Welfare*, 5, 147–169.

Gorman, W.G. (1968). "The structure of utility functions", *Review of Economic Studies*, 35, 367-390.

Ng, Y-K. (1989). "What should we do about future generations? Impossibility of Parfits theory X", *Economics and Philosophy* **5**, 235–253.

Palivos, T., and Yip, C.K. (1993). "Optimal population size and endogenous growth", *Economics Letters* **41**, 107–110.

Parfit, D. (1976). "On doing the best for our children". In *Ethics and Population*, ed. M. Bayles, pp. 100–102 Cambridge: Schenkman.

Parfit, D. (1982). "Future generations, further problems", *Philosophy and Public Affairs* **11**, 113–172.

Parfit, D. (1984). *Reasons and Persons*. Oxford: Oxford University Press.

Razin, A., Yuen, C.-W. (1995). "Utilitarian tradeoff between population growth and income growth", *Journal of Population Economics* **8**, 81–87.

Sider, T.R. (1991). "Might theory X be a theory of diminishing marginal value?", *Analysis* **51**, 265–271.

Zuber, S., Asheim, G.B. (2012). "Justifying social discounting: the rank-discounted utilitarian approach", *Journal of Economic Theory* **147**, 1572–1601.