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Abstract

The Alchian-Allen substitution theorem states that an increase of the prices of two similar goods by the same amount leads to a relative increase in the compensated demand of the more expensive good. In this paper we generalize this theorem to ordinary demand functions and show under which conditions the Alchian-Allen result continues to hold when income and endowment effects are taken into account.

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1. Introduction

The Alchian-Allen theorem (see Alchian and Allen, 1964, p. 74–75) states that an increase of the prices of two similar, that means substitutable goods by the same amount leads to a relative increase in the compensated demand of the more expensive good. This effect, which is concerned with pure substitution effect, has been intensively discussed in the theoretical and empirical literature. [See, for example, Gould and Segall (1969), Borcherding and Silberberg (1978), Umbeck (1980), Bertonazzi, Maloney, and McCormick (1993), Cowen and Tabarrok (1995), Sobel and Garrett (1997), Bauman (2004), Hummels and Skiba (2004), Lawson and Raymer (2006), Nesbit (2007), Saito (2008), and Liu (2011).]

While the Alchian-Allen result has been confirmed empirically for some consumption goods, see, for example, Sobel and Garrett (1997) and Hummels and Skiba (2004),[1] conflicting findings are reported in gasoline markets: Nesbit (2007) confirms the Alchian-Allen theorem, whereas Lawson and Raymer (2006) do not. To see why empirical findings are seemingly contradictory, it is important to realize that in due compliance with the Alchian-Allen theorem, the confirmations of Sobel and Garrett (1997) and Hummels and Skiba (2004) are based on the assumption that income effects are insignificant. While this is a reasonable assumption if we consider (the traditional example of) good and bad apples, income effects are more important for some goods like gasoline.

Moreover, the Alchian-Allen theorem may be applicable to the choice of two types of leisure: a per unit charge (here: the wage rate) onto two substitutable goods (here: two leisure activities, such as pure leisure time and time spent together with the children) lowers the relative price of the more expensive good and therefore raises the relative demand for that good. In this application, the neglect of income effects would deprive the time allocation decision one of its most important features, and thus its singularity. Furthermore, in the time allocation decision, as well as in many other applications, there are also endowment effects which work in opposite to the usual income effect: While a higher price of a consumption good induces a negative income effect (provided the good is normal), a higher price of a good in which the consumer has some initial endowment induces a positive income effect.

[1] Bertonazzi et al. (1993) provide another confirmation, though in a somewhat different context, namely, in a household production framework, which results from an interpretation of the charge in the Alchian-Allen theorem. See Umbeck (1980) and Cowen and Tabarrok (1995) for various interpretations.

Thus, whenever time allocation problems are concerned, substitution effects are not only accompanied by and mixed with income effects, but are also superimposed by endowments effects.

The aim of this work is therefore as follows. We derive a generalized version of the Alchian-Allen theorem so as to account for all of the crucial determinants of observed economic behavior: income effects, endowment effects and substitution effects as well as their mutual interactions. To achieve at this, we build on the work of Gould and Segall (1969), Borcherding and Silberberg (1978), and Bauman (2004) who have previously extended the Alchian-Allen theorem to the case of three or more goods, in terms of compensated demand functions.[2] In our approach, though, we use ordinary instead of compensated demand functions to capture both income and endowment effects, and show that the Alchian-Allen result may be extended to the presence of income and endowment effects (under rather mild qualifications). In this sense, the Alchian-Allen theorem survives a generalization. In particular, our generalized Alchian-Allen formula concurs with the pure substitution version provided by Borcherding and Silberberg (1978), if either the income elasticities for the goods under consideration coincide or if "on average" (that is, aggregated over all goods) a consumer consumes his/her endowments.

2. The Model

Consider the standard model of consumer behavior with three goods. Suppose that the consumer's preferences may be represented by a differentiable, monotonic, strictly quasi-concave utility function $u : \mathbb{R}^3_+ \to \mathbb{R} : (x_1, x_2, x_3) \mapsto u(x_1, x_2, x_3)$. We allow for the consumer to have non-negative initial endowments, denoted by $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ for the three goods respectively and a positive money income m. Moreover, let p_1, p_2, and p_3 denote the respective (before-tax) prices, where by assumption $p_1 > p_2$. For the purpose of a more compact notation, we define the following vectors $\mathbf{x} := (x_1, x_2, x_3)^\mathsf{T}$, $\bar{\mathbf{x}} := (\bar{x}_1, \bar{x}_2, \bar{x}_3)^\mathsf{T}$ and $\mathbf{p} := (p_1, p_2, p_3)$. (We write commodity vectors as column and price vectors as row vectors.)

By assumption, a fixed charge t is added to the prices of the first two goods, such that gross consumer prices may be written as $q_1 \equiv p_1 + t$, $q_2 \equiv p_2 + t$, and $q_3 \equiv p_3$; or more compactly $\mathbf{q} := \mathbf{p} + \mathbf{t}$ with $\mathbf{q} := (q_1, q_2, q_3)$, $\mathbf{t} := (t, t, 0)$, where consumer prices are assumed to be positive. The consumer's total income (or wealth) level

[2] Saito (2008) considers a version of the Alchian-Allen theorem with (zero) income effects under some specific assumptions.

given by $\mathbf{q}\cdot\bar{\mathbf{x}}+m$ depends on the price vector and the money income and thus we may define a function $I(\mathbf{q},m) := \mathbf{q}\cdot\bar{\mathbf{x}}+m$. Then, as usual, the consumer's utility maximization problem is given by

$$\max_{x_1,x_2,x_3} \; u(\mathbf{x}) \quad \text{s. t.} \quad \mathbf{q}\cdot\mathbf{x} \leqq \mathbf{q}\cdot\bar{\mathbf{x}}+m, \tag{1}$$

yielding ordinary demand functions $\mathbf{x}^o(\mathbf{q}, I(\mathbf{q},m))$. Taking into account that consumer prices depend on the charge t, we define demand as a function of t (and exogenous income m) rather than of prices (and full income I): $\hat{\mathbf{x}}^o(t,m) := \mathbf{x}^o(p_1+t, p_2+t, p_3, I(p_1+t,p_2+t,p_3,m)) \equiv \mathbf{x}^o(\mathbf{q}(t), I(\mathbf{q}(t),m))$.

Correspondingly, the expenditure minimization problem may be expressed as

$$\min_{x_1,x_2,x_3} \; \mathbf{q}\cdot[\mathbf{x}-\bar{\mathbf{x}}] \quad \text{s. t.} \quad u(\mathbf{x}) \geqq v, \tag{2}$$

the solution of which is given by the compensated demand function $\mathbf{x}^*(\mathbf{q},v)$. Similarly, it is helpful to write compensated demand as a function of the charge t (and v): $\hat{\mathbf{x}}^*(t,v) := \mathbf{x}^*(p_1+t,p_2+t,p_3,v) \equiv \mathbf{x}^*(\mathbf{q}(t),v)$.

In the remainder of our analysis, we use the following notation. The compensated price elasticity of good i with respect to the consumer price of good j is defined as $\varepsilon^*_{ij}(\mathbf{q},v) := (q_j/x^*_i(\mathbf{q},v))(\partial x^*_i(\mathbf{q},v)/\partial q_j)$. The income elasticity of good i is defined by $\varepsilon_{iI}(\mathbf{q},I) := (I/x^o_i(\mathbf{q},I))(\partial x^o_i(\mathbf{q},I)/\partial I)$. Whenever it is clear at which point of the domain these elasticities are evaluated, we suppress their arguments; a similar hint applies to consumer prices where we frequently suppress the argument t and simply write q_i or $\mathbf{q}$. Likewise, in order to save notational effort we shall henceforth simply write x_i and $\mathbf{x}$ to denote the demand level (or the image) of the demand function under consideration, rather than the generic consumption level. Since in the subsequent analysis we exclusively deal with the solution of the consumer choice problem only, no ambiguity should arise, though.

3. Results

Before we derive our central result, Proposition 2, it is advantageous to reformulate a result of Borcherding and Silberberg (1978), which can also be found in e. g., Silberberg and Suen (2000, p. 339), in our notation. Following our notational convention, we here write $x_i = \hat{x}^*_i(t,v)$, ε^*_{ij} instead of $\varepsilon^*_{ij}(\mathbf{q}(t),v)$ and q_i instead of $q_i(t)$. With this note of caution we arrive at

Proposition 1 (Borcherding/Silberberg, 1978).

$$\frac{\partial}{\partial t}\frac{\hat{x}_1^*}{\hat{x}_2^*}(t,v) = \frac{x_1}{x_2}\frac{1}{q_2}\left[(\varepsilon_{11}^* - \varepsilon_{21}^*)\left(\frac{q_2}{q_1} - 1\right) + (\varepsilon_{23}^* - \varepsilon_{13}^*)\right].$$

Proof. As long as compensated demand functions are concerned, the demonstration of Borcherding and Silberberg holds without modification except for the use of consumer prices rather than before-tax prices. □

Proposition 1 implies that if the two goods are not perfect complements (i. e., $\varepsilon_{11}^* < \varepsilon_{21}^*$) and good 1 is not a much stronger substitute for the third good than is good 2 (i. e., $\varepsilon_{23}^* - \varepsilon_{13}^* > -\alpha$ for some small positive α), then the Alchian-Allen result, that the partial derivative $(\partial/\partial t)(\hat{x}_1^*/\hat{x}_2^*)$ is positive, generalizes to this more general framework. (Recall that we assumed $p_1 > p_2$ and thus $q_1 > q_2$.) In addition, negligible price differences between good 1 and good 2 (i. e., $p_1 \approx p_2$ and hence $q_1 \approx q_2$) will weaken the substitution effect. Borcherding and Silberberg (1978) argue that if the two goods under consideration are close substitutes for each other, then they should be similarly related to the third good, thus, the term $\varepsilon_{23}^* - \varepsilon_{13}^*$ should be small.[3]

In order to formulate Proposition 2 it is expedient to write demand as a function of $\mathbf{q}$ and m: $\tilde{\mathbf{x}}^o(\mathbf{q}, m) := \mathbf{x}^o(\mathbf{q}, I(\mathbf{q}, m))$. Similarly, we define the expenditure function $E(\mathbf{q}, v) := \mathbf{q} \cdot [\mathbf{x}^*(\mathbf{q}, v) - \bar{\mathbf{x}}]$. It then follows as a familiar duality result that $x_i = x_i^*(\mathbf{q}, v) \equiv \tilde{x}_i^o(\mathbf{q}, E(\mathbf{q}, v))$, or more compactly $\mathbf{x} = \mathbf{x}^*(\mathbf{q}, v) \equiv \tilde{\mathbf{x}}^o(\mathbf{q}, E(\mathbf{q}, v))$. Alternatively, this identity may also be expressed as $\mathbf{x} = \hat{\mathbf{x}}^*(t, v) \equiv \hat{\mathbf{x}}^o(t, E(\mathbf{q}(t), v))$. We are now prepared to derive a version of the Alchian-Allen theorem which accounts for both income and endowment effects.[4]

Proposition 2 (The Generalized Alchian-Allen Formula).

$$\frac{\partial}{\partial t}\frac{\hat{x}_1^o}{\hat{x}_2^o}(t,m) = \frac{\partial}{\partial t}\frac{\hat{x}_1^*}{\hat{x}_2^*}(t,v) + \frac{x_1}{x_2}\frac{1}{I}(\varepsilon_{1I} - \varepsilon_{2I})\sum_{j=1}^{2}(\bar{x}_j - x_j).$$

The proof of Proposition 2 makes use of the following lemma, which is a generalization of the well-known Hicks-Slutsky equation when endowment effects are present, that is when the consumer's income is given by the value of its initial endowment (in some or all of the commodities) in addition to money income.

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[3]Minagawa (2012) points out that there may be an exception to this argument.

[4]We here remark that as usual, by the duality approach, we can equally work with the indirect utility function $u^o(\mathbf{q}, m) := u(\tilde{\mathbf{x}}^o(\mathbf{q}, m))$, instead of the expenditure function.

Lemma 1 (The Hicks-Slutsky equation with endowment effects)**.**

$$\frac{\partial x_i^*(\mathbf{q},v)}{\partial q_j} \;=\; \frac{\partial \tilde{x}_i^o(\mathbf{q},m)}{\partial q_j} - (\bar{x}_j - x_j)\,\frac{\partial x_i^o(\mathbf{q},I(\mathbf{q},m))}{\partial I}.$$

Proof of Lemma 1. Differentiating both sides of $x_i^*(\mathbf{q},v) \equiv \tilde{x}_i^o(\mathbf{q},E(\mathbf{q},v))$ with respect to q_j yields

$$\frac{\partial x_i^*(\mathbf{q},v)}{\partial q_j} \;=\; \frac{\partial \tilde{x}_i^o(\mathbf{q},m)}{\partial q_j} + \frac{\partial \tilde{x}_i^o(\mathbf{q},m)}{\partial m}\frac{\partial E(\mathbf{q},v)}{\partial q_j}. \tag{3}$$

Applying Shephard's lemma: $\partial E(\mathbf{q},v)/\partial q_j = x_j^*(\mathbf{q},v) - \bar{x}_j$ (see Cornwall (1984, p. 747)) and noting $\partial \tilde{x}_i^o/\partial m = \partial x_i^o/\partial I$, we obtain the required result. □

Proof of Proposition 2. By definition we have $\hat{x}_i^*(t,v) = x_i^*(\mathbf{q}(t),v)$ and thus

$$\frac{\partial \hat{x}_i^*(t,v)}{\partial t} \;=\; \sum_{j=1}^{2} \frac{\partial x_i^*(\mathbf{q},v)}{\partial q_j}\,\frac{dq_j(t)}{dt}. \tag{4}$$

Using $dq_j(t)/dt = 1$ and applying Lemma 1, we obtain

$$\frac{\partial \hat{x}_i^*(t,v)}{\partial t} \;=\; \sum_{j=1}^{2} \left[\frac{\partial \tilde{x}_i^o(\mathbf{q},m)}{\partial q_j} - (\bar{x}_j - x_j)\,\frac{\partial x_i^o(\mathbf{q},I(\mathbf{q},m))}{\partial I}\right]. \tag{5}$$

Now it is straightforward to calculate the partial derivative of the demand ratio

$$\frac{\partial}{\partial t}\frac{\hat{x}_1^*}{\hat{x}_2^*}(t,v) = \frac{1}{(x_2)^2}\left[\frac{\partial \hat{x}_1^*(t,v)}{\partial t}x_2 - x_1\frac{\partial \hat{x}_2^*(t,v)}{\partial t}\right]. \tag{6}$$

Substituting for the partial derivatives $\partial \hat{x}_i^*(t,v)/\partial t$, $i = 1, 2$ and expressing in terms of income elasticities, we arrive at

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\hat{x}_1^*}{\tilde{x}_2^*}(t,v) = \frac{1}{(x_2)^2}&\left\{\sum_{j=1}^{2}\left[\frac{\partial \tilde{x}_1^o(\mathbf{q},m)}{\partial q_j}\right]x_2 - x_1\sum_{j=1}^{2}\left[\frac{\partial \tilde{x}_2^o(\mathbf{q},m)}{\partial q_j}\right]\right\}\\ &- \frac{x_1}{x_2}\frac{1}{I}\left[\varepsilon_{1I}(\mathbf{q},I(\mathbf{q},m)) - \varepsilon_{2I}(\mathbf{q},I(\mathbf{q},m))\right]\sum_{j=1}^{2}(\bar{x}_j - x_j). \end{aligned} \tag{7}$$

Since $\partial \hat{x}_i^o(t,m)/\partial t = \sum_{j=1}^{2}[\partial \tilde{x}_i^o(\mathbf{q},m)/\partial q_j]$, we know that the first part on the right hand side equals $(\partial/\partial t)(\hat{x}_1^o(t,m)/\hat{x}_2^o(t,m))$. Finally, transposing the last term to the left hand side completes the proof. □

Remark 1. ¿From our formulation of Proposition 2 and Lemma 1 it is apparent that our model covers the baseline scenario where the consumer has no initial endowments and income is thus equal to exogenous monetary income (i. e., $I \equiv m$). In this case, endowment effects do not appear and only ordinary income effects are present. If, however, income is price dependent, endowment effects arise reflecting the fact that higher prices of these goods directly increase the consumer's income.

Suppose that the summation term $\sum_{j=1}^{2}(\bar{x}_j - x_j)$, representing the household's aggregate excess supply, is positive.[5] Then, Proposition 2 implies that if good 2 does not feature much stronger income effects than good 1 (i. e., $\varepsilon_{1I} - \varepsilon_{2I} > -\beta$ for some small positive β), the Alchian-Allen result, that $(\partial/\partial t)(\hat{x}_1^*/\hat{x}_2^*)$ is positive, continues to hold — and may even strengthen — if we account for both income and endowment effects; that is, we have $(\partial/\partial t)(\hat{x}_1^o/\hat{x}_2^o) > (\partial/\partial t)(\hat{x}_1^*/\hat{x}_2^*) > 0$. In particular, when both goods feature identical income elasticities, the generalized Alchian-Allen formula, given in Proposition 2, coincides with the substitution result of Borcherding and Silberberg (1978), stated in Proposition 1, irrespective of whether consumption falls short of or exceeds initial endowment. Similarly, if the household's aggregate excess supply is sufficiently small, i. e., "on average" the household consumes its initial endowment, both formulae also coincide — and the Alchian-Allen result continues to hold under the same qualifications given above (see our discussion of Proposition1).

4. Conclusion

In this paper we have contributed to the discussion and the advancements of the Alchian-Allen theorem (see Alchian and Allen, 1964), which has been discussed intensively in both the past and the presence. In particular, based on the work of Borcherding and Silberberg (1978), we accomplished to derive a version of this theorem which uses uncompensated rather than compensated demand, and which, in addition to usual income effects, also accounts for endowment effects. Our generalization of the Alchian-Allen theorem shows that the Alchian-Allen result for compensated demand continues to hold for uncompensated demand, unless the income elasticity of the lower priced good substantially exceeds the income elasticity of the higher priced good provided that consumption does not exceed initial endowment (or reverse if consumption exceeds the initial endowment).

[5] While we refer here to the case where the sum is positive, we may also discuss the case where the sum is negative. However, since this discussion and its implications are obvious, they are omitted.

We believe, as mentioned in the Introduction, that the generalization is applicable for a variety of economic decisions. For example, time allocation decisions appear to be well suited for an application of the generalized Alchian-Allen theorem: A parent may spend her/his non-working time for, at least, two different modes of leisure, such as leisure time without the child (pure leisure time) and leisure time together with the child (parental child care); and since both types of activity are arguably close substitutes, their opportunity costs include a common component (the foregone wage), and the price of pure leisure time is higher than the price of parental child care by the cost of non-parental (i. e., external) child care, this seems to be a suitable and relevant case for the generalized Alchian-Allen theorem.

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