# A Nested Contest:
# Tullock Meets the All-Pay Auction

## J. Atsu Amegashie

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## Abstract

I present a two-player *nested* contest which is a convex combination of two widely studied contests: the Tullock (lottery) contest and the all-pay auction. A Nash equilibrium exists for all parameters of the nested contest. If and only if the contest is sufficiently asymmetric, then there is an equilibrium in pure strategies. In this equilibrium, individual and aggregate efforts are lower relative to the efforts in a Tullock contest. This leads to the surprising result that if aggregate efforts in the all-pay auction are higher than the aggregate efforts in the Tullock contest, then aggregate efforts in the nested contest may not lie between aggregate efforts in the all-pay auction and aggregate efforts in the Tullock contest. When the contest is symmetric or asymmetric, I find a mixed-strategy equilibrium and describe some properties of the equilibrium distribution function; I also find the equilibrium payoffs and expected bids. When the weight on the all-pay auction component of this nested contest lies in an intermediate range, then there exist multiple *non-payoff-equivalent* equilibria such that there is an *all-pay auction equilibrium* as defined in Alcade and Dahm (2010) and another equilibrium which is not an all-pay auction equilibrium; these equilibria cannot be ranked using the Pareto criterion. If the goal of a contest-designer is to reduce aggregate effort (i.e., wasteful rent-seeking efforts), then this nested contest may be better than both the Tullock contest and the all-pay auction.

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## 1. Introduction

The literature on contests is voluminous and still flourishing. The most studied contests are the all-pay auction and the Tullock (lottery) contest (see, for example, Congleton, Hillman, and Konrad, 2008a, 2008b; Konrad, 2009; Nitzan, 1994). The equilibrium properties of these *separate* contests, including the aggregate and individual efforts, have been compared in Fang (2002), Epstein, Mealem, and Nitzan (2011), and Franke et al (2012).[1] In this paper, I present a contest which is a combination of these two widely popular contests. Such a nested contest may arise in the situations described below.

In the wake of a judging controversy at the Winter 2002 Olympic games, the governing council of the International Skating Union (ISU) scrapped its judging system. The ISU adopted a new judging system called the Code of Points (COP). In 2006, the COP took effect and became mandatory at all international competitions including the Winter Olympics.

The ISU's "Code of points" is a two-stage process. But for my purposes, it suffices to note that a panel of twelve judges award a mark for each skater's grade of execution (GOE) . The GOE value from the twelve judges is then averaged by *randomly* selecting nine judges, discarding the highest and lowest values, and averaging the remaining seven. Therefore, this is a contest in which the players (i.e., the skaters) do not know which members of the awarding committee have the power to make the final decision. In a related context, Epstein et al. (2007, p. 114) observed that "... in some bureaucratic organizations, not only is the distribution of power among the acting figures within the system unknown, but even the set of potential decision makers may only be partially known. In particular, the contestants may not know the identity of the "wire-puller" who controls the decision-making system, possibly from behind the scenes."

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[1] See also Epstein and Nitzan (2006a) for a comparison of the all-pay auction and Tullock contest from the standpoint of a politician who wants to maximize some social welfare function. For an approach using standard tools of mechanism design, see Polishchuk and Tonis (2012).

This type of contest is examined in Amegashie (2006) and Epstein et al. (2007). However, in these papers the players can direct their lobbying effort to specifically target potential judges and each potential judge or rent-giver awards the prize based on an imperfectly-discriminating contest success function (e.g., the Tullock function).[2] In this paper, a potential judge may base his decision on a perfectly-discriminating contest success function (i.e., he awards the prize to the player with the highest effort) or on an imperfectly-discriminating success function. In this sense, the contest is a combination of the all-pay auction and the Tullock contest. Also, each player's effort is simultaneously observed by all potential judges.

As another example, consider a committee with $N \geq 3$ members who must award a single prize in a contest. Suppose that two members of the committee have very strong personalities and different views of the world. Call them H and L. These members could also be the two most senior members of the committee. Suppose the other members of the committee are followers, so H and L compete with each other to get as many members as possible to their different points of view. With probability, $\lambda$, H can sway a majority of the committee members to his point of view while L can do so with probability $1 - \lambda$, where $0 < \lambda < 1$. Then we have a contest in which the contest success function can be modeled as a convex combination of two different success functions. [3]

Furthermore, committees may be prone to *groupthink*. In his influential book, Irving Janis (1972) defined groupthink as a "... mode of thinking that people engage in when they are

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[2] See also Alcade and Dahm (2008) who briefly examine a contest which is a combination of the Tullock contest success function and another *imperfectly-discriminating* success function (i.e., the serial contest function; see also Alcalde and Dahm, 2007).

[3] Amegashie (2002, 2006) and Congleton (1984) consider contests with an N-member rent-giving committee, where $N \geq 3$. These models are related to but *different* from the model in this paper because unlike the present model, the votes of more than one member of the committee counts and *all* the members act independently or are not influenced by other committee members. In Congleton (1984), the rent is awarded by a three-member committee where each committee member votes for the players who lobbies him the most. He showed that there is no equilibrium in pure strategies. Whether there exists a mixed-strategy equilibrium in that game has not, to the best of my knowledge, been investigated.

deeply involved in a cohesive in-group, when the members' strivings for unanimity override their motivation to realistically appraise alternative courses of action." Such a committee may be interested in reaching unanimous decisions because a divided decision makes them (i.e., the committee members) look bad. Therefore, members of the committee may try to minimize conflict and reach a consensus decision without critical evaluation of alternative viewpoints. In such situations, only a few alternatives (e.g., only *two*), for evaluating performance may be considered in any decision-making. These two alternatives may correspond to the views of only two committee members and, with some known probability, one of these two people will be successful in convincing the entire group.

In this paper, I use the Tullock success function with returns parameter, R = 1. This is the most popular version of the Tullock success function in the literature. As mentioned above, I combine it with the auction success function. In the next section, I investigate the equilibria of this nested contest and compare it to the Tullock contest and the all-pay auction. Section 3 discusses the results.


## 2. A nested contest

Consider a complete-information contest in which the prize will be awarded by one of two people (i.e., judges, bureaucrats, decision-makers, etc): call them H and L. There are two risk-neutral players, 1 and 2, who compete for the prize. Player j values the prize at $V_j > 0$, j = 1, 2. Suppose that $V_1 \geq V_2 > 0$.

Player j's effort is $e_j \geq 0$, j = 1,2. Each judge, if he is chosen as the decision-maker, votes for the contestant with the higher output (or performance). Judge H is chosen with probability $\lambda$ while L is chosen with probability $1 - \lambda$, where $\lambda \in (0,1)$. The players must choose their efforts

before knowing which judge will be chosen as the decision-maker and they cannot separately target judges. This is, for example, the case when the judges simultaneously observe the efforts of the players and no additional interaction with any judge is allowed. Of course, one does need two different (potential) judges. It suffices to have a single judge whose type, H or L, is drawn from a  binary distribution where $pr(H) = \lambda$, $pr(L) = 1 - \lambda$, and $\lambda \in (0,1)$.

Suppose that judge H observes effort without noise such that he sees player j's output as $y_j^H = e_j$, $j = 1, 2$. Judge L observes effort with noise such that he sees player j's output as $y_j^L = g(e_j, \varepsilon)$, where $\varepsilon$ is a random variable with a continuous distribution, $j = 1,2$. These may reflect differences in ability in evaluating performance. [4]

Alternatively, we could assume that either judge observes effort *without* noise but judge L, in addition to the players' efforts, takes other factors into account in his decision-making but these factors are not known to the players (Clark and Riis, 1996; Corchon and Dahm, 2010).[5] This interpretation is consistent with Corchon and Dahm (2010, p. 85) who assume that "... contestants are uncertain about a characteristic of the decider that is relevant for his decision. So contestants exert effort without knowing the realization of the characteristic." For two players

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[4]Such differences in evaluation may be seen in contests such as reality TV shows like American Idol and "Dancing with the Stars" where the votes of expert judges and viewers count (e.g., see Amegashie, 2009)  or elections in which some voters are more informed than others. Expert judges are likely to view effort with little or no noise while the viewers are likely to do so with noise. In Amegashie (2009) and all the papers cited here, it is assumed that both groups observe effort with noise, although the noise of one group (e.g., the experts) has a smaller (positive) variance.

[5] For example, the members of a recruitment committee may all care about a candidate's technical ability (e.g., in academia, this may be whether he can publish in good journals). All members of the committee can evaluate technical ability perfectly but while some people care about only technical ability, others may also care the social skills of a candidate (e.g., is he a team player? will he be a good citizen of the department?).

and a uniform distribution of the judge's characteristic, they show that one can obtain the Tullock CSF.[6]

Based on previous works, we may assume that the functional form g(.) and the distribution of $\varepsilon$ are such that prob($y_j^L > y_k^L$) gives the Tullock contest success function, $j \neq k$, $j = 1,2; k = 1, 2.$[7] Or, as mentioned above, one could follow the interpretation in, for example, Corchon and Dahm (2010) where the judge observes effort without noise but has a characteristic that is not known by the contestants.[8] Accordingly, it follows that if judge L is the decision-maker, he will choose player j as the winner with probability.

$$P_j^L(e_j, e_k) = \begin{cases} e_j/(e_j + e_k), & \text{if} \quad \max[e_j, e_k] > 0 \\ \\ 0.5, & \text{if} \quad e_j = e_k = 0 \end{cases}, \tag{1}$$

where $j \neq k, j = 1,2; k = 1, 2$. The Tullock CSF with the returns parameter, $R = 1$ (i.e., the CSF in (1)), is the most popular version of the Tullock function that is used in the literature.

Since judge H observes effort with no noise, the probability that he will vote for player j is given by

$$P_j^H(e_j, e_k) = \begin{cases} 1, & \text{if} \quad e_j > e_k \\ 0.5, & \text{if} \quad e_j = e_k, \\ 0, & \text{if} \quad e_j < e_k \end{cases} \tag{2}$$

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[6] Demonstrating this result is straightforward. Suppose that the rent-giver's payoff is $U_1(\theta, e_1) = (1 - \theta)f(e_1)$ if he awards the prize to player 1 and his payoff is $U_2(\theta, e_2) = \theta f(e_2)$ if he awards the prize to player 2, where $f(e_j)$ is a strictly increasing function and $\theta$ is a random characteristic of the rent-giver that is uniformly distributed on [0,1], j = 1, 2. Then the probability that he will award the prize to player 1 is prob($U_1(\theta, e_1) > U_2(\theta, e_2)$) and is equal to prob($\theta < \hat{\theta}$) = $f(e_1)/[f(e_1) + f(e_2)]$, where $\hat{\theta} \equiv f(e_1)/[f(e_1) + f(e_2)]$. Then the Tullock CSF with returns parameter equal to 1 is obtained if $f(e_j) = e_j$, j = 1, 2.
[7] See, for example, Hillman and Riley (1989), Jia (2008), Fu and Lu (2012) and the surveys on the stochastic derivation of contest success functions in Konrad (2009), Jia and Skaperdas (2011), and Jia, Skaperdas, and Vaidya (2011). Che and Gale (2000) discuss and analyze difference-form CSFs and their connection to the all-pay auction.
[8] This micro-foundation is not crucial to my argument. What matters is that judge H votes for the player who exerts the higher effort while judge L might not.

where $j \neq k, j = 1,2; k = 1, 2.$

Then when player j chooses his effort (bid), he believes that he will win the contest with probability

$$P_j = \lambda P_j^H + (1-\lambda)P_j^L , \qquad (3)$$

$j = 1, 2.$

The contest success function (CSF) in (3), a nested CSF, is a convex combination of a Tullock success function, $P_j^L$, and an auction success function, $P_j^H$, $j = 1,2$. To the best of my knowledge, all previous works have considered only one of these two popular CSFs at a time.[9] But in contests as in Amegashie (2006) and Epstein et al. (2007) where the identity of the ultimate decision-maker is not known and where the potential decision-makers have different abilities in or different approaches to evaluating performance, it is reasonable to nest these two CSFs. Furthermore, given that these are the two predominant CSFs in the literature,[10] it is interesting to examine a nested CSF which is a convex combination of these two CSFs.

Player j chooses $e_j$ to maximize

$$\Pi_j(e_j,e_k) = P_j V_j - e_j, \qquad (4)$$

where $j \neq k; j = 1,2; k = 1,2.$ In the present model with two players, the solution of this contest when $\lambda = 0$ (Tullock contest) or $\lambda = 1$ (all-pay auction) is well known. The Tullock contest has a unique equilibrium in pure strategies[11] while the all-pay auction has a unique equilibrium in

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[9] The Tullock contest is sometimes referred to as a lottery or an imperfectly discriminating contest while the all-pay auction is referred to as the perfectly discriminating contest. As noted earlier, the aggregate and individual efforts in these separate contests have been compared in Fang (2002), Epstein, Mealem, and Nitzan (2011), and Franke et al (2012).

[10] For example, Alcade and Dahm (2010, p. 5) observe that "[A] common modelling approach is to consider the two polar cases of Tullock's rent seeking game with exponent equal to one and the limiting case of the all-pay auction in which the exponent goes to infinity."

[11] For a discussion of general problems of existence in the Tullock contest, see, for example, Baye et al. (1994) and Konrad (2009).

mixed strategies.[12] My goal is to characterize the equilibria in the intermediate case of $\lambda \in (0,1)$ and compare it to the cases of $\lambda = 0$ and $\lambda = 1$.

## 2.1 Equilibrium analysis

### 2.1.1 Pure-strategy equilibria

Define $\hat{V}_j = (1-\lambda)V_j$ and $\Delta_j = P_j^L \hat{V}_j - e_j$, $j = 1, 2$. Note that $\Delta_j$ is the expected payoff of player

j in a Tullock contest in which his valuation is $\hat{V}_j$, $j = 1,2$. We can rewrite (4) as

$$\Pi_j(e_j, e_k) = \Delta_j + \lambda P_j^H V_j, \tag{5}$$

$j \neq k$; $j = 1,2$; $k = 1,2$.

I state the following proposition:

**Proposition 1:** Suppose that $V_1 > V_2$ and $\hat{\lambda} \equiv \dfrac{(V_1 - V_2)^2}{3V_1^2 + 3V_2^2 + 2V_1 V_2}$. If $\lambda \in (0, \hat{\lambda}]$, then (i) there

exists an equilibrium in pure strategies in which player 1 bids $e_1^* = \dfrac{(1-\lambda)V_1^2 V_2}{(V_1 + V_2)^2}$ and player 2

bids $e_2^* = \dfrac{(1-\lambda)V_2^2 V_1}{(V_1 + V_2)^2}$; the expected payoffs are $\Pi_1^* = \dfrac{V_1(V_1^2 + 2\lambda V_1 V_2 + \lambda V_2^2)}{(V_1 + V_2)^2}$ and

$\Pi_2^* = \dfrac{(1-\lambda)V_2^3}{(V_1 + V_2)^2}$, and (ii) this equilibrium is the only pure-strategy equilibrium.[13]

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[12] See Hillman and Riley (1989) and Baye et al. (1996). When there are more than two players, the all-pay auction may not have a unique mixed strategy equilibrium (Baye et al., 1996). Siegel (2009) presents a general analysis of a class of contests of which the complete-information all-pay auction is a special case. The contest in this paper does not belong to the class of contests studied in Siegel (2009).
[13] Of course, this proposition also holds in the well-known case of $\lambda = 0$.

**Proof**: Given previous results on the equilibrium of the Tullock game when $R = 1$ (e.g., Nti,

1999), we know that $\dfrac{\partial \Delta_1}{\partial e_1} = \dfrac{\partial \Delta_2}{\partial e_2} = 0$ if and only if $e_1^* = \dfrac{\hat{V}_1^2 \hat{V}_2}{\left(\hat{V}_1 + \hat{V}_2\right)^2} = \dfrac{(1-\lambda)V_1^2 V_2}{(V_1 + V_2)^2}$ and

$e_2^* = \dfrac{(1-\lambda)V_2^2 V_1}{(V_1 + V_2)^2}$. Since $V_1 > V_2$, it follows that $e_1^* > e_2^* > 0$. So in equilibrium, player 1's payoff

is

$$\Pi_1^* = \left(\lambda + (1-\lambda)\frac{V_1}{V_1 + V_2}\right)V_1 - e_1^* = \frac{V_1(V_1^2 + 2\lambda V_1 V_2 + \lambda V_2^2)}{(V_1 + V_2)^2}, \tag{6}$$

and player 2's equilibrium payoff is

$$\Pi_2^* = \frac{V_2}{V_1 + V_2}(1-\lambda)V_2 - e_2^* = \frac{(1-\lambda)V_2^3}{(V_1 + V_2)^2}. \tag{7}$$

We can rewrite the players' equilibrium payoffs as:

$$\Pi_1^* = \lambda V_1 + \Delta_1^*, \tag{6a}$$

and

$$\Pi_2^* = \Delta_2^*. \tag{7a}$$

Since $\partial \Delta_1/\partial e_1 = 0$ and $e_1^* > e_2^*$ in equilibrium, player 1 has no incentive to bid more than

his equilibrium bid because this reduces $\Delta_1$ without increasing $\lambda V_1$. If he deviates to a lower bid

he again reduces $\Delta_1$ without increasing $\lambda V_1$. Hence player 1 will not deviate.

Now consider player 2. If he deviates to a lower bid, he reduces $\Delta_2$ without changing his

probability of winning the contest when judge H is the decision maker. Note that $e_1^* < V_2$. If

player 2 deviates to $e_1^*$, he reduces $\Delta_2$ but increases the probability that judge H will vote for him

from 0 to 0.5. His payoff is

$$\Pi_2^d = \frac{1}{2} V_2 - e_1^* . \qquad (8)$$

Then player 2 will not deviate to $e_1^*$ if $\Pi_2^* \geq \Pi_2^d$ .

Finally, suppose player 2 deviates to a bid greater than $e_1^*$, say $e_1^* + \beta$ , where $\beta > 0$. Then

he reduces $\Delta_2$ but increases the probability that judge H will vote for him from 0 to 1. The payoff

from this deviation is:

$$\widetilde{\Pi}_2^d(\beta) = \left( \lambda + (1 - \lambda) \frac{e_1^* + \beta}{e_1^* + \beta + e_1^*} \right) V_2 - (e_1^* + \beta) . \qquad (10)$$

Clearly, there exists a small but positive value of $\beta$ which ensures that $\widetilde{\Pi}_2^d(\beta) > \Pi_2^d$ . Hence

deviating to $e_1^* + \beta$ dominates deviating to $e_1^*$ . Therefore, to show that player 2 has no profitable

deviation, it suffices to prove that $\Pi_2^* \geq \widetilde{\Pi}_2^d(\beta)$ given the stated conditions in the proposition.

Define $\widetilde{\Pi}_2^d(0) \equiv \left( \lambda + (1 - \lambda) \frac{e_1^*}{e_1^* + e_1^*} \right) V_2 - e_1^*$ . This is player 2's payoff if he were to get

judge H's vote with certainty by only matching player 1's bid, $e_1^*$ . In choosing $\beta$, player 2

effectively chooses $e_1^* + \beta \equiv e_2^d$ . Given that player 1 bids $e_1^*$, we know that $\dfrac{\partial \widetilde{\Pi}_2^d}{\partial e_2^d} = 0$ , if and only

if $e_2^d = e_2^*$ .[14] Given $e_1^* + \beta \equiv e_2^d > e_2^*$ and the strict concavity of the payoff function, it follows

that

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[14] Note that the term $\lambda V_2$ drops out when the payoff function in (10) is differentiated with respect to player 2's effort.

$$\left.\frac{\partial \tilde{\Pi}_2^d}{\partial e_2^d}\right|_{e_2^d=e_1^*+\beta} < 0. \tag{11}$$

Then given (11), $\Pi_2^* \geq \tilde{\Pi}_2^d(0)$ implies that $\Pi_2^* \geq \tilde{\Pi}_2^d(\beta)$ for all $\beta > 0$. It can be shown that

$\Pi_2^* \geq \tilde{\Pi}_2^d(0)$ if

$$\lambda \leq \hat{\lambda} \equiv \frac{(V_1 - V_2)^2}{3V_1^2 + 3V_2^2 + 2V_1V_2}. \tag{12}$$

To see that the equilibrium in proposition 1 is the only pure-strategy equilibrium, recall

that $\frac{\partial \Delta_1}{\partial e_1} = \frac{\partial \Delta_2}{\partial e_2} = 0$ if and only if $(e_1, e_2) = (e_1^*, e_2^*)$. It follows that any pair of pure-strategy

bids $(e_1, e_2) \neq (e_1^*, e_2^*)$ implies that $\frac{\partial \Delta_j}{\partial e_j} \neq 0$ for, at least, one player, $j = 1, 2$. Then there is a

player j who can *marginally* deviate from his bid and be better off because he can increase $\Delta_j$

without affecting $\lambda P_j^H V_j$ if $P_j^H \in \{0,1\}$ or can increase $\lambda P_j^H V_j$ if $P_j^H = 0.5$, $j = 1,2$. **QED.**

Note that $\hat{\lambda} < 1/3$.[15] Then given the inequality condition in proposition 1 and the fact that

the equilibrium is the only pure-strategy equilibrium, we get the following result:

**Corollary 1:** There is no pure-strategy equilibrium of the nested contest if (i) $\lambda \geq 1/3$, and/or (ii)

$V_1 = V_2 = V$.

If $\lambda = 0$ (i.e., a Tullock contest), then aggregate effort is $E_{TC}^* = \frac{V_1 V_2}{V_1 + V_2}$ and if $\lambda = 1$ (an

all-pay auction), then aggregate effort is $E_{APA}^* = \frac{V_2(V_1 + V_2)}{2V_1}$. And for $\lambda \in (0, \hat{\lambda}]$, aggregate

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effort in the nested contest is $E_n^* = e_1^* + e_2^* = \dfrac{(1-\lambda)V_1V_2}{V_1 + V_2}$. Hence if proposition 1 holds, then

given $\lambda \in (0, \hat{\lambda}]$, $E_n^* = (1-\lambda)E_{TC}^* < E_{TC}^*$. It follows that if $E_{APA}^* > E_{TC}^*$ and proposition 1

holds, we get the surprising result that $E_n^* \notin [E_{TC}^*, E_{APA}^*]$.[16]

This leads to the following proposition:

**Proposition 2**: Suppose that the nested contest has a pure-strategy equilibrium and aggregate

efforts in the all-pay auction are higher than the aggregate efforts in the Tullock contest. Then

aggregate efforts in the pure-strategy equilibrium of the nested contest -- a convex combination

of the all-pay auction and the Tullock contest -- may not lie between aggregate efforts in the all-

pay auction and aggregate efforts in the Tullock contest.[17]

As an example, suppose that $V_1 = 10$, $V_2 = 4.15$, and $\lambda = 0.01$. Then proposition 1 holds

and $E_n^* = 2.90335$. We get $E_{TC}^* = 2.93286$ and $E_{APA}^* = 2.93612$. Clearly,

$E_n^* \notin [E_{TC}^*, E_{APA}^*]$.[18]

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[16] Note that $E_{APA}^* > E_{TC}^*$ if $V_2^2 + 2V_1V_2 - V_1^2 > 0$.

[17] I use "may" because as shown below, there could be multiple equilibria in which one of the equilibria of the nested contest gives the same expected aggregate efforts as the expected efforts in the standard all-pay auction.

[18] Using the parameters $V_1 = 25$, $V_2 = 4$, and $\lambda = 0.2$ gives $E_{APA}^* = 2.3200 < E_{TC}^* = 3.44828$ and $E_n^* = 2.75862$.

Therefore, in this case $E_n^* \in [E_{APA}^*, E_{TC}^*]$.

*2.1.2 Mixed-strategy equilibria*

**Lemma 1**: If $V_1 = V_2 = V > 0$, then there exists a non-degenerate symmetric mixed-strategy

Nash equilibrium in the nested contest for all $\lambda \in (0,1)$. It has no atoms at points of discontinuity.

**Proof:** The proof comes from noting that when $V_1 = V_2 = V$, this discontinuous game has no

pure-strategy equilibrium (i.e., corollary 1 above) and satisfies Theorem 6 in Dasgupta and

Maskin (1986).[19] To apply Theorem 6 in Dasgupta and Maskin (1986), a player must choose his

action (bid) from a non-empty and compact set. In the nested contest, this set is $[0, V]$. The

game must also satisfy the following conditions: first, the sum of the payoffs must be upper

semi-continuous. Since $\Pi_1 + \Pi_2 = V - e_1 - e_2$ is continuous, it follows that the sum of the

payoffs is upper semi-continuous. Second, $\Pi_j$ must be bounded, $j = 1, 2$. This holds since

$-V \le \Pi_j \le V$ for $e_j \in [0, V]$, $j = 1, 2$. Third, the discontinuity set must be defined such that

player j's payoff is discontinuous only if j's strategy is related to player k's strategy by some

function, $f_{jk}(\cdot)$, such that $e_k = f_{jk}(e_j)$, $j \ne k$, $j = 1,2$, $k = 1,2$. Since the players' payoffs are only

discontinuous at symmetric bids, it follows that the identity function satisfies the desired

requirement. Finally, the game must satisfy a property called property ($\alpha$). To see that this

property is satisfied, consider an arbitrary point of discontinuity, $e_1 = e_2 = \hat{e} \ge 0$.

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[19] The reasoning is the same as the proof of Lemma 2.3 in Alcade and Dahm (2010) who apply Theorem 6 in Dasgupta and Maskin (1986) to their game (contest). Baye et al. (1994) were the first to apply existence theorems in Dasgupta and Maskin (1994) to a contest, in particular, the Tullock contest.

Then property ($\alpha$) is satisfied since

$$\lim_{e_j \to^+ \hat{e}} \inf \Pi_j(e_j, \hat{e}) = [\lambda + 0.5(1 - \lambda)]V - \hat{e} > 0.5V - \hat{e} = \Pi_j(\hat{e}, \hat{e}), \qquad (*)$$

for all $\lambda \in (0,1)$, $j = 1, 2$.[20] **QED**.

**Lemma 2:** If $V_1 \geq V_2 > 0$, then there exists a mixed-strategy Nash equilibrium in the nested contest for all $\lambda \in (0,1)$.

**Proof:** Theorem 5 in Dasgupta and Maskin (1986) guarantees the existence of a mixed-strategy equilibrium. The proof that this theorem is applicable is the same as the proof of lemma 1 except that V is replaced by $V_j$ and the players' payoff functions must be weakly lower semi-continuous at points of discontinuity, $j = 1, 2$. That is, the strict inequality in (*) is replaced with a weak inequality.[21] This, of course, holds in this game for all $\lambda \in (0,1)$. **QED**.

**Proposition 3:** Suppose that $V_1 \geq V_2 = V > 0$ and $\lambda \in [0.25,1)$. Then then there exists a non-degenerate mixed-strategy equilibrium in the nested contest defined implicitly by equations (A.6) and (A.7) below. In equilibrium, player 1's distribution function, G(e) is (i) continuous over its entire support, $[0, \bar{v}]$, (ii) differentiable on $(0, \bar{v})$, (iii) strictly increasing on $(0, \bar{v}]$, and (iv) strictly convex on a subset of its support, where $\bar{v} \in (\lambda V, V)$. Player 2 abstains from the contest with probability $(1 - V_2/V_1)$ but uses the same distribution function, G(e) whenever he

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[20] The corresponding condition in Alcade and Dahm (2010) is equivalent to assuming that $\lambda = 1$ in (*). Note that part (b) of the DS property in Alcade and Dahm (2010) requires that at a point of discontinuity, a bidder's success probability must jump to 1. However, at a symmetric *positive* bid, the CSF in this paper is such that an increase in a player's bid leads to a jump in his success probability from 0.5 to $0.5(1 + \lambda) < 1$ given that $\lambda \in (0,1)$. Nevertheless, the CSF in this paper satisfies the spirit of the DS property in Alcade and Dahm (2010) because it pays for a player to increase his bid slightly at points of discontinuity. This is the crucial requirement of the DS property that is relevant for the proof of Lemma 2.3 in Alcade and Dahm (2010). Part (b) of the DS property in Alcade and Dahm (2010) is required to ensure that property ($\alpha$) in Theorem 6 of Dasgupta and Maskin (1986) is satisfied.
[21] Property ($\alpha$) is a stronger version of the requirement of weakly lower semi-continuity. It is property ($\alpha$) which ensures that the players' equilibrium symmetric mixed strategies have no atoms at points of discontinuity.

participates. The equilibrium expected payoffs and expected bids are $\hat{\Pi}_1 = V_1 - V_2$, $\hat{\Pi}_2 = 0$,

$E(e_1) = V_2/2$, and $E(e_2) = (V_2)^2/2V_1$.

**Proof:** Note that if the equilibrium in proposition 3 exists, then it must be a non-degenerate

mixed-strategy equilibrium because the equilibrium in proposition 1 is the only pure-strategy

equilibrium and it has a different set of equilibrium payoffs from the payoffs in proposition 3.

The claims about the equilibrium expected payoffs and bids can be shown by applying

Theorem 3.2 in Alcade and Dahm (2010). This requires showing that the nested contest success

function satisfies the anonymity and elasticity properties of their theorem. They also use a

property called DS in their proof. But the DS property is only required to show that Theorem 6

of Dasgupta and Maskin (1986) is applicable to their contest (game).[22] Accordingly, I shall only

prove that the nested contest success function satisfies the anonymity and elasticity conditions of

Alcade and Dahm (2010) when $\lambda \geq 0.25$.

First, it is obvious that the nested contest success function satisfies the anonymity

property because a contestant's success probability is independent of his identity; it only depends

on the vector of bids.

The elasticity condition has two parts: E1 and E2. Let $\{x, G\} \in \mathbf{N}_+$. Define $\eta_j(e_j, e_k)$ as the

elasticity of a contestant's success probability with respect to his effort, $j = 1, 2$; $k = 1, 2$. Given

the anonymity condition, there is no loss of generality in dropping subscripts. Condition E1 in

Alcalde and Dahm (2010) is satisfied if

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[22] See note 20 of this paper.

$$\eta\left(\frac{x}{G}, \frac{y}{G}\right) \geq \eta\left(\frac{x}{G}, \frac{x}{G}\right), \tag{13}$$

for all $x \in \{0, 1, ..., \bar{x}\}$, $y \in \{x+1, ..., \bar{x}+1\}$, and $\bar{x}$ satisfies $\bar{x} \leq GV_1 < \bar{x} + 1$.

Based on the results in Alcade and Dahm (2010), we know that the Tullock contest success function and the all-pay auction (APA) success function each satisfies (13). Therefore, we can write:

$$\eta\left(\frac{x}{G}, \frac{y}{G}\right)_{Tullock} \geq \eta\left(\frac{x}{G}, \frac{x}{G}\right)_{Tullock} \quad \text{and} \quad \eta\left(\frac{x}{G}, \frac{y}{G}\right)_{APA} \geq \eta\left(\frac{x}{G}, \frac{x}{G}\right)_{APA}. \tag{14}$$

Then multiplying the first inequality in (14) by $(1 - \lambda)$, the second inequality by $\lambda$, and adding gives

$$(1-\lambda)\eta\left(\frac{x}{G}, \frac{y}{G}\right)_{Tullock} + \lambda\eta\left(\frac{x}{G}, \frac{y}{G}\right)_{APA} \geq (1-\lambda)\eta\left(\frac{x}{G}, \frac{x}{G}\right)_{Tullock} + \lambda\eta\left(\frac{x}{G}, \frac{x}{G}\right)_{APA}. \tag{15}$$

Then the inequality in (15) implies that the nested contest success function satisfies condition E1 in Alcade and Dahm (2010). Finally, we have to show that when $\lambda \geq 0.25$, the nested CSF satisfies condition E2.[23] The nested CSF satisfies condition E2 in Alcade and Dahm (2010) if

$$\lambda + \frac{(1-\lambda)(x+1)}{2x+1} \geq \frac{x+2}{2(x+1)}. \tag{16}$$

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[23] According to Alcade and Dahm (2010, p. 4), to ensure that a contest has an all-pay auction equilibrium, condition E2 in their paper "... specifies a minimum win probability that outbidding the opponent by the minimum amount must yield, implying that the CSF must be sufficiently discriminating in favor of the higher bidder. This specifies a lower bound on how much the extreme case of the APA (all-pay auction), in which the higher bidder definitely wins the contest, can be relaxed." parenthesis mine.

The inequality in (16) holds for x = 0. For x ≥ 1, it can be rewritten as $\lambda \geq 0.5/(x + 1)$.[24] The denominator of the expression on the right hand side is strictly decreasing in x and so it is maximized at x = 1 where it attains the value of 0.25. Hence the nested CSF satisfies condition E2 if $\lambda \geq 0.25$. This completes the proof that when $\lambda \geq 0.25$, the nested contest has an *all-pay auction equilibrium*.[25]

Regarding the other claims in proposition 3, we apply Lemma B.2 in Alcade and Dahm (2010). This lemma says that when a symmetric two-player contest has a symmetric equilibrium (possibly in mixed strategies) and there is complete rent dissipation in this equilibrium, then the asymmetric version of the contest (i.e., players with non-identical valuations) has an equilibrium which is the same as the equilibrium of the symmetric contest except that the player with the smaller valuation abstains from the contest with a positive probability. Accordingly, I investigate the symmetric version of this contest in appendices A and B and demonstrate the other claims in proposition 3. **QED**

Propositions 1 and 3 imply the following corollary:

**Corollary 2:** If $\lambda \in [0.25, \hat{\lambda}]$, then there exist two *payoff-non-equivalent* equilibria in the nested contest: a pure-strategy non-all-pay auction equilibrium as in proposition 1 and a mixed-strategy all-pay auction equilibrium as in proposition 3.

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[24]Define $\kappa \equiv (x + 1)/(2x + 1)$. Then (18) can be rewritten as $\lambda \geq [0.5(x + 2)/(x + 1) - \kappa]/(1 - \kappa)$ . Note that $\kappa = 1$ if x = 0. Therefore, $[0.5(x + 2)/(x + 1) - \kappa]/(1 - \kappa)$ is undefined if x = 0. If x > 0, $0 < \kappa < 1$ and we get $\lambda \geq 0.5/(x + 1)$.

[25] An *all-pay auction equilibrium* in Alcade and Dahm (2010) is defined as an equilibrium with the *same* expected equilibrium payoffs and total efforts (but not necessarily the same distribution functions) as the standard all-pay auction in Baye et. al (1996) and Hillman and Riley (1989). Theorem 3.2 in Alcalde and Dahm (2010) states sufficient conditions that a CSF must satisfy in order to guarantee the existence of an all-pay auction equilibrium.

The equilibria in corollary 2 cannot be ranked using the Pareto criterion. Obviously, player 2 is better off in the equilibrium in proposition 1 since $\Pi_2^* = \dfrac{(1-\lambda)V_2^3}{(V_1+V_2)^2} > 0$. However, player 1 is worse off. To see this, note that

$$\Pi_1^*(\lambda) - (V_1 - V_2) = \frac{V_2[(2\lambda-1)V_1^2 + (1+\lambda)V_1V_2 + V_2^2]}{(V_1+V_2)^2}, \tag{17}$$

where the dependence of $\Pi_1^*$ on $\lambda$ is made explicit. The expression in (17) is strictly increasing in $\lambda$. Consider the highest value of $\lambda$ in corollary 2. This is, of course, $\hat{\lambda}$ since $\lambda \in [0.25, \hat{\lambda}]$ in corollary 2. This gives the highest possible value for (17). Putting $\hat{\lambda}$ into (17) gives

$$\Pi_1^*(\hat{\lambda}) - (V_1 - V_2) = \frac{V_2(3V_2^2 - V_1^2)}{3V_1^2 + 2V_1V_2 + 3V_2^2}. \tag{18}$$

Towards a contradiction, suppose that when corollary 2 holds, player 1 is not worse off in the equilibrium in proposition 1 relative to the equilibrium in proposition 3. Then the expression in (18) is non-negative. This requires $V_1^2 / V_2^2 \le 3$. And for both equilibria to exist (i.e., for corollary 2 to hold), we require $\hat{\lambda} \ge 0.25$. This holds if

$$V_1^2 - 10V_1V_2 + V_2^2 \ge 0. \tag{19}$$

It is obvious that there exists pairs of valuations $(V_1, V_2)$ which satisfy (19).

Let $V_1^2 = \theta V_2^2$. Therefore, $V_1 = V_2\sqrt{\theta}$. Then we require $\theta \in (1,3]$ because $V_1 = V_2\sqrt{\theta}$ and $V_1 > V_2$ imply that we require $\sqrt{\theta} > 1$ or $\theta > 1$. Put $V_1 = V_2\sqrt{\theta}$ into (19) and simplify to get

$$(1 + \theta - 10\sqrt{\theta})V_2^2 \ge 0. \tag{20}$$

There exists no $\theta \in (1,3]$ which satisfies (20). So when (18) is non-negative, the inequality in (19) does not hold. This contradicts the claim that corollary 2 holds and player 1 is not worse off in the equilibrium in proposition 1 relative to the equilibrium in proposition 3. Accordingly, if corollary 2 holds, then player 1 must be worse off in the equilibrium in proposition 1.

## 3. Discussion

Corollary 2 is interesting. To appreciate this point, Baye et. al (1996) showed that in an all-pay auction with *three or more* players, there could be multiple equilibria. However, these equilibria are all *payoff-equivalent*. Alcalde and Dahm (2010) also find a similar result for imperfectly discriminating contests. In a Tullock contest with *three or more* players, Ewerhart (2012) shows that there are multiple *non-payoff equivalent* equilibria if the returns parameter of the Tullock CSF is sufficiently high.[26,27] Corollary 2 is interesting because it shows that with only *two* players, there are multiple *non-payoff equivalent* equilibria in the nested contest. Also, it is interesting that one equilibrium does not Pareto dominate the other because player 1 prefers one equilibrium while player 2 prefers the other equilibrium.

If proposition 1 holds, then the nested contest gives smaller individual and aggregate efforts than a standard Tullock contest. Hence if the goal is to reduce aggregate effort (i.e., wasteful rent-seeking efforts), then this contest is better than a Tullock contest. Intuitively, the pure-strategy equilibrium in proposition exists if a sufficiently high weight is put on the Tullock component of the nested contest. Since $\partial \Pi_1^* / \partial \lambda > 0$ but $\partial \Pi_2^* / \partial \lambda < 0$, it follows that in the

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[26] Chowdhury and Sheremeta (2011) find multiple equilibria in Tullock contests.  But in their model, the valuations of the players are *endogenous* (i.e., they are functions of the players' efforts). The current paper and the cited papers consider *exogenous* valuations.

[27] Note that the nested contest is not a log-supermodular contest as defined in Ewerhart (2012).

equilibrium in proposition 1, the stronger player is better off when a bigger weight is put on the all-pay auction component of the contest while the weaker player is worse off.

Proposition 2 is particularly striking because it shows that a contest which is a convex combination of the all-pay auction and the Tullock contest could yield an aggregate effort which is *not* a convex combination of the aggregate efforts in the Tullock contest and all-pay auction. Thus the nested contest could have different properties from its underlying contests. In fact, when proposition 2 holds, then aggregate effort in the nested contest is smaller than aggregate effort in the Tullock contest and it is also smaller than aggregate effort in the all-pay auction. If the goal is to reduce aggregate effort (i.e., wasteful rent-seeking efforts), then this contest may be better than both the Tullock contest and the all-pay auction.

The intuition for proposition 2 requires an understanding of the intuition for the equilibrium in proposition 1. First, a pure-strategy equilibrium exists if the weight on the all-pay auction is sufficiently small. Then the asymmetry in valuations induces a competition for only the Tullock-contest part of the rent because by maximizing his payoff based on only the Tullock part, the strong contestant exerts a higher effort which is enough to guarantee the all-pay auction part of the rent. Since the all-pay auction part of the rent is sufficiently small, it does not pay for the weak contestant to outbid the strong contestant in order to get the all-pay auction part especially when the asymmetry in valuations is sufficiently high. So he also maximizes his payoff based on the Tullock part of the rent. And since their valuations in the Tullock part is smaller than their valuation if the game was only a Tullock contest and also given that their valuations is reduced by the *same* proportion (i.e., from $V_j$ to $(1 - \lambda)V_j$), it follows that each player's effort is smaller.[28]

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[28] The equal proportional reduction in their valuations is crucial because reductions in their valuations that are not proportionally the same could lead to a rise in aggregate efforts (see Epstein and Nitzan, 2006b).

Unlike the two-player all-pay auction, proposition 3 implies that neither player randomizes uniformly over a given support in a mixed-strategy equilibrium. Finally, because the auction success function is an explicit component of the nested CSF in (3), we can say more about the properties of the distribution function (in a mixed-strategy equilibrium) than is possible in Alcade and Dahm (2010) and Baye et al. (1994).

## 4. Conclusion

The Tullock contest and all-pay auction have been examined separately in the literature on contests. This paper has taken the first step of examining a contest which is a nested version of these two popular contests. Some properties of the equilibria of this nested contest differ from all-pay auction and Tullock contests. In addition to Alcade and Dahm (2010), this approach may be an alternative way of studying departures from the all-pay auction. For example, one could use this approach to study the robustness of the "exclusion principle" in Baye et al. (1993). It remains to *describe the properties* of an equilibrium to the nested contest when $\lambda \in (\hat{\lambda}, 0.25)$. By lemma 2, we know that an equilibrium exists and given part (ii) of proposition 1, it will be a non-degenerate mixed strategy because $\lambda > \hat{\lambda}$. I conjecture that it will be a non-all-pay auction equilibrium. Experimental studies of this nested contest will also be interesting. All these extensions are left for future investigations.

**Appendix A**

Following up on lemma 1, we can establish a bit more about the nature of the symmetric

mixed-strategy equilibrium of this contest. Accordingly, I undertake this investigation.

Let $G(e_j)$ be the *common* cumulative distribution function (cdf) with support $[\underline{v}, \overline{v}]$ by

player j in a symmetric mixed-strategy equilibrium, j = 1, 2. Since there is no pure-strategy

equilibrium in the symmetric case, it follows that a mixed-strategy equilibrium will necessarily

be non-degenerate. Therefore, $\overline{v} > \underline{v}$.

Define

$$\Omega_j(e_j, \hat{V}, \underline{v}, \overline{v}) = \hat{V} \int_{\underline{v}}^{\overline{v}} \frac{e_j}{e_j + e_k} dG(e_k) - e_j, \tag{A.1}$$

where $\hat{V} = (1 - \lambda)V$, j = 1,2; k = 1,2; j ≠ k.

Then suppressing $\underline{v}, \overline{v}$, and $\hat{V}$ in the expression for $\Omega_j$, we may write player j's payoff as:

$$\hat{\Pi}_j(e_j) = \Omega_j(e_j) + \lambda G(e_j)V, \tag{A.2}$$

j = 1, 2.

Let

$$\tilde{e}_j = \arg\max_{e_j} \Omega_j(e_j). \tag{A.3}$$

Since in a non-degenerate mixed-strategy equilibrium, no player puts a probability mass equal to 1 on a bid of zero,[29] it follows that $e_j = \overline{v} \geq V$ gives $\hat{\Pi}_j(e_j) < 0$, $j = 1,2$. Then given that a player can guarantee himself a minimum payoff of zero, it follows that $\overline{v} < V$. Similarly, equation (A.1) implies that $\Omega_j(e_j) < 0$ if $e_j \geq \hat{V} \equiv (1-\lambda)V$. Therefore, $e_j \in [0, \hat{V})$, $j = 1, 2$. By strict concavity of $\Omega_j(e_j)$, $\tilde{e}_j = \tilde{e}_j(\hat{V}, \underline{v}, \overline{v}) \geq 0$ is unique, $j = 1, 2$.

I shall now prove some lemmas by drawing on arguments that are similar to those in Hillman and Riley (1989) and Baye et al. (1996) but differ from them in some respects because the nested contest has a Tullock component. In what follows, I note that in a symmetric equilibrium, $\tilde{e}_j = \tilde{e}$, $j = 1, 2$.

**Lemma 3**: *Neither player will be bid a positive bid with positive probability; equilibrium strategies (above zero) are continuous mixed strategies.*

**Proof:** To see this, suppose (without loss of generality) that player 1 bids $e_1 = \hat{e}_1 > 0$ with positive probability. Then the probability that player 2 wins rises discontinuously as function of his bid at $e_2 = \hat{e}_1$. Hence there exists $\varepsilon > 0$ such that player 2 will bid on the interval $[\hat{e}_1 - \varepsilon, \hat{e}_1]$ with zero probability.[30] But then player 1 is better off bidding $\hat{e}_1 - \varepsilon$ instead of $\hat{e}_1$ since his probability of winning is the same. This contradicts the hypothesis that bidding $e_1 = \hat{e}_1 > 0$ with positive probability is an equilibrium strategy. **QED**

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[29] We shall show that there is no atom on zero.

[30] This is also true in the case of non-identical players. To elaborate, note that for $e_1 > 0$, $e_2/(e_1 + e_2)$ is strictly increasing in $e_2$. Suppose (without loss of generality) that player 1 bids $e_1 = \hat{e}_1 > 0$ with probability $\delta > 0$. Let $\rho \in [0, 1 - \delta]$ be the probability that player 1 bids $e_1 < \hat{e}_1$. Then if player 2 also bids $\hat{e}_1$, his expected payoff is $[\lambda(\delta/2 + \rho) + (1-\lambda)\int \frac{\hat{e}_1}{\hat{e}_1 + e_1} dG_1(e_1)] V_2 - \hat{e}_1$. If instead player 2 were to bid $\hat{e}_1 + \varepsilon$, where $\varepsilon > 0$ but *very small*, his expected payoff will be, *at least*, $[\lambda(\delta + \rho) + (1-\lambda)\int \frac{\hat{e}_1}{\hat{e}_1 + e_1} dG_1(e_1)] V_2 - (\hat{e}_1 + \varepsilon)$. Then since there exists

**Lemma 4:** $\underline{v} = \widetilde{e}$ .

**Proof:** Since negative bids are not possible, we require $\underline{v} \geq 0$. Towards a contradiction, suppose that $\underline{v} \neq \widetilde{e}$. In particular, suppose that $\underline{v} > \widetilde{e} \geq 0$. Then $\underline{v} > 0$. Given lemma 3, it follows that the probability of a tie at a positive bid is zero. Therefore, $G(\underline{v}) = 0$. Then bidding $\underline{v} > 0$ gives

$\hat{\Pi}_j(\underline{v}) = \Omega_j(\underline{v})$, j = 1, 2. Then by deviating and reducing his bid to $\widetilde{e}$, player j increases $\Omega_j(e_j)$ without reducing $\lambda G(e_j)V$, j = 1, 2. Now suppose instead that $\underline{v} < \widetilde{e}$. Then bidding $\widetilde{e}$ (a higher bid) dominates bidding $\underline{v}$ because player j increases $\Omega_j(e_j)$ without reducing $\lambda G(e_j)V$, j = 1, 2.

**QED.**

**Lemma 5:** *In a symmetric equilibrium, no player puts an atom on $\underline{v}$.*

**Proof:** In a *symmetric* equilibrium, the lower bound of the support of each player's mixed strategy is $\underline{v}$. Suppose that $\underline{v} = \widetilde{e} > 0$. Then lemma 3 implies that there cannot be an atom at $\underline{v}$. Suppose instead that $\underline{v} = \widetilde{e} = 0$. Towards a contradiction, suppose that player 1 puts an atom on $\underline{v} = 0$ in a symmetric equilibrium. Then there exists some $\varepsilon > 0$ such that player 2 is better off bidding $\underline{v} + \varepsilon > 0$ than bidding $\underline{v}$ since his success probability increases discontinuously.[31] So player 2's minimum bid is $\underline{v} + \varepsilon \neq \underline{v} = 0$. Then the equilibrium is not symmetric, a

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$\varepsilon \in (0, 0.5\lambda\delta V_2)$, player 2 is better off bidding $\hat{e}_1 + \varepsilon$ than bidding $\hat{e}_1$ if player 1 bids $\hat{e}_1$ with positive probability. If player 2 bids $\hat{e}_1 - \varepsilon' \geq 0$, his *maximum* expected payoff is $[\lambda\rho + (1 - \lambda)\int_{\hat{e}_1 + e_1}^{\hat{e}_1} dG_1(e_1)] V_2 - (\hat{e}_1 - \varepsilon')$, where $\varepsilon' > 0$ but *very small*. Then bidding $\hat{e}_1 + \varepsilon$ dominates bidding $\hat{e}_1 - \varepsilon'$ since there exists $(\varepsilon + \varepsilon') \in (0, \lambda\delta V_2)$. Hence there exists $\varepsilon' > 0$ such that player 2 will bid on the interval $[\hat{e}_1 - \varepsilon', \hat{e}_1]$ with zero probability.

[31] This implies that in the asymmetric case, both players cannot have atoms at zero. Only one player could possibly have an atom at zero.

contradiction.[32] Alternatively, since a bid of zero is a point of discontinuity, Theorem 6 in

Dasgupta and Maskin (1986) implies that there is no atom at zero. **QED**.

**Lemma 6:** *The equilibrium distribution, G(e), is (i) differentiable on* $(\underline{v}, \overline{v})$*, (ii) strictly*

*increasing on* $(\underline{v}, \overline{v}]$*, and (iii) strictly convex on a subset of its support*.

**Proof:** For parts (i) and (ii), see appendix B. Consider part (iii). Towards a contradiction,

suppose that $G(e_j)$ is concave on its entire support, $j = 1, 2$. Then we cannot construct a non-

degenerate mixed-strategy equilibrium because, given that $\Omega(e_j)$ is a strictly concave function,

$\hat{\Pi}(e_j) = \Omega(e_j) + \lambda G(e_j)V$, will also be strictly concave in $e_j$ and so a unique $e_j$ maximizes each

player's payoff given the strategy of the other player, $j = 1, 2$. Thus a necessary condition for

constructing a non-degenerate mixed-strategy equilibrium is that $G(e_j)$ is strictly convex on, at

least, a subset of its support. **QED**.

In a symmetric mixed-strategy equilibrium, $\hat{\Pi}(\underline{v}) = \hat{\Pi}(\overline{v})$. Lemmas 3 and 5 imply that

the players randomize their bids *continuously* on $[\underline{v}, \overline{v}]$. Therefore, $G(\underline{v}) = 0$ and $G(\overline{v}) = 1$. Then

$\hat{\Pi}(\underline{v}) = \hat{\Pi}(\overline{v})$ gives

$$\Omega(\underline{v}) = \lambda V + \Omega(\overline{v}), \tag{A.4}$$

where $\Omega(\underline{v}) > \Omega(\overline{v})$.

Expanding (A.4) and simplifying gives

$$(1 - \lambda)V \int_{\underline{v}}^{\overline{v}} \frac{e}{(\overline{v} + e)(\underline{v} + e)} dG(e) = 1 - \frac{\lambda V}{\overline{v} - \underline{v}}. \tag{A.5}$$

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[32] Recall that the present game satisfies Theorem 6 in Dasgupta and Maskin (1986). This theorem does not rule out the existence of asymmetric equilibria in symmetric discontinuous games. However, by appealing to this theorem, we know that a symmetric mixed-strategy exists in this game, as stated in lemma 1. Therefore, given that our focus is on symmetric equilibria of the game, it is correct to claim that the players' mixed strategy must have the *same* support in equilibrium.

The left hand side of (A.5) is positive since $0 < \lambda < 1$. Therefore, the right hand side

must also be positive. This gives $\bar{v} > \underline{v} + \lambda V$. Recall that $\tilde{e} < \hat{V} \equiv (1-\lambda)V$. Otherwise, the

optimized value of $\Omega(e)$ will be negative. Therefore, $\underline{v} = \tilde{e} < (1-\lambda)V$.

Given (A.4), we require that in a symmetric mixed-strategy equilibrium,

$\lambda V G(e) + \Omega(e) = \Omega(\underline{v})$ for all $e \in [\underline{v}, \bar{v}]$. Expanding and simplifying gives

$$\lambda V G(e) + (1-\lambda)V(e-\underline{v}) \int_{\underline{v}}^{\bar{v}} \frac{e_k}{(e+e_k)(\underline{v}+e_k)} dG(e_k) - (e-\underline{v}) = 0, \tag{A.6}$$

for all $e \in [\underline{v}, \bar{v}]$.

Given lemma 4, we rewrite equation (A.3) as

$$\underline{v} = \arg\max_{e} \left( (1-\lambda)V \int_{\underline{v}}^{\bar{v}} \frac{e}{e+e_k} dG(e_k) - e \right). \tag{A.7}$$

As in Alcade and Dahm (2010) and Baye et al. (1994), an explicit solution for the

distribution function, $G(e)$, given that the strategy space is continuous, is either impossible or

very difficult.[33] In the present game, the desired cumulative distribution function implicitly

satisfies equations (A.6) and A.7). Its existence is guaranteed by Theorem 6 in Dasgupta and

Maskin (1986).[34] Like Alcalde and Dahm (2010) and Baye et al. (1994), I am unable to prove

that it is unique.

Finally, the following observations are helpful. If $\underline{v} = \tilde{e} = 0$ and player 1 bids this

amount, then $\hat{\Pi}_1(\underline{v}) = \Omega_1(\underline{v}) = 0$ because $G(\underline{v}) = 0$ and $\tilde{e}/(\tilde{e} + e_2) = 0$ (there are no ties at any

bid). Therefore, if $\underline{v} = \tilde{e} = 0$ in a symmetric mixed-strategy equilibrium, then $\hat{\Pi}(\underline{v}) = \Omega(\underline{v}) = 0$

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[33]To the best of my knowledge, this has not been done for the Tullock contest studied in Baye et al (1994).

[34] Lemma 4 says that $\tilde{e}(\hat{V}, \underline{v}, \bar{v}) = \underline{v}$. This requires finding a fixed point for $\underline{v}$. Obviously, this fixed point exists if $\tilde{e} = 0$.

(i.e., complete rent dissipation). By applying theorem 3.2 in Alcalde and Dahm (2010), we have already shown that such an equilibrium indeed exists if $\lambda \geq 0.25$. Hence, if $\lambda \geq 0.25$, then it follows that $\underline{v} = \tilde{e} = 0$. Then applying lemma B.2 and Theorem 3.2 in Alcade and Dahm (2010) proves all the claims in proposition 3 except the claims about the monotonicity and differentiability of the equilibrium distribution function, $G(e)$.[35]

## Appendix B

**Part (i) of lemma 6: proof that the equilibrium distribution function, $G(e)$, is differentiable on $(\underline{v}, \overline{v})$.**

Without loss of generality, consider player 1. Fix some $e_1 \in (\underline{v}, \overline{v})$. Consider a sequence $e_1^n$ with limit $e_1$, where $e_1^n \in (\underline{v}, \overline{v}) \ \forall n$.

Define $d(e_1) = \int_{\underline{v}}^{\overline{v}} \frac{e_1}{e_1 + e_2} dG(e_2)$ and $d(e_1^n) = \int_{\underline{v}}^{\overline{v}} \frac{e_1^n}{e_1^n + e_2} dG(e_2)$.

In a mixed-strategy equilibrium, we must have

$$\lambda V G(e_1) + (1 - \lambda) V d(e_1) - e_1 = \lambda V G(e_1^n) + (1 - \lambda) V d(e_1^n) - e_1^n, \tag{B.1}$$

for all n. Then we get

$$\frac{G(e_1) - G(e_1^n)}{e_1^n - e_1} = \frac{1}{\lambda V} + \frac{(1 - \lambda)}{\lambda} [d(e_1) - d(e_1^n)]. \tag{B.2}$$

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[35]I am unable to prove that an equilibrium with $\underline{v} = \tilde{e} > 0$ exists. If it did, it will imply partial rent dissipation. To see this, note that $\hat{\Pi}(\underline{v}) = \Omega(\underline{v}) + \lambda G(\underline{v}) V = \Omega(\underline{v})$ given that $G(\underline{v}) = 0$. Since a player can guarantee himself a payoff of zero by bidding zero and $\tilde{e}$ is the unique maximizer of $\Omega(e)$, it follows that $\underline{v} = \tilde{e} > 0$ necessarily implies that $\hat{\Pi}(\underline{v}) = \Omega(\underline{v}) > 0$ (i.e., partial rent dissipation).

Note that $\lim_{n\to\infty} e_1^n = e_1$ implies that $\lim_{n\to\infty}[d(e_1) - d(e_1^n)] = 0$. Then given $\lambda \in (0,1)$, we get

$$G'(e_1) = \lim_{n\to\infty}\left(\frac{G(e_1) - G(e_1^n)}{e_1^n - e_1}\right) = \frac{1}{\lambda V}. \qquad (B.3)$$

Since for $e_1 \in (\underline{v}, \overline{v})$, $G'(e_1)$ exists and is finite, it follows that $G(e_1)$ is differentiable on $(\underline{v}, \overline{v})$. By symmetry, this is true for player 2. **QED.**


**Part (ii) of lemma 6: proof that G(e) is strictly increasing on $(\underline{v}, \overline{v}]$.**

We note that in a mixed-strategy equilibrium, a player's payoff is constant for all e in the support of his mixed strategy. Without loss of generality, consider player 1. We can write

$$\lambda V G(e_1) + \Omega(e_1) = \Omega(\underline{v}). \qquad (B.4)$$

Then given that $G(e_1)$ is differentiable, we can take the derivative of (B4) with respect to $e_1$ and rearrange to get:

$$\lambda V G'(e_1) = -\frac{\partial\Omega}{\partial e_1}. \qquad (B.5)$$

Since $\Omega(e_1)$ is strictly concave and $\partial\Omega/\partial e_1 \leq 0$ at $e_1 = \tilde{e} = \underline{v} \geq 0$ where $\tilde{e} = \underline{v}$ is the unique maximizer of $\Omega(e_1)$, it follows that $\partial\Omega/\partial e_1 \neq 0$ at $e_1 \neq \tilde{e} = \underline{v}$.[36] Therefore, $G'(e_1) \neq 0$ for all $e_1 \neq \tilde{e} = \underline{v}$. But since a cumulative distribution function is non-decreasing, it follows that $G'(e_1) > 0$ on $(\underline{v}, \overline{v}]$. **QED.**

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[36] If $\partial\Omega/\partial e_1 = 0$ at the optimum, this point is obvious. If $\partial\Omega/\partial e_1 < 0$ at the optimum, then strict concavity of $\Omega(e_1)$ and $e_1 \geq 0$ rules out $\partial\Omega/\partial e_1 = 0$ at any other point.

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