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Abstract

In this paper, we study a two-country dynamic setup with environmental externalities and potential model misspecification in relation to this public good. Under model uncertainty, robust policies help to correct the inefficiencies associated with free riding on public good provision, implying that there are welfare benefits from robust policies even when the fear of model misspecification proves to be unfounded. However, the incentive to free ride on the precautionary policies of the other country may discourage the implementation of robust policies, despite their welfare superiority.

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1 Introduction

Public goods are under-provided in the absence of coordination. This situation arises because of free riding incentives. In this paper, we revisit this classic problem in a context where there is also uncertainty about the correct model specification regarding the public good. Environmental quality provides a good example of such a public good, since the true model that generates environmental damage is hardly known. Our focus is on the decision about whether or not to adopt robust policy when there is such model uncertainty and on the implications of this decision for the provision of the public good.

There is an extensive literature that examines an individual's voluntary contribution to the provision of a public good in a setting of uncertainty (see e.g. Cornes and Sandler (1996, ch. 6)). The general result is that a mean-preserving increase in the variance of the quantity of the public good supplied by others can increase its provision in equilibrium, assuming a convex marginal utility function. In this literature, however, it is assumed that individuals know the properties of the model that generates the public good. Under model uncertainty, by contrast, the decision maker worries about potential misspecification of the conditional mean of the public good in the model.

The literature on robust decision-making has formalised a conservative approach to choice under model uncertainty. This is achieved by focusing on the implementation of a *max-min* method, whereby optimisation is carried out under a worst case scenario (see e.g. Hansen and Sargent (2008)). The general recommendation of robust policy-making is to adopt precautionary policies that ensure a minimum level of welfare.[1]

To analyse the interplay between public good provision and robust decision-making, we use a two-country model where world-wide environmental quality plays the role of a public good in the sense that provision of this good in one country affects the welfare of the other country. Cross-border externalities of this type lead to the standard incentive to free ride on one another's contribution to the public good by over-producing and over-polluting. To this well-recognised setup, we add model uncertainty. In particular, there is uncertainty about the evolution of environmental quality. Fear of model misspecification resulting from this extension may lead countries to adopt robust policies; this decision depends, among other things, on whether other countries adopt similar policies. In particular, by solving for a non-cooperative

[1] See e.g. the applications in Hansen and Sargent (2008 and 2010), Dennis (2010), Athanassoglou and Xepapadeas (2012) and Svec (2012) for examples of precautionary behaviour in asset pricing, as well as in monetary, fiscal and environmental policy.

(Nash) equilibrium, we show that model uncertainty generates an additional type of interaction in the sense that robust policies adopted in one country will affect, via the public good channel, welfare in the other country. This leads to an incentive to free ride on the robust policies of the other country, which is in addition to the standard free riding incentive with public goods.

In the literature to date, the selection of robust or non-robust policies under model uncertainty has generally been treated as an exogenous preference parameter, determined by Nature (or the modeller) before agents make their choices. However, economic agents are faced with a dilemma when choosing whether or not to implement robust policies under model uncertainty. If they decide to implement *optimal* policies by trusting the available model, they maximise their welfare if the model is correct, but expose themselves to "bad scenaria" if the model is misspecified. On the other hand, if they decide to implement *robust* policies by planning for bad scenaria, they may incur unnecessary costs in the case where fear of model misspecification proves to be unfounded. *Ex ante*, i.e. before agents know whether fears of model misspecification are founded, welfare comparisons under robust and non-robust policies are generally not conclusive and thus the decision about whether to implement robust policies depends on preferences and subjective evaluations. At policy level, this inconclusiveness results in debates regarding the adoption of robust policies.[2]

In this paper, we suggest that robust policy can be *ex ante* welfare-improving when there are market imperfections, such as the under-provision of public goods, yet, despite its welfare superiority, robust policy is not necessarily chosen. To illustrate this situation within our two-country model, we consider decision-making that takes place in two stages. In the first stage, each country chooses to implement either robust or non-robust policy. In the second stage, each country chooses its economic and environmental allocations. Throughout the paper, we focus on non-cooperative Nash behaviour by countries. Solving the problem by backward induction, we first solve for the second stage for any degree of robustness. This contributes to the literature by solving for robust policies under externalities in a dynamic Nash game between two agents. We then solve for the first stage. Assuming that countries can choose either optimal or robust policies given the potentially

[2]For instance, this is evident in the discussions about whether or not to adopt precautionary policies relating to environmental protection and climate change, health risk and disease spread, defence systems, financial regulation, etc. Proponents of precautionary measures highlight the potentially huge costs of model misspecification if societies are not prepared for a "bad scenario". On the other hand, opponents highlight the very large, potentially unnecessary costs of adopting such measures, if the fear of the "bad scenario" proves unfounded.

misspecified model, this stage is solved like a prisoner's dilemma problem. The solution to this stage contributes to the literature by making explicit, and endogenous to the agent, the choice between robust and non-robust behaviour.

We show that the presence of public good type externalities leads to welfare gains from robust policies, even when fear of model misspecification is unfounded. This happens because fear of model misspecification creates an incentive to decrease the depletion of the environmental good which works in the opposite direction from the standard incentive to free ride on the other's contribution to the public good. Compared to the literature on public good provision under uncertainty (see above), in our case, the precautionary behaviour is associated with robust decision making as a result of fear of misspecifying the conditional means of the model's stochastic state variables and does not rely on a convex marginal utility function (i.e. assumptions about third-order derivatives).

Therefore, robust policies work as a substitute for cooperation in public good provision. As a result, under public good externalities, robust policy is welfare superior even when its raison d'être (namely, fear of model misspecification) proves to be unfounded. This is because conservatism helps to correct for the underlying imperfection in the model. The intuition is consistent with that of Dennis (2010) who shows that the existence of a policy imperfection (lack of commitment on the part of the central bank) implies that robust policy plays an additional corrective role, improving outcomes even when fear of model misspecification is unfounded.

However, we also show that, although model uncertainty increases public good provision to the extent that robust policies are implemented, it also generates further incentives for free riding that may discourage the implementation of robust policies in the first place. In particular, although each country has an incentive to follow robust policy as a way of protecting itself against model misspecification, it also has an incentive to free ride on the robust policies of others. As a result, whether robust policies emerge in equilibrium depends on the evaluation of this trade-off by the agents, given their beliefs regarding possible outcomes under model misspecification. These new incentives may discourage the implementation of robust policies, despite the fact that the equilibrium with robust policies is *ex ante* welfare superior. In particular, robust policy is chosen only when agents attach a sufficiently high weight to bad outcomes resulting from model misspecification. If this is not the case, the incentive to free ride on the conservatism of the other agents dominates and so robust behaviour is not an equilibrium strategy. Therefore, model uncertainty by itself cannot ensure that public good provision will be increased.

The paper is organised as follows. In the next section, we describe the model, first in a narrative form and then by presenting the formal setup. This is followed in section 3 by the solution of the second stage of the game and in section 4 by the solution of the first stage. Findings are summarised in section 5. The dynamic Nash solution with robust policies is discussed in more detail in the Appendix.

2 A model with externalities and model uncertainty

2.1 Description of the model

There are two agents or countries called home (h) and foreign (f) which, for simplicity, are symmetric. Without cooperation, cross-border environmental externalities imply that each country does not internalise how its own decisions affect the other country's environmental quality and welfare. The countries are also concerned that the environmental processes in their model can be misspecified (for model uncertainty in models with environmental public goods, see e.g. Athanassoglou and Xepapadeas (2012)). In the presence of such model uncertainty, agents (countries) may also find it preferable to follow robust policies. This creates an extra type of free-riding incentives, since one country's decision to follow robust, or non-robust, policy affects environmental quality and hence the welfare of the other country. The final outcome will depend on the interplay of these two types of incentives.

There are two stages of decision making. In the first stage, the two countries decide whether to follow robust or non-robust policy. In the second stage, the two countries choose their economic and environmental allocations. We solve this problem by backward induction. In each stage, we focus on non-cooperative behavior.

We first solve for the second stage. In particular, we solve a Nash game in Markov strategies for economic and environmental allocations, for any degree of robustness. The latter is summarised, as we shall see below and as is also the case in the robustness literature, by a parameter θ^i, where $i = h, f$, which measures the extent of fear of model misspecification in each country i. The solution of this stage allows us to characterise the lifetime welfare of each country as a function of the current state and the value of θ^i in each country, given the process for model misspecification.

In turn, in the first stage, taking all this into account, countries choose the type of their policy, namely, their own θ^i, by comparing outcomes under robust and non-robust policies for different potential realisations of model uncertainty. For simplicity, working as in the prisoner's dilemma problem, we assume that this is a discrete choice. In particular, there are two polar values of θ^i, low and high, meaning respectively robust and non-robust policy.[3] We also assume that the choice of θ^i is made once-and-for-all.[4] The assumption that the choice of robustness type, as summarised by θ^i, is made once-and-for-all is similar conceptually to e.g. Persson and Tabellini (1994), where voters choose the type of the policy-maker in models of representative democracy. In Persson and Tabellini (1994), while there is no commitment in the choice of future policy (this is like our second stage), there is commitment in the choice of the type of political representatives who will in turn choose policy (this is like our first stage).

2.2 Model setup

Each country is populated by one representative agent, who consumes, produces and pollutes the environment. Pollution occurs as a by-product of production.

We assume that the only way that the two countries are linked is via environmental quality, which is an international public good. In particular, the agent in each country values economy-wide, or world, environmental quality, defined as the weighted average of environmental quality in each country. But the agent in each country is also uncertain about the true process that generates pollution, or equivalently environmental quality, and he thus fears that the model he uses for the purpose of decision making is a potentially misspecified approximation to the true process. We follow Hansen and Sargent (2008) in defining model uncertainty and using robust policies.

We first present the problems of the home and foreign country, h and f respectively, and then discuss environmental externalities and model uncertainty.

[3] In other words, in this paper we are interested in the qualitative decision of "robust or non-robust policies", rather than the quantitative decision of "how much robustness". This is similar to the discrete choices analysed in, for instance, the "cooperative" versus "non-cooperative" solution in the standard prisoner's dilemma problem.

[4] In terms of the literature on robust control, this is equivalent to assuming that the agents do not "learn" the true model and so reduce model uncertainty over time. The common justification for this assumption is that the true and the approximating model are close enough that the decision maker cannot distinguish realisations of model misspecification from genuine randomness (see e.g. Hansen and Sargent (2008), who also examine learning under model uncertainty).

2.2.1 Home country

The representative agent in the home country derives utility from consumption, c_t^h, where the superscript h denotes outcomes in the home country, and environmental quality. The latter is a weighted sum of environmental quality at home, $Q_t^{h,h}$, and abroad, $Q_t^{h,f}$ (the meaning of the double superscript will be discussed below). Formally, the utility of the agent in h is given by:

$$U_t^h = -\left(\mu(c_t^h - c^*)^2 + (1-\mu)\left(Q_t^{h,h} + \xi Q_t^{h,f} - (1+\xi)\, Q^*\right)^2\right)$$

where c^* and Q^* are utility bliss points for consumption and environmental quality respectively.[5] The parameter $0 < \xi < 1$ measures the extent of environmental externalities from one country to the other and the parameter $0 < \mu < 1$ measures the weight given to consumption, relative to environmental quality in determining utility in country h.

The agent produces output by using a linear AK-type technology[6] and decides how much of the output produced to consume and how much to invest in capital k_{t+1}^h, which will be used for production in the next time period. Substituting the resource constraint and the production function in the capital evolution equation, the "economic model" in country h is given by:

$$k_{t+1}^h - (1-\delta)k_t^h + c_t^h = Ak_t^h$$

where $0 < \delta < 1$ is a depreciation rate and $A > 0$ is technology scale factor.

The agent is, however, uncertain about the environmental model, that is, about the process of environmental quality. In defining model uncertainty, we follow Hansen and Sargent (2008). In particular, the country/agent believes that a "good" approximation of the motion for environmental quality is:

$$Q_{t+1}^{h,h} = (1-\rho^Q)Q^h + \rho^Q Q_t^{h,h} - \varphi k_t^h + \sigma^Q \widehat{\varepsilon}_{t+1}^{h,Q}$$

$$Q_{t+1}^{h,f} = (1-\rho^Q)Q^f + \rho^Q Q_t^{h,f} - \varphi k_t^f + \sigma^Q \widehat{\varepsilon}_{t+1}^{f,Q},$$

where $0 < \rho^Q < 1$ measures the persistence of environmental quality, $\varphi > 0$ measures the extent to which economic activities damage environmental quality and $\widehat{\varepsilon}_{t+1}^{h,Q}$ and $\widehat{\varepsilon}_{t+1}^{f,Q}$ are Gaussian variables distributed identically and independently through time with zero mean and unit variance. In this formulation, σ^Q scales the variance of $\widehat{\varepsilon}_{t+1}^{h,Q}$ and $\widehat{\varepsilon}_{t+1}^{f,Q}$, and hence measures the size of the shocks.

[5] The specific functional form is chosen so that the problem can be written in Linear-Quadratic (LQ) form (see also Hansen and Sargent (2008, ch. 10), for a similar example).

[6] Again, the AK production technology is chosen so that the the problem can be written in Linear-Quadratic (LQ) form. Using an AK specification is common in the literature on growth and the environment (see e.g. Economides and Philippopoulos (2008)).

Note that the environmental quality in the home country depends on economic activity, as summarised by k_t^h, in this country and similarly environmental quality in the foreign country depends on economic activity k_t^f, in that country, where the capital evolution in the foreign country follows (the superscript f denotes quantities in the foreign country):

$$k_{t+1}^f - (1-\delta)k_t^f + c_t^f = Ak_t^f.$$

However, the agent believes that the above model is misspecified and that the "true" model for the motion of environmental quality is:

$$Q_{t+1}^{h,h} = (1-\rho^Q)Q^h + \rho^Q Q_t^{h,h} - \varphi k_t^h + \sigma^Q(\varepsilon_{t+1}^{h,Q} + w_{t+1}^{h,h})$$

$$Q_{t+1}^{h,f} = (1-\rho^Q)Q^f + \rho^Q Q_t^{h,f} - \varphi k_t^f + \sigma^Q(\varepsilon_{t+1}^{f,Q} + w_{t+1}^{h,f}),$$

where $\varepsilon_{t+1}^{h,Q}$ and $\varepsilon_{t+1}^{f,Q}$ are *i.i.d.* Gaussian variables, distributed with zero mean and unit variance, but $w_{t+1}^{h,h}$ and $w_{t+1}^{h,f}$ are unknown statistical perturbations to the "environmental model".[7] The true model is a "distorted" or "perturbed" version of the approximating model. Note that the agent in the home country is uncertain about both environmental processes, in both countries. Thus, $w_{t+1}^{h,h}$ captures model misspecification, as feared by country h, for the environmental process of country h. Similarly, $w_{t+1}^{h,f}$, captures model misspecification, as feared by country h, for the environmental process of country f. This justifies the presence of both $Q_t^{h,h}$ and $Q_t^{h,f}$ in the objective function above.

To measure the difference between the approximating and the distorted model for the home country, let x_t^h denote the vector of state variables $\left(1, k_t^h, k_t^f, Q_t^{h,h}, Q_t^{h,f}\right)$, f_0^h the one-step transition density associated with the approximating model and f^h the one-step transition density associated with the true, or distorted, model. We use *conditional relative entropy*, defined as the expected log-likelihood ratio of the two models, evaluated with respect to the true model, to measure the statistical discrepancy between the two models of the transition from x_t^h to x_{t+1}^h:

[7] This specification for the true model allows for misspecifications to the approximating model that occur only as a distortion to the conditional mean of the innovation to the state and leaves the conditional volatility of the shock, as parametrised by σ^Q, unchanged. This is for computational convenience. As shown in Hansen and Sargent (2008), as long as we stay within the linear quadratic framework with Gaussian distributions for the approximating model, allowing for a more general class of misspecifications to the approximating model does not change the policy function and the worst case shock under robust decision making.

$$I(f_0, f)(x^h) = \int \log\left(\frac{f(x_{t+1}^h \mid x_t^h)}{f_0(x_{t+1}^h \mid x_t^h)}\right) f(x_{t+1}^h \mid x_t^h) dx_{t+1}^h.$$

Following Hansen and Sargent (2008), it can be shown that within each period:

$$I(f_0, f)(x^h) = 0.5\left[\left(w_{t+1}^{h,h}\right)^2 + \left(w_{t+1}^{h,f}\right)^2\right],$$

and that, in turn, the quantity $2E_0 \sum_{t=0}^{\infty} \beta^{t+1} I(f_0, f)(x^h)$ can be considered as an intertemporal measure of model misspecification for the home country, where $0 < \beta < 1$ is a subjective rate of time preference and the mathematical expectation is evaluated with respect to the distorted model.

The two models (the approximating and the distorted) are close in the statistical sense that:

$$E_0 \sum_{t=0}^{\infty} \beta^{t+1}\left[\left(w_{t+1}^{h,h}\right)^2 + \left(w_{t+1}^{h,f}\right)^2\right] \leq \eta^h,$$

where η^h measures the extent of model misspecification, or "fear of model misspecification". When $\eta^h = 0$, the problem collapses to the case where there is no model uncertainty or else the approximating model is also the true model.

To summarise, the home country chooses the paths of its own consumption, capital and environmental quality, by taking the actions of the foreign country as given, according to the following optimisation problem:

$$\max_{\{c_t^h\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} -\beta^t \left(\mu(c_t^h - c^*)^2 + (1-\mu)\left(Q_t^{h,h} + \xi Q_t^{h,f} - (1+\xi)\, Q^*\right)^2\right) \quad (1)$$

where the end-of-period values of the state variables are given by:

$$\begin{aligned} k_{t+1}^h - (1-\delta)k_t^h + c_t^h &= Ak_t^h \\ k_{t+1}^f - (1-\delta)k_t^f + c_t^f &= Ak_t^f \\ Q_{t+1}^{h,h} &= (1-\rho^Q)Q^h + \rho^Q Q_t^{h,h} - \varphi k_t^h + \sigma^Q(\varepsilon_{t+1}^{h,Q} + w_{t+1}^{h,h}) \\ Q_{t+1}^{h,f} &= (1-\rho^Q)Q^f + \rho^Q Q_t^{h,f} - \varphi k_t^f + \sigma^Q(\varepsilon_{t+1}^{f,Q} + w_{t+1}^{h,f}), \end{aligned}$$

and $w_{t+1}^{h,h}$ and $w_{t+1}^{h,f}$ are unknown processes satisfying the constraint:

$$E_0 \sum_{t=0}^{\infty} \beta^{t+1}\left[\left(w_{t+1}^{h,h}\right)^2 + \left(w_{t+1}^{h,f}\right)^2\right] \leq \eta_0^h$$

2.2.2 Foreign country

The problem for the foreign country is analogous to the problem of the home country. Hence, we can summarise it as:

$$\max_{\{c_t^f,,k_{t+1}^f,Q_{t+1}^{f,f}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} -\beta^t \left(\mu(c_t^f - c^*)^2 + (1-\mu)\left(Q_t^{f,f} + \xi Q_t^{f,h} - (1+\xi)\,Q^*\right)^2\right) \tag{2}$$

where the end-of-period values of the state variables are given by:

$$\begin{aligned} k_{t+1}^h - (1-\delta)k_t^h + c_t^h &= Ak_t^h \\ k_{t+1}^f - (1-\delta)k_t^f + c_t^f &= Ak_t^f \\ Q_{t+1}^{f,f} &= (1-\rho^Q)Q^{ff} + \rho^Q Q_t^{f,f} - \varphi k_t^f + \sigma^Q(\varepsilon_{t+1}^{f,Q} + w_{t+1}^{f,f}) \\ Q_{t+1}^{f,h} &= (1-\rho^Q)Q^h + \rho^Q Q_t^{f,h} - \varphi k_t^h + \sigma^Q(\varepsilon_{t+1}^{h,Q} + w_{t+1}^{f,h}), \end{aligned}$$

and $w_{t+1}^{f,f}$ and $w_{t+1}^{f,h}$ satisfy the constraint:

$$E_0 \sum_{t=0}^{\infty} \beta^{t+1} \left[\left(w_{t+1}^{f,f}\right)^2 + \left(w_{t+1}^{f,h}\right)^2\right] \leq \eta^f.$$

Note that the agent in the foreign country is also uncertain about both environmental processes, in both countries. Thus, $w_{t+1}^{f,f}$ captures model misspecification, as feared by country f, for the environmental process of country f. Similarly, $w_{t+1}^{f,h}$, captures model misspecification, as feared by country f, for the environmental process of country h. This representation allows us to consider the effects on policy-making of different fears of model misspecification across the two countries. It does not imply that there are actually two different processes for environmental quality for each country, but that each country designs its policies assuming different environmental quality processes, if there are different fears of model misspecification. In the special case where $\eta^h = \eta^f = 0$, then $w_{t+1}^{h,h} = w_{t+1}^{f,h} \equiv w_{h,t+1}^h = 0$ and $w_{t+1}^{f,f} = w_{t+1}^{h,f} \equiv w_{f,t+1}^f = 0$ and the problem collapses to a standard, no model uncertainty case, where $Q_{t+1}^{h,h} = Q_{t+1}^{f,h} \equiv Q_{t+1}^h$ and $Q_{t+1}^{f,f} = Q_{t+1}^{h,f} \equiv Q_{t+1}^f$. For the more general case where robust policy making is considered, when we simulate the solution to the model, there is always one process for $Q_{t+1}^{h,h} = Q_{t+1}^{f,h} \equiv Q_{t+1}^h$ and one process for $Q_{t+1}^{f,f} = Q_{t+1}^{h,f} \equiv Q_{t+1}^f$ *ex post*, as there is a single exogenous realisation of the $w_{h,t+1}^h$ and $w_{f,t+1}^f$ processes respectively.

3 Decision-making for any degree of robustness

In this section, we solve for economic and environmental allocations. This will be for any degree of robustness in each country as captured by η^h and η^f (where non-robust policies can be obtained as a special case by setting η^h and η^f to be zero). In particular, we solve a Nash game between the two countries with each country choosing its own economic and environmental variables, while taking the associated decisions of the other country, as well as the degree of robustness in each country, η^h and η^f, as given (they will be chosen in the next stage below). Following the literature, we solve for Markov strategies given η^h and η^f.

Before we solve the game between the two countries in subsection 3.2, we formalise the concept of robustness in the two-country case. The representative agent in each country obtains decision rules that are robust to model misspecification, in the sense that they give him good results even in unfavourable w shocks, where w can be defined to include the unknown statistical perturbations that are relevant for each agent. Hence, for the home country, w^h would include $w^{h,h}$ and $w^{h,f}$, whereas, for the foreign country, w^f would include $w^{f,f}$ and $w^{f,h}$. In doing so, each country takes the quantities determined by the other country as given.

In order for his decision rule to assure him a lower bound on utility in an unfavourable environment, the agent makes his choices as if the w process that is relevant for his problem follows a worst-case scenario. In particular, he pretends that w is chosen by a fictional malevolent agent, whose objective is to minimise his (the agent's) objective. By planning against such a worst-case process, he designs a decision rule that performs well under a set of perturbed models.[8] In other words, the representative agent uses the malevolent agent as a device to achieve robustness. This implies that effectively each agent/country solves a *maxmin* problem.

Note that in the Nash game we consider, both countries are robust decision makers, possibly with different degrees of fear of model misspecification, but each country's fear of misspecification and the resulting robust policies are known to the other country. Thus, each country designs its robust policies by taking the robust policies of the other country and, indeed, the whole problem, i.e. the *maxmin* game of the other country, as given. This implies that, for instance, when the home country solves its *maxmin* problem, it

[8] In particular, robust choices will be optimal under the worst-case scenario, but, more generally, such precautious policies are also expected to out-perform non-robust policies in other bad outcomes resulting from model uncertainty as well.

takes both the maximising and minimising choices resulting from the foreign country's *maxmin* problem as given. Therefore, for country h, w^f is given from the *maxmin* problem of country f, while w^h is assumed to be chosen by a malevolent agent who wishes to minimise its objective.

In this situation, different fears of model misspecification will imply that robust policies in each country will de designed under different perceived w^h and w^f processes, or, in other words, under different perceived, or "feared", environmental processes. Hence, for the purposes of robust decision making, we need to allow for $w_{t+1}^{h,h} \neq w_{t+1}^{f,h}$ and $w_{t+1}^{f,f} \neq w_{t+1}^{h,f}$, when there are different fears of model misspecification. This implies that $Q_{t+1}^{h,h} \neq Q_{t+1}^{f,h}$ and $Q_{t+1}^{f,f} \neq Q_{t+1}^{h,f}$ under differences in the perceived worst-case scenaria. Of course, given that all w represent a fictional device in the robust problem, used only to determine the choices for the variables under the control of the maximising agents, the actual process for environmental quality satisfies $Q_{t+1}^{h,h} = Q_{t+1}^{f,h} \equiv Q_{t+1}^{h}$ and $Q_{t+1}^{f,f} = Q_{t+1}^{h,f} \equiv Q_{t+1}^{f}$ *ex post*.

3.1 Linear-quadratic representation of the problem

3.1.1 Home country

We first present the problem for the home country. By defining $\widetilde{c}_t^h \equiv c_t^h - c^*$, $\widetilde{c}_t^f \equiv c_t^f - c^*$, $\widetilde{Q}_t^{h,h} \equiv Q_t^{h,h} - Q^*$, $\widetilde{Q}_t^{h,f} \equiv Q_t^{h,f} - Q^*$, $\widetilde{Q}_t^{f,f} \equiv Q_t^{f,f} - Q^*$ and $\widetilde{Q}_t^{f,h} \equiv Q_t^{f,h} - Q^*$, we can rewrite the problem as:

$$\max_{\{\widetilde{c}_t^h\}_{t=0}^{\infty}} \min_{\{w_{t+1}^{h,h}, w_{t+1}^{h,f}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} -\beta^t \left(\mu (\widetilde{c}_t^h)^2 + (1-\mu) (\widetilde{Q}_t^{h,h} + \xi \widetilde{Q}_t^{h,f})^2 \right) \quad (3)$$

where the end-of-period values of the state variables are given by:

$$k_{t+1}^h = -c^* + (A + 1 - \delta) k_t^h - \widetilde{c}_t^h \quad (4)$$

$$k_{t+1}^f = -c^* + (A + 1 - \delta) k_t^f - \widetilde{c}_t^f \quad (5)$$

$$\widetilde{Q}_{t+1}^{h,h} = (1-\rho^Q) Q^h + \left(\rho^Q - 1\right) Q^* + \rho^Q \widetilde{Q}_t^{h,h} - \varphi k_t^h + \sigma^Q (\varepsilon_{t+1}^{h,Q} + w_{t+1}^{h,h}) \quad (6)$$

$$\widetilde{Q}_{t+1}^{h,f} = (1-\rho^Q) Q^f + \left(\rho^Q - 1\right) Q^* + \rho^Q \widetilde{Q}_t^{h,f} - \varphi k_t^f + \sigma^Q (\varepsilon_{t+1}^{f,Q} + w_{t+1}^{h,f}) \quad (7)$$

$$\widetilde{Q}_{t+1}^{f,f} = (1-\rho^Q) Q^f + \left(\rho^Q - 1\right) Q^* + \rho^Q \widetilde{Q}_t^{f,f} - \varphi k_t^f + \sigma^Q (\varepsilon_{t+1}^{f,Q} + w_{t+1}^{f,f}) \quad (8)$$

$$\widetilde{Q}_{t+1}^{f,h} = (1-\rho^Q) Q^h + \left(\rho^Q - 1\right) Q^* + \rho^Q \widetilde{Q}_t^{f,h} - \varphi k_t^h + \sigma^Q (\varepsilon_{t+1}^{h,Q} + w_{t+1}^{f,h}), \quad (9)$$

and

$$E_0 \sum_{t=0}^{\infty} \beta^{t+1} \left[\left(w_{t+1}^{h,h} \right)^2 + \left(w_{t+1}^{h,f} \right)^2 \right] \leq \eta^h \quad (10)$$

Letting $\widetilde{h}$, $n^{h,\max}$, $n^{h,\min}$, $n^{f,\max}$, $n^{f,\min}$ and n^{ex} equal the number of states, $\left(1, k_t^h, k_t^f, \widetilde{Q}_t^{h,h}, \widetilde{Q}_t^{h,f}, \widetilde{Q}_t^{f,f}, \widetilde{Q}_t^{f,h}\right)$, the number of controls, $(\widetilde{c}_t^h)$, for the maximising agent for the home country, the number of controls, $\left(w_{t+1}^{h,h}, w_{t+1}^{h,f}\right)$, for the minimising agent for the home country, the number of controls, $(\widetilde{c}_t^f)$, for the maximising agent for the foreign country, the number of controls, $\left(w_{t+1}^{f,h}, w_{t+1}^{f,f}\right)$, for the minimising agent for the foreign country and the number of exogenous shocks $\left(\varepsilon_{t+1}^{h,Q}, \varepsilon_{t+1}^{f,Q}\right)$ we can now write the linear constraints in (4)-(9) above in matrix form as:

$$x_{t+1} = Ax_t + B^h u_t^h + B^f u_t^f + C\varepsilon_{t+1} + D^h w_{t+1}^h + D^f w_{t+1}^f$$

where

$$x_t = \begin{bmatrix} 1 & k_t^h & k_t^f & \widetilde{Q}_t^{h,h} & \widetilde{Q}_t^{h,f} & \widetilde{Q}_t^{f,f} & \widetilde{Q}_t^{f,h} \end{bmatrix}';$$

$$u_t^h = \left[\widetilde{c}_t^h\right]'; \quad u_t^f = \left[\widetilde{c}_t^f\right]';$$

$$w_{t+1}^h = \begin{bmatrix} w_{t+1}^{h,h} & w_{t+1}^{h,f} \end{bmatrix}'; \quad w_{t+1}^f = \begin{bmatrix} w_{t+1}^{f,f} & w_{t+1}^{f,h} \end{bmatrix}';$$

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{t+1}^{h,Q} & \varepsilon_{t+1}^{f,Q} \end{bmatrix}';$$

$$A_{(\widetilde{h}x\widetilde{h})} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c^* & A+1-\delta & 0 & 0 & 0 & 0 & 0 \\ -c^* & 0 & A+1-\delta & 0 & 0 & 0 & 0 \\ (1-\rho^Q)Q^h + \left(\rho^Q - 1\right)Q^* & -\varphi & 0 & \rho^Q & 0 & 0 & 0 \\ (1-\rho^Q)Q^f + \left(\rho^Q - 1\right)Q^* & 0 & -\varphi & 0 & \rho^Q & 0 & 0 \\ (1-\rho^Q)Q^f + \left(\rho^Q - 1\right)Q^* & 0 & -\varphi & 0 & 0 & \rho^Q & 0 \\ (1-\rho^Q)Q^h + \left(\rho^Q - 1\right)Q^* & -\varphi & 0 & 0 & 0 & 0 & \rho^Q \end{bmatrix};$$

$$B^h_{(\widetilde{h}xn^{h,\max})} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad B^f_{(\widetilde{h}xn^{f,\max})} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$C_{(\tilde{h}xn^{ex})} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma^Q & 0 \\ 0 & \sigma^Q \\ 0 & \sigma^Q \\ \sigma^Q & 0 \end{bmatrix} ; \quad D^h_{(\tilde{h}xn^{h,\min})} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma^Q & 0 \\ 0 & \sigma^Q \\ 0 & 0 \\ 0 & 0 \end{bmatrix} ; \quad D^f_{(\tilde{h}xn^{f,\min})} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma^Q & 0 \\ 0 & \sigma^Q \end{bmatrix}$$

Following Hansen and Sargent (2008), we can add the non-linear entropy constraint, (10), to the objective function with a time-invariant multiplier, denoted as θ^h, and rewrite the linear-quadratic problem of the home country as:

$$\max_{\{u_t^h\}_{t=0}^\infty} \min_{\{w_{t+1}^h\}_{t=0}^\infty} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left(u_t^h\right)' R_u^h u_t^h + \left(w_{t+1}^h\right)' R_w^h w_{t+1}^h + x_t' Q^h x_t \right\} \tag{11}$$

subject to:

$$x_{t+1} = Ax_t + B^h u_t^h + B^f u_t^f + C\varepsilon_{t+1} + D^h w_{t+1}^h + D^f w_{t+1}^f \tag{12}$$

where

$$R^h_{u(n^{h,\max}xn^{h,\max})} = [-\mu] ; \quad R^h_{w(n^{f,\min}xn^{f,\min})} = \begin{bmatrix} \beta\theta^h & 0 \\ 0 & \beta\theta^h \end{bmatrix} ;$$

$$Q^h_{(\tilde{h}x\tilde{h})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\mu) & -((1-\mu)\xi) & 0 & 0 \\ 0 & 0 & 0 & -((1-\mu)\xi) & -(1-\mu)\xi^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonnegative multiplier, θ^h, is a penalty on the minimising agent for choosing policies that reduce welfare for the economic (maximising) agent and can be used as a measure of the degree of robustness. The value of θ^h is inversely related to the value of η^h, so the lower is θ^h, the higher the degree of robustness. Notice that, since this is a multiplier associated with an intertemporal (or present-value) constraint, it is time invariant or flat when the problem is solved at $t = 0$.

For the problem to be well defined, the objective function (11) needs to be concave with respect to u_t^h and convex with respect to w_{t+1}^h. We require R_u^h to be negative definite, Q^h to be negative semi-definite and R_w^h to be positive definite (see also Anderson *et al.* (1996), and Hansen and Sargent (2008), for assumptions regarding the coefficient matrices for linear quadratic problems).

To solve each country's maxmin problem, we solve for Markov strategies in a Nash game between the maximising and the minimising agent (see Hansen and Sargent, (2008, Definition 7.4.1)).[9] Since the game between the maximising and the minimising agent in each country is a dynamic zero-sum game, the order of optimisation does not affect the solution. In other words, the *Bellman-Isaacs* condition holds (see e.g. Hansen and Sargent (2008, chapter 7); see also Basar and Olsder (1999, ch. 5 and 6) for dynamic, zero-sum open-loop and feedback Nash equilibria). Therefore, to solve the problem in (11)-(12), we can stack the first order conditions of the maximising and the minimising agents.

To implement this solution, we follow Hansen and Sargent (2008, ch. 2) and note that, in (11)-(12), the first order conditions of the maximising agent with respect to u_t^h and of the minimising agent with respect to w_{t+1}^h, are the same as the first order conditions of an ordinary (i.e. non-robust) optimal linear regulator (OLR) who chooses $\widetilde{u}_t^h$, where $\widetilde{u}_{t(\widetilde{n}x1)}^h = \begin{pmatrix} u_t^{h\prime} & w_{t+1}^{h\prime} \end{pmatrix}'$. Hence, we write the *extremisation*[10] problem in (11)-(12) as:

$$\underset{\{\widetilde{u}_t^h\}_{t=0}^{\infty}}{ext} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left(\widetilde{u}_t^h\right)' R^h \widetilde{u}_t^h + x_t' Q^h x_t \right\} \tag{13}$$

$$x_{t+1} = A x_t + \widetilde{B}^h \widetilde{u}_t^h + \widetilde{B}^f \widetilde{u}_t^f + C\epsilon_{t+1}, \tag{14}$$

where

$$R_{(\widetilde{n}x\widetilde{n})}^h = \begin{bmatrix} \left[R_u^h\right] & 0_{(n^{h,\max} x n^{h,\min})} \\ 0_{(n^{h,\min} x n^{h,\max})} & \left[R_w^h\right] \end{bmatrix}$$

and $\widetilde{B}_{(\widetilde{h}x\widetilde{n})}^h = \begin{bmatrix} B^h & D^h \end{bmatrix}$, $\widetilde{u}_{t(\widetilde{n}x1)}^f = \begin{pmatrix} u_t^{f\prime} & w_{t+1}^{f\prime} \end{pmatrix}'$, $\widetilde{B}_{(\widetilde{h}x\widetilde{n})}^f = \begin{bmatrix} B^f & D^f \end{bmatrix}$ and $\widetilde{n} = n^{h,\max} + n^{h,\min} = n^{f,\max} + n^{f,\min}$.

3.1.2 Foreign country

Working as above, the linear-quadratic representation of the extremisation problem for the foreign country is given by:

[9] The definition of the *Markov Perfect Equilibrium* discussed here describes equilibria that are, under certainty, consistent with memoryless feedback equilibria (see e.g. Basar and Olsder (1999, ch. 6)).

[10] Following Whittle (1990), extremisation denotes joint maximisation and minimisation.

$$\underset{\{\widetilde{u}_t^f\}_{t=0}^{\infty}}{ext}\ E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left(\widetilde{u}_t^f\right)' R^f \widetilde{u}_t^f + x_t' Q^f x_t \right\}$$

$$x_{t+1} = Ax_t + \widetilde{B}^h \widetilde{u}_t^h + \widetilde{B}^f \widetilde{u}_t^f + C\epsilon_{t+1},$$

where the matrices A, $\widetilde{B}^h$, $\widetilde{B}^f$and C and the vectors x_t, $\widetilde{u}_t^h$, $\widetilde{u}_t^f$ and ϵ_{t+1} are as above and:

$$R^f_{(\widetilde{n}x\widetilde{n})} = \begin{bmatrix} \left[R_u^f\right] & 0_{(n^{f,\max}xn^{f,\min})} \\ 0_{(n^{f,\min}xn^{f,\max})} & \left[R_w^f\right] \end{bmatrix},$$

$$R^f_{u(n^{f,\max}xn^{f,\max})} = [-\mu]\,; \quad R^f_{w(n^{f,\min}xn^{f,,\min})} = \begin{bmatrix} \beta\theta^f & 0 \\ 0 & \beta\theta^f \end{bmatrix}\;;$$

$$Q^f_{(\widetilde{h}x\widetilde{h})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-\mu) & -((1-\mu)\,\xi) \\ 0 & 0 & 0 & 0 & 0 & -((1-\mu)\,\xi) & -(1-\mu)\,\xi^2 \end{bmatrix}$$

3.2 Nash game in Markov strategies (for any degree of robustness)

We are now ready to solve for the Nash game between countries in the second stage. As said, each country takes the other country's actions as given when choosing its consumption, capital and environmental quality. Also as said, this is given the degree of robustness in each country, which will be chosen in the first stage. We solve for Markov strategies in this Nash game between the countries.

We first solve the problem for the home country. The solution for the foreign county is symmetric to that of the home country. Then, the Nash

equilibrium of this stage of the game is obtained by combining the first-order conditions for both countries.

3.2.1 Bellman equations

We want to find a time-invariant *policy function* h^h mapping the state x_t into the control $\widetilde{u}_t^h$, such that the sequence of controls $\{\widetilde{u}_t^h\}_{t=0}^{\infty}$ generated by iterating the two functions

$$
\begin{aligned}
\widetilde{u}_t^h &= h^h(x_t), \\
x_{t+1} &= Ax_t + \widetilde{B}^h \widetilde{u}_t^h + \widetilde{B}^f \widetilde{u}_t^f + C\epsilon_{t+1},
\end{aligned} \tag{15}
$$

starting from an initial condition x_0 at $t = 0$, for given $\widetilde{u}_t^f$, solves the original problem. We make use of a type of *certainty equivalence*, which applies to the class of linear quadratic games relevant here (see e.g. Hansen and Sargent, 2008, ch. 2). In particular, the decision rules for both the maximising and the minimising agents in the *maxmin* game are the same in a particular non-stochastic version of the problem, i.e. where $\varepsilon_{t+1} = 0$. Therefore, we focus on the problem:

$$
\underset{\{\widetilde{u}_t^h\}_{t=0}^{\infty}}{ext} \sum_{t=0}^{\infty} \beta^t \left\{ \left(\widetilde{u}_t^h\right)' R^h \widetilde{u}_t^h + x_t' Q^h x_t \right\} \tag{16}
$$

$$
x_{t+1} = Ax_t + \widetilde{B}^h \widetilde{u}_t^h + \widetilde{B}^f \widetilde{u}_t^f. \tag{17}
$$

The Bellman equation for the extremisation problem is:

$$
v(x \mid \theta^h, \theta^f) = \underset{\widetilde{u}^h}{ext} \left\{ \left(\widetilde{u}^h\right)' R^h \widetilde{u}^h + x' Q^h x + \beta v(\widetilde{x} \mid \theta^h, \theta^f) \right\}, \tag{18}
$$

where $v(x \mid \theta^h, \theta^f)$ is the value function, x is current period's state, $\widetilde{x}$ is next period's state and the notation suggests that this is conditional on the choices of the atemporal variables θ^h and θ^f which will be chosen in the first stage of the game. Equation (18) is a *functional* equation, to be solved for the pair of unknown functions $v(x \mid \theta^h, \theta^f)$, h^h.

We guess that the value function for the extremisation problem is quadratic:

$$
v(x \mid \theta^h, \theta^f) = x' P^h x, \tag{19}
$$

where P^h is a negative semidefinite symmetric $(\widetilde{h} \times \widetilde{h})$ matrix. Note that, in the solution, P^h will be a function of θ^h and θ^f, so that P^h could be written as $P^h\left(\theta^h, \theta^f\right)$. To simplify notation, we do not denote this explicit dependence of P^h on θ^h and θ^f.

Substituting the guess function in (19) the Bellman equation gives:

$$
v(x \mid \theta^h, \theta^f) = \underset{\widetilde{u}^h}{ext} \left\{ \left(\widetilde{u}^h\right)' R^h \widetilde{u}^h + x' Q^h x + \beta \widetilde{x}' P^h \widetilde{x} \right\}. \tag{20}
$$

The Bellman equation for the foreign country is symmetric to the above, where it suffices to substitute the superscript h with f and f with h.

3.2.2 Recursive solution

We describe in detail in the Appendix how to obtain the solution for this problem, first, by combining the first-order conditions for the two countries and solving the system to obtain the Nash equilibrium, given the guesses for P^h and P^f, and, then, by obtaining P^h and P^f and verifying the guessed form for the solution and iterating on the resulting system of Riccati equations in (41)-(42) the Appendix. The solution is summarised by the following recursive, state-space form:

$$\begin{aligned} \widetilde{u}^h &= Fx \\ \widetilde{u}^f &= Kx \\ x_{t+1} &= Ax_t + \widetilde{B}^h \widetilde{u}_t^h + \widetilde{B}^f \widetilde{u}_t^f + C\epsilon_{t+1}, \text{ or} \end{aligned} \tag{21}$$

$$\begin{aligned} \widetilde{u}^h &= Fx \\ \widetilde{u}^f &= Kx \\ x_{t+1} &= \left(A + \widetilde{B}^h F + \widetilde{B}^f K\right) x_t + C\epsilon_{t+1}, \end{aligned} \tag{22}$$

The stability of the solution is guaranteed if the eigenvalues of $\left(A + \widetilde{B}^h F + \widetilde{B}^f K\right)$ are all less than one in absolute value.[11] Working backwards, an appropriate partitioning of the matrix F will give us the policy function for u_t^h and the worst case scenario for w_{t+1}^h:

$$\begin{aligned} u_t^h &= F^u x_t \\ w_{t+1}^h &= F^w x_t, \end{aligned} \tag{23}$$

and, similarly, an appropriate partitioning of the matrix K will give us the policy function for u_t^f and the worst case scenario for w_{t+1}^f:

$$\begin{aligned} u_t^f &= K^u x_t \\ w_{t+1}^f &= K^w x_t. \end{aligned} \tag{24}$$

Note that this is a recursive solution, in the sense that the current economic and environmental choices of the two countries-agents depend only on the current value of the state variables and the time-invariant or flat values of θ^h and θ^f. Notice that θ^h and θ^f appear in the definition of the matrices that are part of the problem set up and thus, through the solution of the system

[11] Of course, except for the one corresponding to the constant term, which will be exactly unity.

of Riccati equations in (41)-(42), determine F and K. To make clearer the dependence of the solution upon the choice of the robustness type of the two countries, we summarise the solution as:

$$u_t^h = F^u\left(\theta^h, \theta^f\right) x_t \quad (25)$$

$$w_{t+1}^h = F^w\left(\theta^h, \theta^f\right) x_t \quad (26)$$

$$u_t^f = K^u\left(\theta^h, \theta^f\right) x_t \quad (27)$$

$$w_{t+1}^f = K^w\left(\theta^h, \theta^f\right) x_t \quad (28)$$

$$x_{t+1} = \left(A + B^h F^u\left(\theta^h, \theta^f\right) + B^f K^u\left(\theta^h, \theta^f\right)\right) x_t + +D^h w_{t+1}^h + D^f w_{t+1}^f + C\epsilon_{t+1} \quad (29)$$

4 Choosing the degree of robustness

The above was for any degree of robustness, as captured by θ^h and θ^f. We now move to the first stage of the game, regarding the competition of the two countries over the choice of θ^h and θ^f.

When choosing its own θ, each country knows that the actual process for w_{t+1}^h and w_{t+1}^f will not necessarily follow the worst-case scenario in (26) and (28). As discussed earlier, the malevolent agent, or else the worst-case scenario, is only used as a fictional device by the conservative decision-maker in order to achieve robustness. Thus, the true model will be given in general by:

$$u_t^h = F^u\left(\theta^h, \theta^f\right) x_t \quad (30)$$

$$u_t^f = K^u\left(\theta^h, \theta^f\right) x_t \quad (31)$$

$$x_{t+1} = \left(A + B^h F^u\left(\theta^h, \theta^f\right) + B^f K^u\left(\theta^h, \theta^f\right)\right) x_t + +D^h w_{t+1}^h + D^f w_{t+1}^f \quad (32)$$

where w_{t+1}^h and w_{t+1}^f follow unknown processes.[12] Therefore, a conservative decision-maker who fears that model misspecification can only result in negative outcomes, needs to evaluate his choice of robust or non-robust policies under different possible negative outcomes for the unknown w_{t+1}^h and w_{t+1}^f, by taking the case where the worst-case scenario is the true model as the lower bound, and the case where the approximating model is the true model, as the upper bound (see subsection 4.1 below for scenaria studied).

[12] The representation in (30)-(32) encompasses (25)-(29) as a special case, in particular when the unknown processes in (30)-(32) follow the feared worst-case scenario in (25)-(29). For simplicity, we turn off all random shocks associated with the known distribution ε_{t+1}.

The countries choose whether to follow robust policies by comparing lifetime utility under robust and non-robust policies for different potential realisations of model uncertainty (see below). In each country, the lifetime utility in each period is a function of the current value of the state variables and the values of θ^h and θ^f, given the process for model misspecification. Thus, since the objective is lifetime utility, the optimal values of θ^h and θ^f will be given by atemporal conditions, which means that θ^h and θ^f will be functions of the current value of the state variables, given the process for model misspecification.

Without cooperation, each country chooses its optimal degree of robustness taking as given the degree of robustness of the other country. Here, working as in the prisoner's dilemma problem, we focus on two polar choices, which correspond to robust and non-robust or optimal policy in the literature.[13] These two cases are achieved respectively by setting a low value of θ, $\theta = 0.1$, and a high value of θ, $\theta = 1,000,000$. To evaluate the implications of robust versus non-robust policies, we assume that in time period 0 the equilibrium is given by the steady state of the Nash equilibrium under model certainty and then simulate the solution following (30)-(32) under robust (i.e. $\theta = 0.1$) and non-robust (i.e. $\theta = 1,000,000$) policies. The time horizon for the simulations is 300 years.

The parameter values used for the numerical solution of the model below are given in Table 1. The economic and environmental parameters are similar to those commonly used in the literature (see e.g. Angelopoulos *et al.* (2012) and the references therein). The target values for consumption and environmental quality and the productivity parameters are chosen to ensure interior, well-defined solutions for the economic and environmental variables in the model. We report that our findings are robust to changes in these parameters.

Table 1: Parameter values

β	μ	c^*	Q^*	ξ	δ	A	φ	Q^h	Q^f	ρ^Q	σ^Q
0.97	0.7	1.5	2	0.8	0.1	2	0.05	1	1	0.95	0.01

[13] Note that the value function is an implicit function of θ. Therefore, in a more general setup, each country could evaluate its welfare by varying its own θ, given the other country's choice for θ, under different possible negative outcomes for the unknown w_{t+1}^h and w_{t+1}^f. This would then determine the outcome for θ in a Nash equilibrium. By focusing on a prisoner's dilemma-type discrete choice of θ, we can demonstrate the effects of competition regarding the choice of robustness in a simpler setup.

4.1 Scenaria studied

We examine equilibrium outcomes under different scenaria. In particular, we examine outcomes under two examples of "bad scenaria", i.e. under negative realisations of model uncertainty. We also examine outcomes in the limiting case where the fear of model misspecification is unfounded.[14] These examples of outcomes of model misspecification are sufficient to demonstrate the two key results in this model, namely the welfare superiority of robust policies and the incentive to free ride on the robust policies of others.

In the case of bad scenaria, we present examples when the unknown distributions associated with model uncertainty always take negative values. In particular, for the results presented in the Tables below, we assume that the values for the model misspecification variables do not follow the worst-case scenario, as given by (26) and (28), but are equal to the absolute of values drawn from a standard normal, multiplied by $-2 * \sigma^Q$ to obtain a "very bad scenario" and by $-\sigma^Q$ to obtain a "bad scenario". We thus simulate the model as given in (30)-(32), where w_{t+1}^h and w_{t+1}^f take negative values as explained above. We examine cases of such "bad scenaria" and not the "worst-case" scenario, as, under the worst-case scenario, robust policy is optimal by definition. The examples considered here serve to capture the potential benefits of robust, relative to non-robust, policy-making in adverse outcomes of model misspecification.

In the case of unfounded fear of model misspecification, we present results by simulating robust and non-robust policies under the approximating model. This is obtained by setting $w_{t+1}^h = w_{t+1}^f \equiv 0$ in equations (30)-(32). The latter implies that the approximating model is the correct one but the agents solve for robust decision rules (see e.g. Hansen and Sargent (2008, ch. 2)). This serves to capture the potential cost of robust policy-making when the fear of model misspecification proves to be unfounded, or else when there is an unnecessary "robustness premium".

4.2 Robust policies are welfare superior

To better understand the costs and benefits of robust policies under externalities, we first examine the benchmark case where the two countries are of the same type. In other words, they share the same θ in each period,

[14]Consistent with the literature on robust control under model uncertainty, we restrict our interest in decision making by conservative agents, who are only concerned about the downside risk of model misspecification. Thus, we do not examine outcomes in "good" scenaria that may result from potential model misspecification.

$\theta^h = \theta^f \equiv \theta$.[15] This provides a benchmark that facilitates the subsequent analysis of competition over θ.

Table 2 presents results for the home country (results for the foreign country are identical since the countries are symmetric) under negative realisations of model uncertainty and when the fear is unfounded, for both robust and non-robust policies. We report outcomes for consumption ($\overline{c}$), capital ($\overline{k}$), output ($\overline{y}$), environmental quantity ($\overline{Q}$) and utility in the long run ($\overline{u}$) and lifetime welfare (U) under all cases.

Table 2: Nash equilibrium with similar robust policies ($\theta^h = \theta^f \equiv \theta$)

	Fear is founded (very bad scenario)		Fear is founded (bad scenario)		Fear is unfounded	
	$\theta = 0.1$	$\theta = 10^6$	$\theta = 0.1$	$\theta = 10^6$	$\theta = 0.1$	$\theta = 10^6$
$\overline{c}$	1.042	1.063	1.063	1.083	1.084	1.104
$\overline{k}$	0.549	0.560	0.559	0.570	0.570	0.581
$\overline{y}$	1.097	1.119	1.119	1.141	1.141	1.162
$\overline{Q}$	0.145	0.134	0.288	0.277	0.430	0.419
$\overline{u}$	-3.491	-3.518	-2.985	-3.009	-2.518	-2.539
U	-105.105	-105.479	-94.365	-94.693	-84.341	-84.624

Starting with outcomes under bad scenaria, we see that there are indeed gains from following robust policies, both in the long run and over the lifetime. As can be seen, the preference for robustness triggers a form of precautionary behaviour, in the form of better environmental protection at the cost of output, and this acts as a buffer against the bad environmental realisations (see also e.g. Vardas and Xepapadeas (2010) and Athanassoglou and Xepapadeas (2012) on precautionary environmental behaviour resulting from robust decision making).

A more interesting result is obtained when we evaluate outcomes and welfare when the fear is unfounded. In particular, there are welfare gains by following robust policies even when the fear of model misspecification is unfounded. In this case, non-robust policies in the Nash equilibrium, although rational and consistent with individual optimisation, are no longer optimal, as the countries fail to account for environmental externalities. In such a second-best environment, robust policy-making is not redundant and

[15] We have also solved the model under the assumption that the countries can cooperate on economic and environmental policies, so that the externalities are internalised. This first-best case, as expected, results in higher welfare compared to the Nash equilibrium. Moreover, it reproduces the standard results from the literature on robust control, i.e. there are welfare gains from following robust policies under bad outcomes of nature, and robustness premia when fears of model misspecification are unfounded.

outperforms non-robust policy, even when its raison d'être, namely, model misspecification, is not fulfilled. This is because the precautionary principle corrects for the under-provision of the public good. Environmental protection is inefficiently low in a Nash equilibrium and robust policy-making helps to remedy this, irrespective of whether fears of model misspecification are founded or not. To put it differently, robust behaviour works as a substitute of cooperation. This is consistent with the analysis in Dennis (2010), where the cost of robustness, even when the fear of model misspecification is unfounded, is eliminated because robust policy-making serves to correct for a policy failure, namely, the lack of commitment on the part of monetary authorities. Similarly, in our model, robust policy-making serves to correct for a market failure, namely, under-provision of public goods.

The public economics literature has discussed extensively the conditions under which uncertainty about the quantity of the public good supplied by others can lead a risk-averse individual to increase his own contribution to the public good (see e.g. Cornes and Sandler (1996, ch. 6)). In general, such a type of uncertainty is not enough to guarantee the emergence of precautionary behaviour and a rise in individual contributions. In particular, in this literature, the precautionary behaviour needed for the voluntary provision of the public good is similar to the conventional form of precautionary savings and arises from a convex marginal utility function and a mean-preserving increase in the variance of the exogenous processes (see e.g. Leland (1968) and Sandmo (1970)).[16] In our model, however, we study a different form of uncertainty that relates to the correct specification of the model of the public good, as opposed to an increase in the variance of the quantity of the public good only, and we find that robust policies always increase the provision of the public good. The precautionary behaviour emerging from robust decision making is a result of fear of mis-specifying the conditional means of the model's stochastic state variables and does not require a convex marginal utility function. Furthermore, as we discuss below, although model uncertainty creates incentives for the individual agent to increase the provision of the public good by following robust policies, it also creates incentives to free ride on the robust behaviour of others.

To further evaluate the effect of the precautionary principle on economic and environmental outcomes, we plot, in Figure 1, the transition path of the system from time 0, when robust policies start to be implemented, until the system converges to the new steady state under the assumption that fear of model misspecification is unfounded.[17]

[16] These exogenous processes can, for instance, refer to the provision of the public good by others, or to the productivity processes.

[17] Plotting the paths under the case of unfounded fear of model misspecification allows

[Figure 1 here]

As can be seen, the economy starts from the non-robust equilibrium under model certainty, as given by the last column in Table 2 and converges towards the steady state captured by the immediately preceding column in Table 2. The incentive for robust policies is manifested by increases over time in environmental quality, which reflect the desire to create a buffer stock of environmental capital, as discussed above. This is achieved at the expense of lower production, as can be seen in Figure 1.

4.3 Free-riding on robustness

The previous analysis suggested that if countries could choose the same θ, they would always choose robust policies in a model with environmental externalities, even when the fear of model misspecification is unfounded.

We now examine the choice of θ in the absence of an institutional mechanism that guarantees such a cooperation in θ. In this case, each country chooses its own θ by comparing outcomes for high and low θ, taking the actions of the other country as given. This can be thought as a prisoner's dilemma problem. However, recall that there is an extra dimension here, as each country evaluates the outcomes from different choices of θ under different realisations of model misspecification.

In Table 3, we present lifetime welfare under all possible combinations, in a usual extensive format, under bad outcomes from model misspecification and when the fear is unfounded. The first number inside each parenthesis in Table 3 represents the lifetime welfare of the home country, while the second number the lifetime welfare of the foreign country. The fact that the setup is symmetric is obviously reflected in the results. Note that the welfare outcomes when both countries choose either robust or non-robust policies are identical with those in Table 2. The differences emerge when one of the countries follows robust policy, while the other one follows non-robust policy.

us to isolate the effects of robust policy on economic and environmental choices, since, under negative realisations of model uncertainty, the outcomes include both the effects of robustness and of model misspecification.

Table 3: The choice of robust policies:
Lifetime welfare (Home, Foreign)

Fear is founded (very bad scenario)

	Robust ($\theta^f = 0.1$)	Non-robust ($\theta^f = 10^6$)
Robust ($\theta^h = 0.1$)	(-105.105,-105.105)	(-105.478,-105.106)
Non-robust ($\theta^h = 10^6$)	(-105.106,-105.478)	(-105.479,-105.479)

Fear is founded (bad scenario)

	Robust ($\theta^f = 0.1$)	Non-robust ($\theta^f = 10^6$)
Robust ($\theta^h = 0.1$)	(-94.365,-94.365)	(-94.707,-94.350)
Non-robust ($\theta^h = 10^6$)	(-94.350,-94.707)	(-94.693,-94.693)

Fear is unfounded

	Robust ($\theta^f = 0.1$)	Non-robust ($\theta^f = 10^6$)
Robust ($\theta^h = 0.1$)	(-84.341,-84.341)	(-84.654,-84.310)
Non-robust ($\theta^h = 10^6$)	(-84.310,-84.654)	(-84.624,-84.624)

The first result that appears in Table 3 is that under "very bad" realisations of model misspecification, the choice of robust policies is welfare improving irrespective of the other country's choice of θ. Thus, under the "very bad scenario", robust policy is the dominant strategy. However, the situation is reversed when the scenario is just "bad", or, in the limit, when the fear of model misspecification turns out to be unfounded. In these cases, the dominant strategy becomes the choice of non-robust policies. This happens because the existence of externalities implies that each country has an incentive to free-ride on the precautionary or robust behaviour of the other country. This can be better understood by looking at the dynamic transition for the two countries when the fear of model misspecification is unfounded in Figure 2. The plots in this Figure assume that the home country follows robust policies and the foreign non-robust.

[Figure 2 here]

As can be seen, the home country, by following a conservative policy, increases its environmental stock at the cost of lower output and consumption over time. The foreign country, instead, free-rides on the higher environmental quality offered by the increase in the environmental stock of the home country, and lets its own environmental stock decrease, while enjoying higher output and consumption.

The results in Table 3 suggest that when the countries believe that model misspecification can lead to very bad outcomes of nature, robust policies will emerge in a Nash equilibrium. This happens because the incentive to protect oneself against possible bad outcomes of nature, resulting from model misspecification, dominates the incentive to free ride on the other country's robustness. Hence, robustness is the equilibrium strategy in such cases. In turn, robustness serves as a substitute for cooperation in environmental policy, since it works in the opposite direction from the incentive to free ride on the other country's contribution to the international public good. On the contrary, when the countries believe that the environmental model they use does not allow for large unfavourable model misspecification, robust policies cannot arise in equilibrium. This happens because now the incentive to free-ride on the other country's precautionary policy dominates. In turn, in the absence of robust behaviour, there is no mechanism to mitigate the incentive to free ride on each other's contribution to the international public good and this results in inefficient outcomes.

Our analysis therefore shows that robust policies are not guaranteed to be the equilibrium outcome, since there is a subset of potential realisations of model uncertainty, for which the incentive to free-ride on the robust policy of the other country dominates. Hence, whether robust policies arise in equilibrium, or not, depends on the preferences and beliefs of the decision maker, which seems to be as in the standard model of robust control under model uncertainty considered in the literature. However, in the standard model, the welfare comparison of robust versus non-robust policies is inconclusive *ex ante*, i.e. before model uncertainty is actually resolved. By contrast, in our model, the equilibrium with robust policies is welfare superior to the equilibrium without robust policies *ex ante*, because it corrects for an underlying imperfection in the model. But robust policies may not arise because of free riding.

5 Conclusions

In this paper, we studied the benefits from, and the choice of, robust policy in a two-country setup with environmental externalities and fear of model misspecification regarding the environmental process. Solving a dynamic Nash game, we showed that robust policies can out-perform non-robust policies, even when the fear of model misspecification is unfounded. This happens because the precautionary principle, associated with robustness, corrects for the inefficiencies caused by environmental externalities. The choice of robust versus non-robust policies was then examined as in a prisoner's dilemma setup, and it was shown that the externalities create an incentive to free ride on others' robustness.

Hence, we presented a case where, although robust policy is *ex ante* welfare superior, it is not necessarily chosen because each agent/country has an incentive to free ride on the conservatism of the other agent/country. Robust policy can be chosen only when agents attach a sufficiently high weight to damaging outcomes resulting from model misspecification. If this is not case, the incentive to free ride on the conservatism of the other agent/country dominates and so robustness does not emerge as an equilibrium strategy. These results suggest that, at policy level, reputational or institutional mechanisms that encourage the adoption of robust decision-making, where appropriate, are important.

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6 Appendix

To obtain the solution in (21), we first use (20) for the home country and the transition law in (17) to eliminate next period's state:

$$
\begin{aligned}
v(x \mid \theta^h, \theta^f) = \underset{\widetilde{u}^h}{ext}\{ & \left(\widetilde{u}^h\right)' R^h \widetilde{u}^h + x' Q^h x + \\
& \beta \left(A x_t + \widetilde{B}^h \widetilde{u}_t^h + \widetilde{B}^f \widetilde{u}_t^f\right)' P^h \left(A x_t + \widetilde{B}^h \widetilde{u}_t^h + \widetilde{B}^f \widetilde{u}_t^f\right)\}
\end{aligned}
$$

The above equation implies:

$$
\begin{aligned}
v(x \mid \theta^h, \theta^f) = \underset{\widetilde{u}^h}{ext}\{ & \left(\widetilde{u}^h\right)' R^h \widetilde{u}^h + x' Q^h x + \beta x' A' P^h A x + \beta x' A' P^h \widetilde{B}^h \widetilde{u}_t^h + \\
& \beta x' A' P^h \widetilde{B}^f \widetilde{u}_t^f + \beta \left(\widetilde{u}^h\right)' \left(\widetilde{B}^h\right)' P^h A x + \beta \left(\widetilde{u}^h\right)' \left(\widetilde{B}^h\right)' P^h \widetilde{B}^h \widetilde{u}_t^h + \\
& \beta \left(\widetilde{u}^h\right)' \left(\widetilde{B}^h\right)' P^h \widetilde{B}^f \widetilde{u}_t^f + \beta \left(\widetilde{u}^f\right)' \left(\widetilde{B}^f\right)' P^h A x + \\
& \beta \left(\widetilde{u}^f\right)' \left(\widetilde{B}^f\right)' P^h \widetilde{B}^h \widetilde{u}_t^h + \beta \left(\widetilde{u}^f\right)' \left(\widetilde{B}^f\right)' P^h \widetilde{B}^f \widetilde{u}_t^f\}.
\end{aligned} \tag{33}
$$

The Bellman equation for the foreign country is symmetric to the above, where it suffices to substitute the superscript h with f and f with h.

6.1 FOCs (home country)

The first order condition with respect to $\widetilde{u}^h$ that is necessary for the maximum problem on the right side of (33) implies:

$$
\begin{aligned}
\widetilde{u}^h = & -\beta \left(R^h + \beta \left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1} \left(\widetilde{B}^h\right)' P^h A x - \\
& \beta \left(R^h + \beta \left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1} \left(\widetilde{B}^h\right)' P^h \widetilde{B}^f \widetilde{u}_t^f.
\end{aligned} \tag{34}
$$

In the solution, $\widetilde{u}_t^f = Kx$, where K is an undetermined matrix. Thus, (34) can be written as:

$$
\begin{aligned}
\widetilde{u}^h = & -\beta \left(R^h + \beta \left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1} \left(\widetilde{B}^h\right)' P^h A x - \\
& \beta \left(R^h + \beta \left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1} \left(\widetilde{B}^h\right)' P^h \widetilde{B}^f K x, \text{ or}
\end{aligned}
$$

$$\widetilde{u}^h = Fx, \text{ where} \tag{35}$$

$$\begin{aligned} F &= -\beta\left(R^h + \beta\left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1}\left(\widetilde{B}^h\right)' P^h A - \\ &\quad \beta\left(R^h + \beta\left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1}\left(\widetilde{B}^h\right)' P^h \widetilde{B}^f K. \end{aligned} \tag{36}$$

6.1.1 Nash equilibrium

Note that the solution for the problem of the foreign country is exactly symmetric. Hence, we obtain:

$$\widetilde{u}^f = Kx, \text{ where} \tag{37}$$

$$\begin{aligned} K &= -\beta\left(R^f + \beta\left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1}\left(\widetilde{B}^f\right)' P^f A - \\ &\quad \beta\left(R^f + \beta\left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1}\left(\widetilde{B}^f\right)' P^f \widetilde{B}^h F. \end{aligned} \tag{38}$$

The Nash equilibrium (for given matrices P^h and P^f) is obtained by solving the system in (36) and (38). This gives:

$$\begin{aligned} F &= -\begin{pmatrix} I - \beta^2\left(R^h + \beta\left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1}\left(\widetilde{B}^h\right)' \times \\ P^h \widetilde{B}^f\left(R^f + \beta\left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1}\left(\widetilde{B}^f\right)' P^f \widetilde{B}^h \end{pmatrix}^{-1} \times \\ &\quad \beta\left(R^h + \beta\left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1}\left(\widetilde{B}^h\right)' P^h A + \\ &\quad \begin{pmatrix} I - \beta^2\left(R^h + \beta\left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1}\left(\widetilde{B}^h\right)' P^h \widetilde{B}^f \times \\ \left(R^f + \beta\left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1}\left(\widetilde{B}^f\right)' P^f \widetilde{B}^h \end{pmatrix}^{-1} \times \\ &\quad \beta^2\left(R^h + \beta\left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1}\left(\widetilde{B}^h\right)' P^h \widetilde{B}^f \times \\ &\quad \left(R^f + \beta\left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1}\left(\widetilde{B}^f\right)' P^f A \end{aligned} \tag{39}$$

and

$$
\begin{aligned}
K &= -\left(\begin{array}{c} I - \beta^2 \left(R^f + \beta \left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1} \left(\widetilde{B}^f\right)' P^f \widetilde{B}^h \times \\ \left(R^h + \beta \left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1} \left(\widetilde{B}^h\right)' P^h \widetilde{B}^f \end{array} \right)^{-1} \times \\
&\quad \beta \left(R^f + \beta \left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1} \left(\widetilde{B}^f\right)' P^f A + \\
&\quad \left(\begin{array}{c} I - \beta^2 \left(R^f + \beta \left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1} \left(\widetilde{B}^f\right)' P^f \widetilde{B}^h \times \\ \left(R^h + \beta \left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1} \left(\widetilde{B}^h\right)' P^h \widetilde{B}^f \end{array} \right)^{-1} \times \qquad (40) \\
&\quad \beta^2 \left(R^f + \beta \left(\widetilde{B}^f\right)' P^f \widetilde{B}^f\right)^{-1} \left(\widetilde{B}^f\right)' P^f \widetilde{B}^h \times \\
&\quad \left(R^h + \beta \left(\widetilde{B}^h\right)' P^h \widetilde{B}^h\right)^{-1} \left(\widetilde{B}^h\right)' P^h A
\end{aligned}
$$

6.1.2 Verifying the guesses

If the guesses are correct, then the solution must satisfy the Bellman equations for the two countries. We thus first substitute (35) and (37) in (33) and solve for the matrices P^h and P^f that satisfy the resulting equation. This gives:

$$
\begin{aligned}
P^h &= Q^h + F'R^hF + \beta A'P^hA + \beta A'P^h\widetilde{B}^hF + \beta A'P^h\widetilde{B}^fK + \\
&\quad \beta F'\left(\widetilde{B}^h\right)' P^hA + \beta F'\left(\widetilde{B}^h\right)' P^h\widetilde{B}^hF + \beta F'\left(\widetilde{B}^h\right)' P^h\widetilde{B}^fK \qquad (41) \\
&\quad + \beta K'\left(\widetilde{B}^f\right)' P^hA + \beta K'\left(\widetilde{B}^f\right)' P^h\widetilde{B}^hF + \beta K'\left(\widetilde{B}^f\right)' P^h\widetilde{B}^fK.
\end{aligned}
$$

Working similarly, we obtain for the foreign country:

$$
\begin{aligned}
P^f &= Q^f + K'R^fK + \beta A'P^fA + \beta A'P^f\widetilde{B}^fK + \beta A'P^f\widetilde{B}^hF + \\
&\quad \beta K'\left(\widetilde{B}^f\right)' P^fA + \beta K'\left(\widetilde{B}^f\right)' P^f\widetilde{B}^fK + \beta K'\left(\widetilde{B}^f\right)' P^f\widetilde{B}^hF \qquad (42) \\
&\quad + \beta F'\left(\widetilde{B}^h\right)' P^fA + \beta F'\left(\widetilde{B}^h\right)' P^f\widetilde{B}^fK + \beta F'\left(\widetilde{B}^h\right)' P^f\widetilde{B}^hF.
\end{aligned}
$$

Figure 1: The effects of robust policy

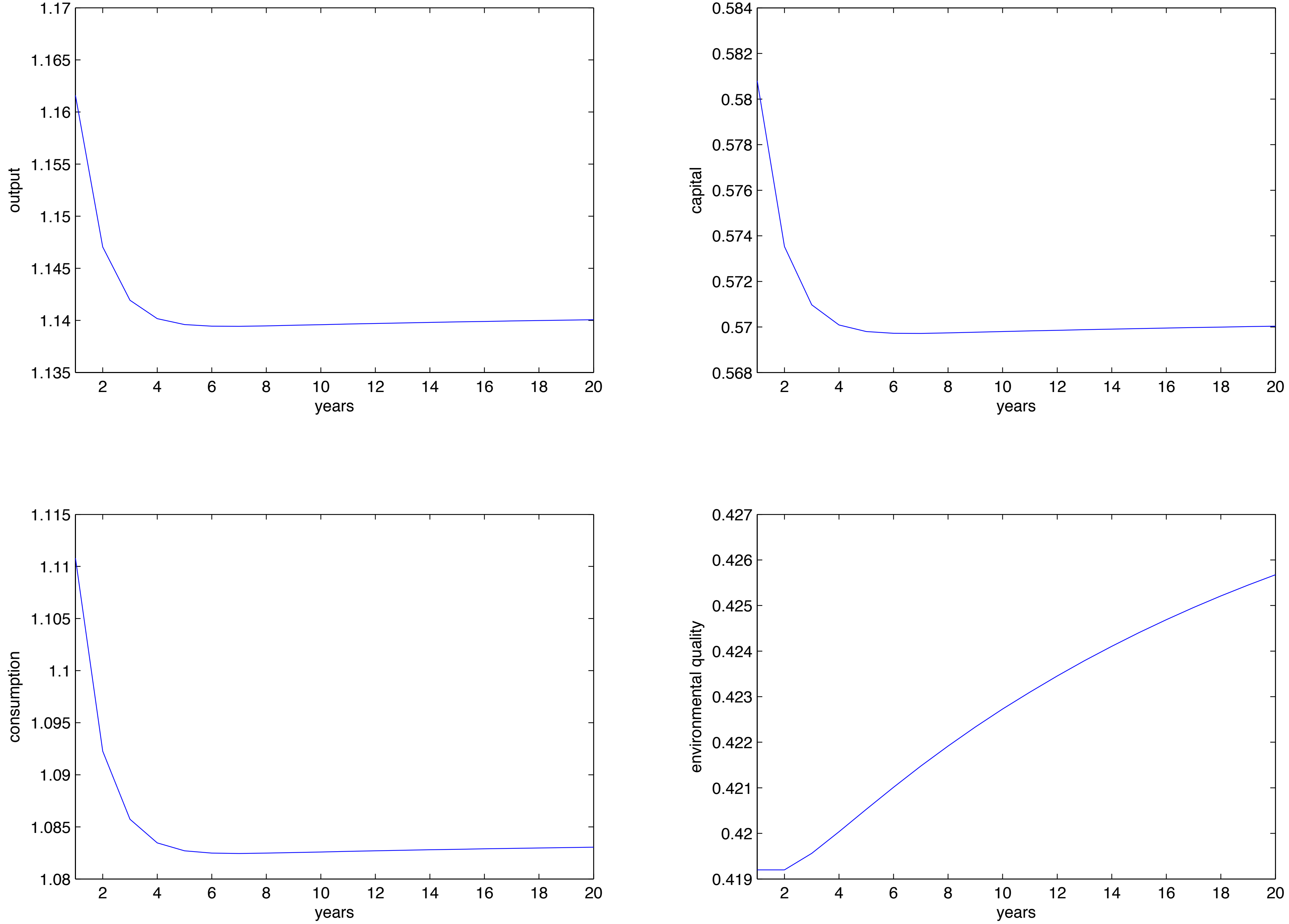

Figure 2: The effects of free riding in robust policy