



# Working Papers

[www.cesifo.org/wp](http://www.cesifo.org/wp)

## Asset Pricing with Uncertain Betas: A Long-Term Perspective

Christian Gollier

CESIFO WORKING PAPER NO. 4072

CATEGORY 11: INDUSTRIAL ORGANISATION

JANUARY 2013

*An electronic version of the paper may be downloaded*

- *from the SSRN website:* [www.SSRN.com](http://www.SSRN.com)
- *from the RePEc website:* [www.RePEc.org](http://www.RePEc.org)
- *from the CESifo website:* [www.CESifo-group.org/wp](http://www.CESifo-group.org/wp)

CESifo

Center for Economic Studies & Ifo Institute

# Asset Pricing with Uncertain Betas: A Long-Term Perspective

## Abstract

How should one evaluate investment projects whose CCAPM betas are uncertain? This question is particularly crucial for projects yielding long-lasting impacts on the economy, as is the case for example for many green investments. We define the notion of a certainty equivalent beta. We show that its term structure is not constant and that, for short maturities, it equals the expected beta. If the expected beta is larger than a threshold (which is negative and large in absolute value in all realistic calibrations), the term structure of the certainty equivalent beta is increasing and tends to its largest plausible value. If current beliefs concerning the asset's beta are represented by a normal distribution, the certainty equivalent beta becomes infinite for finite maturities.

JEL-Code: G110, G120, E430, Q540.

Keywords: asset prices, term structure, risk premium, certainty equivalent beta.

*Christian Gollier*  
*Toulouse School of Economics*  
*LERNA / University of Toulouse*  
*christian.gollier@tse-fr.eu*

December 2012

I am indebted to John Campbell, Jim Hammitt and Marty Weitzman for useful discussions and helpful comments on this paper during my research visit at the Harvard economics department in the Fall semester of 2012. I also thank Pierre Fery and Stéphane Gallon for useful comments on an earlier version of this paper. Julien Sauvagnat has been my research assistant for this paper. The research leading to these results has received funding from the Chairs “Risk Markets and Value Creation” and “Sustainable Finance and Responsible Investments” at TSE, and from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) Grant Agreement no. 230589.

## 1. Introduction

How should we evaluate our efforts in favor of future generations? This question is central in many current public policy debates, from fighting climate change to investing in biotechnologies, and depleting non-renewable resources, for example. Economic theory provides strong normative arguments in favor of using the Net Present Value criterion as a decision tool, with a discount rate that reflects both the opportunity cost of capital and the citizens' propensity to invest for the future. Under the standard assumptions of the Consumption-based Capital Asset Pricing Model (CCAPM, Lucas (1978)), this discount rate  $r = r_f + \beta\pi$  is the sum of a risk-free rate  $r_f$  and a risk premium  $\beta\pi$ . Since Weitzman (1998), various authors have recommended to use a decreasing term structure for the risk-free discount rate, thereby placing more weight on long-term riskless impacts in the evaluation process.<sup>2</sup>

The development of this literature has mostly been devoted to the evaluation of safe projects. This focus on the risk free discount rate is quite surprising, because most actions involving the distant future have highly uncertain impacts. For example, in spite of intense research efforts around the world over the last two decades, the socioeconomic impacts of climate change are still highly uncertain. We have learned from the normative version of the CCAPM that what matters to evaluate risky projects is their impact on the aggregate risk in the economy. This is evaluated by their parameter  $\beta$ , which measures the elasticity of the logarithm of their net benefits with respect to changes in the logarithm of aggregate consumption  $c_t$ . Projects with a larger beta will have a larger positive impact on the aggregate risk in the economy. They should be penalized by being discounted at a larger rate. On the contrary, a project with a negative beta reduces the aggregate risk, which implies that it should be discounted at a rate smaller than the risk free rate. More generally, if two projects yield the same flow of expected benefits, the one with the smaller beta should have a larger social value.

An important problem is that socioeconomic betas are difficult to estimate. Large companies and assets funds tend to use them with parsimony. For example, Krueger, Landier and Thesmar (2012) demonstrate that conglomerates generally use a unique discount rate to evaluate different

---

<sup>2</sup> See for example Weitzman (2001, 2007, 2009), Gollier (2002, 2008, 2012a), Newell and Pizer (2003), and Groom, Koundouri, Panopoulou and Pantelidis, (2007).

projects rather than project-specific ones. This may be due to the complexity of estimating project-specific betas. Whatever the reason, it tends low-beta conglomerates to overvalue high-beta projects, and to undervalue low-beta projects. An even more upsetting example is related to public policy evaluations in the western world. Up to our knowledge, except France and Norway, all countries evaluate their actions using a unique discount rate independent of the uncertainty affecting their impacts. For example, a unique rate of 7% is used in the United States since 1992. It was argued at that occasion that the “7% is an estimate of the average before-tax rate of return to private capital in the U.S. economy” (OMB (2003)). In 2003, the OMB also recommended the use of a discount rate of 3%, in addition to the 7% mentioned above as a sensitivity. This new rate of 3% was justified as follows: “This simply means the rate at which society discounts future consumption flows to their present value. [...]the real rate of return on long-term government debt may provide a fair approximation” (OMB, (2003)). In short, the OMB does not recommend evaluators to estimate the beta of the policy under scrutiny. Rather, it recommends estimating the policy’s NPV using two discount rates, corresponding to a beta of zero or one, respectively. From our experience of advising public institutions in their evaluation of environmental policies, we believe that this is due to the complexity of estimating the beta of flows of (non-traded) socioeconomic benefits, often disseminated over a long period of time.

For an investment project whose cash flows share characteristics of those of some traded asset, one should use deleveraged market betas of these assets to compute the NPV of the project. This method is not without deficiencies. It is for example often the case that the resemblance between the cash flows of the project and those of the traded asset is weak, and that it is limited to a short period of time. We should also add to this picture the well-known failure of the CCAPM to predict market prices from the assets’ betas. Finally, markets do not price the typical global, long-term externalities that motivated this paper, as those associated to climate change or genetic manipulations for example. For these reasons, the potential errors in the estimation of the project’s beta should be taken into account when evaluating its social value.

In this paper, we propose to reconsider the CCAPM by explicitly recognizing that betas are uncertain. We consider any project whose beta is constant but unknown to the evaluator. Our beliefs about the true value of the project’s beta is given by some distribution function for  $\beta$ . We maintain the other classical assumption of the model. In particular, we assume that the

representative agent has a constant relative risk aversion, and that log consumption follows an arithmetic Brownian motion. In this context, we show that the classical asset pricing formula of the CCAPM is robust to the introduction of this parametric uncertainty. More precisely, it does not affect the basic message of the CCAPM contained in the pricing formula  $r = r_f + \beta\pi$ . However, the uncertainty affecting the beta of the project necessitates to replace to uncertain  $\beta$  in this formula by a Certainty Equivalent Beta (CEB). This paper is about the characterization of the CEB.

Two interpretations of existing pricing theories are shown to be fallacious in this paper. The first fallacy is based on the assumption that the beta is not correlated to the growth of aggregate consumption, i.e., the “beta of the  $\beta$ ” is zero. In spite of this fact, it is not true that the risk on the project’s beta should not be priced. This fallacy is due to the fact that the uncertainty on  $\beta$  is not additive. However, we show in this paper that the CEB tends to the expected beta of the project for short maturities. In other words, the risk on beta is not priced for small maturities. This is not true for longer maturities.

The second fallacy is based on the potential use of the ideas around “Gamma discounting” developed by Weitzman (1998, 2001, 2010). Roughly speaking, because the discount factor  $\exp-(r_f + \beta\pi)t$  is decreasing and convex in  $\beta$ , taking the expectation of the discount factor to compute the present value of a unit future benefit in  $t$  years would be equivalent to using a CEB which is smaller than the mean beta, and which tends to the smallest plausible beta for large maturities. The idea is that, contrary to the random walk of the growth rate of consumption, the risk on beta is persistent. Compounding returns over many periods implies that, in the long run, the smallest plausible beta will drive the level of the discount factor. In this paper, we call this the “Weitzman effect”, which tends to raise the present value of the benefit. Although it brings some insights to the term structure of the CEB, this line of reasoning is also misleading. This is because the expected benefit of the project which has to be discounted is also sensitive to the beta. In general, if the beta is not too negative, a larger beta yields a larger expected benefit. This implies that there is a negative correlation between the discount factor and the expected benefit to be discounted. This negative correlation reduces the present value of the cash flow, in particular for the longest maturities. We call this the “benefit-growth effect”. We show in Section 3 of this

paper that this effect dominates the Weitzman effect in most circumstances. In other words, the term structure of the CEB is in general increasing, and it tends to the largest plausible beta for very large maturities. The term structure of the discount rates of risky assets inherits this upward-sloping property of the CEB.

In Section 4, we show that an analytical solution exists if our current beliefs about the project's beta are normally distributed. In that case, the CEB and the associated discount rate using the CCAPM formula exist and are bounded only for relatively short maturities. The critical maturity is equal to the inverse of the product of the variance of the economic growth rate and of the beta. For example, if we assume that the volatility of the economic growth rate is 4% per annum and that the standard deviation of the beta equals 1, this critical maturity above which the project's discount rate becomes infinite is equal to  $T=625$  years. Whether this is plus or minus infinity depends upon whether the benefit-growth effect dominates the Weitzman effect. When the benefit-growth effect dominates, the CEB tends to infinity when the maturity tends to  $T$ . This means that all benefits occurring at or after  $T$  are completely irrelevant for the decision. This would be true independent of the potentially fabulous size of these benefits. Suppose alternatively that the Weitzman effect dominates. Then, the CEB and the discount rate tends to minus infinity for maturities tending to  $T$ . This means that the existence of any plausible positive net benefit occurring at or after  $T$  should trigger the decision to invest, whatever the cost.

In Section 5, we apply these theoretical results to different contexts. We first show that the long-term beta of an environmental asset is equal to the inverse of the elasticity of substitution between this asset and consumption. We use time series data to estimate the elasticity of the demand for residential land in the United States. We show that the beta to be used for projects whose social benefit is to expand residential land should be increasing with maturity. We also measure the degree of uncertainty affecting socioeconomic and financial betas of different industries in France and in the United States.

Finally, we show in Section 6 that our model can be reinterpreted by assuming that the project is a portfolio of various projects or assets with different (sure) betas. We also present a class of projects for which the benefit-growth effect is switched off by a dynamic rebalancing strategy for this portfolio, so that the CEB has a downward-sloping term structure.

## 2. The model

We determine the present value  $PV$  today ( $t=0$ ) of a project that yield a single net benefit  $F_t$  occurring in year  $t$ . To do this, we examine how this project affects the standard utilitarian social welfare function

$$W = \sum_{t=0} e^{-\delta t} E u(c_t), \quad (1)$$

where  $\delta$  is the rate of pure preference for the present, and  $\{c_t\}_{t \geq 0}$  is the flow of consumption of the representative agent. Because the project is marginal, this present value must be equal to

$$PV = E \left[ e^{-\delta t} \frac{u'(c_t)}{u'(c_0)} F_t \right]. \quad (2)$$

We can rewrite this equation as follows:

$$PV = e^{-r_t} E F_t \quad (3)$$

with

$$r_t = \delta - \frac{1}{t} \ln \frac{E F_t u'(c_t)}{u'(c_0) E F_t}. \quad (4)$$

Equation (3) identifies the value of the project to the present value of the flow of expected future benefits, using  $r_t$  as the rate at which net benefit  $E F_t$  is discounted. This discount rate is defined in equation (4). It depends upon the risk characteristics of the net benefit at date  $t$ . We assume that

$$F_t = f_t e^{\varepsilon_t} c_t^\beta, \quad (5)$$

where  $\varepsilon_t$  has a zero mean and is independent of  $c_t$ . Parameter  $f_t \in \mathbb{R}$  is free and normalized to unity in the next two sections. We assume that  $\varepsilon_0 = 0$ , i.e.,  $F_0 = c_0^\beta$ . Parameter  $\beta$  measures the sensitiveness of the net benefit of the project to changes in macroeconomic conditions. When  $\beta = 0$ , the project has just an idiosyncratic risk component  $\xi$  that is not priced because of the

second-order nature of risk aversion. When  $\beta = 1$ , the project duplicates a stake in the economy as a whole. As suggested by Campbell (1986), a project with  $\beta > 1$  can be seen as a leveraged claim on the economy. On the contrary, a project with  $\beta < 0$  offers a hedge for macroeconomic shocks. As we will see later on, parameter  $\beta$  can also be interpreted as the CCAPM beta of the project.

In this paper, we generalize the CCAPM framework by allowing the beta of the asset to be uncertain. Let  $Q$  denote the cumulative distribution of  $\beta$ . We assume that  $\beta$  is independent of the growth process.

Except for the uncertainty of the beta, our model duplicates the classical CCAPM model. We assume that relative risk aversion is a constant  $\gamma > 0$ , so that the utility function of the representative agent is  $u(c) = c^{1-\gamma} / (1-\gamma)$ . We also assume that the growth of log consumption defined as  $g_t = \ln c_t / c_{t-1}$  follows a random walk, so that  $(g_1, g_2, \dots)$  is an i.i.d. process. Finally, we assume that the growth  $g_t$  of log consumption is normally distributed with mean  $\mu_g$  and volatility  $\sigma_g$ . This implies that we can rewrite equation (4) as follows:

$$\begin{aligned}
r_t - \delta &= \frac{1}{t} \ln E \left( \frac{c_t}{c_0} \right)^\beta - \frac{1}{t} \ln E \left( \frac{c_t}{c_0} \right)^{\beta-\gamma} \\
&= t^{-1} \ln E e^{\beta(g_1 + \dots + g_t)} - t^{-1} \ln E e^{(\beta-\gamma)(g_1 + \dots + g_t)} \\
&= t^{-1} \ln E \left[ E \left[ \left( e^{\beta g} \right)^t \mid \beta \right] \right] - t^{-1} \ln E \left[ E \left[ \left( e^{(\beta-\gamma)g} \right)^t \mid \beta \right] \right] \\
&= t^{-1} \ln E e^{t\chi(\beta, g)} - t^{-1} \ln E e^{t\chi(\beta-\gamma, g)},
\end{aligned} \tag{6}$$

where  $\chi(a, x) = \ln E \exp(ax)$  is the Cumulant-Generating Function (CGF) associated to random variable  $x$  evaluated at  $a \in \mathbb{R}$ . The CGF, if it exists, is the log of the better known moment-generating function. In expected utility theory,  $\chi(a, x)$  is the certainty equivalent of  $ax$  under constant absolute risk aversion equaling  $-1$ . CGF has recently been used by Martin (2012b) to explore asset prices under non Gaussian economic growth processes. Equation (6) is equivalent to:

$$r_t(\beta) = \delta + t^{-1} \chi(t, \chi(\beta, g)) - t^{-1} \chi(t, \chi(\beta - \gamma, g)). \tag{7}$$



The expression  $\chi(t, \chi(\beta, g))$  contains a sequence of two CGFs. The first CGF,  $\chi(\beta, g)$ , computes the certainty equivalent of  $\beta g$  conditional to  $\beta$ . The second CGF computes the certainty equivalent of  $t\chi(\beta, g)$  using the distribution of  $\beta$ . A similar process appears also in the last term  $\chi(t, \chi(\beta - \gamma, g))$  of this equation.

In this paper as in Gollier (2012b), we use the following properties of CGF (see Billingsley (1995)).

**Lemma 1** : *If it exists, the CGF function  $\chi(a, x) = \ln E \exp(ax)$  has the following properties:*

- i.  $\chi(a, x) = \sum_{n=1}^{\infty} \kappa_n(x) a^n / n!$  where  $\kappa_n(x)$  is the  $n$ th cumulant of random variable  $x$ . If  $m_n^x$  denotes the centered moment of  $x$ , we have that  $\kappa_1(x) = Ex$ ,  $\kappa_2(x) = m_2^x$ ,  $\kappa_3(x) = m_3^x$ ,  $\kappa_4(x) = m_4^x - 3(m_2^x)^2, \dots$
- ii. The most well-known special case is when  $x$  is  $N(\mu, \sigma^2)$ , so that  $\chi(a, x) = a\mu + 0.5a^2\sigma^2$ .
- iii.  $\chi(a, x + y) = \chi(a, x) + \chi(a, y)$  when  $x$  and  $y$  are independent random variables.
- iv.  $\chi(0, x) = 0$  and  $\chi(a, x)$  is infinitely differentiable and convex in  $a$ .
- v.  $a^{-1}\chi(a, x)$  is increasing in  $a$ , from  $Ex$  to the supremum of the support of  $x$  when  $a$  goes from zero to infinity.

Property ii of this Lemma directly implies

$$\chi(\beta, g) = \beta\mu_g + 0.5\beta^2\sigma_g^2 \quad \text{and} \quad \chi(\beta - \gamma, g) = (\beta - \gamma)\mu_g + 0.5(\beta - \gamma)^2\sigma_g^2. \quad (8)$$

Using property i and iii of Lemma 1, this yields the following pricing formula examined in this paper:

$$r_t(\beta) = r_f + t^{-1}\chi(t, \beta\mu_g + 0.5\beta^2\sigma_g^2) - t^{-1}\chi(t, \beta\mu_g + 0.5\beta^2\sigma_g^2 - \beta\pi), \quad (9)$$

where  $r_f = \delta + \gamma\mu_g - 0.5\gamma^2\sigma_g^2$  is the CCAPM risk free rate and  $\pi = \gamma\sigma_g^2$  is the CCAPM macro risk premium. In the benchmark CCAPM model, parameter  $\beta$  is a known constant, so this equation implies that

$$r_t(\beta) = r_f + (\beta\mu_g + 0.5\beta^2\sigma_g^2) - (\beta\mu_g + 0.5\beta^2\sigma_g^2 - \beta\pi) = r_f + \beta\pi. \quad (10)$$

Equation (10) reminds us three important features of the benchmark model. First, the term structures of the risk free rate and of the risk premium is flat. This is a consequence of the assumption that the growth process is i.i.d.. Second, the project-specific risk premium is proportional to the project's beta. This is the consequence of the assumed Gaussian distribution of changes in log consumption (Martin (2012b)). Third, this equation also confirms that parameter  $\beta$  can be interpreted as the CCAPM beta of the project.

In the remainder of this paper, we generalize equation (10) to the case of an uncertain beta. When the beta of the project is ambiguous, one can define a ‘‘Certainty Equivalent Beta’’ (CEB)  $\hat{\beta}_t(\beta)$  so that the rate to be used to discount today a cash flow occurring at date  $t$  is  $r_f + \hat{\beta}_t(\beta)\pi$ , by analogy to the CCAPM equation (10). Equation (9) tells us that this CEB is defined as follows:

$$\hat{\beta}_t(\beta) = \frac{1}{\pi t} \left( \chi(t, \beta\mu_g + 0.5\beta^2\sigma_g^2) - \chi(t, \beta\mu_g + 0.5\beta^2\sigma_g^2 - \beta\pi) \right). \quad (11)$$

Keep in mind that  $\beta\mu_g + 0.5\beta^2\sigma_g^2$  is the growth rate of the expected net benefit, whereas  $\beta\mu_g + 0.5\beta^2\sigma_g^2 - \beta\pi$  is the growth rate of the risk-neutral expectation of the net benefit. Equation (11) means that the certainty equivalent risk premium is the annualized difference between the CGFs of these two uncertain growth rates.

### 3. General results

In this section, we characterize the certainty equivalent beta without making any assumption about the distribution of  $\beta$ . Equation (11) defines the CEB essentially as the annualized difference between two CGFs. One can use the properties described in Lemma 1 to derive various properties of the CEB. Let us first exploit property  $v$ . Because  $t^{-1}\chi(t, x)$  tends to  $Ex$  when  $t$  tends to zero, equation (11) implies that

$$\lim_{t \rightarrow 0} \hat{\beta}_t(\beta) = \frac{1}{\pi} \left( E(\beta\mu_g + 0.5\beta^2\sigma_g^2) - E(\beta\mu_g + 0.5\beta^2\sigma_g^2 - \beta\pi) \right) = E\beta = \mu_\beta. \quad (12)$$

It yields the following proposition.

**Proposition 1:** *The CEB  $\hat{\beta}_t(\beta)$  tends to the mean beta  $\mu_\beta$  when the maturity  $t$  tends to zero.*

Thus, the parametric uncertainty affecting the beta has no effect on the discount rate for short maturities. For short maturities, in a fashion similar to additive diversifiable risks, this uncertainty should not be priced. Proposition 1 also tells us that, for short maturities, the following two assets should have exactly the same value: Asset A has an ambiguous beta of mean 0.5. Asset B is a portfolio that contains 50% of the risk free asset and 50% of the market portfolio (an asset with  $\beta = 1$ ).

Lemma 1 is also useful to explore the term structure of the CEB. Property  $v$  tells us that the CEB is the difference of two increasing functions of  $t$ . We can also infer from property  $i$  that

$$\lim_{t \rightarrow 0} \frac{\partial}{\partial t} \hat{\beta}_t(\beta) = \frac{1}{2\pi} \left( \text{Var}(\beta\mu_g + 0.5\beta^2\sigma_g^2) - \text{Var}(\beta\mu_g + 0.5\beta^2\sigma_g^2 - \beta\pi) \right). \quad (13)$$

This observation is important. It shows that the slope of the term structure of the CEB is determined by the relative uncertainty affecting the growth rates of respectively the objective and risk-neutral expectations of the net benefit. This is due to the persistency of the impact of the beta on these growth rates. The slope of the term structure is determined by how the uncertainty of  $\beta$  is transmitted to these two growth rates. Because  $\mu_g$  is larger than  $\mu_g - \pi$ , it is intuitive that the risk-neutral expectation has a smaller variance than the objective expectation.

The problem is that one cannot ignore the terms that are quadratic in  $\beta$  in the formulas of the expected growth rate of benefits. To illustrate this, suppose that the support of  $\beta$  is in a small neighborhood of  $\beta_0 = (\gamma/2) - (\mu_g / \sigma_g^2)$ . Because  $\text{Var}(f(\beta)) \simeq |f'(\beta_0)| \text{Var}(\beta)$ , we can derive the following approximation to equation (13):

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\partial}{\partial t} \hat{\beta}_t(\beta) &\simeq \frac{\text{Var}(\beta)}{2\pi} \left( \left| \mu_g + \beta_0 \sigma_g^2 \right| - \left| \mu_g - \pi + \beta_0 \sigma_g^2 \right| \right) \\ &= \frac{\text{Var}(\beta)}{2\pi} (|0.5\pi| - |-0.5\pi|) = 0 \end{aligned} \quad (14)$$

This exercise illustrates the fact that it is not always the case that the variance of the objective expectation of the growth rate of benefits is larger than the variance of the corresponding risk-

neutral expectation. This implies that the slope of the term structure of the CEB is generally ambiguous.

To remove this ambiguity, we can rearrange the RHS of equation (13) to obtain

$$\lim_{t \rightarrow 0} \frac{\partial}{\partial t} \hat{\beta}_t(\beta) = -\frac{\pi}{2} \text{Var}(\beta) + \frac{1}{\pi} \text{Cov}(\beta \mu_g + 0.5 \beta^2 \sigma_g^2, \beta \pi). \quad (15)$$

The first term in the RHS of this equation is “à la Weitzman”: When the discount rate is uncertain, compounding this rate over different maturities tends to generate a decreasing term structure. The second term comes from the fact that the cash flow to be discounted has an expectation that depends upon  $\beta$ . It is easy to check that

$$\frac{1}{\pi} \text{Cov}(\beta \mu_g + 0.5 \beta^2 \sigma_g^2, \beta \pi) = (\mu_g + \sigma_g^2 E\beta) \text{Var}(\beta) + 0.5 \sigma_g^2 \text{Skew}(\beta). \quad (16)$$

Suppose for now that the distribution of  $\beta$  is symmetric. Equation (16) then means that the second term in the RHS of equation (15) is positive if  $\mu_g + \sigma_g^2 E\beta \geq 0$ , i.e., if the expected growth rate of the benefit is increasing in  $\beta$  when evaluated at  $E\beta$ . Under this condition, the uncertainty affecting the expected cash flow tends to make the term structure increasing. We refer to this effect as the “benefit-growth effect”. In general, the Weitzman effect and the benefit-growth effect go in opposite directions. We summarize these findings in the following proposition.

**Proposition 2:** *The CEB satisfies the following property:*

$$\lim_{t \rightarrow 0} \frac{\partial}{\partial t} \hat{\beta}_t(\beta) = \left[ \mu_g + \sigma_g^2 E\beta - \frac{\pi}{2} \right] \text{Var}(\beta) + \frac{\sigma_g^2}{2} \text{Skew}(\beta). \quad (17)$$

It implies that, under a symmetric distribution for  $\beta$ , the term structure is increasing if and only if the benefit-growth effect ( $\mu_g + \sigma_g^2 E\beta$ ) dominates the Weitzman effect ( $\pi/2$ ). This proposition also tells us how the asymmetry in the distribution of beta affects the term structure for small maturities. Namely, a negative skewness in the distribution of beta tends to reduce the slope of term structure of the CEB.

One can also use property *i* of Lemma 1 to characterize the subsequent derivatives of the CEB and of the discount rate with respect to the maturity. It yields

$$\lim_{t \rightarrow 0} \frac{\partial^n}{\partial t^n} \hat{\beta}_t(\beta) = \frac{1}{(n+1)\pi} \left( \kappa_{n+1}(\beta\mu_g + 0.5\beta^2\sigma_g^2) - \kappa_{n+1}(\beta\mu_g + 0.5\beta^2\sigma_g^2 - \beta\pi) \right), \quad (18)$$

where  $\kappa_{n+1}(x)$  is the  $(n+1)$ th cumulant of random variable  $x$ . For example, the curvature ( $n=2$ ) of the CEB will involve in the right-hand side of this equation the centered moments of  $\beta$  up to the fifth order.

One can finally use property  $\nu$  to determine the asymptotic value of the CEB. We know that, when it exists,  $t^{-1}\chi(t, x)$  converges to the supremum of the support of  $x$ . Applying this property to both CGFs that appear in equation (11) implies that

$$\sup(\beta\mu_g + 0.5\beta^2\sigma_g^2) = \begin{cases} \beta_{\min}\mu_g + 0.5\beta_{\min}^2\sigma_g^2 & \text{if } \mu_g + \beta^*\sigma_g^2 \leq 0 \\ \beta_{\max}\mu_g + 0.5\beta_{\max}^2\sigma_g^2 & \text{if } \mu_g + \beta^*\sigma_g^2 > 0, \end{cases} \quad (19)$$

and

$$\sup(\beta(\mu_g - \pi) + 0.5\beta^2\sigma_g^2) = \begin{cases} \beta_{\min}(\mu_g - \pi) + 0.5\beta_{\min}^2\sigma_g^2 & \text{if } \mu_g + \beta^*\sigma_g^2 \leq \pi \\ \beta_{\max}(\mu_g - \pi) + 0.5\beta_{\max}^2\sigma_g^2 & \text{if } \mu_g + \beta^*\sigma_g^2 > \pi, \end{cases} \quad (20)$$

where  $\beta^* = 0.5(\beta_{\min} + \beta_{\max})$  is the center of the support of  $\beta$ . This yields the following proposition.

**Proposition 3:** *If we suppose that the support of  $\beta$  is  $[\beta_{\min}, \beta_{\max}]$ , the CEB has the following property:*

$$\lim_{t \rightarrow \infty} \hat{\beta}_t(\beta) = \begin{cases} \beta_{\min} & \text{if } \mu_g + \beta^*\sigma_g^2 \leq 0 \\ \beta_{\min} + (\beta_{\max} - \beta_{\min}) \left( \frac{\mu_g + \beta^*\sigma_g^2}{\pi} \right) & \text{if } 0 < \mu_g + \beta^*\sigma_g^2 \leq \pi \\ \beta_{\max} & \text{if } \mu_g + \beta^*\sigma_g^2 > \pi, \end{cases} \quad (21)$$

with  $\beta^* = 0.5(\beta_{\min} + \beta_{\max})$ .

Remember that the sign of  $\mu_g + \beta^* \sigma_g^2$  tells us whether the expected growth rate of benefits is locally increasing in the beta of the project, evaluated at the center of its support. If it is negative, the CEB tends to the smallest plausible beta. On the contrary, if it is larger than the aggregate risk premium  $\pi$ , the CEB tends to the largest plausible beta. In between, the CEB converges toward a linear interpolation of the two bounds of the support of the plausible betas. An interesting feature is that it is the position of the center  $\beta^*$  of the support of  $\beta$  that determines the CEB for long maturities. This should be compared to our result in Proposition 2 in which the slope of the CEB is determined by the position of  $\mu_g + E\beta\sigma_g^2$  relative to  $\pi/2$ . The different possible cases are presented in Table 1.

	$\beta^* \leq -\frac{\mu_g}{\sigma_g^2}$	$-\frac{\mu_g}{\sigma_g^2} < \beta^* \leq \gamma - \frac{\mu_g}{\sigma_g^2}$	$\beta^* > \gamma - \frac{\mu_g}{\sigma_g^2}$
$E\beta \leq 0.5\gamma - \frac{\mu_g}{\sigma_g^2} - 0.5 \frac{Skew(\beta)}{Var(\beta)}$	$\lim_{t \rightarrow 0} \hat{\beta}_t \leq 0$ $\lim_{t \rightarrow \infty} \hat{\beta}_t = \beta_{\min}$	$\lim_{t \rightarrow 0} \hat{\beta}_t \leq 0$ $\lim_{t \rightarrow \infty} \hat{\beta}_t \in [\beta_{\min}, \beta_{\max}]$	$\lim_{t \rightarrow 0} \hat{\beta}_t \leq 0$ $\lim_{t \rightarrow \infty} \hat{\beta}_t = \beta_{\max}$
$E\beta > 0.5\gamma - \frac{\mu_g}{\sigma_g^2} - 0.5 \frac{Skew(\beta)}{Var(\beta)}$	$\lim_{t \rightarrow 0} \hat{\beta}_t \geq 0$ $\lim_{t \rightarrow \infty} \hat{\beta}_t = \beta_{\min}$	$\lim_{t \rightarrow 0} \hat{\beta}_t \geq 0$ $\lim_{t \rightarrow \infty} \hat{\beta}_t \in [\beta_{\min}, \beta_{\max}]$	$\lim_{t \rightarrow 0} \hat{\beta}_t \geq 0$ $\lim_{t \rightarrow \infty} \hat{\beta}_t = \beta_{\max}$

Table 1: Shape of the term structure of the CEB for different values of the mean  $E\beta$  and of the center  $\beta^*$  of the support of the project's beta.

It is useful to compute the order of magnitude of these thresholds. A relative risk aversion of  $\gamma = 2$  is usually considered as reasonable in the macro and finance literature. The average growth rate of consumption in the western world over the last two centuries has been around  $\mu_g = 2\%$ , whereas its mean volatility can be approximated at  $\sigma_g = 4\%$  (see for example Maddison (1991)). Let us also assume that the distribution describing our beliefs about  $\beta$  is symmetric, so that  $E\beta = \beta^*$  and  $Skew(\beta) = 0$ . In that case, the CEB is increasing in  $t$  for low  $t$  if and only if  $E\beta$  is larger than  $-11.5$ . Moreover, the CEB tends to  $\beta_{\max}$  if and only if  $E\beta = \beta^*$  is larger than  $-10.5$ . This South-East corner of Table 1 thus covers a vast majority of investment projects in the real world. Observe also that a negative skewness for  $\beta$  may help to reverse this conclusion.

#### 4. The Gaussian beta case

In this section, we characterize the certainty equivalent beta  $\hat{\beta}_t(\beta)$  in the special case in which the distribution of  $\beta$  is normal with mean  $E\beta = \mu_\beta$  and variance  $Var(\beta) = \sigma_\beta^2$ . As it clearly appeared in the previous section, an important difficulty comes from the fact that equation (11) contains two CGFs of a quadratic function of the random variable  $\beta$ . This is why we first describe the following technical result, which is proved in the Appendix.

**Lemma 2:** *Suppose that random variable  $z$  is normally distributed with mean  $\mu_z$  and standard deviation  $\sigma_z$ . Consider any pair  $(a, b) \in \mathbb{R}^2$  such that  $b < 1/(2\sigma_z^2)$ . Then, we have that*

$$E \exp(az + bz^2) = (1 - 2b\sigma_z^2)^{-1/2} \exp\left(\frac{a\mu_z + 0.5a^2\sigma_z^2 + b\mu_z^2}{1 - 2b\sigma_z^2}\right). \quad (22)$$

This lemma has a well-known special case corresponding to  $b = 0$ , which corresponds to property *ii* of Lemma 1. One can use this for  $z = \beta$ ,  $b = \sigma_g^2 t$  and respectively  $a = \mu_g t$  and  $a = (\mu_g - \pi)t$  in equation (11). It implies the following proposition, which describes the analytical solution for the CEB in the Gaussian case.

**Proposition 4:** *Suppose that the beta of the project is normally distributed with mean  $\mu_\beta$  and variance  $\sigma_\beta^2$ . Then, for all maturities  $t < T = 1/\sigma_g^2\sigma_\beta^2$ , the Certainty Equivalent Beta  $\hat{\beta}_t(\beta)$  of the project is defined as follows:*

$$\hat{\beta}_t(\beta) = \frac{\mu_\beta + t\sigma_\beta^2(\mu_g - 0.5\gamma\sigma_g^2)}{1 - t\sigma_g^2\sigma_\beta^2}. \quad (23)$$

Proof: Lemma 2 implies that if we assume that  $0.5\sigma_g^2 t < 1/(2\sigma_\beta^2)$ , i.e.,  $t < T$ , both CGFs in equation (11) are finite. Applying this lemma twice allows us to rewrite equation (11) as follows:

$$\begin{aligned}
\hat{\beta}_t(\beta) &= \frac{1}{\pi t} \ln \frac{\exp\left(\frac{\mu_g \mu_\beta t + 0.5 \mu_g^2 \sigma_\beta^2 t^2 + 0.5 \sigma_g^2 \mu_\beta^2 t}{1 - \sigma_g^2 \sigma_\beta^2 t}\right)}{\exp\left(\frac{(\mu_g - \pi) \mu_\beta t + 0.5 (\mu_g - \pi)^2 \sigma_\beta^2 t^2 + 0.5 \sigma_g^2 \mu_\beta^2 t}{1 - \sigma_g^2 \sigma_\beta^2 t}\right)} \\
&= \frac{1}{\pi} \left[ \frac{\mu_g \mu_\beta + 0.5 \mu_g^2 \sigma_\beta^2 t + 0.5 \sigma_g^2 \mu_\beta^2}{1 - \sigma_g^2 \sigma_\beta^2 t} - \frac{(\mu_g - \pi) \mu_\beta + 0.5 (\mu_g - \pi)^2 \sigma_\beta^2 t + 0.5 \sigma_g^2 \mu_\beta^2}{1 - \sigma_g^2 \sigma_\beta^2 t} \right] \quad (24) \\
&= \frac{\mu_\beta + \mu_g \sigma_\beta^2 t - 0.5 \pi \sigma_\beta^2 t}{1 - \sigma_g^2 \sigma_\beta^2 t}.
\end{aligned}$$

This concludes the proof of Proposition 4.  $\square$

Observe first that Proposition 4 generalizes the CCAPM. Indeed, suppose that the distribution of  $\beta$  is degenerated, i.e.,  $\mu_\beta = \beta$  and  $\sigma_\beta = 0$ . Proposition 4 implies that  $\hat{\beta}_t(\beta) = \beta$  and  $r_t(\beta) = r_f + \beta\pi$ . In this case, the term structure of the discount rate is flat and well defined for all maturities, i.e.,  $T = +\infty$ .

When beta is normally distributed, the CEB defined by equation (23) has its own term structure, which is inherited by the term structure of the risky discount rate after multiplying by the constant aggregate risk premium  $\pi$  and adding the constant risk free rate  $r_f$ . Observe that, as in the general case, the CEB tends to  $E\beta$  when the maturity tends to zero. Observe also that the term structure of the CEB is monotone. It is increasing if and only if the expected beta is larger than  $0.5\gamma - \mu_g / \sigma_g^2$ , as stated in Proposition 2.

In the Gaussian case, the CEB is defined for maturities below an upper limit  $T = 1 / \sigma_g^2 \sigma_\beta^2$ . In fact, for maturities approaching this upper limit from below, the CEB and the associated discount rate become unbounded. This is due to the fact that the normality assumption allows for extremely large and extremely low plausible betas. For large maturities, the exponentially decreasing probability of these extreme events is compensated by the exponentially increasing nature of compounded returns. In fact, Lemma 2 tells us that both terms in the RHS of equation (11), i.e.,  $EF_t$  and  $EF_t u'(c_t) / u'(c_0)$ , tend to infinity when  $t$  tends to  $T$ . Under the plausible



assumption  $\mu_\beta \geq 0.5\gamma - \mu_g / \sigma_g^2$ , the CEB and the associated discount rate tend to plus infinity. In that case,  $T$  can be interpreted as a “blind maturity” such that one should be completely blind relative to all benefits of the project occurring above this maturity. Under the opposite assumption  $\mu_\beta < 0.5\gamma - \mu_g / \sigma_g^2$ , the CEB tends to minus infinity. In this alternative case,  $T$  defines a critical maturity so that if some positive expected benefit are generated by the project above this maturity  $T$ , then the project should be implemented at *any* cost. This critical maturity is equal to the inverse of the product of the variances of the consumption growth and of the beta. If we retain the calibration with  $\sigma_g = 4\%$  per annum as above, it equals 625 times the precision of  $\beta$ . For a standard deviation of  $\beta$  between 0.01 and 2, we obtain a critical maturity between  $T=156$  years and  $T=6\,250\,000$  years. Thus, this critical maturity is well above the typical maturities for assets that are traded on financial markets. However, it is well in the range of some of the environmental projects currently debated in different countries, as those associated to climate change or to the management of nuclear waste for example.

Let us calibrate this model with  $\mu_g = 0.5\%$ ,  $\sigma_g = 4\%$ , and  $\gamma = 2$ . If we assume further that the beta of the project is normally distributed with mean  $\mu_\beta = 0.5$  and standard deviation  $\sigma_\beta = 2$ , the CEB has an increasing term structure ( $\mu_\beta > 0.5\gamma - (\mu_g / \sigma_g^2) = -2.125$ ), and it tends to  $+\infty$  for maturities tending to  $T = 156.25$  years. This term structure corresponds to the convex curve in Figure 1. We now show that this result is radically modified when we truncate the distribution of  $\beta$ . Suppose first that this truncation be symmetric around the mean, with  $\beta$  being the truncation of  $N(\mu_\beta, \sigma_\beta^2)$  in the support  $[\mu_\beta - k\sigma_\beta, \mu_\beta + k\sigma_\beta]$ . In Figure 1, we draw the CEB for different values of  $k$ . Because the center of these supports is  $\mu_\beta$ , which is larger than  $\gamma - (\mu_g / \sigma_g^2) = -1.125$ , all these calibrations belong to the South-East corner of Table 1. Because the mean is not affected by the truncation, the CEB remains equal to  $\mu_\beta$  for small maturities, and it is locally increasing. However, the CEBs remain finite for all maturities. They diverge from the non-truncated CEB at relatively small maturities to converge asymptotically to  $\beta_{\max} = \mu_\beta + k\sigma_\beta$ .

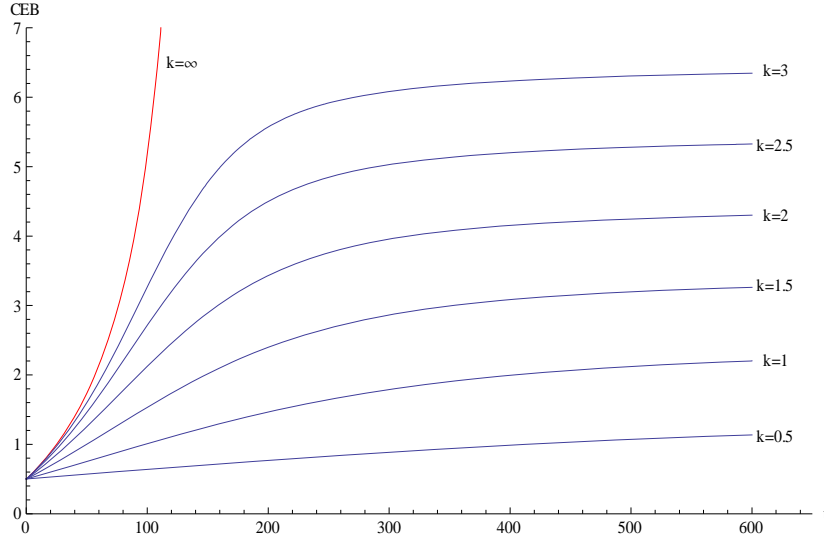


Figure 1: Term structure of the CEB with  $\mu_g = 0.5\%$ ,  $\sigma_g = 4\%$ , and  $\gamma = 2$ . The upper convex curve corresponds to  $\beta$  being normally distributed with mean  $\mu_\beta = 0.5$  and standard deviation  $\sigma_\beta = 2$ . The other curves correspond to the truncated version of this normal distribution in support  $[\mu_\beta - k\sigma_\beta, \mu_\beta + k\sigma_\beta]$ .

Let us alternatively assume that the normal distribution of  $\beta$  is asymmetrically truncated in interval  $[\beta_{\min}, \beta_{\max}]$ , with  $\beta_{\max} = 3$ . Figure 2 depicts the term structure of the CEB for  $\beta_{\min} = -6, -7, \dots, -10, -20$ , thereby yielding increasingly negative skewness. This numerical exercise brings various interesting insights to this work. First, the CEB at low maturities is reduced by the truncation. This is due to the asymmetric cuts of the two tails, which reduces the expected beta from 0.5 to approximately 0.1. From Proposition 1, this reduces the CEB at low maturities. Second, the term structure of the CEB in the truncated cases is increasing because, from Proposition 2, the benefit-growth effect dominates the Weitzman effect. Moreover, the term structure of the CEB is almost linear for a wide range of maturities, which implies that equation (17) provides a good basis to determine the CEB within this range of maturities. Third, in spite of the fact that the truncations only affect the long tails of the distribution of the beta, they have radical effects on the CEB for long maturities. These results are in line with the observation by Martin (2012a) that the value of long-term assets is mostly driven by the possibility of extreme events. In particular, the term structure of the CEB is decreasing at long maturities. Because  $-\mu_g / \sigma_g^2$  is equal to  $-3.125$ , Proposition 3 tells us for example that the CEB tends asymptotically

to  $\beta_{\min}$  for all calibrations with  $\beta_{\min} \leq -9.25$ . The bifurcation from the linear term structure is particularly impressive for the most asymmetric truncations. In spite of the fact that the beta of the project is very unlikely to be negative and large in absolute value, the mere plausibility of this hypothesis drives the choice of the discount rate for long maturities.

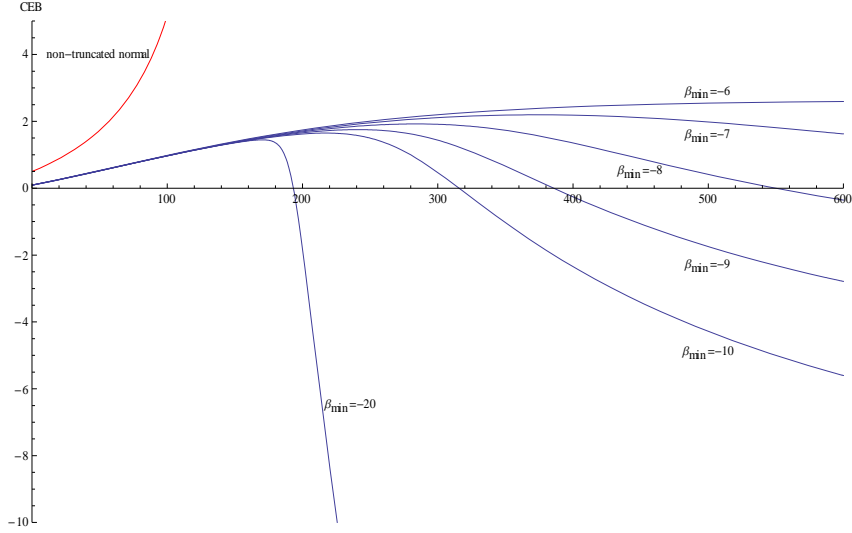


Figure 2: Term structure of the CEB with  $\mu_g = 0.5\%$ ,  $\sigma_g = 4\%$ , and  $\gamma = 2$ . The red curve corresponds to  $\beta$  being normally distributed with mean  $\mu_\beta = 0.5$  and standard deviation  $\sigma_\beta = 2$ . The other curves correspond to the truncated version of this normal distribution with  $\beta_{\max} = 3$  and various  $\beta_{\min}$ .

### 5. Measuring the uncertainty affecting the beta of an asset

In this section, we show how our methodology can be used in different contexts. The applications that we examine here are about evaluating an asset, which yields a flow of net benefits  $\{F_t\}_{t \geq 0}$ . Its social value  $V_0$  must be equal to the present value of this flow. In the absence of uncertainty about the beta of the benefits, we obtain

$$V_0 = \sum_{t=1} e^{-rt} EF_t = F_0 \sum_{t=1} e^{(-r + \beta\mu_g + 0.5\beta^2\sigma_g^2)t} = F_0 \left( e^{r - \beta\mu_g - 0.5\beta^2\sigma_g^2} - 1 \right)^{-1} = kF_0, \quad (25)$$

with  $r = r_f + \beta\pi$ . Similarly,  $V_1 = kF_1$ . This implies that the rate of return of holding the asset in the first period is equal to

$$R_1 = \ln \frac{F_1 + V_1}{V_0} = \ln \frac{1+k}{k} + \ln \frac{F_1}{F_0} = a + \beta g + \varepsilon_1. \quad (26)$$

This confirms that parameter  $\beta$  is the CCAPM beta of the asset. This section is about the measure of the uncertainty affecting the beta of various assets.

### 5.1. The beta of environmental assets

Guesnerie (2004), Hoel and Sterner (2007), Sterner and Persson (2008), Gollier (2010) and Traeger (2011) have shown that the evolution of relative prices and substitutability are crucial in the evaluation of environmental policies. Environmental assets that cannot be substituted by other goods in the economy and whose supply is constant over time have a social value which will be highly sensitive to economic growth. Their beta will thus be relatively large. Our objective in this subsection is to clarify the link between the beta of environmental assets and their degree of substitutability.

Consider an economy with 2 goods, a numeraire good  $c$ , and an environmental asset that yields a net benefit  $x$ . The investment project under scrutiny is aimed at increasing the quantity of  $x$ . Following the authors mentioned above, the instantaneous utility function of the representative consumer is assumed to belong to the CES family, with

$$U(x, c) = \frac{1}{1-\gamma} y^{1-\gamma}, \quad \text{with } y = [\alpha x^{1-\beta} + (1-\alpha)c^{1-\beta}]^{\frac{1}{1-\beta}} \quad (27)$$

where  $y$  is an aggregate good,  $\gamma$  is the aversion to risk on this aggregate good, and  $\alpha \neq 1$  and  $\beta \in \mathbb{R}^+$  are two scalars.<sup>3</sup> Parameter  $\beta$  is the inverse of the elasticity of substitution. The marginal benefit of increasing the consumption of good  $x$  expressed in the numeraire is equal to

$$F = -\frac{dc}{dx}\bigg|_U = \frac{\alpha}{1-\alpha} \left(\frac{c}{x}\right)^\beta \quad (28)$$

If we assume that the environmental asset yields a flow of  $x$  that is constant through time, equation (28) for the sensitivity of the cash flow to aggregate consumption is equivalent to

---

<sup>3</sup> When  $\beta = 1$ , we get a Cobb-Douglas function with  $y = c^\alpha x^{1-\alpha}$

equation (5), where the beta of the project is equal to the inverse of the elasticity of substitution between good  $x$  and the numeraire.

In our model with a random walk for changes in log consumption, the value  $V_t$  of the environmental asset is proportional to its current net benefit  $F_t$  expressed in the numeraire, as shown by equation (25). Thus, equation (28) implies that the underlying asset must have a social value that is proportional to  $(c/x)^\beta$ . The simplest method to estimate the beta in this context is thus to observe that the value  $V$  of the environmental asset must satisfy the following dynamic relationship:

$$g_V = \beta(g_c - g_x), \quad (29)$$

where  $g_x$  is the change in the log of  $x$ . In other words, the beta of the project under scrutiny is equal to the ratio of the growth rate of the relative price of good  $x$  to the difference between the growth rates of  $c$  and  $x$ . Inspired by Hoel and Sterner (2007), one can illustrate this method by applying to residential land. Suppose that the supply of residential land is fixed ( $g_x = 0$ ). Davis and Heathcote (2007) provide data on the real price of residential land in the United States over the period 1975Q1-2012Q1. Using the yearly version of their data, one can estimate the parameters of the following linear regression:

$$g_V = a + \beta g_c + \varepsilon. \quad (30)$$

The OLS estimator of  $b$  equals  $\mu_\beta = 2.84$ , with a large standard error  $\sigma_\beta = 1.27$ . This suggests a small elasticity of substitution of residential land and other goods in the economy. Observe also that the standard deviation of the beta is large. Under the normality assumption, there is a 1% probability that the true beta be in fact negative. Suppose also that  $\mu_g = 2\%$ ,  $\sigma_g = 4\%$  and  $\gamma = 2$ . Because  $\mu_\beta = 2.87 > -11.5 = 0.5\gamma - (\mu_g / \sigma_g^2)$ , Proposition 2 tells us that the term structure of the CEB is increasing. Moreover, under the assumption that  $\beta \sim N(\mu_\beta, \sigma_\beta^2)$ , the CEB tends to plus infinity for finite maturities ( $T = 387$  years). The CEB equals 8 or 18 respectively for a maturity of 100 years or 200 years.

## 5.2. The socioeconomic and financial betas in various economic sectors of the economy

In this subsection, we examine the uncertainty of the OLS estimation of the beta in (30) when one uses the traditional method based on the time series of returns and growth rates. Let us contemplate an investment project that is aimed to contribute to the development of a specific industry. This could for example take the form of an expansion of the electricity sector by using the current technology mix observed in that sector. If we assume that the economies of scale are approximately constant, and in the absence of innovation, one can use macroeconomic data measuring the creation of social value of the electricity sector to estimate the social benefit of such an investment. The French INSEE provides yearly data about the real value added produced by different sectors of the French economy.<sup>4</sup> The value added of a sector is defined as the value of production minus intermediate consumption. It must therefore be noticed that this data set does not take account of the externalities generated by these sectors, for example in the agricultural sector or in R&D. Table 2 summarizes the OLS estimation of equation (30) for a subset of the sectors listed in this database for period 1975-2011, where  $g_v$  is the yearly growth rate of real value added of the sector under scrutiny.

The standard error of the estimator of the beta lies between a low  $\sigma_\beta=0.15$  for the education sector and a relatively large  $\sigma_\beta =0.81$  for the agricultural sector. If we suppose as before that  $\mu_g = 2\%$ ,  $\sigma_g = 4\%$  and  $\gamma = 2$ , we obtain that the OLS estimator  $\mu_\beta$  is always larger than the threshold  $0.5\gamma - (\mu_g / \sigma_g^2) = -11.5$  defined in Corollary 2, so that the term structure of the CEB to be used to evaluate such investment projects is increasing for all sectors listed in Table 2. This table also provides the sectoral CEB for the 0, 50, 100 and 200 maturities.

The advantage of the value added approach is that it takes into account of the entire social value creation, with the exception of non-internalized externalities. Thus, the estimations described in Table 2 are about “socioeconomic” CCAPM betas. One could alternatively examine the “financial” CAPM betas, in which only the fraction of the value added accruing to investors is taken into account, and in which the factor is the market return rather than the rate of growth of

---

<sup>4</sup> See data set « 6.202 Valeur ajoutée brute par branche en volume aux prix de l'année précédente chaînés » on the INSEE website [http://www.insee.fr/fr/themes/comptes-nationaux/tableau.asp?sous\\_theme=5.2.2&xml=t\\_6202d](http://www.insee.fr/fr/themes/comptes-nationaux/tableau.asp?sous_theme=5.2.2&xml=t_6202d). This approach is inspired from Pierre Fery's appendix of Gollier (2011), which is a report to the French government on the economic evaluation of public policies under uncertainty.

consumption. In Table 3, we report OLS estimations of the CAPM betas for the two-digit Fama-French industry (FF48) of the U.S. economy, using yearly data from 1927 to 2011. Observe that the average standard deviation of 0.12 is much smaller than in the case of the socio-economic beta. This implies that the slopes of the CEB term structures are also smaller. The industry with the most uncertain beta is sector 27 (precious metals) with a standard deviation of 0.282, so that the CEB goes from 0.42 for short maturities to 0.73 for maturity  $t=200$  years, and to infinity for blind maturity  $T=7883$  years. The other CEBs have a less upward-sloping term structure, and a latter blind maturity.

These examples are illustrative of the difficulty to estimate betas with enough accuracy. The problem is usually made more complex than described above because most investment projects have a risk profile that does not correspond to the risk profile of the industry in which these investments will be implemented. To illustrate, it would make little sense to use the beta of utilities in the U.S. to evaluate an investment project in photovoltaic solar panels. In the same vein, this sectoral beta would not be useful to evaluate the project to build a high-voltage connection between Canada and the U.S. to make the two national electricity networks more resilient to asymmetric demand shocks. The evaluation of such an investment project would require estimating the elasticity of the demand for insurance against electricity outages to changes in GDP. The standard deviation associated to such estimations is likely to be larger than those described in Tables 2 and 3 of this subsection.

## ***6. Alternative interpretations of the model and extensions***

The assumption of our model is that there exists a linear relationship between the social return of the investment project and the growth rate of the economy, as expressed in equation (5). But the  $\beta$  of this linear relationship is initially unknown to the evaluator. There exist two other possible interpretations to this model which are alternative to the uncertainty affecting the project's beta.

### *6.1. Reinterpretation 1: Valuation of payoffs that have non-constant betas*

The first reinterpretation is based on the following rewriting of equation (5):

$$F_t = f_t \xi_t \int \exp(\beta \ln c_t) q(\beta) d\beta. \quad (31)$$

The integral in the right-hand-side of this equality can be interpreted as the Laplace transform of function  $q$  evaluated at  $\ln c_t$ . Thus, our results can be used to evaluate any investment project whose cash flows are related to log consumption through a Laplace transform of a distribution function. The CCAPM is limited to the evaluation of cash flows that are linked to log consumption through an *exponential* function, as is implicitly stated in equation (5).

### 6.2. Reinterpretation2: Valuation of portfolios

Let us alternatively modify equation (5) using a discrete distribution  $(\beta_1, q_1; \dots; \beta_n, q_n)$  for  $\beta$ :

$$F_t = f_t e^{-\varepsilon_t} \sum_{\theta=1}^n q_\theta c_t^{\beta_\theta}. \quad (32)$$

In this model, we allow the distribution of the beta to vary across different maturities. Observe that  $F_t$  can be reinterpreted as the cash flow of a portfolio of  $n$  different assets indexed by  $\theta = 1, \dots, n$ . Asset  $\theta$  has a sure constant beta equaling  $\beta_\theta$ . Thus, our results are useful to evaluate conglomerates composed of different investments, each with each own beta. Krueger, Landier and Thesmar (2012) have examined the investment strategy of such conglomerates in the US over the last three decades.

### 6.3. Extension: Valuation of projects whose expected payoffs are independent of beta

Up to now, we considered a benefit  $F_t$  whose expectation conditional to  $(c_t, \beta)$  is proportional to  $f_t c_t^\beta$ , where  $f_t$  is independent of  $\beta$ . Without any uncertainty on the asset's beta, it yields the traditional CCAPM pricing formula (10) in which  $\beta$  is the OLS estimator of equation (26). Under uncertainty, we have shown that this implies that  $E[F_t | \beta]$  is uncertain, a phenomenon which is at the origin of the complexity of this paper. Let us alternatively consider projects with the following risk structure:



$$F_t | \beta = f_t e^{\varepsilon_t} \frac{c_t^\beta}{E c_t^\beta}, \quad (33)$$

where  $\varepsilon_t$  has a zero mean and is independent of  $c_t$ , and  $f_t \in \mathbb{R}^+$ . Under certainty about  $\beta$ , this alternative model is indistinguishable from the more natural one that we examined in this paper. But, under uncertainty, the crucial difference of this risk structure is that the expected benefit conditional to  $\beta$  is independent of  $\beta$ . For this class of projects, a larger beta means a larger systematic risk, but not a larger expected payoff. This switches off the benefit-growth effect. From equation (4), it implies the following characterization of the CEB for this class of projects:

$$\begin{aligned} r_t - \delta &= -\frac{1}{t} \ln \frac{E F_t u'(c_t)}{u'(c_0) E F_t} = -\frac{1}{t} \ln E \left[ \frac{(c_t / c_0)^{\beta - \gamma}}{E (c_t / c_0)^\beta} \right] \\ &= -\frac{1}{t} \ln E \frac{\exp((\beta - \gamma)tg)}{E \exp(\beta tg)} = -\frac{1}{t} \ln E \frac{\exp((\beta - \gamma)\mu_g + 0.5(\beta - \gamma)^2 \sigma_g^2)t}{\exp(\beta\mu_g + 0.5\beta^2 \sigma_g^2)t} \\ &= \gamma\mu_g - 0.5\gamma^2 \sigma_g^2 - t^{-1} \ln E \exp(-\beta\pi t). \end{aligned} \quad (34)$$

This can be rewritten as follows:

$$r_t(\beta) = r_f - t^{-1} \chi(t, -\beta\pi), \quad (35)$$

which implies the following definition of the CEB for this class of projects:

$$\hat{\beta}_t(\beta) = -\frac{1}{\pi t} \chi(t, -\beta\pi). \quad (36)$$

This confirms that only the Weitzman effect appears in the pricing of this class of projects. Lemma 1 applied to this result directly implies the following properties. First, as before the CEB is equal to the mean beta for small maturities. Second, the CEB has a decreasing term structure (Weitzman effect). Third, it tends to the smallest plausible beta for maturities tending to infinity. Finally, if we assume that our beliefs about the true beta are normally distributed, then the CEB  $\hat{\beta}_t(\beta)$  is equal to  $\mu_\beta - 0.5\pi\sigma_\beta^2 t$ , which decreases linearly with the maturity.

We believe that most real projects are such that larger betas yield larger systematic risk and larger expected payoffs, as it appears in the traditional CCAPM formulation  $F_t = c_t^\beta$ . However, some projects may be better modeled with the alternative formulation  $F_t = c_t^\beta / E c_t^\beta$ . This is mostly an

empirical question. Notice that Weitzman (2012) discusses the discount rate to evaluate a portfolio that contains two assets, the first being safe ( $\beta_1 = 0$ ), and the other being the aggregate portfolio ( $\beta_2 = 1$ ). He assumes that the portfolio is rebalanced through time to maintain the share of the flows of the two assets. Given the equivalence between the portfolio interpretation of our theory and its parametric uncertainty interpretation, this discussion allows us to conclude that this is this rebalancing of the portfolio that switches off the benefit-growth effect which plays a prominent role in this paper. It explains why we obtain radically different results. The downward-sloping property of the term structure of the CEB in Weitzman's model and its generalization (33) presented in this section is driven by the fact that the composition of the portfolio is continuously and massively rebalanced towards the components with the smallest betas.

## **7. Conclusion**

The starting point of this research is that CCAPM betas are often difficult to estimate. This is likely to be the main reason why the standard toolbox for public investment and policy evaluation does not say much about how risk should be integrated in the benefit-cost analysis. In fact, believe it or not, three decades after the discovery of the normatively-appealing CCAPM, evaluators at U.S. Environmental Protection Agency or at the World Bank, to give two prominent examples, are still requested to use a single discount rate independent of the project-specific risk profile. This implies that we collectively invest too much in projects that raise the macroeconomic risk, and too little in projects that insure us against it. In this paper, we have taken seriously the origin of the problem by explaining how one should take into account of the potential errors in the estimation of the betas in cost-benefit analysis.

To each probability distribution describing the uncertainty associated to a project, we have defined and characterized a “certainty equivalent beta” that should be used to determine the rate at which this project should be discounted. We have shown that two effects are at play in this context. The Weitzman effect comes from the power of discounting over long maturities. A large beta implies a large risk premium and a large discount rate. Contrary to the i.i.d. risks on the growth rate of consumption, the uncertainty on beta is persistent. As shown for example by Weitzman (1998) and Weitzman and Gollier (2010), this persistency implies a decreasing term

structure for the CEB. However, there is also a benefit-growth effect that comes from the fact that the expected benefit is most often increasing in the beta. This negative correlation between the discount rate and the cash flow to be discounted reduces its present value, i.e., it increases the CEB, and it does so more strongly at higher maturities. We have shown in this paper that this effect usually dominates the Weitzman effect, so that the term structure of the rate at which expected benefit should be discounted is in general upward-sloping. The global effect of the uncertainty affecting the beta is particularly strong when we assume that our beliefs can be represented by a normal distribution, since the CEB goes to infinity for finite maturities in that case.

This research opens new paths for exploration. On the empirical dimension, it would be interesting to test the hypothesis that long-dated traded assets with a more uncertain beta have a smaller market value. On the theoretical dimension, we have often assumed in this paper that the growth rate of consumption follows an arithmetic Brownian motion. This implies that the risk free rate and the systematic risk premium have a flat term structure. It also implies that our results are subjects to the standard critiques of the risk free rate puzzle and of the equity premium puzzle. If we allow for parametric uncertainty about the stochastic process of economic growth, the risk free rate and the systematic risk premium will have respectively a decreasing and an increasing term structure, as shown by Gollier (2012b). It would be interesting to explore a model in which the parametric uncertainties about economic growth and about the project's beta are combined.

## References

- Billingsley, P., (1995), *Probability and Measure*, Wiley, New York.
- Campbell, J.Y., (1986), Bond and stock returns in a simple exchange model, *Quarterly Journal of Economics*, vol. 101(4), 785-804.
- Davis, M.A. and J. Heathcote, (2007), The price and quantity of residential land in the United States," *Journal of Monetary Economics*, vol. 54 (8), p. 2595-2620. Data located at Land and Property Values in the U.S., Lincoln Institute of Land Policy <http://www.lincolninst.edu/resources/>
- Gollier, C., (2002), Time horizon and the discount rate, *Journal of Economic Theory*, 107, 463-473.
- Gollier, C., (2008), Discounting with fat-tailed economic growth, *Journal of Risk and Uncertainty*, 37, 171-186.
- Gollier, C., (2010), Ecological discounting, *Journal of Economic Theory*, 145, 812-829.
- Gollier, C., (2011), *Le calcul du risque dans les investissements publics*, Centre d'Analyse Stratégique, Rapports & Documents n°36, La Documentation Française.
- Gollier, C., (2012a), *Pricing the planet's future: The economics of discounting in an uncertain world*, Princeton University Press, Princeton.
- Gollier, C., (2012b), Evaluation of long-dated investments under uncertain growth trend, volatility and catastrophes, mimeo, Toulouse School of Economics.
- Gollier, C., and M.L. Weitzman, (2010), How Should the Distant Future be Discounted When Discount Rates are Uncertain?, *Economic Letters*, 107(3), 350-353.
- Groom, B., P. Koundouri, E. Panopoulou and T. Pantelidis, (2007), An Econometric Approach to Estimating Long-Run Discount Rates. *Journal of Applied Econometrics*, 22, 641-656.
- Guesnerie, R., (2004), Calcul économique et développement durable, *Revue Economique*, 55, 363-382.

- Hoel, M., and T. Sterner, (2007), Discounting and relative prices, *Climatic Change*, DOI 10.1007/s10584-007-9255-2, March 2007.
- Kimball, M.S., (1990), Precautionary savings in the small and in the large, *Econometrica*, 58, 53-73.
- Krueger, P., A. Landier, and D. Thesmar, (2012), The WACC fallacy: The real effects of using a unique discount rate, mimeo, Toulouse School of Economics.
- Lucas, R., (1978), Asset prices in an exchange economy, *Econometrica*, 46, 1429-46.
- Maddison, A., (1991), *Phases of Economic Development*, Oxford Economic Press.
- Martin, I., (2012a), On the valuation of long-dated assets, *Journal of Political Economy*, 120, 346-358.
- Martin, I., (2012b), Consumption-based asset pricing with higher cumulants, *Review of Economic Studies*, forthcoming.
- Newell, R. and W. Pizer, (2003), Discounting the Benefits of Climate Change Mitigation: How Much Do Uncertain Rates Increase Valuations? *Journal of Environmental Economics and Management*, 46 (1), 52-71.
- Sterner, T. and M. Persson, (2008), An Even Sterner Report": Introducing Relative Prices into the Discounting Debate, *Review of Environmental Economics and Policy*, vol 2, issue 1.
- Traeger, C.P., (2011), Sustainability, limited substitutability and non-constant social discount rates, *Journal of Environmental Economics and Management* 62(2), 215-228.
- US Office of Management and Budget, (2003), Circular N. A-4 To the Heads of Executive Department Establishments, Subject: Regulatory Analysis. Washington: Executive Office of the President.
- Weitzman, M.L., (1998), Why the far-distant future should be discounted at its lowest possible rate?, *Journal of Environmental Economics and Management*, 36, 201-208.
- Weitzman, M.L., (2001), Gamma discounting, *American Economic Review*, 91, 260-271.

Weitzman, M. L., (2007), Subjective expectations and asset-return puzzle, *American Economic Review*, 97, 1102-1130.

Weitzman, M. L., (2009), On Modeling and Interpreting the Economics of Catastrophic Climate Change, *Review of Economics and Statistics*, 91 (1), 1-19.

Weitzman, M.L., (2010), Risk-adjusted gamma discounting, *Journal of Environmental Economics and Management*, 60, 1-13.

Weitzman, M.L., (2012), Rare disasters, tail-hedged investments, and risk-adjusted discount rate, NBER WP 18496.

## Appendix: Proof of Lemma 2

We have that

$$E \exp(az + bz^2) = \frac{1}{\sigma_z \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(az + bz^2 - \frac{(z - \mu_z)^2}{2\sigma_z^2}\right) dz. \quad (37)$$

After rearranging terms in the integrant, this is equivalent to

$$E \exp(az + bz^2) = \frac{\exp\left(-\frac{\mu_z^2}{2\sigma_z^2} - y\right)}{\sigma_z / \hat{\sigma}} \left[ \frac{1}{\hat{\sigma} \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(z - \hat{\mu})^2}{2\hat{\sigma}^2}\right) dz \right], \quad (38)$$

with

$$y = \frac{\left(a + (\mu_z / \sigma_z^2)\right)^2}{4b - (2 / \sigma_z^2)},$$

$$\hat{\mu} = -\frac{a + (\mu_z / \sigma_z^2)}{2b - (1 / \sigma_z^2)},$$

and

$$-\frac{1}{2\hat{\sigma}^2} = b - \frac{1}{2\sigma_z^2}.$$

Notice that  $\hat{\sigma}$  exists only if we assume that  $b < 1/(2\sigma_z^2)$ . Notice also that the bracketed term in equation (38) is the integral of the density function of the normal distribution with mean  $\hat{\mu}$  and variance  $\hat{\sigma}^2$ . This must be equal to unity. This equation can thus be rewritten as

$$\begin{aligned} E \exp(az + bz^2) &= \frac{\hat{\sigma}}{\sigma_z} \exp\left(-\frac{\mu_z^2}{2\sigma_z^2} - \frac{\left(a + (\mu_z / \sigma_z^2)\right)^2}{4b - (2 / \sigma_z^2)}\right) \\ &= (1 - 2b\sigma_z^2)^{-1/2} \exp\left(\frac{a\mu_z + 0.5a^2\sigma_z^2 + b\mu_z^2}{1 - 2b\sigma_z^2}\right). \end{aligned} \quad (39)$$

This concludes the proof of Lemma 2.  $\square$

Table 2: OLS estimation of the  $\beta$  in equation (30), where  $g_v$  is the yearly growth rate of real added value of the sector, and  $g_c$  is the growth rate of consumption. Data set: France, 1975-2011, INSEE 6.202.

Sector	$\sigma_\beta$	Certainty equivalent beta for different maturities			
		$\mu_\beta = \hat{\beta}_0$	$\hat{\beta}_{50}$	$\hat{\beta}_{100}$	$\hat{\beta}_{200}$
Agriculture	0.81	0.67	1.34	2.10	3.90
Electricity	0.49	1.93	2.19	2.47	3.05
Water management	0.31	0.41	0.50	0.60	0.79
Electronic equipment	0.56	1.93	2.28	2.64	3.42
Electrical equipment	0.51	2.81	3.11	3.43	4.11
Textiles	0.39	1.72	1.88	2.05	2.40
Paper and printing	0.27	0.89	0.96	1.04	1.19
Chemicals	0.61	0.93	1.31	1.72	2.61
Pharmaceutics	0.54	1.35	1.66	1.98	2.67
Steel works	0.32	1.25	1.36	1.46	1.68
Construction	0.30	1.28	1.37	1.47	1.66
Transportation	0.23	1.53	1.58	1.64	1.75
Restaurants, hotels	0.25	0.73	0.79	0.85	0.98
Communication	0.55	1.47	1.79	2.13	2.86
Finance and insurance	0.37	0.10	0.23	0.36	0.63
Real estate	0.19	0.64	0.68	0.71	0.78
R&D	0.42	0.02	0.18	0.35	0.71
Arts and entertainment	0.28	0.40	0.48	0.55	0.71
Education	0.15	0.51	0.53	0.55	0.60
Healthcare	0.21	0.14	0.18	0.22	0.31



Table 3: OLS estimation of the  $\beta$  in equation (30), where  $g_v$  is the yearly real rate of return of the industry, and  $g_c$  is the yearly market real rate of return. Data set: Kenneth French's website for average annual rate of return of the two-digit Fama-French industry (FF48) from 1927 to 2011.

[http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data\\_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).

FF48	Description	$\sigma_\beta$	Certainty equivalent beta for different maturities			
			$\mu_\beta = \hat{\beta}_0$	$\hat{\beta}_{50}$	$\hat{\beta}_{100}$	$\hat{\beta}_{200}$
1	Agriculture	0.123	0.87	0.88	0.90	0.93
2	Food Products	0.069	0.70	0.71	0.71	0.72
3	Candy & Soda	0.184	0.80	0.83	0.87	0.94
4	Beer & Liquor	0.131	1.20	1.22	1.24	1.27
5	Tobacco Products	0.111	0.50	0.51	0.52	0.54
6	Recreation	0.155	1.23	1.26	1.28	1.33
7	Entertainment	0.121	1.32	1.34	1.35	1.38
8	Printing and Publishing	0.119	1.13	1.15	1.16	1.19
9	Consumer Goods	0.062	0.88	0.89	0.89	0.90
10	Apparel	0.104	1.02	1.03	1.04	1.06
11	Healthcare	0.275	1.11	1.18	1.26	1.42
12	Medical Equipment	0.109	0.77	0.79	0.80	0.82
13	Pharmaceutical Products	0.087	0.73	0.74	0.74	0.76
14	Chemicals	0.072	1.04	1.05	1.06	1.07
15	Rubber and Plastic Products	0.115	1.15	1.17	1.18	1.21
16	Textiles	0.108	1.24	1.25	1.26	1.29
17	Construction Materials	0.061	1.16	1.17	1.17	1.18
18	Construction	0.142	1.27	1.29	1.31	1.35
19	Steel Works Etc	0.095	1.28	1.29	1.30	1.32
20	Fabricated Products	0.150	1.00	1.02	1.04	1.09
21	Machinery	0.065	1.17	1.17	1.18	1.19
22	Electrical Equipment	0.067	1.17	1.18	1.18	1.19
23	Automobiles and Trucks	0.110	1.45	1.46	1.48	1.50
24	Aircraft	0.171	1.42	1.45	1.48	1.54
25	Shipbuilding, Railroad Equip	0.132	1.06	1.08	1.10	1.13
26	Defense	0.182	0.44	0.48	0.51	0.57
27	Precious Metals	0.282	0.42	0.49	0.57	0.73
28	Non-Metallic and Metal Mining	0.111	1.07	1.08	1.09	1.12
29	Coal	0.167	0.77	0.79	0.82	0.88

30	Petroleum and Natural Gas	0.075	0.84	0.85	0.85	0.86
31	Utilities	0.089	0.72	0.72	0.73	0.75
32	Communication	0.074	0.68	0.68	0.69	0.70
33	Personal Services	0.195	0.80	0.83	0.87	0.95
34	Business Services	0.104	1.32	1.34	1.35	1.37
35	Computers	0.104	1.23	1.25	1.26	1.28
36	Electronic Equipment	0.103	1.47	1.48	1.50	1.52
37	Measuring and Control Equip	0.088	0.93	0.93	0.94	0.96
38	Business Supplies	0.108	1.29	1.30	1.32	1.34
39	Shipping Containers	0.078	0.87	0.88	0.88	0.89
40	Transportation	0.069	1.03	1.03	1.04	1.05
41	Wholesale	0.109	1.14	1.15	1.16	1.19
42	Retail	0.075	1.01	1.02	1.02	1.03
43	Restaurants, Hotels, Motels	0.127	1.17	1.18	1.20	1.23
44	Banking	0.118	1.09	1.10	1.12	1.15
45	Insurance	0.101	0.85	0.86	0.87	0.89
46	Real Estate	0.149	1.45	1.48	1.50	1.55
47	Trading	0.085	1.25	1.26	1.27	1.28
48	Almost Nothing	0.124	1.12	1.14	1.16	1.19

---