On the Efficient Provision of Public Goods by Means of Lotteries

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Abstract

We provide a solution to the free-rider problem in the provision of a public good. To this end we define a biased indirect contribution game which provides the efficient amount of the public good in non-cooperative Nash equilibrium. No confiscatory taxes or other means of coercion are used. We rather extend the model of Morgan (2000), who used fair raffles as voluntary contribution schemes, to unfair or biased raffles, which we show to be equivalent to fair raffles whose tickets are sold to consumers at different individual prices. We give a detailed account of the solution for the case of two different consumers and discuss its implications for the general case of many consumers.

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1 Introduction

There is a huge literature on the provision of public goods. Morgan (2000), however, was the first to link the old institution of lotteries directly to the theory of the provision of public goods. He views a lottery, part of whose proceeds go to the provision of a public good, as a voluntary contribution scheme (and not merely a substitute for confiscatory tax schemes by the state). And, indeed, many private charities or institutions that lack any taxing power use lotteries to generate revenues for their respective aims.

Lotteries are a worldwide phenomenon (for an extensive study of US state lotteries see e.g. Clotfelter and Coak, 1991) and the traditional view of them as inefficient and regressive instruments for raising surrogate tax money is being challenged by Morgan's question, whether they rather may constitute an effective contribution scheme towards the supply of public goods.

Morgan (2000) showed that the provision of a public good can be enhanced by the use of lotteries or - in Morgan's terms - fixed prize raffles. In such a raffle a pre-announced fixed price of size/value R is offered by the prospective provider of the public good; e.g. a charity, and awarded to the lucky buyer of the winning lottery ticket. The prize R itself is financed out of the proceeds from ticket sales; i.e. if total ticket sales amount to S only the amount of $S - R$ can be used for financing the public good. In this way the standard voluntary provision problem for a public good is amended by a lottery that is tied to the financing of the public good; i.e. the utility derived from buying a lottery ticket is not only determined by the probability of winning of the prize but also - and additionally - by the amount of public good provided through the lottery. Morgan (2000) shows that if the public good is socially desirable, then a raffle with fixed prize R can always generate revenue from ticket sales in excess of R. Moreover, this *excess* amount always exceeds the amount that would be collected in a voluntary contribution scheme for the public good. Hence the amount of public good provided by means of a raffle exceeds the amount provided through voluntary contribution schemes; raffles are welfare enhancing. The reason for this interesting result is that the positive externality of a contribution on others in the pure voluntary provision scheme is now counteracted to some extent by the negative externality on others of buying additional lottery tickets (which lowers their probability of winning the contest for R). As a result contributions towards financing the public good increase in comparison to voluntary contribution. This type of result is also robust with respect to different modifications and extensions of the model, for instance, risk-averse consumers considered in Duncan (2000), increasing group size analyzed in Pecorino and Temimi (2007), endogenized prize value introduced in Lange (2006), and incomplete information (under simplified linear public good preferences) with one prize in Goeree et al. (2005) and multiple prizes in Faravelli (2011). However, most of these extensions share one limitation with the framework of Morgan (2000): any prize sum of finite value is never sufficient to finance and provide the *efficient amount* of the public good.

In this regard Morgan's model, as well as most of the mentioned extensions, can be viewed as a fund-raising model as he is not primarily interested in efficient provision of the public good. For the comparison of two lottery types in a model where consumers also obtain entertainment through lottery participation see Maeda (2008), and for the fund-raising capacities of contests Franke et al. (2012).

The present paper is directly concerned with the efficient provision of a public good by means of lotteries or raffles in the presence of heterogeneous consumers who - in particular - value the public good differently. As the simple lottery from Morgan (2000) results in underprovision of the public good, one remedy would be to modify the lottery such that incentives to contribute to the lottery are further increased. One alternative to achieve this is to design a lottery function that overweights large contributions (in this case the lottery is transformed into a so called Tullock contest with exponent $r > 1$). In a context where the prize sum is financed by lump-sum taxes this has been analyzed in Kolmar and Wagener (2012), while Giebe and Schweinzer (2012) consider a similar idea in a model with consumption taxes (i.e., individual consumption of the private good is taxed with a sales tax). The prize sum is then either entirely financed by the lump-sum tax in Kolmar and Wagener (2012), or partially by the consumption tax in Giebe and Schweinzer (2012), where the remaining sum is transformed into the public good. A further difference between those two papers is that the contest in Giebe and Schweinzer (2012) is nominally 'fought' with the expenditures of consumers for the private good; i.e. the individual winning probabilities for the prize share of tax revenue are automatically determined by the individual expenditures for private good consumption; the consumer who consumes more of the private good (and hence pays more tax) has a higher chance of winning the contest prize. There is *no* lottery or raffle held but the authors attribute 'a lottery feel' (ibid., p. 2) to their mechanism of direct taxation. However, in both models a combined tax and lottery/contest mechanism can be derived that achieves an efficient allocation of the private and the public good.

In contrast we retain Morgan's (2000) original setup, which has the merit of being completely free of coercion (through taxation or otherwise). This feature of the model makes it especially attractive for applications in the context of charities and other non-public organizations that usually lack coercive taxing or transfer power (neither in Giebe and Schweinzer (2012) nor in Kolmar and Wagener (2012) consumers can avoid paying taxes and contributing to the public good provision.)

To this end we introduce *biased* raffles into Morgan's model and ask whether this can even further increase revenue from the raffle and hence provision of the public good, ideally up to the optimal level. We give a first answer in the affirmative; i.e. we achieve a solution of the free-rider problem on a completely voluntary basis in non-cooperative Nash equilibrium. We also show that a biased raffle of our type is equivalent to a fair raffle of Morgan's type whose tickets are sold at individual prices to consumers. Hence biased raffles imply a Lindahl-like pricing idea for the public good provision. Indeed, one can view such a raffle (see Morgan (2000), section 5) as an impure public good with private characteristics (in the raffle prize dimension) and public characteristics (in the provided public good dimension). Note then that price discrimination is applied to the private good component of the impure public good by charging different ticket prices.

The paper is organized as follows: Section 2 introduces the model of a biased indirect contribution game and derives equilibrium existence and uniqueness results for the general case of n heterogeneous consumers. It is further shown that these equilibria correspond one-to-one to Morgan's unbiased indirect contribution game with price discrimination. Section 3 demonstrates existence of efficient biased raffles for the case of two consumers, while section 4 gives a characterization of efficient raffles. Section 5 gives an example and section 6 concludes.

2 The Model

Our economy exists of n, $i = 1, \ldots, n$, consumers, who each has a quasi-linear utility function of the type

$$u_i(w_i, G) = w_i + h_i(G)$$

with w_i summarizing the wealth of i and G denoting the amount of the public good provided economy wide. It is standard to assume that $h_i^{'} > 0$ and $h_i^{''} < 0$, $i = 1, \ldots, n$. Wealth can be transformed into public good by using the production function $f(w) = w$; i.e. one unit of (private) wealth can be transformed into one unit of the public good. All consumers are (expected) utility maximizers.

A social planner would like to implement the socially optimal amount of the public good (which coincides in this quasi-linear framework with the efficient allocation); i.e. he would choose to provide G^* units of the public good, where G^* maximizes

$$\text{(SO)} \quad \begin{aligned} W &= \textstyle\sum_{i=1}^n u_i(w_i, G) - G \\ &= \textstyle\sum_{i=1}^n (w_i + h_i(G)) - G \end{aligned}$$

With non-binding wealth constraints the optimal amount G^* has to satisfy the well-known Samuelson condition:

$$\sum_{i=1}^{n} h_i'(G^*) = 1.$$

If, for simplicity, we also assume that $h_i'(0) > 1$, then the public good is always desirable and $G^* > 0$ should be provided.
The following facts are well-know:

i) *Voluntary contribution schemes provide less than G^* of the public good:* In such a scheme consumers directly contribute an amount x_i, $i = 1, \ldots, n$ to the provision of the public good. Consumer i determines his contribution by maximizing

$$u_i\left(w_i - x_i, \sum_{j=1}^{n} x_j\right) = w_i - x_i + h_i\left(\sum_{j=1}^{n} x_j\right)$$

As Bergstrom et al. (1986) show, this results in $\sum_{j=1}^{n} x_j^* < G^*$ in any Nash equilibrium $x^* = (x_1^*, \ldots, x_n^*)$ of the contribution game.

ii) *Fixed-prize raffles provide less than G^* of the public good:* Morgan (2000) showed that the provision of the public good can be enhanced by the use of special lotteries, which he termed fixed-prize raffles. In such a raffle a pre-announced fixed prize of value R is offered by the prospective provider of the public good; e.g. the government or a charity institution, and awarded to the lucky buyer of the winning lottery ticket. The prize R itself has to be financed out of the proceeds from ticket sales S. So, if ticket sales amount to S, only the amount $S - R$ can be used towards financing the public good. A consumer is hence not asked directly to contribute to the provision of the public good, but indirectly through the purchase of lottery tickets for R (with the remaining proceeds being transformed into G, the public good). Consumer i consequently maximizes

$$Eu_i(x_i, x_{-i}) = w_i - x_i + \frac{x_i}{\sum_{j=1}^{n} x_j} R + h_i\left(\sum_{j=1}^{n} x_j - R\right).$$

This indirect voluntary provision game always has a unique Nash equilibrium (Morgan (2000), Proposition 2). Moreover, the amount of the public good provided in this equilibrium always exceeds the amount provided by the voluntary contribution scheme of case i). Note, that this means that ticket sales not only exceed R, and hence the prize R can be provided, but also that they exceed R *plus* the amount provided by the voluntary contribution scheme. The positive externality in providing amounts of the public good (by privately buying lottery tickets) onto others is now automatically combined with a negative externality onto others as this reduces their chances to win the prize R. Overall the individual incentives to free-ride are sufficiently reduced to provide *more* of the public good *and* - at the same time - retrieve the cost of the prize R (Morgan (2000), Theorem 1). However, even in this situation the efficient amount of the public good cannot be implemented with a finite prize sum; that is, efficient public good provision can only be achieved in the limit for a prize of infinite value which requires consequently unlimited wealth of consumers (Morgan (2000), Theorem 2). This holds also true for more general utility functions than the ones used by Morgan (see Duncan, 2002).

Recent advances in contest theory allow to exploit the underlying heterogeneity of contestants in order to extract higher total efforts of contestants by introducing biased contest success functions, see Franke, Kanzow, Leininger and Schwartz (2011, 2012). This is achieved by favoring specific (weak) contests through the bias (which amounts to biased lotteries or raffles) to induce a more balanced playing field among contestants. This leads to the question whether in the indirect provision game with a fixed-prize raffle of Morgan *biased* raffles could be used to further increase ticket sales and hence increase potential provision of the public good such that the efficient amount G^* can even be implemented with a finite prize sum R.

Franke et al. (2011) consider general biased contest success functions of the Tullock type, which determine individual winning probability in a contest of n contestants as a function of individual efforts by

$$p_i(x_1, \ldots, x_n) = \frac{\alpha_i x_i}{\sum_{j=1}^n \alpha_j x_j} \quad i = 1, \ldots, n; \alpha = (\alpha_1, \ldots, \alpha_n) > 0.$$

This form of bias was already introduced by Tullock himself in his seminal paper (Tullock, 1980). Franke et al. (2011) solve the contest design problem of finding the optimal individual bias weights α_i, $i = 1, \ldots, n$, if the aim of the contest organizer is to maximize total effort $\sum_{i=1}^n x_i$ in Nash equilibrium. Heterogeneity of contestants is expressed by different valuations of the prize at stake in the contest; hence each individual contestant with wealth w_i and valuation R_i wants to maximize his expected payoff of

$$u_i(x_1, \ldots, x_n) = w_i - x_i + \frac{\alpha_i x_i}{\sum_{j=1}^n \alpha_j x_j} R_i, \quad i = 1, \ldots, n.$$

Franke et al. (2011) give a complete solution in closed form and compare it to the unbiased contest solution with $\alpha_i = \alpha = 1$, $i = 1, \ldots, n$. They show that the bias weights can be used in two ways to increase competition between contestants, which leads to higher efforts. Firstly, competition between active contestants (with different valuations) in the unbiased case can be increased by *effectively* narrowing down the difference of their valuations and, secondly, inactive contestants in the unbiased contest can be incentivized to become active by a sufficiently high bias weight. Both of these channels can, however, not be exploited if contestants are homogeneous: in this case all contestants will already be active in the unbiased contest and narrowing down their effective differences in valuation is unfeasible *per se*. I.e. the optimal weights in the homogeneous case are given by $\alpha_i = 1$, $i = 1, \ldots, n$, and the unbiased contest is the optimal one.

In the present context heterogeneity among consumers stems from their different valuations for the public good as expressed by their valuation functions $h_i(G)$, $i = 1, \ldots, n$. Hence a *biased* raffle of the above Tullock type with fixed prize R, a value *common* to all consumers, can still be used with the hope of increasing ticket sales by exploiting *this* heterogeneity. This leads to the following biased indirect contribution game. Each consumer i determines the amount of ticket purchases x_i by maximizing the expected payoff

$$\text{(BR)} \qquad u_i(x_1, \ldots, x_n) = w_i - x_i + \frac{\alpha_i x_i}{\sum_{j=1}^n \alpha_j x_j} R + h_i\left(\sum_{j=1}^n x_j - R\right) \quad i = 1, \ldots, n.$$

A Nash equilibrium $x^* = (x_1^*, ..., x_n^*)$ is given by a vector of ticket purchases such that

$$\begin{aligned} & w_i - x_i^* + \frac{\alpha_i x_i^*}{\sum_{j=1}^n \alpha_j x_j^*} R + h_i\Big(\sum_{j=1}^n x_j^* - R\Big) \\ \geq \; & w_i - x_i + \frac{\alpha_i x_i}{\alpha_i x_i + \sum_{j \neq i} \alpha_j x_j^*} R + h_i\Big(x_i + \sum_{j \neq i} x_j^* - R\Big) \;\; \text{for all } x_i \in [0, w_i]\,, i = 1, \ldots, n \end{aligned}$$

Our first result states that our biased indirect contribution game has an equilibrium:

Proposition 2.1 *A Nash equilibrium of the biased indirect contribution game always exists.*

Proof. We can use the classic existence results by Debreu, Glicksberg and Fan (see Osborne and Rubinstein (1994), Theorem 20.3, p.20), which states that a game with compact and convex strategy sets for all players and continuous payoff functions for all players, which in addition are quasi-concave in a player's own strategy, admits a Nash equilibrium: strategy spaces are compact and convex; and although payoff functions are not continuous at $x = (0, \ldots, 0)$, this is inessential as the equilibrium must be interior; i.e., $x_i > 0$ at least for some $i = 1, \ldots, n$. Equally, it holds that for any $x = (x_1, \ldots, x_n)$ the second derivative of the payoff function u_i is negative as $h_i(\cdot)$ *and* $\frac{\alpha_i x_i}{\sum_{j=1}^n \alpha_j x_j} R$ are strictly concave functions of x_i, $i = 1, \ldots, n$. Hence $u_i(x_1, \ldots, x_n)$ itself is concave in x_i, which implies quasi-concavity in x_i. □

Some more work is required to prove that equilibrium in a biased indirect contribution game, in fact, is unique. Morgan (2000) proves this for the unbiased case, while Cornes and Hartley (2005) prove uniqueness of equilibrium for the biased Tullock contest. Combining the methods of these two papers should yield the result. Alternatively, one can take up Morgan's (2000) brief observation that a raffle in his sense can be regarded as an impure public good with a private and a public good component. Our model (as does Morgan's) then fits the setup of Kotchen (2007), who provides a uniqueness proof for equilibrium in impure public good models. Hence we state

Proposition 2.2 *The equilibrium in a biased indirect contribution game is unique.*

Note that - although all raffle tickets are sold at the same prize 1 (in terms of wealth units) - raffle tickets are converted into different winning probabilities for the prize R due to the different bias weights applied to the buyers. Hence the *private* good component of the impure public good is effectively sold at different *individualized* prizes. This is in the spirit of Lindahl-prices in a decentralized voluntary contribution scheme, but applied to the private good component of an artificially created impure public good. Can a biased raffle likewise be used to achieve the same effect; namely, the provision of the efficient amount G^* of the public good, albeit in non-cooperative Nash equilibrium?

Indeed, Franke et al. (2011) observe that the biased raffle, resulting in winning probabilities of the type $p_i(x_1, \ldots, x_n) = \frac{\alpha_i x_i}{\sum_{j=1}^n \alpha_j x_j}$, $i = 1, \ldots, n$, is equivalent to an unbiased raffle $p_i(y_1, \ldots, y_n) = \frac{y_i}{\sum_{j=1}^n y_j}$ with individual ticket prices $p_i = \frac{1}{\alpha_i}$, $i = 1, \ldots, n$, which implies that the raffle organizer can price-discriminate between consumers when selling the raffle tickets. To be more precise: the problem (BR) of a consumer in a biased indirect contribution game can be transformed into the following problem *where the equilibrium remains invariant to this transformation.*

Define $y_i := \alpha_i x_i$, $i = 1, \ldots, n$. Then (BR) can be written as

$$\text{(PD)} \qquad \max_{y_i} w_i - \frac{1}{\alpha_i} y_i + \frac{y_i}{\sum_{j=1}^n y_j} R + h_i\left(\sum_{j=1}^n \frac{1}{\alpha_j} y_j - R\right),$$

and we see that the prize R is now awarded in a fair raffle with ticket price $p_i = \frac{1}{\alpha_i}$ for consumer i, $i = 1, \ldots, n$. Also note, that revenue from ticket sales now amounts to $\sum_{i=1}^n p_i y_i = \sum_{i=1}^n \frac{1}{\alpha_i} y_i$; i.e. accounting for the cost of the raffle prize leads to a supply of the public good of $\sum_{i=1}^n \frac{1}{\alpha_i} y_i - R$.

The identity of the solutions of the biased indirect contribution game defined by (BR) and the unbiased indirect contribution game with price discrimination defined by (PD) is now immediate: (BR) leads to the first-order conditions

$$-1 + \frac{\alpha_i \sum_{j \neq i} \alpha_j x_j}{(\sum_{j=1}^n \alpha_j x_j)^2} R + h_i^{'} \left(\sum_{j=1}^n x_j - R \right) = 0, \quad i = 1, \ldots, n;$$

while (PD) results in

$$-\frac{1}{\alpha_i} + \frac{\sum_{j \neq i} y_j}{(\sum_{j=1}^n y_j)^2} R + \frac{1}{\alpha_i} h_i^{'} \left(\sum_{j=1}^n \frac{1}{\alpha_j} y_j - R \right) = 0, \quad i = 1, \ldots, n.$$

If we multiply the latter by α_i, $i = 1, \ldots, n$, they read

$$-1 + \frac{\alpha_i \sum_{j \neq i} y_j}{(\sum_{j=1}^n y_j)^2} R + h_i^{'} \left(\sum_{j=1}^n \frac{1}{\alpha_j} y_j - R \right) = 0, \quad i = 1, \ldots, n.$$

So, if $y = (y_1, \ldots, y_n)$ solves the latter equations, then $x = (x_1, \ldots, x_n) = \left(\frac{1}{\alpha_1} y_1, \ldots, \frac{1}{\alpha_n} y_n\right)$ must solve the first system (and vice versa). Hence a corollary to Proposition 2.1 says:

Corollary 2.3 *An unbiased indirect contribution game with price discrimination has a (unique) equilibrium for any price vector $p = (p_1, \ldots, p_n) > 0$ which is equivalent to the (unique) equilibrium of the biased indirect contribution game where $\alpha_i = \frac{1}{p_i}$ for all $i = 1, \ldots, n$.*

Moreover, we see that adding up the first-order conditions of either problem leads to

$$-n + \frac{\sum_{i=1}^n \alpha_i (\sum_{j \neq i} \alpha_j x_j)}{(\sum_{j=1}^n \alpha_j x_j)^2} R + \sum_{i=1}^n h_i^{'} \left(\sum_{j=1}^n x_j - R \right) = 0$$

resp.

$$-n + \frac{\sum_{i=1}^n \alpha_i (\sum_{j \neq i} y_j)}{(\sum_{j=1}^n y_j)^2} R + \sum_{i=1}^n h_i^{'} \left(\sum_{j=1}^n \frac{1}{\alpha_j} y_j - R \right) = 0$$

From this we immediately see that an unbiased raffle cannot supply the efficient amount of G^* of the public good, because if $\sum_{j=1}^n x_j = G^* + R$ resp. $\sum_{j=1}^n \frac{1}{\alpha_j} y_j = G^* + R$ and $\alpha_i = 1$, $i = 1, \ldots, n$, both equations would require (as $\sum h_i^{'} = 1$) that $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j = R$, which contradicts that the efficient amount is supplied. We summarize as follows:

Lemma 2.4 (Morgan, 2000) *In the indirect contribution game a fair raffle with uniform ticket prices cannot provide the efficient amount G^* of the public good in equilibrium.*

3 Efficient provision of the public good for n = 2

We now consider the simple case of two consumers, 1 and 2, with $u_1(w_1, G) = w_1 + h_1(G)$ and $u_2(w_2, G) = w_2 + h_2(G)$. Samuelson's optimality condition for G^* hence simplifies to

$$h_1'(G^*) + h_2'(G^*) = 1.$$

The biased indirect contribution game for the public good implies that the two consumers solve the following maximization problems:

$$\max_{x_1} w_1 - x_1 + \frac{x_1}{x_1 + \alpha x_2} R + h_1(x_1 + x_2 - R)$$

$$\max_{x_2} w_2 - x_2 + \frac{\alpha x_2}{x_1 + \alpha x_2} R + h_2(x_1 + x_2 - R).$$

Without loss of generality we have normalized the bias such that $\alpha_1 = 1$ and $\alpha_2 = \alpha$. (We could also choose without loss of generality the normalization $\alpha_1 + \alpha_2 = 1$ to reduce the problem to a single parameter α, see Franke et al. (2011)). The two first-order conditions for the interior equilibrium hence read:

i) $\quad -1 + \frac{\alpha x_2}{(x_1 + \alpha x_2)^2} R + h_1'(x_1 + x_2 - R) = 0$, and

ii) $\quad -1 + \frac{\alpha x_1}{(x_1 + \alpha x_2)^2} R + h_2'(x_1 + x_2 - R) = 0.$

The second order conditions for a maximum always hold. As before i) + ii) then yields

iii) $\quad -2 + \frac{\alpha(x_1 + x_2)}{(x_1 + \alpha x_2)^2} R + h_1'(x_1 + x_2 - R) + h_2'(x_1 + x_2 - R) = 0$

We directly ask whether the provision of the socially optimal amount G^* of the public good is compatible with equilibrium of the biased indirect contribution game. So suppose that the Samuelson's optimality condition applies; i.e. $h_1'(G) + h_2'(G) = 1$. Equation iii) then reduces to

$$\text{(EE)} \qquad \frac{\alpha(x_1 + x_2)}{(x_1 + \alpha x_2)^2} R = 1, \quad \text{and} \quad x_1 + x_2 = G^* + R.$$

The efficient amount $G^* > 0$ of the public good is solely determined by the valuation functions $h_1(\cdot)$ and $h_2(\cdot)$. The question then is whether we can find values for the bias α and the prize R such that (EE) holds rendering the equilibrium efficient.

Recall from Lemma 1 that a fair raffle (with identical weights) cannot do so for any two consumers. Conversely, we now show that identical consumers with $h_1(\cdot) = h_2(\cdot)$ will not provide the efficient public good in any indirect contribution game independently of the bias α and prize sum R: for this observe that the first-order conditions i) and ii) imply that $x_1 = x_2 = x$ for two identical consumers. But we have

Lemma 3.1 *(EE) does not have a symmetric solution $x_1 = x_2 = x$.*

Proof. Let $x_1 = x_2 = x$. Then (EE) reduces to:

$$\frac{2\alpha}{(1+\alpha)^2 x}R = 1, \text{ and } 2x = G^* + R,$$
$$\Rightarrow \quad G^* + R = \frac{4\alpha}{(1+\alpha)^2}R$$
$$\Rightarrow \quad G^* = \left(\frac{4\alpha}{(1+\alpha)^2} - 1\right)R = \frac{-(1-\alpha)^2}{(1+\alpha)^2}R < 0.$$

The last inequality is a contradiction. □

An important corollary to the above Lemma hence is:

Lemma 3.2 *Identical consumers will not provide the efficient amount G^* of the public good in any indirect contribution game, i.e., independently of the bias α and prize sum R.*

The reason for this result can already be found in Franke et al. (2011): in the case of homogenous contestants the highest total effort is obtained by an *unbiased* Tullock contest success function. Likewise here, consumers with identical valuations of the public good purchase the highest number of tickets in an *unbiased* raffle, which however cannot provide G^*. However, the slightest degree of heterogeneity (at the margin of G^*) can be exploited and amplified by a suitable prize R to implement G^*. To be more precise we define:

Definition 3.3 *Two consumers are heterogenous (at the margin of G^*), if $h_1'(G^*) \neq h_2'(G^*)$ and homogenous otherwise.*

An efficient equilibrium (that has been shown to be interior) must solve the following two equations resulting from the respective first order conditions:

iv) $\frac{\alpha x_2}{(x_1+\alpha x_2)^2}R = 1 - h_1'(G^*)$

v) $\frac{\alpha x_1}{(x_1+\alpha x_2)^2}R = 1 - h_2'(G^*)$.

Moreover, by definition the following equation has to be satisfied: $x_1 + x_2 = G^* + R$. Combining eq. iv) and v) leads to:

$$\frac{\alpha x_2}{1 - h_1'(G^*)}R = (x_1 + \alpha x_2)^2 = \frac{\alpha x_1}{1 - h_2'(G^*)}R,$$

and hence, from the equality of the first and last terms:

$$(1 - h_2^{'}(G^*))x_2 = (1 - h_1^{'}(G^*))x_1$$
$$\Rightarrow \boxed{x_1 = \frac{1-h_2^{'}(G^*)}{1-h_1^{'}(G^*)}x_2}$$

It is convenient to define $h^* = \frac{1-h_2^{'}(G^*)}{1-h_1^{'}(G^*)} > 0$ and thus $x_1 = h^* x_2$. Note that $h^* \neq 1$ if and only if consumers are heterogenous. Now substitute into (EE) to get

$$(h^* + 1)x_2 = G^* + R$$
$$\Rightarrow \quad x_2^* = \frac{G^* + R}{h^* + 1} \quad \text{and} \quad x_1^* = \frac{h^*}{h^* + 1}(G^* + R)$$

This determines the relative contributions in efficient equilibrium, if also the first equation of (EE) holds:

$$\frac{\alpha(h^* + 1)x_2}{(h^* + \alpha)^2 x_2^2}R = 1 \Leftrightarrow x_2 = \frac{\alpha(h^* + 1)}{(h^* + \alpha)^2}R$$

which implies that the following equation must hold:

$$\frac{G^* + R}{h^* + 1} = \frac{\alpha(h^* + 1)}{(h^* + \alpha)^2}R$$

Further manipulations yield

$$\begin{aligned}
G^* &= \frac{\alpha(h^* + 1)^2}{(h^* + \alpha)^2}R - R \\
&= \left[\frac{\alpha(h^* + 1)^2}{(h^* + \alpha)^2} - 1\right]R \\
&= \left[\frac{\alpha(h^* + 1)^2 - (h^* + \alpha)^2}{(h^* + \alpha)^2}\right]R \\
&= \frac{\alpha h^{*2} + 2\alpha h^* + \alpha - h^{*2} - 2\alpha h^* - \alpha^2}{(h^* + \alpha)^2}R \\
&= \frac{(\alpha - 1)(h^{*2} - \alpha)}{(h^* + \alpha)^2}R
\end{aligned}$$

Hence, the raffle (α, R) has to satisfy the following equation to induce efficient public good provision in equilibrium:

$$(\overline{\text{EE}}) \qquad G^* = \frac{(\alpha - 1)(h^{*2} - \alpha)}{(h^* + \alpha)^2}R$$

Obviously, ($\overline{\text{EE}}$) can always be satisfied with a sufficiently large R *provided* that the factor in front of R is positive; i.e. if

$$\text{(I)} \qquad (\alpha - 1)(h^{*2} - \alpha) > 0.$$

Observe that both factors in eq. (I) are positive if $1 < \alpha < h^*$, and both factors are negative if $h^* < \alpha < 1$. Moreover, $h^* \neq 1$ because consumers are assumed to be heterogeneous. Consequently, for any $h^* \neq 1$ an $\alpha > 0$ exists such that (I) holds; and for this α, in turn, there exists $R > 0$ such that ($\overline{\text{EE}}$) holds. We have therefore proven the following theorem.

Theorem 3.4 *If consumers are heterogenous there always exists a biased raffle* (α, R)*, which in equilibrium of the biased indirect contribution game provides the efficient level* G^* *of the public good.*

In the next section we look at efficient raffles in some more detail.

4 Efficient raffles

In this section we ask what restrictions on a raffle (α, R) are imposed by Theorem 1; in particular, for which R does there exist a suitable bias α such that the raffle (α, R) implements the efficient amount $G^\star$ of public good provision.

For this we recall ($\overline{\text{EE}}$) and define the function

$$f(\alpha) = \frac{(\alpha - 1)(h^{*2} - \alpha)}{(h^* + \alpha)^2} \text{ such that } G^* = f(\alpha) \cdot R.$$

We first note that $f'(\alpha) = \frac{(h^{*2}+1-2\alpha)(h^*+\alpha)-2(\alpha-1)(h^{*2}-\alpha)}{(h^*+\alpha)^3} = \frac{(h^*-\alpha)(h^*+1)^2}{(h^*+\alpha)^3}$ which implies that $f'(\alpha) > 0$ for $\alpha < h^*$ and $f'(\alpha) < 0$ for $\alpha > h^*$. Hence, $f(\alpha)$ is pseudo-concave (single-peaked) and the unique maximum of f occurs at $\alpha_{max} = h^* > 0$. Moreover, $f(\alpha)$ attains its maximum in the interval $[1, h^{*2}]$ if $h^* > 1$, and in the interval $[h^{*2}, 1]$ if $h^* < 1$.
The maximand $\alpha_{\max} = h^*$ gives the maximum of $f(\alpha)$ as

$$f(\alpha_{\max}) = f(h^*) = \frac{(h^* - 1)^2}{4h^*} > 0.$$

According to ($\overline{\text{EE}}$) we have to satisfy $f(\alpha) \cdot R = G^*$. Hence, R must exceed $R_{\min} = \frac{G^*}{f(\alpha_{\max})}$:

$$\text{(M)} \qquad R \geq R_{\min} = \frac{G^*}{\frac{(h^*-1)^2}{4h^*}} = \frac{4h^*}{(h^* - 1)^2} \cdot G^* > 0.$$

We have shown the following proposition:

Proposition 4.1 *Consider the equilibrium of the indirect contribution game.*

i) Let (α, R) be a biased raffle that provides the efficient amount G^* of the public good. Then $R \geq R_{\min}$ must hold.

ii) For any $R \geq R_{\min}$ there exists $\alpha > 0$ such that (α, R) yields the efficient amount G^* of the public good.

The second statement in Proposition 4.1 follows from the fact, that - given G^* and R - $f(\alpha)$ can be continuously varied between $f(\alpha_{\max})$ and 0.

The derivation of Proposition 4.1 suggests that for the efficient raffle there exists a trade-off between balancing the heterogeneity of the consumers and the necessary prize sum. To get further insights into the nature of this trade-off the relation between $R_{\min}$ and h^* has to be analyzed.

Note that for homogeneous consumers $h^* = 1$ holds, while for heterogeneous consumers $h^* \neq 1$. We define $h =\mid h^* - 1 \mid$ as an index of heterogeneity among consumers as it measures "distance" from the homogeneous case $h^* = 1$. From (M) we know that $R_{\min} = g(h^*) \cdot G^*$ with $g(h^*) = \frac{4h^*}{(h^*-1)^2}$. We hence have to study $g(h^*)$, which is positive for all $0 < h^* \neq 1$ and not defined at $h^* = 1$. Moreover, $\lim_{h^* \to 0} g(h^*) = \lim_{h^* \to \infty} g(h^*) = 0$ and

$$g'(h^*) = \frac{4(h^* - 1)^2 - 4h^* \cdot 2(h^* - 1)}{(h^* - 1)^4} = -\frac{4(h^* + 1)}{(h^* - 1)^3}.$$

The last expression implies that $g(h^*)$ is monotonically increasing in h^* for $h^* < 1$, and monotonically decreasing for $h^* > 1$ with a pole at $h^* = 1$. Consequently, g is monotonically decreasing in $h =\mid h^* - 1 \mid$, our index of heterogeneity. Hence, if the heterogeneity between consumers increases then the relation between the minimal prize sum R_{min} and the efficient provision level G^* decreases:

Lemma 4.2 *The relation $\frac{R_{\min}}{G^*}$ is monotonically decreasing in the heterogeneity of the consumers.*

The last result does not mean that the absolute value of $R_{\min}$ must behave monotonically in the index of heterogeneity. Consider a change in $h =\mid h^* - 1 \mid$. Then a change in h^* as defined above may also change G^*. So $R_{\min} = g(h^*) \cdot G^*$ could still behave non-monotonically. The following example shows that this indeed, can occur.

5 An example

Suppose $h_1(G) = b \cdot G^{\frac{1}{2}}$ and $h_2(G) = G^{\frac{1}{2}}$ with $b > 0$. Then the utility functions of the two consumers in the indirect biased provision game are

$$u_1(x_1, x_2) = w_1 - x_1 + \tfrac{x_1}{x_1+\alpha x_2} R + b \cdot (x_1 + x_2 - R)^{\frac{1}{2}}$$

$$u_2(x_1, x_2) = w_2 - x_2 + \tfrac{\alpha x_2}{x_1+\alpha x_2} R + (x_1 + x_2 - R)^{\frac{1}{2}}$$

The optimal amount G^* of the public good is characterized by the Samuelson condition $b \cdot \frac{1}{2} G^{-\frac{1}{2}} + \frac{1}{2} \cdot G^{-\frac{1}{2}} = 1$, which is solved by $G^* = \frac{(1+b)^2}{4}$. Consequently, $h_1'(G^*) = \frac{b}{1+b}$ and $h_2'(G^*) = \frac{1}{1+b}$. We calculate $h^* = \frac{1-\frac{1}{1+b}}{1-\frac{b}{1+b}} = b$ which gives $\alpha_{\max} = b$. With $f(\alpha_{max}) = \frac{(b-1)^2}{4b}$ it follows from (M) that

$$R_{\min}(b) = g(b) \cdot G^* = \frac{4b}{(b-1)^2} \cdot \frac{(1+b)^2}{4} = \frac{b(1+b)^2}{(b-1)^2}.$$

Note that $R_{\min}(b)$ is not monotonic in b:

$$R'_{\min} = \frac{(1 + 4b + 3b^2)(b-1)^2 - b(1+b)^2 2(b-1)}{(b-1)^4} = \frac{(b+1)(b^2 - 4b - 1)}{(b-1)^3}$$

The last expression is negative for $b \in (1, 2 + \sqrt{5})$, and positive for $b \in (0, 1) \cup (2 + \sqrt{5}, \infty)$. This leads to the following result:

Lemma 5.1 *The minimal prize sum $R_{\min}$ need not be monotonic in the heterogeneity of the consumers.*

For our specific setup considered here the dependence between the minimal prize sum $R_{\min}$ and the heterogeneity factor b can be analyzed further. Inspection of $R'_{\min}(b)$ over the interval $(1, \infty)$ reveals that there is exactly one root as

$$b^2 - 4b - 1 = 0.$$

This follows from $b_{1,2} = 2 \pm \sqrt{5}$ with $b_1 > 1$ and $b_2 < 0$. At $b_1 = 2 + \sqrt{5}$ there is a local minimum of $R_{\min}(b)$ as Figure 1 shows. The increase of $R_{\min}(b)$ for $b > b_1$ is explained by the increase of $G^* = \frac{(1+b)^2}{4}$ in b, which overcompensates the decrease in $g(b)$ according to Lemma 4; in fact, while $g(b)$ is decreasing, $G^*(b)$ is increasing.

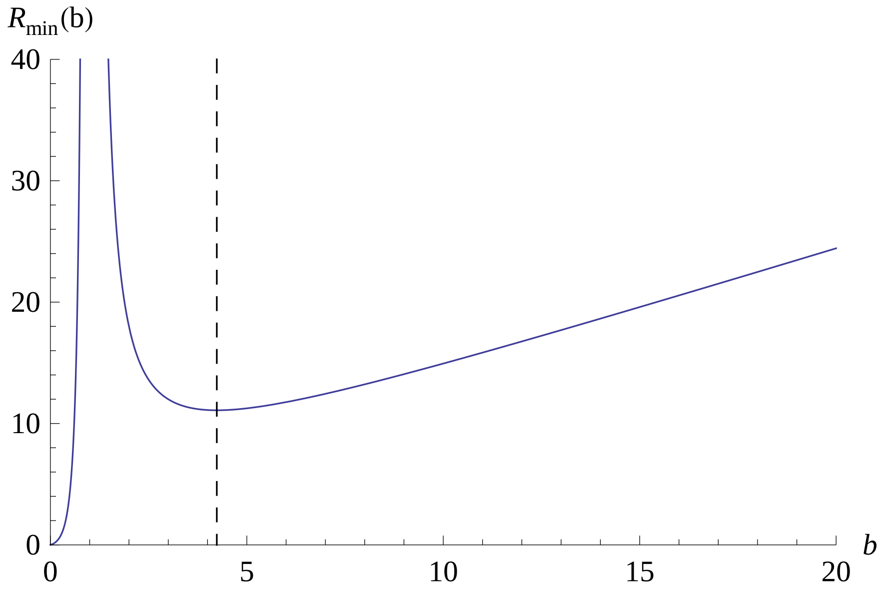

Figure 1: Minimal prize sum $R_{\min}(b)$ for different heterogeneity levels b

We are now going to show how the efficient raffle with the minimal prize sum can be determined explicitly for the specific case $b = 2$. A first consequence of this specification is that G^* must satisfy

$$G^{-1/2} + \frac{1}{2}G^{-1/2} = 1, \quad \text{and thus } G^* = \frac{9}{4}, \quad \text{and, as a consequence, } h^* = 2.$$

Based on this expression we can first derive the minimal prize sum $R_{\min}$ and then determine the corresponding weights $(\alpha_1, \alpha_2) = (1, \alpha^{max})$ such that in equilibrium $x_1^* + x_2^* = R_{\min} + \frac{9}{4}$ holds. From (M) the minimal prize sum is $R = 18$ and the corresponding $\alpha_{max} = h^* = 2$. The first order conditions then read:

$$\begin{aligned} -1 + \frac{2x_2}{(x_1 + 2x_2)^2} \cdot 18 + (x_1 + x_2 - 18)^{-1/2} &= 0 \\ -1 + \frac{2x_1}{(x_1 + 2x_2)^2} \cdot 18 + \frac{1}{2}(x_1 + x_2 - 18)^{-1/2} &= 0, \end{aligned}$$

and the solution is $x_1^* = \frac{27}{2}$ and $x_2^* = \frac{27}{4}$. Hence, we have that

$$x_1^* + x_2^* = \frac{81}{4} = 18 + \frac{9}{4} = R + G^*.$$

Alternatively, a fair lottery with prize discrimination ($p = (1, p_2)$) leads to the following utility functions:

$$(\bar{U}) \qquad \begin{aligned} \bar{u}_1(x_1, x_2) &= w_1 - x_1 + \frac{x_1}{x_1 + x_2} R + 2\,(x_1 + p_2 x_2 - R)^{1/2} \\ \bar{u}_2(x_1, x_2) &= w_2 - p_2 x_2 + \frac{x_2}{x_1 + x_2} R + (x_1 + p_2 x_2 - R)^{1/2} \end{aligned}$$

Choose, again, $R = 18$ and let $p = (1, p_2) = (1, \frac{1}{2})$ in line with Corollary 2.3, then the first-order conditions of $\bar{U}$ become:

$$\begin{aligned} -1 + \frac{x_2}{(x_1 + x_2)^2} 18 + \left(x_1 + \frac{1}{2} x_2 - 18\right)^{-1/2} &= 0 \\ -\frac{1}{2} + \frac{x_1}{(x_1 + x_2)^2} 18 + \frac{1}{4} \left(x_1 + \frac{1}{2} x_2 - 18\right)^{-1/2} &= 0, \end{aligned}$$

with the solution $x_1^* = x_2^* = \frac{27}{2}$ and hence we have:

$$x_1^* + \frac{1}{2} x_2^* = \frac{81}{4} = 18 + \frac{9}{4} = R + G^*.$$

6 Concluding Remarks

We have considered an indirect biased contribution game for public good provision and examined its (Nash) equilibrium properties. In this game contributions to the provision of a public good are elicited from consumers by offering a biased raffle with a fixed prize R. The prize R is financed out of the proceeds from raffle ticket sales and only the surplus revenue over R goes towards provision of the public good. Equilibrium exists for any number of consumers (Proposition 2.1) and is unique (Proposition 2.2). Moreover, the equilibrium outcome can always be implemented with a fair raffle whose tickets are sold at different prices to different consumers; holding a biased raffle can be understood as holding a fair raffle with a price-discriminating ticket seller.

We then concentrate on the case of two consumers and show that biased raffles can provide the optimal amount of the public good in equilibrium (Theorem 3.4) with a finite prize sum. This is in contrast to fair raffles, which always provide less than the optimal amount of the public good. This solution of the free-rider problem is achieved on a purely voluntary basis of consumer behavior: participation in the biased raffle is voluntary; no confiscatory taxes are needed on the part of the provider. All he has to do is organize a biased raffle with a fixed prize R. We determine the minimally necessary prize R to finance the optimal amount of the public good and how this depends on the bias α and heterogeneity of consumers (Proposition 4.1).

The treatment of the general case with more than two consumers should preserve the efficiency result of Theorem 3.4. However, the difficulty is the following: while in the case of two consumers both always want to buy tickets for the biased raffle, not all n consumers necessarily want to do so, if $n > 2$. The real difficulty is to determine the participation constraints of consumers in equilibrium. Franke et al. (2011) have given a complete solution of this problem in the pure contest case of a biased raffle; i.e. when raffle expenditures constitute foregone efforts of contestants, which do not generate any further utility to them. In future research we will adapt the methods developed in Franke et al. (2011) to the present case when efforts are not foregone, but also feed supply of a public good.

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