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Optimal Student Loans and Graduate Tax under Moral Hazard and Adverse Selection

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Abstract

We completely characterize the set of second-best optimal "menus"of student-loan contracts in a simple economy with risky labour-market outcomes, adverse selection, moral hazard and risk aversion. The model combines structured student loans and an elementary optimal income-tax problem à la Mirrlees. This combination can be called a graduate tax. There are two categories of second-best optima: the equal treatment and the separating allocations. The equal treatment case is obtained when the social weights of student types are close to their population frequencies; the expected utilities of different types are then equalized, conditional on the event of success on the labor market. But individuals are ex ante unequal because of differing probabilities of success, and ex post unequal, because the income tax trades o¤ incentives and insurance (redistribution). In separating optima, the talented types bear more risk than the less-talented ones; they arise only if the social weight of the talented types is sufficiently high. The second-best optimal graduate tax provides incomplete insurance because of moral hazard; it typically involves cross-subsidies; generically, it cannot be decomposed as the sum of an optimal income tax depending only on earnings, and a loan repayment, depending only on education. Therefore, optimal loan repayments must be income-contingent.

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1   Introduction

The importance of student loans for the accumulation of human capital, economic growth and welfare cannot be overestimated. In the United States, the total amount of outstanding student debt has reached \$1 trillion at the end of 2011. In Great Britain, the rise of tuition fees seems to have caused a sharp increase in average student debt[1]. With the recent economic downturn, it became clear that an increasing number of students experience difficulties to repay their loans[2]. Student loans pose interesting financial engineering and regulation problems. There are many discussions on the optimal design of these loans: for instance, the UK and Australia have a form a income-contingent repayment system, since loan repayments are based on the graduate's monthly earnings, just like income tax, and interest rates are subsidized[3]. In some continental European countries, student loans play a negligible *role* but, given the severe shortage of public funds, they could go hand in hand with a substantial raise in tuition fees, and become a new source of funds for universities[4].

There is an important econometric literature on the impact of credit constraints on university or college attendance[5]. For recent quantitative studies of alternative student-loan policies in the US, see, *e.g.*, Ionescu (2009), Lochner and Monge-Naranjo (2010). These questions are hotly debated, yet, to the best of our knowledge, the micro-economic theory of student loans is still underdeveloped[6]. In particular, we need a normative foundation for the intuition that income-contingent loans are the appropriate solution, when informational asymmetries between lenders and borrowers are involved in the allocation and design of loans. In the following, we propose a simple model of student loans, under the combined effects of risk aversion, moral hazard and adverse selection. We explore the structure of the set of second-best optimal (or interim incentive-efficient) allocations of credit to risk-averse

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[1] The average student debt is predicted to be around 50,000 pounds, on leaving the university, for those starting in 2012. for details, see *http://www.slc.co.uk/statistics*.

[2] See, for instance, *The Economist*, October 29th, 2011, p17 and P 73. In the US and in 2009, the default rate on student loans has reached 8.8%.

[3] See, *e.g.*, Barr and Johnston (2010).

[4] See, *e.g.*, Jacobs and Van der Ploeg (2006).

[5] See, for instance, Carneiro and Heckman (2002), Keane and Wolpin (2001), Stinebrickner and Stinebrickner (2008). For a survey, see Lochner and Monge-Naranjo (2011).

[6] See our discussion of the literature below.

students in an economy in which individual talents and efforts are not observed by the lender, future earnings are subject to risk and incomes can be taxed.

Our main results are the following. We consider an economy with two unobservable types of students, the talented (or low-risk) and the less-talented (high-risk) students, where risk affects earnings. The distribution of individual earnings is affected by *ex ante* and *ex post* effort choices. To fix ideas, *ex ante* effort is exerted during college years, while *ex post* effort is exerted on the labor market. An event called *success* is randomly drawn once *ex ante* efforts have been chosen. The government observes individual earnings and success, but observes neither *ex ante* nor *ex post* effort. The probability of success depends on *ex ante* effort, while the *ex post* effort choices of workers determine earnings, conditional on success. The talented types and the hard-working individuals, that is, those who choose a high *ex ante* effort, will obtain high-wage job opportunities with a higher probability than the low-effort or the less-talented types. Since *ex post* efforts are not observed, the student-loan problem is combined with an elementary form of the Mirrleesian optimal income-tax problem. We describe the set of second-best Pareto-optima by letting the social weight of types vary in the social welfare function (*i.e.*, a standard weighted average of expected utilities). The second-best optima can be implemented by a combination of structured loans and taxes.

We find that there are two broad categories of second-best optima, namely, the *separating* and *equal treatment* optima. When the social weight of types is in the neighborhood of their frequencies in the population of students, and therefore, in the vicinity of the standard utilitarian case, the second-best optimal menu of contracts exhibits a form of *pooling*, called *equal treatment*: the students' *expected utilities*, net of loan repayments, are equal, *conditional* on the random individual outcome called success. In other words, the expected utility of net earnings, as a function of individual "success" or "failure" on the labour market, should be independent of the student's unobservable type. But of course, in spite of being treated equally in this particular sense, students are *ex ante* unequal, since the talented types have a greater probability of success, and they are *ex post* unequal, since the optimal income tax trades off the provision of *insurance* (and redistribution) against that of *incentives*. This first type of solution is also characterized by *bunching* in the sense that it remains constant as a function of social weights on an interval. Hence, in the vicinity

of the standard utilitarian case, equal treatment (as defined above) is incentive compatible and second-best optimal. This is not the textbook allocation of insurance under adverse selection à la Rothschild-Stiglitz[7]. To obtain the familiar *separating* menu of loan contracts as a second-best optimum, we need to increase the social weight of the talented types relative to their natural frequency in the population.

The optimal menu of contracts exhibits *incomplete insurance*: this is mainly due to moral hazard. In the case of a separating optimum, both types are incompletely insured but the talented types bear more risk than the less-talented. The students obtain the maximal amount of income insurance compatible with high-effort incentives among the less-talented. The talented types are therefore subjected to an inefficiently large amount of income risk.

As a by-product, we find that *second-best optimal loan contracts are always income-contingent*, even in the presence of an income tax, and there are no bankruptcies. To be more precise, we find that the second-best cannot be implemented by the sum of an income tax, that depends only on observed earnings, and a loan repayment, that depends only on the quality of education (or years of education). It must be that the income tax, either depends on education, or that the loan repayments depend on income (this is why loans must be "structured"). Finally, the budget is by construction balanced (we did not explore subsidies that would be financed by means of external sources of funds) but the second-best optima typically exhibit *cross-subsidies* between types: the talented repay more and subsidise the less-talented. The second-best solution can be interpreted as a *graduate tax*, with a certain degree of progressivity.

It is well-known that microeconomic models of insurance and models of banking are formally close. Rothschild and Stiglitz's approach to screening in insurance markets has been applied to banking, albeit with adaptations (see, *e.g.*, Bester (1985)). Classic theories of credit contracts typically treat adverse selection and moral hazard separately (see Freixas and Rochet (1998)). A contribution of the present paper is to propose a study of the structure of second-best optima in a screening model à la Rothschild-Stiglitz, but with the added complication of moral hazard, since outcome probabilities also depend on hidden actions[8].

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[7]See Rothschild and Stiglitz (1976), Wilson (1977), Miyazaki (1977), Spence (1978).

[8]The structure of second-best optima in insurance markets with pure adverse selection has been studied by Crocker and Snow (1985) and Henriet and Rochet (1990).

Student loans are a very natural application for the theory of incentives or Mechanism Design under hidden actions and hidden types. The general theory of optimum (or equilibrium) contracts under moral hazard, adverse selection *and* risk aversion is known to be a very hard problem (see Arnott (1991) for comments and further references to unpublished essays on this question, see also the more recent synthesis of Boadway and Sato (2012) on optimal taxation with uncertain earnings). Solutions can be exhibited when principal and agent are both risk-neutral (see, *e.g.*, Picard (1987) and Caillaud, Guesnerie and Rey (1992); see also the discussion in Laffont and Martimort (2002, chapter 7)). In the field of optimal regulation theory, a few contributions have dealt with special cases (see, *e.g.*, McAfee and McMillan (1986), Baron and Besanko (1987), Laffont and Rochet (1998)). An extension of Rothschild and Stiglitz's insurance market model to moral hazard, and hence the study of equilibria in such an extended model, is proposed in the often quoted, but unpublished manuscript of Chassagnon and Chiappori (1997). Our model is close to that of the latter contribution, but Chassagnon and Chiappori did not study cross-subsidies between types and the set of Pareto optima. Recent work on the Principal-Agent model in the case at hand required advanced mathematical optimisation techniques (see Faynzilberg and Kumar (2000)) or used stochastic calculus, as in the asset-pricing, continuous-time finance literature (see, *e.g.*, Sung Jaeyoung (2005)). These intimidating technicalities mainly explain why we study a simple textbook model here, but it conveys, we think, the essential intuitions and ideas (and yet, some of the proofs are not straightforward). Chatterjee and Ionescu (2011) propose a quantitative analysis of a model of student loans with moral hazard, exploring the feasibility of offering insurance against college-failure risk, but they do not rely on *Mechanism Design* techniques as we do here. Finally, some contributions have been devoted to education in an optimal income-taxation model. This has been done in static and two-period settings, see, *e.g.*, Anderberg (2009), Bovenberg and Jacobs (2005), De Fraja (2002), Fleurbaey *et al.* (2002). More recently, and closer to the present contribution, Findeisen and Sachs (2012) have studied the combination of an income-tax and income-contingent loans in an optimal tax model with endogenous investment in human capital; they use a more complicated model than us, with a continuum of types, but have recourse to numerical simulations. They reach similar conclusions about the usefulness of income-contingent reimbursement.

In the following, Section 2 describes the model and studies first-best optima. Section 3 is devoted to a preliminary analysis of the asymmetric information case and of incentive constraints. Section 4 presents the main results, characterizing the set of second-best optima and discusses implementation by means of structured student loans and the income tax. Concluding remarks are in Section 5. The long proofs are presented in the appendix.

2 A Simple Model

2.1 Basic Assumptions

We consider a population of students with the same von Neumann-Morgenstern utility $u(.)$. There are two types of students, indexed by $i = 1, 2$. Each type of student choses a quality of education q_i. Assume that q_i is a nonnegative real number. Each student is *successful* or *fails*. The individuals of both types have independent probabilities of success, that depend on individual effort, denoted e_i, and on the type itself. Let $\mathbf{p}_i(e_i)$ denote the probability of success of a type i student exerting effort e_i.

The distribution of earnings takes the form of a two-stage lottery: success is drawn first with probability $\mathbf{p}_i$, and the wage w itself is the result of another lottery, given success. We assume that, in case of success, *i.e.*, with probability $\mathbf{p}_i(e_i)$, type i obtains a random wage $\widetilde{w}$ on the labor market, with a probability distribution that depends on the quality of education q. In case of failure, *i.e.*, with probability $1 - \mathbf{p}_i(e_i)$, the student gets the basic income $w_0 \leq \widetilde{w}$. In other words, the event of success on the labour market yields a wage always greater than w_0. A possible interpretation is the following: the probability of success or failure depends on factors known to the student *ex ante, i.e.*, before the beginning of work experience, (call them cognitive skills to fix ideas); this defines the student *type*. Ultimately, the wage w also depends on independent factors (say, on non-cognitive skills) that are not known by the student *ex ante,* and not observed by the government. This specific structure is a way of combining a student-loan model with an elementary form of the optimal income-taxation problem in a simple and relatively tractable manner, as will be explained below.

To keep the analysis of the model relatively simple, we assume that *ex ante* effort can take only two values, *high* or *low*, or $e_i \in \{0,1\}$ for all i. Effort has a cost $c_i e_i$ for type i, where $c_i > 0$ is a parameter. Given these assumptions, to simplify notation, we denote $P_i = \mathbf{p}_i(1)$ and $p_i = \mathbf{p}_i(0)$. We assume that effort e raises the probability of success for each type. Type 2, the "talented type", is more likely to succeed given the effort level. Formally, we assume,

Assumption 1. $0 < p_i < P_i < 1$, $i = 1, 2$, and $P_2 > P_1$; $p_2 > p_1$.

Assume now that type 1 students have chosen an education of quality q_i, and, to fix ideas, that we have $q_2 > q_1$. We assume that the distribution of wages $\widetilde{w}$, given success, takes the following form. An education of quality q_2 normally leads to a position with a wage $w(q_2)$, where $w(.)$ is an increasing function of quality, but there is some probability π that the student finds a job with a wage $w(q_1) < w(q_2)$. In other words, a type 2 student meets expectations with probability $1 - \pi$ but gets the same wage as a type 1 with probability π. A possible interpretation is that this student lacks the necessary skills to really occupy a position commanding the high wage. We have in mind that π is small and in any case, we assume that $\pi < 1/2$. Symmetrically, a student of type 1, with education q_1, gets a wage $w(q_1)$ with probability $(1-\pi)$ and "escapes her fate", that is, obtains a job with a wage $w(q_2)$, with probability π. The probability π is independent of effort e_i. We could assume that π depends on type i but that would not lead to any interesting insight. These assumptions are a simple way of creating an informational problem for the government, because any type i can end up with a high wage $w(q_2)$, a middle-range wage $w(q_1)$ or a low income w_0, albeit with different probabilities. We also could assume that the low income depends on type i, and denote it w_{0i}, with $w_{01} \neq w_{02}$, but again, that would not add anything substantial to the analysis.

The total cost of education is simply $\gamma_i q$ for quality q; the unit cost γ_i is positive and depends on type. In addition, we assume that all students are strictly risk averse and that there are diminishing returns to education on the labor market.

Assumption 2.

   a) $u(.)$ and $w(.)$ are strictly increasing, strictly concave and continuously differentiable as a function of w and q, respectively.

   b) $w(q) \geq w_0$ for all $q \geq 0$.

   c) $\gamma_1 \geq \gamma_2$

We assume that types i and efforts e_i are not observable, but that the government observes the individual's wage w, the quality of education q, which is recorded, and the event of success or failure in the first random draw. In other words, the governement knows if $w = w_0$ or if w is greater than w_0, the result of the first lottery, but doesn't know the result of the second lottery. This assumption leads to some simplifications, because it allows us to reduce the number of *ex post* incentive constraints taken into consideration, again without losing anything essential in the analysis.

   By definition, a student loan covers the cost of education. So, the amount of a loan to type i is $\gamma_i q_i$. Reimbursement is contingent on earnings and on the quality of education q_i, both observed by the government. Let (R_i, R_i', r_i) denote the repayment profile of a loan to type i. A type i student choosing education q_i earns a net income $w(q_i) - R_i$ with probability $\mathbf{p}_i(e_i)(1 - \pi)$, a net income $w(q_j) - R_i'$, $j \neq i$, with probability $\mathbf{p}_i(e_i)\pi$ and net income $w_0 - r_i$ with probability $1 - \mathbf{p}_i(e_i)$.

   We now nest an elementary income-taxation problem in this model. Assume that, once on the labour market, a successful student can decide to exert *ex post* efforts $\varepsilon_i \in \{0, 1\}$, and $\eta_i \in \{0, 1\}$. Define a *completely successful* type 2 as a student who "wins" both lotteries: this student can occupy a high-pay job. The completely successful type 2 can choose to reduce her effort *ex post* to $\eta_2 = 0$ and earn only $w(q_1)$. In doing so, this type 2 agent would change the cost of her effort by $b_{21} < 0$, or equivalently, reduce the disutility of her work by $|b_{21}|$. In other words, $|b_{21}|$ is the increase in utility due to a smaller effort on the labour market for a successful type 2 behaving as a type 1.

   Similarly, define a *completely successful* type 1 as an individual with exceptional skills, who can for that reason occupy a high-wage position (in spite of being a type 1). We assume that a completely successful type 1 can save a disutility $|b_{11}|$ by choosing to earn just the

middle wage $w(q_1)$, exerting *ex post* effort $\varepsilon_1 = 0$. Hence, $b_{11} < 0$ is the (negative) cost of a completely successful type 1 behaving as an ordinary type 1.

In addition, we assume that a type 1 (respectively, a type 2) with an unlucky draw of skills in the second lottery could earn the high wage, but at the cost of a very high disutility $b_{12} > 0$ (respectively $b_{22} > 0$). For the sake of notational elegance and simplicity, we choose efforts $\eta_1 = 0$ (respectively, $\varepsilon_2 = 0$) to be the effort choices that permit a student with unsuccessful draws of the second lottery to pose as a successful individual.

These assumptions create a Mirrleesian taxation problem, because the most lucky types can decide to reduce their effort *ex post* if their income is taxed too heavily, that is, in our case, if repayments R_2 and R'_1 are too high. The allocation of student loans is obviously constrained by adverse selection (due to unobserved types), and by a non-trivial moral hazard problem (due to unobserved effort). These difficulties are themselves combined with an optimal taxation problem, posed by the design of the contingent repayment schedule (due to unobserved *ex post* efforts). Given our assumptions, the cost of effort of a type i student can be defined as follows,

$$C_i(e_i, \varepsilon_i, \eta_i) = c_i e_i + \mathbf{p}_i(e_i)[(1-\pi)(1-\eta_i)b_{ij} + \pi(1-\varepsilon_i)b_{ii}]. \tag{1}$$

Remark that when all efforts are equal to 1, *i.e.*, if students do the "right thing", given our conventions, we have $C_i(1,1,1) = c_i$.

By definition, in this economy, an allocation, is an array $\{(e_i, q_i, R_i, R'_i, r_i, \varepsilon_i, \eta_i)\}_{i=1,2}$. A menu of contracts is an array $\{(q_i, R_i, R'_i, r_i)\}_{i=1,2}$. Let λ_i denote the frequency of type i in the student population, with $\lambda_1 + \lambda_2 = 1$. Assume that a public lending authority distributes all loans. Given the above assumptions, if (q_i, R_i, R'_i, r_i) is chosen by type i only, the *per capita* resource constraint imposes,

$$\Sigma_i \lambda_i \{\mathbf{p}_i(e_i)[(1-\pi)(\eta_i R_i + (1-\eta_i)R'_i) + \pi(\varepsilon_i R'_i + (1-\varepsilon_i)R_i)] + (1-\mathbf{p}_i(e_i))r_i - \gamma_i q_i\} \geq 0. \tag{RC}$$

If all effort variables are equal to 1, that is, if $(e_1, e_2) = (1,1)$, $(\varepsilon_1, \varepsilon_2) = (1,1)$, $(\eta_1, \eta_2) = (1,1)$, the resource constraint boils down to

$$\Sigma_i \lambda_i \{P_i(1-\pi)R_i + P_i \pi R'_i + (1-P_i)r_i - \gamma_i q_i\} \geq 0. \tag{2}$$

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2.2   First-Best Optimality

Denote

$$U_i = \pi v_i + (1 - \pi)V_i,$$

the conditional expected utility of a successful type i, where, by definition,

$$V_i = u[\eta_i(w(q_i) - R_i) + (1 - \eta_i)(w(q_j) - R_i')], \qquad (3)$$

$$v_i = u[\varepsilon_i(w(q_j) - R_i') + (1 - \varepsilon_i)(w(q_i) - R_i)],$$

where $j = i + (-1)^{i+1}$. The utility of an unsuccessful type i is denoted u_i, where by definition,

$$u_i = u(w_0 - r_i). \qquad (4)$$

The *ex ante* expected utility of a type i student is simply

$$\mathbf{p}_i(e_i)U_i + (1 - \mathbf{p}_i(e_i))u_i - C_i(e_i, \varepsilon_i, \eta_i). \qquad (5)$$

Let α_1 and α_2 be the weights of type 1 and type 2 in the welfare function. We assume $\alpha_1 + \alpha_2 = 1$ without loss of generality, and $\alpha_i > 0$ for all i. A first-best optimum can be obtained as a solution of the following problem,

$$\text{Maximize} \quad \Sigma_i \alpha_i[\mathbf{p}_i(e_i)U_i + (1 - \mathbf{p}_i(e_i))u_i - C_i(e_i, \varepsilon_i, \eta_i)] \qquad (6)$$

with respect to $\{(q_i, R_i, R_i', r_i, e_i, \varepsilon_i, \eta_i)\}_{i=1,2}$, subject to the resource constraint RC, and $(e_i, \varepsilon_i, \eta_i) \in \{0, 1\} \times \{0, 1\} \times \{0, 1\}$.

To determine the first-best effort vector $(e_i^*, \varepsilon_i^*, \eta_i^*)$, we need to compute the optimal allocation of utility in $2^4 = 16$ cases, that is, consider in turn each possible vector of efforts and compare the value of welfare for each of these combinations. The only really interesting case is when $(e_i, \varepsilon_i, \eta_i) = (1, 1, 1)$ for all i, that is, high effort e_i on the part of both types is required. It can be shown that $e = (1, 1)$ is optimal if the effort costs c_1, c_2 are small enough and if the difference $P_i - p_i$ is large enough, that is, if effort is sufficiently effective in increasing the probabilities of success in the first lottery. We assume that this is indeed the case.

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In addition, it is intuitively reasonable to assume that a type i with an unlucky drawing of the second lottery, that is, a type 2 who lacks the skills to occupy a high-salary job, or an ordinary type 1, could earn the high wage (and pretend to be completely successful), but would incur a very high *ex post* effort cost. In other words, we assume that b_{12} and b_{22} are very large. This means that $\eta_1 = 1$ and $\varepsilon_2 = 1$ are always optimal, and the unsuccessful types will never be tempted to pose as completely successful ones. We also assume that the other two parameters, b_{21} and b_{11}, are not too large in absolute value, and that the difference between $w(q_1)$ and $w(q_2)$ is sufficiently large. Under these assumptions, it is socially efficient to require that the completely successful types occupy high-pay jobs. We therefore assume that it is socially efficient to set all effort variables equal to one, for both types.

If all effort variables are equal to 1, we have $V_i = u[w(q_i) - R_i]$, $v_i = u[w(q_j) - R'_i]$, with $j = i + (-1)^{i+1}$. Define the inverse utility function

$$z(x) = u^{-1}(x). \tag{7}$$

We obtain

$$R_i = w(q_i) - z(V_i), \qquad R'_i = w(q_j) - z(v_i), \qquad \text{and} \qquad r_i = w_0 - z(u_i), \tag{8}$$

with $j \neq i$. With these definitions, the first-best optimality problem can be rewritten as follows. Eliminating R_i, R'_i and r_i from the objective and the resource constraint RC and denoting $P_i = \mathbf{p}_i(1)$, we obtain,

$$\text{Maximize} \quad \sum_i \alpha_i [P_i U_i + (1 - P_i) u_i - c_i] \tag{9}$$

with respect to $(q_i, V_i, v_i, u_i)_{i=1,2}$, subject to the resource constraint,

$$\sum_i \lambda_i \{B_i(q) - P_i E(z_i) - (1 - P_i) z(u_i)\} \geq 0, \tag{RC'}$$

where, by definition,

$$B_i(q) = P_i[(1 - \pi) w(q_i) + \pi w(q_j)] + (1 - P_i) w_0 - \gamma_i q_i, \tag{10}$$

$$E(z_i) = (1 - \pi) z(V_i) + \pi z(v_i), \tag{11}$$

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$j \neq i$ and $q = (q_1, q_2)$. The function $B_i(q)$ is the expected surplus of education for type i.

It is easy to show that the efficient choice of q must maximize $\Sigma_i \lambda_i B_i(q)$. Assuming that the solution is interior, we necessarily have $q_i = q_i^*$, where q_i^* solves,

$$[P_i(1 - \pi) + (\lambda_j/\lambda_i)P_i\pi]w'(q_i^*) = \gamma_i, \tag{12}$$

for all $i = 1, 2$, $j \neq i$, and where $w'(.)$ denotes the derivative of $w(.)$. This condition is necessary and sufficient since $w(.)$ is concave. It is easy to check that, for a sufficiently small value of π, we have $q_2^* > q_1^*$ and $B_2(q^*) > B_1(q^*)$, since we assumed $P_2 > P_1$ and $\gamma_1 \geq \gamma_2$. We would like to ignore some uninteresting corner solutions; we therefore state the following technical assumption.

Assumption 3. $q_2^* > q_1^* > 0$ and $B_i(q^*) > w_0$ for all i.

Thus, we assume that the first-best education is interior and that the efficient amount of education is profitable, on average, for both types.

Under these assumptions, the first-best problem becomes easy to solve. Note first that it is a convex programming problem, since $z(.)$ is a convex function and the objective is a linear function of utility levels V_i, v_i, and u_i. To write the first-order necessary conditions for optimality, let β denote the Lagrange multiplier of the resource constraint. We find, for $i, j = 1, 2$,

$$0 = \beta \Sigma_i \lambda_i (\partial B_i(q)/\partial q_j)$$
$$\alpha_i = \beta \lambda_i z'(V_i) = \beta \lambda_i z'(v_i) = \beta \lambda_i z'(u_i)$$

for all i. This immediately yields,

$$\frac{\alpha_i}{\beta^* \lambda_i} = z'(V_i^*) = z'(v_i^*) = z'(u_i^*), \tag{13}$$

$$\frac{z'(V_1^*)}{z'(V_2^*)} = \frac{\alpha_1/\lambda_1}{\alpha_2/\lambda_2} = \frac{z'(v_1^*)}{z'(v_2^*)} = \frac{z'(u_1^*)}{z'(u_2^*)}, \tag{14}$$

$$\beta^* = \frac{1}{\Sigma_i \lambda_i z'(u_i^*)} > 0, \tag{15}$$

and $q = q^*$. It follows that first-best optimality implies *full insurance*, that is, for all i,

$$V_i^* = v_i^* = u_i^*, \tag{16}$$

and the resource constraint must be binding. If, in addition, $\alpha_i = \lambda_i$, we get *full equality*, *i.e.*, $V_1^* = V_2^*$, $v_1^* = v_2^*$ and $u_1^* = u_2^*$. These results are standard consequences of risk aversion. Students are fully insured against labour market risk, but also against the risk of being of type 1.

The above results describe an extremely idealized situation in which any degree of redistribution is possible, and politically acceptable. Note that full insurance implies $w(q_i^*) - R_i^* = w(q_j^*) - R_i'^* = w_0 - r_i^*$. Under Assumption 2, this implies $R_i^* - r_i^* = w(q_i^*) - w_0 > 0$, or $w(q_i^*) - R_i^* - w_0 = -r_i^*$. So, if we require $w(q_i^*) - R_i^* \geq w_0$, *i.e.*, if we want individuals to receive weakly more in case of success than if they were not educated, full insurance implies $r_i^* \leq 0$. We also have $w(q_2^*) - w(q_1^*) = R_2^* - R_2'^* = R_1'^* - R_1^* > 0$.

In addition, note that there doesn't exist an unconstrained optimum with $r_i^* \geq 0$ for all i. If such an optimum did exist, then, because of full insurance, we would have $w(q_i^*) - R_i^* = w(q_j^*) - R_i'^* = w_0 - r_i^* \leq w_0$ and therefore, $\Sigma_i \lambda_i (P_i E z_i^* + (1 - P_i) z(u_i^*)) \leq w_0 < \Sigma_i \lambda_i B_i(q^*)$, a contradiction, since resources would then be wasted.

If we do not permit negative repayments (*i.e.*, if the banker is not an insurer), optimality implies $r_i^* = 0$: we find a *contingent reimbursement loan*, in the ordinary sense that no repayment is required in case of "failure".

The logic of political acceptability of the loan and transfer schemes should also lead to consideration of *individual rationality* constraints for each type. We take these constraints to be *interim* participation constraints, that is, for all i,

$$P_i[(1 - \pi)u(w(q_i) - R_i) + \pi u(w(q_j) - R_i')] + (1 - P_i)u(w_0 - r_i) - c_i \geq u(w_0). \qquad (\mathrm{IR}_i)$$

IR_i means that type i prefers to participate in the loan scheme with education over earning the basic wage w_0 for sure. In the following, we will implicitly restrict the discussion of Pareto optima to allocations satisfying IR_i for all i. In practice, this means that we do not consider values of α_2 too close to 0 or to 1.

3 Asymmetric Information and Second-Best Optima

Let us now study the case in which types are not observed by public authorities. By definition, second-best optimal (or *interim efficient*) allocations maximize a weighted sum of the student's expected utilities, subject to resource-feasibility and incentive-compatibility constraints. Students self-select in a menu of contracts proposed by the public authorities. The allocation determines *ex post* utility values (V_i, v_i, u_i) and a quality of education q_i for each type i. Although in principle, second-best effort levels could be different from their first-best counterparts, we now assume that effort levels equal to 1 are second-best optimal. This is at the same time a reasonable assumption and the only interesting case here, given that effort variables are discrete. Other cases, in which some or all of the types exert zero effort, could be studied in a very similar way as the one analyzed below. Again, high efforts will be optimal if the ratios $c_i/(P_i - p_i)$ and if the disutility costs b_{11} and b_{21} of *ex post* effort are not too large, for otherwise, the social cost of providing incentives could be higher than the benefits of effort in terms of aggregate surplus. The social benefits of effort are clearly the increased probabilities of success and the increased productivity of agents on the labour market. In this model, the social cost of providing incentives is due to the addition of the degree of inequality and risk that the latter impose, on top of the direct disutility of effort itself.

3.1 Incentive constraints

We consider first the *ex post* incentives. The students know their types. Assume that they reveal their types by choosing the quality of their education, that is $q_1 < q_2$, and type i students choose education q_i. This result will be obtained if the menu of loans $\{(q_i, R_i, R'_i, r_i)_{i=1,2}\}$ is incentive compatible. The completely successful students of type 1 will not decide to choose a job with a middle-range salary *ex post* if and only if,

$$v_1 \geq V_1 - b_{11}. \tag{ICX_1}$$

Similarly, a completely successful type 2 student will not be tempted to behave as a type 1 *ex post* if and only if,

$$V_2 \geq v_2 - b_{21}. \tag{ICX_2}$$

We will see below that these constraints must be binding at a second-best optimum. The other constraints, that is, $V_1 \geq v_1 - b_{12}$ and $v_2 \geq V_2 - b_{22}$ will always be satisfied since b_{12} and b_{22} are very large, as assumed above.

Given values of (V_i, v_i) satisfying ICX_1 and ICX_2, we now have a *generalized Principal-Agent problem* in the sense of Myerson (1982). See also Laffont and Martimort (2002). We apply the *extended revelation principle* . The constraints bearing on *ex ante* utilities and efforts are *revelation* and *obedience* constraints: the students should simultaneously self-select by choosing the right contract in the menu of loans *and* exert the right amount of effort. Since we assume that high effort is efficient, we can now write the incentive constraints as follows:

$$P_i U_i + (1 - P_i) u_i \;\; \geq \;\; P_i U_j + (1 - P_i) u_j, \qquad\qquad (\overline{IC}_i)$$

$$P_i U_i + (1 - P_i) u_i - c_i \;\; \geq \;\; p_i U_i + (1 - p_i) u_i, \qquad\qquad (MH_i)$$

$$P_i U_i + (1 - P_i) u_i - c_i \;\; \geq \;\; p_i U_j + (1 - p_i) u_j, \qquad\qquad (\underline{IC}_i)$$

where by definition,

$$U_i = \pi V_i + (1 - \pi) v_i. \qquad\qquad (EU)$$

for all $i = 1, 2$ and $j \neq i$, and recall that $p_i = \mathbf{p}_i(0)$. The *self-selection* constraint $\overline{IC}_i$ says that type i should not be tempted to pose as type j *ex ante* while exerting high effort. The *moral hazard* constraint MH_i says that type i should prefer to exert high effort over low effort and honestly revealing her (his) type *ex ante*. In addition, constraint $\underline{IC}_i$ says that type i prefers high effort to low effort and posing as type j *ex ante*.

The second-best optimality problem is the following: maximize $\Sigma_i \alpha_i [P_i U_i + (1 - P_i) u_i]$ subject to, RC', $\overline{IC}_i$, MH_i, $\underline{IC}_i$, ICX_i and EU. When the problem is posed in this form, it is immediate that the variables (V_i, v_i) appear only in ICX_1, ICX_2 and in the resource constraint RC'. We can therefore decompose the optimization problem as follows. Fix the value of U_i. Then, one can easily prove, using Kuhn-Tucker conditions, that (V_i, v_i) must be chosen in such a way that they minimize $(1 - \pi) z(V_i) + \pi z(v_i)$ subject to ICX_i, $i = 1, 2$ and EU. Intuitively, this is just the least costly way of providing the incentives ex ante, given the *ex post* effort constraints. It is also easy to check that the social planner should choose

the minimal level of risk compatible with ICX_i. It follows that both ICX_i constraints must be binding at the second-best optimum. We can state this result formally.

Result 1. The *ex post* incentive constraints ICX_i, $i = 1, 2$, must be binding at any second-best optimum, that is,

$$v_1 = V_1 - b_{11}, \qquad \text{and} \qquad v_2 = V_2 + b_{21}. \tag{17}$$

Given this result and constraint EU, the (V_i, v_i) variables can be completely eliminated from the welfare maximization problem. To simplify notation, define the new variables:

$$b_1 = -b_{11} > 0 \qquad \text{and} \qquad b_2 = b_{21} < 0. \tag{18}$$

Given this definition, we have the convenient expression of ex ante utility,

$$U_i = V_i + \pi b_i. \tag{19}$$

Eliminating (V_i, v_i) from RC', we obtain the modified resource constraint,

$$\sum_i \lambda_i \{B_i(q) - P_i[(1 - \pi)z(U_i - \pi b_i) + \pi z(U_i + (1 - \pi)b_i)] - (1 - P_i)z(u_i)\} \geq 0, \qquad (\overline{RC})$$

We now study the consequences of the other, *ex ante* incentive constraints.

It is not difficult to see that MH_i can be rewritten as, $(P_i - p_i)(U_i - u_i) \geq c_i$, or

$$U_i - u_i \geq K_i \tag{MH_i}$$

where by definition,

$$K_i = \frac{c_i}{P_i - p_i}. \tag{20}$$

Moral hazard will thus force a gap between the reward of success and that of failure. It is natural to assume that type 2 is more efficient than type 1 while exerting effort. Formally, we assume the following.

Assumption 4. $\quad K_1 \geq K_2 \geq 0.$

Adding up the $\overline{IC}_i$ constraints immediately yields

$$(P_2 - P_1)(U_2 - U_1) \geq (P_2 - P_1)(u_2 - u_1)$$

and $P_2 > P_1$ implies the property

$$U_2 - u_2 \geq U_1 - u_1. \tag{D}$$

This property has important consequences. If type 1 is insured against failure in the limited sense that $U_1 = u_1$, then, type 2 gets more in the good state, $i.e.$, $U_2 \geq u_2$. But if type 2 is insured against failure in the sense that $U_2 = u_2$, then, type 1 gets more in the bad state, $i.e.$, $u_1 \geq U_1$.

Since $\overline{IC}_i$ can be rewritten $P_i(U_i - U_j) \geq (1 - P_i)(u_j - u_i)$, we get the string of inequalities,

$$\frac{P_2}{1 - P_2}(U_2 - U_1) \geq u_1 - u_2 \geq \frac{P_1}{1 - P_1}(U_2 - U_1). \tag{$\overline{IC}$}$$

An immediate consequence is the following.

Result 2.

$\overline{IC}$ constraints imply

$$U_2 \geq U_1, \qquad \text{and} \qquad u_1 \geq u_2. \tag{21}$$

Proof: Since $P_2 > P_1$, if U_1 was strictly greater than U_2 we would get a contradiction. $\overline{IC}$ above shows that $U_2 \geq U_1$ implies $u_1 - u_2 \geq 0$.

Q.E.D.

Result 3.

a) If $\overline{IC}_1$ and $\overline{IC}_2$ are simultaneously binding, then $U_2 = U_1$ and $u_1 = u_2$: we get *equal treatment* (but not necessarily full insurance).

b) If equal treatment doesn't hold, then, either $\overline{IC}_1$ or $\overline{IC}_2$ is binding or none of them (but not both).

c) Under $\overline{IC}_1$ and $\overline{IC}_2$, then $u_1 = u_2$ if and only if $U_2 = U_1$.

Proof: The proofs of Results 2a and 2b are trivial, since $P_2 > P_1$. Result 2c follows from the fact that $u_1 = u_2$ and $\overline{IC}$ imply $U_2 - U_1 \geq 0 \geq U_2 - U_1$ and therefore $U_2 = U_1$. But we also have that $\overline{IC}$ and $U_2 = U_1$ imply $u_1 = u_2$.

Q.E.D.

We then easily find the following results.

Result 4. Under Assumption 4, if $\overline{IC}_1$, $\overline{IC}_2$ and MH_1 hold, then MH_2 is satisfied.

Proof: From $\overline{IC}_i$, we derive condition D and we obtain the following string of inequalities:

$$U_2 - u_2 \geq U_1 - u_1 \geq K_1 \geq K_2,$$

so MH_2 is satisfied.

Q.E.D.

The $\underline{IC}_i$ constraints are an added difficulty, but we can in fact ignore them, as shown by Result 5.

Result 5. Under Assumption 4,

a) if $\overline{IC}_i$, $i = 1, 2$ and MH_1 hold, then $\underline{IC}_1$ is satisfied.

b) if $\overline{IC}_2$ is satisfied, and if, in addition, $\overline{IC}_1$ and MH_1 are binding, then, $\underline{IC}_2$ is satisfied.

The proof is in the appendix.

3.2   Some useful properties derived from Kuhn-Tucker conditions

The second-best optimality problem can now be further simplified. The benevolent public banker should maximize $\Sigma_i \alpha_i (P_i U_i + (1 - P_i) u_i)$ with respect to (U_i, u_i), subject to $\overline{IC}_i$, MH_1, $\underline{IC}_2$ and $\overline{RC}$, $i = 1, 2$. To study this problem, we will also temporarily ignore (*i.e.*, relax) constraint $\underline{IC}_2$ and check at the end that it is indeed satisfied. Let β, δ, μ_1 and μ_2 be the nonnegative Lagrange multipliers of, respectively, constraints $\overline{RC}$, MH_1, $\overline{IC}_1$ and $\overline{IC}_2$. The

first-order conditions (*i.e.*, Kuhn-Tucker conditions) for the second-best optimality problem are the following. For $i, j = 1, 2$,

$$\beta \Sigma_i \lambda_i \partial B_i(q)/\partial q_j = 0; \tag{FOC0}$$

$$\alpha_1 P_1 + \mu_1 P_1 - \mu_2 P_2 + \delta = \beta \lambda_1 P_1 E z_1'; \tag{FOC1}$$

$$\alpha_2 P_2 + \mu_2 P_2 - \mu_1 P_1 = \beta \lambda_2 P_2 E z_2'; \tag{FOC2}$$

$$\alpha_1 (1 - P_1) + \mu_1 (1 - P_1) - \mu_2 (1 - P_2) - \delta = \beta \lambda_1 (1 - P_1) z'(u_1); \tag{FOC3}$$

$$\alpha_2 (1 - P_2) + \mu_2 (1 - P_2) - \mu_1 (1 - P_1) = \beta \lambda_2 (1 - P_2) z'(u_2); \tag{FOC4}$$

with, by definition,

$$E z_i' = (1 - \pi) z'(U_i - \pi b_i) + \pi z'(U_i + (1 - \pi) b_i), \tag{22}$$

and with the complementary slackness conditions, *i.e.*,

$$\delta(U_1 - u_1 - K_1) = 0, \tag{CS_1}$$

$$\beta\{\Sigma_i \lambda_i [B_i(q) - P_i E z_i - (1 - P_i) z(u_i)]\} = 0, \tag{CS_2}$$

$$\mu_i\{P_i U_i + (1 - P_i) u_i - P_i U_j - (1 - P_i) u_j\} = 0, \tag{CS_{3i}}$$

where $j = i + (-1)^{i+1}$. These conditions are necessary and sufficient for an optimum, because as noted above, the problem is convex. It follows from this that, if we find a solution in which all multipliers are nonnegative, we have found the solution.

We now prove two useful preliminary results, the proof of which relies on first-order conditions.

Result 6.

(a) $\overline{RC}$ is binding at a second-best optimum.

(b) If π is sufficiently small, and if $\overline{IC}_1$ and $\overline{IC}_2$ are binding, then, MH_1 must be binding at a second-best optimum.

The proof is in the appendix.

We then find that, if a single $\overline{IC}$ constraint is binding at the optimum, this constraint must be $\overline{IC}_1$.

Result 7. At a second-best optimum, if $\overline{IC}_2$ is binding, then, if π is small enough, $\overline{IC}_1$ must be binding too.

The proof is in the appendix.

4 Characterization of second-best optima

We will consider two broad cases in turn: *(i)*, the *pure student-loan problem*; *(ii)*, the student loan problem combined with an income-taxation problem, that we shall call the *graduate-tax problem.*

The optima in the pure student-loan problem are obtained when *ex post* efforts can be ignored. One way of doing this is to assume that the second lottery is degenerate, *i.e.*, set $\pi = 0$, and assume in addition that $b_{ij} = 0$. Under these assumptions, the social planner faces an adverse selection problem and an *ex ante* moral hazard problem, but the *ex post* effort constraints can be ignored (or the individual's potential income can be perfectly observed). Recall that, by assumption, the outcome of the first lottery, the mere success or failure, is observed by public authorities.

The *graduate tax problem* is the full problem described above, when the outcome of the second lottery is not observed[9], and $\pi > 0$. Finally, we ask if the optimal graduate tax can be decomposed as the sum of an income tax, depending on earnings only, and a loan repayment that depends only on education: the answer is no in general. To implement the optimal graduate tax, we need a loan repayment schedule that depends on education *and* income: an *income-contingent repayment schedule.*

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[9]Remark that a *pure optimal taxation problem* would be obtained by putting additional constraints on the graduate tax problem, assuming that the loan repayment schedule depends on the observed income, but not on the quality q of education. In other words, if the public authority forgets the education choices of individuals, these past choices cannot be used to discriminate among students. We would obtain this case by assuming $\pi > 0$, $b_{ij} > 0$ and adding the constraints $R'_i = R_j$, $i \neq j$, and $r_1 = r_2$, but this is not feasible in our context (this would violate IC constraints, forcing the pooling of education levels). The allocation problem described here is more flexible, permits more discrimination among individuals than the pure Mirrleesian problem.

4.1 The pure student-loan problem

To study the pure student-loan problem, we assume $\pi = 0$ and $b_{ij} = 0$, so that ICX_i constraints can be ignored. A further study of FOCs yields the following result, that we whall call the *equal treatment* result.

Proposition 1. (*Equal treatment as a second best under moral hazard and adverse selection.*) Assume $\pi = 0$ and $b_{ij} = 0$. Then, there exists an open interval $(\underline{L}_2, \overline{L}_2)$, including λ_2, such that if $\alpha_2 \in (\underline{L}_2, \overline{L}_2)$, then, the second-best optimal solution has the following properties:

$$U_1 = U_2 = \widetilde{U}, \qquad u_1 = u_2 = \widetilde{u} \quad (\text{equal} \quad \text{treatment}),$$

$$\widetilde{U} = \widetilde{u} + K_1 \quad (\text{incomplete} \quad \text{insurance}),$$

$\overline{RC}$, MH_1, $\overline{IC}_1$ and $\overline{IC}_2$ are all binding. If, in addition, $K_1 > K_2$, then, MH_2, $\underline{IC}_1$ and $\underline{IC}_2$ hold as strict inequalities. The quality of education is the first-best allocation q^*.

Proposition 1 is just a corollary of Proposition 3, proved below. We state it here first for the sake of clarity. This proposition means that if the weight α_2 of type 2 in the welfare function is close to the empirical frequency λ_2 of type 2 in the student population, and then possibly for values of α_2 that are greater than λ_2, but not too large, both $\overline{IC}_i$ constraints are binding, and the two types are treated equally in a specific sense. Since $\pi = 0$, we have $V_i = U_i$ and v_i plays no rôle. So, the result means that both types should receive the same payoff, conditional on success (or failure). It naturally follows that $w_1 - R_1 = w_2 - R_2$ and therefore $R_2 - R_1 = w_2 - w_1 > 0$: the talented types will be taxed from the difference $w_2 - w_1$ in case of success. In case of failure, both types get w_0 and repay the same amount $\widetilde{r}$. Thus, the second-best solution says that students may be completely insured against the risk of being of type 1, behind the veil of ignorance. This is of course the result of risk aversion, which translates into inequality aversion of the utilitarian social planner. In this simple case, student loans would be used for redistribution, in a way which is compatible with incentives. But the students of each given type cannot be completely insured, because of moral hazard: this is the main inefficiency here; it must be that all students bear some amount of income risk. Note that since MH_1 is the only binding moral-hazard constraint,

under the reasonable assumption that $K_1 > K_2$, the amount of risk imposed on both types is dictated by that required to induce effort among the less talented types. This is clearly too much risk for the talented types, who are also exploited *ex post* by the repayment schedule. The solution involves a form of exploitation of the talented. The revenues from high repayments are typically used to finance cross-subsidies between types, and therefore to redistribute between types.

But the types remain unequal in an *ex ante* sense, since the probability of success of the talented is greater than that of the less-talented, *i.e.*, $P_2 > P_1$. It follows that the *ex ante* expected utility of the talented, which is equal to

$$Eu_2 = \widetilde{u} + P_2 K_1,$$

is higher than the *ex ante* expected utility of the less-talented, and the difference between the two is, perhaps surprisingly, $Eu_2 - Eu_1 = K_1(P_2 - P_1) > 0$. In a certain sense, the type 2 students benefit from the relatively lower performances of the type 1 students (recall that $K_1 = c_1/(P_1 - p_1)$). If the type 2 students could be separated from type 1 students, they would bear a risk induced by the difference $U_2 - u_2 = K_2 < K_1$, but this is not possible without violating the self-selection constraints, because $\overline{IC}$ and MH_1 imply $U_2 - u_2 \geq U_1 - u_1 = K_1$. In spite of the radical form of *ex post* taxation through repayments, the talented types are better off, on average.

Given that both $\overline{IC}$ constraints are binding, the types are indifferent between the loan contracts in the menu. As usual in these cases, we assume that the type i choose the loan contract (q_i, R_i, R'_i, r_i). It may be for instance, that students have lexicographic preferences on the expected utility of income and the consumption of education, in such a way that, when indifferent between two contracts that have the same expected utility, the type 1 prefer to study less, while the type 2 prefer to study more.

What can we say about individual rationality here? The fact that $Eu_2 > Eu_1$ tells us that IR_1 implies IR_2, and IR_1 holds if $Eu_1 = \widetilde{u} + P_1 K_1 \geq w_0$. Under Assumption 3, this will be true if the surplus B generated by students is large enough (*i.e.*, $w(q_i) - w_0$ is large enough). So, typically, we should not worry about IR constraints in this problem.

To fully characterize second-best optimality in this case, we must find the second-best

optimal solution when $\alpha_2 > \overline{L}_2$. We look for a second-best allocation in which a single $\overline{IC}$ constraint is binding. Then, by Result 7, we know that $\overline{IC}_1$ is the binding $\overline{IC}$ constraint, and this can happen *only if* $\alpha_2 > \lambda_2$.

Proposition 2. (*Separating optima*) If this second-best optimum has only one binding $\overline{IC}$ constraint, then, $\overline{IC}_1$ is binding, $\overline{IC}_2$ is slack, MH_1 and $\overline{RC}$ are binding; we have $U_2 > U_1 > u_1 > u_2$ and necessarily, $\alpha_2 > \lambda_2$. The second-best solution is fully determined by the following 4 equations: $\overline{IC}_1$, MH_1 and $\overline{RC}$, expressed as equalities, and the condition,

$$\frac{\lambda_2}{\lambda_1 \alpha_2} \frac{[P_2(1 - P_\alpha)z'(U_2) - P_\alpha(1 - P_2)z'(u_2)]}{(P_2 - P_1)} = P_1 z'(U_1) + (1 - P_1)z'(u_1), \qquad \text{(F)}$$

where, by definition, $P_\alpha = \alpha_1 P_1 + \alpha_2 P_2$. The quality of education is the first-best allocation q^*.

Again, Proposition 2 is just a corollary of Proposition 4, proved below. Given that, under risk aversion, the function z is strictly convex, the Kuhn-Tucker, first-order necessary conditions for optimality are also sufficient, and the solution must be either of the form given by Proposition 1 (*equal treatment*) or of the form described by Proposition 2 (*separating allocation*). As a corollary, we get that the solution is a separating allocation of Proposition 2 if and only if $\alpha_2 > \overline{L}_2$. In a separating optimum, the talented types bear much more risk than the less-talented, but these optima will typically redistribute less resources from type 2 to type 1, and in spite of the higher risk, the expected utility of type 2 is higher than under equal treatment. Indeed, we have $U_2 > U_1 = K_1 + u_1 > u_1 > u_2$. The expected utilities are such that

$$Eu_2 = u_2 + P_2(U_2 - u_2) > u_1 + P_2(U_1 - u_1) > u_1 + P_1 K_1 = Eu_1,$$

where the first inequality is just $\overline{IC}_2$ while the second one is an immediate consequence of MH_1.

The individual rationality constraint of type 1, IR_1, could be violated when $\alpha_2 \to 1$. We therefore implicitly assume that α_2 is not too close to 1, so that $Eu_1 > u(w_0)$.

4.2   The optimal graduate tax

To study the optimal *graduate tax problem*, we now assume that $\pi > 0$, $b_1 > 0$ and $b_2 < 0$, so that the ICX_i constraints can no longer be ignored. A further study of FOCs yields the following result, that may still be called the *equal treatment* result. The statement is almost the same as that of Proposition 1, but the consequences are very different, because of binding ICX_i constraints.

Proposition 3. (*Equal treatment as a second best under adverse selection, ex ante and ex post moral hazard.*) Assume $\pi > 0$, $b_1 > 0$ and $b_2 < 0$. Then, there exists $\bar{\pi} > 0$ and an open interval $(\underline{\lambda}_2, \bar{\lambda}_2)$, including λ_2, such that if $\alpha_2 \in (\underline{\lambda}_2, \bar{\lambda}_2)$ and $\pi < \bar{\pi}$, then, the second-best optimal solution has the following properties:

$$U_1 = U_2 = U(\pi), \qquad u_1 = u_2 = u(\pi) \quad \text{(equal   treatment)},$$

$$U(\pi) = u(\pi) + K_1 \quad \text{(incomplete   insurance)},$$

$\overline{RC}$, MH_1, $\overline{IC}_1$, $\overline{IC}_2$, ICX_1 and ICX_2 are all binding. If, in addition, $K_1 > K_2$, then, MH_2, $\underline{IC}_1$ and $\underline{IC}_2$ hold as strict inequalities. The quality of education is the first-best value q^*.

For proof, see the appendix.

At this point, several remarks can be made. We first find a bunching property: when α is close to λ, the optimal solution doesn't depend on α.

Corollary 1. (*Bunching with respect to α*) The second-best optimal solution of Proposition 3 is independent of α, when α_2 is small enough, *i.e.*, if $\alpha_2 < \overline{L}_2$, we have,

$$\frac{\partial}{\partial \alpha_2}(U_2, u_2, U_1, u_1) = 0.$$

Proof: The second-best allocation is the solution of a system of four equations with four unknowns: *(i)*, $U_1 = U_2$; *(ii)*, $u_1 = u_2$; *(iii)*, $U_1 = u_1 + K_1$; and *(iv)*, given these constraints, $\overline{RC}$ pins down $u_1 = u(\pi)$. None of these equations involve α.
Q.E.D.

Next, we need to list the *ex post* utilities associated with the solution, and derive the *optimal graduate tax schedule*. Given that $U_1 = U_2$, we have $V_1 + \pi b_1 = V_2 + \pi b_2$ and therefore,

$$V_2 - V_1 = \pi(b_1 - b_2) > 0,$$

(recall that $b_1 > 0$ and $b_2 < 0$). Next, using ICX_i constraints, we derive,

$$v_1 = V_1 + b_1, \qquad \text{and} \qquad v_2 = V_2 + b_2.$$

and

$$v_2 - v_1 = -(1 - \pi)(b_1 - b_2) < 0.$$

Corollary 2. (*Optimal graduate tax*) If $\alpha_2 < \bar{\lambda}_2$ and if π is sufficiently small, the optimal graduate tax has the following properties

$$
\begin{aligned}
u_1 &= u(\pi), \\
u_2 &= u(\pi), \\
V_1 &= u(\pi) + K_1 - \pi b_1, \\
V_2 &= u(\pi) + K_1 - \pi b_2, \\
v_1 &= u(\pi) + K_1 + (1 - \pi)b_1, \\
v_2 &= u(\pi) + K_1 + (1 - \pi)b_2.
\end{aligned}
\tag{23}
$$

It follows that,

(a) $v_1 > V_1$ and $V_2 > v_2$;

(b) if $K_1 > \pi b_1$, we have $V_1 > u_1$;

(c) if $K_1 > (1 - \pi)|b_2|$, we have $v_2 > u_2$;

(d) if $|b_1| > |b_2|$, then $v_1 > V_2$;

(e) if π is sufficiently small, we have $V_1 > v_2$;

(f) $v_1 > V_2 > V_1 > v_2$ if and only if

$$\frac{\pi}{1 - \pi} < \min\left\{\frac{|b_2|}{|b_1|}, \frac{|b_1|}{|b_2|}\right\}.$$

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The proof of this corollary is easy. It follows from this result that, in practice, for sufficiently small values of π, the successful self-made man earns more after tax than the successful Cambridge graduate, since $v_1 > V_2$, and the unsuccessful Cambridge graduate is taxed more than the ordinary student, since $V_1 > v_2$. These consequences of the optimal graduate tax become explicit when we derive the implications in terms of income-contingent repayment (or education-contingent income tax). This means that in the general case, the equal treatment property is achieved by means of an unequal *ex post* treatment of graduates.

Corollary 3. (*Optimal graduate tax repayment schedule*) If $\alpha_2 < \bar{\lambda}_2$ and if π is sufficiently small, the optimal graduate tax has the following properties,

       (a) $u_1 = u_2$ implies $r_1 = r_2$ (*equal treatment in case of failure*);

       (b) $V_2 > V_1$ implies $R_2 - R_1 < w_2 - w_1$ (*the talented cannot be fully exploited*);

       (c) $v_1 > V_2$ implies $R_2 > R_1'$ (*self-made (wo)men repay less*);

       (d) $V_1 > v_2$ implies $R_2' > R_1$ (*unsuccessful high-level graduates repay more*);

Those who studied longer repay more conditional on pre-tax earnings w_i. Remark that in general, this repayment/graduate-tax schedule cannot be implemented by means of the addition of an ordinary income tax, that would depend only on w_i, and of a standard loan repayment, that would depend only on education q_i. In other words, we would like to choose an income tax schedule $T_i = T(w_i)$ and a loan repayment schedule $A_i = A(q_i)$, $i = 1, 2$ such that for any given 4 numbers (R_1, R_2, R_1', R_2'), we have,

$$
\begin{aligned}
R_1 &= T_1 + A_1, \\
R_2 &= T_2 + A_2, \\
R_1' &= T_2 + A_1, \\
R_2' &= T_1 + A_2.
\end{aligned}
\tag{24}
$$

It is not difficult to check that this linear system's determinant is zero. But this system also implies,

$$
\begin{aligned}
R_2 - R_1' &= A_2 - A_1 = R_2' - R_1, \\
R_2 - R_2' &= T_2 - T_1 = R_1' - R_1.
\end{aligned}
$$

For a generic (R_1, R_2, R'_1, R'_2), these relations will not be true. Indeed, it is easy to see that,

$$(R_2 - R'_2) - (R'_1 - R_1) = z(v_1) - z(V_1) + z(v_2) - z(V_2), \tag{25}$$

which is generically different from zero. The above expression would be exactly zero if, for instance, z was linear and $b_1 = -b_2$. But in general, it will not vanish. We can state the following corollary.

Corollary 4. (*Justification of income-contingent loan repayments*) It is generically impossible to decompose the second-best transfers as the sum of an income tax, depending only on earnings, and student-loan repayments, depending only on education. It must be that, either the student-loan repayments are income-contingent, or the income tax is education-contingent.

For the sake of completeness, the remaining question is to find the second-best optimal solutions when $\alpha_2 > \overline{\lambda}_2$. We look for a second-best allocation in which a single $\overline{IC}$ constraint is binding. Then, by Result 7, we know that $\overline{IC}_1$ must be the binding constraint, and this can happen *only if* $\alpha_2 > \lambda_2$.

Proposition 4. If a second-best optimum has only one binding $\overline{IC}$ constraint, then, there exists a $\overline{\pi} > 0$, such that for all $0 \leq \pi < \overline{\pi}$, constraint $\overline{IC}_1$ is binding; $\overline{IC}_2$ is slack; constraints MH_1, $\overline{RC}$, ICX_1 and ICX_2 are binding; we have $U_2 > U_1 > u_1 > u_2$ and necessarily, $\alpha_2 > \lambda_2$. The second-best solution is fully determined by the following 4 equations: $\overline{IC}_1$, MH_1 and $\overline{RC}$, expressed as equalities, and the condition,

$$\frac{\lambda_2}{\lambda_1 \alpha_2} \frac{[P_2(1 - P_\alpha)Ez'_2 - P_\alpha(1 - P_2)z'(u_2)]}{(P_2 - P_1)} = P_1 Ez'_1 + (1 - P_1)z'(u_1) \tag{H}$$

For proof see the appendix.

We have completely characterized the second-best optima under adverse selection, *ex ante* and *ex post* moral hazard, in the optimal graduate tax problem. Again, since the Kuhn-Tucker conditions are necessary and sufficient, we have shown that when $\alpha_2 > \overline{\lambda}_2$, the solution is that described by Proposition 4.

Proposition 3 shows that if the social weights of types are close to their true frequency in the population, the solution exhibits *equal treatment* in the limited sense that $U_1 = U_2$

and $u_1 = u_2$ and *incomplete insurance*, since $U_1 = u_1 + K_1$. The solution typically involves *ex post inequality* since $v_1 > V_2 > V_1 > v_2$. The solution also entails *ex ante inequality* because at a second-best optimum,

$$Eu_2 - Eu_1 = (P_2 - P_1)K_1 > 0.$$

Both types obtain the same expected payment in the event of "success" as well as in the event of "failure" and incomplete insurance takes care of effort incentives, but the talented have a higher probability of success.

Proposition 4 shows that the second-best optimum is a *separating* allocation à la Rothschild-Stiglitz when the social weight of the talented types is sufficiently higher than their frequency in the population, since in this case, $U_2 > U_1 > u_1 > u_2$. In other words, to get a *separating optimum*, the social planner must be willing to markedly favor the highly productive types. It is still true that types are *ex ante* unequal since, using IC and MH constraints, exactly as in the above subsection devoted to the pure student-loan problem, we find that

$$Eu_2 = u_2 + P_2(U_2 - u_2) > u_1 + P_1K_1 = Eu_1.$$

But the talented types are now less well insured in case of failure, since $u_1 > u_2$.

These allocations are trivially not first-best efficient, since first-best efficiency requires full insurance. The solutions potentially entail a limited form of *exploitation of the talented*, by means of cross-subsidies between types, since the less-talented are also producing less surplus *per capita*. In the case described by Proposition 3, this subsidy from the talented survives as a price paid to solve the incentive problem, in particular since $U_1 = U_2$. It is only when the social welfare function sufficiently favors the talented that the incentive problem is solved by means of *screening*, imposing a higher level of risk (and return) on the most productive agents.

5 Conclusion

We have studied optimal student-loan contracts in a simple private information economy with two unobservable types of students. Types differ in the probability distributions of individual labour-market outcomes (adverse selection). Future earnings are risky. Students are risk-averse and choose an *ex ante* effort variable, affecting the probabilities of success, that is not observed by the lender (moral hazard). This poses an optimal insurance problem. Students can also reduce their effort *ex post* and thus reduce their earnings below potential. This poses an additional optimal-taxation problem à la Mirrlees. We completely describe the set of second-best optimal (or interim efficient) incentive-compatible menus of loan contracts. There are two types of optima: the *separating* and *equal treatment* allocations. Equal treatment arises when the social weights of types are in the neighborhood of their frequencies in the student population. In this case, the *expected utility* of students of different types are equalized, conditional on the student's observable success. However, students are *ex ante* unequal since they differ in their probability of success on the labour market. In addition, students are *ex post* unequal since the second-best allocation trades off incentives and insurance-redistribution motives. This type of allocation is different from the familiar menus of separating contracts in screening models à la Rothschild-Stiglitz. The separating menus, in which the talented students bear more risk than the less-talented ones, appear only if the social weight of talented types is sufficiently greater than the latter type's frequency. In both cases, the optimal menus of contracts exhibit incomplete insurance, as a consequence of moral hazard; they typically involve cross-subsidies in favour of the less-talented. The less-talented obtain the maximal amount of insurance, compatible with effort incentives. Optimal student loans are always income-contingent, even in the presence of an income tax. In other words, the second-best transfers cannot be decomposed as the sum of an income tax, depending only on earnings, and a loan repayment, depending only on education. It must be that the optimal loan-repayments are income contingent, or that the income tax is itself education-contingent. The student-loan contracts can be interpreted as a form of *graduate tax*.

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7   Appendix: Proofs

Proof of Result 5:

(a) If $\overline{IC}_1$ holds, then,

$$(1 - P_1)(u_1 - u_2) \geq P_1(U_2 - U_1),$$

and since under IC, $U_2 - U_1 \geq 0$, and we assumed $P_1 > p_1$, we also have $(1 - p_1)(u_1 - u_2) \geq p_1(U_2 - U_1)$. But MH_1 implies $c_1 - (P_1 - p_1)(U_1 - u_1) \leq 0$. This trivially implies

$$(1 - p_1)(u_1 - u_2) \geq p_1(U_2 - U_1) + c_1 - (P_1 - p_1)(U_1 - u_1), \tag{26}$$

and rearranging terms we get the equivalent inequality,

$$P_1U_1 + (1 - P_1)u_1 - c_1 \geq p_1U_2 + (1 - p_1)u_2,$$

but this is exactly $\underline{IC}_1$.

(b) Given that MH_1 is binding, $\underline{IC}_2$ can be expressed as follows,

$$P_2U_2 + (1 - P_2)u_2 - c_2 \geq p_2(u_1 + K_1) + (1 - p_2)u_1 = u_1 + p_2K_1. \qquad (\underline{IC}_2 + MH_1)$$

Combining $\overline{IC}_1$ and MH_1, holding as equalities, we easily obtain,

$$u_1 + P_1K_1 = P_1U_2 + (1 - P_1)u_2. \qquad (\overline{IC}_1 + MH_1)$$

Substituting the value of u_1 derived from $(\overline{IC}_1 + MH_1)$ in $(\underline{IC}_2 + MH_1)$ yields, after some rearrangement of terms,

$$(P_2 - P_1)(U_2 - u_2) \geq c_2 + (p_2 - P_1)K_1.$$

Dividing both sides by $(P_2 - p_2) > 0$ and rearranging terms, we obtain,

$$(U_2 - u_2)\left[\frac{P_2 - P_1}{P_2 - p_2}\right] \geq K_2 + \frac{p_2 - P_1}{P_2 - p_2}K_1.$$

From condition D and MH_1 we know that $U_2 - u_2 \geq U_1 - u_1 = K_1$. In addition, $(P_2 - P_1)/(P_2 - p_2) = 1 + (p_2 - P_1)/(P_2 - p_2) > 0$. Hence, the following string of inequalities:

$$(U_2 - u_2)\left[\frac{P_2 - P_1}{P_2 - p_2}\right] \geq K_1\left[1 + \frac{p_2 - P_1}{P_2 - p_2}\right] \geq K_2 + \frac{p_2 - P_1}{P_2 - p_2}K_1,$$

since, by Assumption 4, $K_1 \geq K_2$. This shows that $\underline{IC_2}$ is satisfied when $\overline{IC_2}$ holds and when $\overline{IC_1}$ and MH_1 are equalities.

Q.E.D.

Proof of Result 6:

(a) Adding equations FOC1 to FOC4, we easily find,

$$1/\beta = \Sigma_i \lambda_i \{ P_i E z_i' + (1 - P_i) z'(u_i) \} > 0,$$

so that $\beta > 0$, since $z'(.) > 0$. Hence, by CS_2, it follows that $\overline{RC}$ is binding.

(b) If $\overline{IC_1}$ and $\overline{IC_2}$ are binding, then $U_1 = U_2 = U$ and $u_1 = u_2 = u$. Suppose that MH_1 is slack, *i.e.*, $U_1 > u_1 + K_1$, at the second-best optimum, then by CS_1, we have $\delta = 0$. With $\delta = 0$, FOC1 and FOC3 form a linear system in (μ_1, μ_2); that is,

$$\mu_1 P_1 - \mu_2 P_2 = P_1 [\beta \lambda_1 E z_1' - \alpha_1];$$
$$\mu_1 (1 - P_1) - \mu_2 (1 - P_2) = (1 - P_1)[\beta \lambda_1 z'(u) - \alpha_1];$$

This system has a nonzero determinant, equal to $P_2 - P_1 > 0$, and a unique solution (μ_1^*, μ_2^*). It is easy to check that,

$$\mu_2^* = \frac{P_1(1 - P_1)}{P_2 - P_1} \beta \lambda_1 \left(z'(u) - E z_1' \right).$$

But,

$$E z_1' = (1 - \pi) z'(U_1 - \pi b_1) + \pi z'(U_1 + (1 - \pi) b_1).$$

Since $b_1 > 0$ and since MH_1 implies $U > u + K_1$, it follows that for a sufficiently small value of $\pi > 0$, by continuity, $E z_1'$ is close to $z'(U)$ and thus, $z'(u) - E z_1' < 0$. We conclude that $\mu_2^* < 0$. This is a violation of Kuhn-Tucker conditions, since all multipliers must be non-negative. We have found a contradiction.

Q.E.D.

Proof of Result 7:

If $\overline{IC}_2$ is binding, and $\overline{IC}_1$ is slack, then, by CS_{31}, we have $\mu_1 = 0$. Using FOC2 and FOC4, we easily obtain, $\beta\lambda_2 z'(u_2) = \alpha_2 + \mu_2 = \beta\lambda_2 E z'_2$ and therefore, $z'(u_2) = E z'_2$. If π is small, $E z'_2$ is close to $z'(U_2)$ and therefore, by continuity of z', u_2 is close to U_2. But $\overline{IC}$ and MH_1 together imply $U_2 \geq U_1 = u_1 + K_1 > u_1 \geq u_2$, so that $U_2 - u_2 > K_1$. It follows that we have a contradiction if π is small enough.

Q.E.D.

Proof of Proposition 3:

The FOCs can be rewritten,

$$\mu_1 P_1 - \mu_2 P_2 + \delta = P_1[\beta\lambda_1 E z'_1 - \alpha_1]; \qquad \text{(FOC1b)}$$

$$\mu_2 P_2 - \mu_1 P_1 = P_2[\beta\lambda_2 E z'_2 - \alpha_2]; \qquad \text{(FOC2b)}$$

$$\mu_1(1 - P_1) - \mu_2(1 - P_2) - \delta = (1 - P_1)[\beta\lambda_1 z'(u_1) - \alpha_1]; \qquad \text{(FOC3b)}$$

$$\mu_2(1 - P_2) - \mu_1(1 - P_1) = (1 - P_2)[\beta\lambda_2 z'(u_2) - \alpha_2]. \qquad \text{(FOC4b)}$$

Our optimum candidate exhibits equal treatment, $U_1 = U_2 = U(\pi)$, $u_1 = u_2 = u(\pi)$, and both $\overline{IC}$ constraints are binding. By Result 6 above, MH_1 must then be binding too. This imposes $U(\pi) = u(\pi) + K_1$. Adding the four FOCb equations easily yields, $\beta = \beta(\pi)$, satisfying the relation,

$$\beta(\pi)[P_1\lambda_1 E z'_1 + P_2\lambda_2 E z'_2 + (1 - P_\lambda)z'(u(\pi))] = 1, \qquad (\Sigma FOC)$$

where, by definition, $P_\lambda = P_1\lambda_1 + P_2\lambda_2$, and where,

$$E z'_i = (1 - \pi)z'(U(\pi) - \pi b_i) + \pi z'(U(\pi) + (1 - \pi)b_i), \qquad (27)$$

for $i = 1, 2$. We know that $\overline{RC}$ is binding and it follows that $(U(\pi), u(\pi))$ is fully determined by the intersection of $\overline{RC}$ and MH_1. It is easy to check that the solution $(U(\pi), u(\pi))$ is a continuous function of π for $\pi \geq 0$. We also find that FOC0 implies $q = q^*$, since $\beta(\pi) > 0$.

We must check that the associated multipliers $\mu_i(\pi)$ are nonnegative. FOC2b and FOC4b provide us with a linear system of equations for (μ_1, μ_2). The determinant of this

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system is $P_2 - P_1 > 0$, so that there is a unique solution $(\mu_1(\pi), \mu_2(\pi))$. We easily derive,

$$\mu_1(\pi) = \frac{P_2(1 - P_2)\beta(\pi)\lambda_2}{(P_2 - P_1)}[Ez_2' - z'(u(\pi))], \tag{28}$$

$$\mu_2(\pi) = \frac{\lambda_2 P_2(1 - P_1)}{(P_2 - P_1)}\left[\beta(\pi)Ez_2' - \frac{\alpha_2}{\lambda_2}\right] + \frac{\lambda_2 P_1(1 - P_2)}{(P_2 - P_1)}\left[\frac{\alpha_2}{\lambda_2} - \beta(\pi)z'(u(\pi))\right]. \tag{29}$$

Remark first that, by continuity of z' and $u(\pi)$, when $\pi \to 0$, we have $Ez_i' \to z'(U(0))$ for all i, and $z'(U(0)) > z'(u(0))$, since z' is strictly increasing and since MH_1 imposes $U(0) = K_1 + u(0)$. It follows that $Ez_i' > z'(u(\pi))$ for sufficiently small values of π. Using these inequalities, expression ΣFOC above, and the fact that $z' > 0$, for sufficiently small π, we derive

$$1 > \beta(\pi)z'(u(\pi)), \qquad \text{and} \qquad 1 < \beta(\pi)\max\{Ez_1', Ez_2'\}. \tag{30}$$

When π is small, then $Ez_1' \simeq Ez_2'$ and thus $1 < \beta(\pi)\min\{Ez_1', Ez_2'\}$. From these remarks, we immediately obtain $\mu_1(\pi) > 0$ for sufficiently small π. Note that this property is true for all values of α_2 in $(0, 1)$, since $\mu_1(\pi)$ doesn't depend on α_2. Given the inequalities derived above, for any $\alpha_2 \leq \lambda_2$, we also have $\beta(\pi)Ez_2' - (\alpha_2/\lambda_2) > 0$, and therefore, using $P_2 > P_1$, we find

$$\mu_2(\pi) > \frac{\lambda_2 P_1(1 - P_2)}{(P_2 - P_1)}\left[\beta(\pi)Ez_2' - \frac{\alpha_2}{\lambda_2} + \frac{\alpha_2}{\lambda_2} - \beta(\pi)z'(u(\pi))\right]$$

$$= \frac{\lambda_2\beta(\pi)P_1(1 - P_2)}{(P_2 - P_1)}[Ez_2' - z'(u(\pi))] > 0. \tag{31}$$

In particular, if $\lambda_2 = \alpha_2$, then $\mu_2(\pi) > 0$. By continuity, there exists an interval of values of α_2, including λ_2, such that $\mu_2(\pi)$ is positive. By Result 5, we know that both $\underline{IC_i}$ constraints are satisfied. By Result 4, we know that MH_2 is also satisfied.

Finally, we must check that the multiplier $\delta = \delta(\pi) > 0$. Adding FOC3b and FOC4b, we derive an expression for $\delta(\pi)$, that is,

$$\delta(\pi) = 1 - P_\alpha - \beta(\pi)(1 - P_\lambda)z'(u(\pi)), \tag{32}$$

where by definition, $P_\alpha = \alpha_1 P_1 + \alpha_2 P_2$. This can be rewritten

$$\frac{\delta(\pi)}{(1 - P_\lambda)} = \frac{(1 - P_\alpha)}{(1 - P_\lambda)} - \beta(\pi)z'(u(\pi)). \tag{33}$$

Since for sufficiently small π, we have $1 > \beta(\pi)z'(u(\pi))$, this means that there is an interval of values of α_2, around λ_2, such that $\delta(\pi) > 0$. In addition, $\delta(\pi) > 0$ for all values of $\alpha_2 \leq \lambda_2$

since $\alpha_2 \to 0$ implies $P_\alpha \to P_1 < P_\lambda$. We conclude that there exists an interval $(\underline{\lambda}_2, \overline{\lambda}_2)$ of values of α_2, including λ_2, such that all Lagrange multiplers are positive, provided that π is small enough. Therefore, we have found the optimal solution for these values of α_2 and π. Q.E.D.

Proof of Proposition 4:

If a second-best optimum has only one binding $\overline{IC}$ constraint, then, by Result 7, $\overline{IC}_1$ must be binding and $\overline{IC}_2$ is slack. Hence, $\mu_2 = 0$. By Results 4 and 5, we can neglect MH_2 and $\underline{IC}_i$ constraints. Adding equations FOC1 to FOC4, we easily obtain, $1/\beta = \Sigma_i \lambda_i (P_i E z_i' + (1 - P_i) z'(u_i)) > 0$. Hence, $\overline{RC}$ is binding. Suppose now that MH_1 is slack. Then, $\delta = 0$. From FOC1b and FOC3b, we easily derive,

$$\beta \lambda_1 z'(u_1) = \alpha_1 + \mu_1 = \beta \lambda_1 E z_1'.$$

Hence, $z'(u_1) = E z_1'$. If π is sufficiently small, by continuity, this implies $E z_1' \simeq U_1$ and $U_1 - u_1 < K_1$, a contradiction, since MH_1 imposes $u_1 < U_1$. Thus, MH_1 is binding and $U_1 = u_1 + K_1$. From FOC2b, and the requirement that $\mu_1 \geq 0$, we derive

$$\mu_1 = \frac{P_2}{P_1} [\alpha_2 - \beta \lambda_2 E z_2'] \geq 0. \tag{A6}$$

From FOC1b, we derive,

$$\mu_1 + \frac{\delta}{P_1} = \beta \lambda_1 E z_1' - \alpha_1 \geq 0. \tag{B6}$$

Combining A6 and B6, we obtain,

$$\frac{\alpha_2}{\lambda_2} \geq \beta E z_2' \qquad \text{and} \qquad \beta E z_1' \geq \frac{\alpha_1}{\lambda_1}. \tag{C6}$$

We must compare $E z_2'$ and $E z_1'$. It is easy to see that $E z_2' > E z_1'$ if and only if,

$$(1 - \pi)[z'(U_2 - \pi b_2) - z'(U_1 - \pi b_1)] > \pi[z'(U_1 + (1 - \pi)b_1) - z'(U_2 + (1 - \pi)b_2)]. \tag{D6}$$

Since $U_2 > U_1$ and $b_1 > 0 > b_2$, we have $z'(U_2 - \pi b_2) > z'(U_1 - \pi b_1)$, and (D6) is obviously true, by continuity, for sufficiently small π. It follows from (C6) and (D6) that we must have $\alpha_2(1 - \lambda_2) > (1 - \alpha_2)\lambda_2$, or $\alpha_2 > \lambda_2$.

Combining A6 and B6 again, assuming that $\delta > 0$, we obtain

$$\delta = \lambda_1 P_1[\beta E z_1' - (\alpha_1/\lambda_1)] + P_2 \lambda_2[\beta E z_2' - (\alpha_2/\lambda_2)] > 0,$$

which implies,

$$\Sigma_i \lambda_i P_i E z_i' > \frac{P_\alpha}{\beta}. \tag{E6}$$

The allocation U_1, U_2, u_1, u_2 is determined by a system of four equations, the first three are obviously $\overline{IC}_1$, MH_1 and $\overline{RC}$, expressed as equalities. To find the fourth equation, we eliminate Lagrange multipliers from FOC1b, FOC3b and FOC4b. More precisely, adding FOC1b and FOC3b yields

$$\mu_1 = \lambda_1 \beta[P_1 E z_1' + (1 - P_1)z'(u_1)] - 1 + \alpha_2. \tag{F6}$$

On the other hand, substituting A6 yields,

$$\frac{P_2}{P_1}[\alpha_2 - \beta \lambda_2 E z_2'] = \lambda_1 \beta[P_1 E z_1' + (1 - P_1)z'(u_1)] - (1 - \alpha_2),$$

or equivalently,

$$\frac{P_\alpha}{\beta} = \lambda_2 P_2 E z_2' + \lambda_1 P_1[P_1 E z_1' + (1 - P_1)z'(u_1)]. \tag{G6}$$

Note that if G6 holds, since $E z_1' > z'(\widetilde{u}_1)$, then, necessarily, (E6) holds: this confirms that $\delta > 0$. Substituting the expression for β, derived above, in (G6), and rearranging terms, yields the fourth equation that we need to solve the problem,

$$\frac{\lambda_2}{\lambda_1 \alpha_2} \frac{[P_2(1 - P_\alpha)E z_2' - P_\alpha(1 - P_2)z'(u_2)]}{(P_2 - P_1)} = P_1 E z_1' + (1 - P_1)z'(u_1) \tag{H}$$

The second-best optimum (U_1, u_1, U_2, u_2) is fully determined by $\overline{RC}$, MH_1, $\overline{IC}_1$ expressed as equalities and condition H. The condition $\mu_1 \geq 0$ yields a lower bound on the values of α_2 that can be derived from (A6), that is, equivalently, from

$$\frac{\alpha_2}{\lambda_2} \geq \beta E z_2'.$$

(Note that $\beta E z_2' > 1$, so that $\alpha_2 > \lambda_2$ is required for this type of solution to be optimal).

Q.E.D.