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Abstract

This paper evaluates direct and structural discrimination as a means of increasing efforts in the most widely studied contests. We establish that a designer who maximizes efforts subject to a balanced-budget constraint prefers dual discrimination, namely, change of the contestants' prize valuations as well as bias of the impact of their efforts. Optimal twofold discrimination is often superior to any single mode of discrimination under any logit CSF. Our main result establishes that, surprisingly, from the designer's point of view, dual discrimination can yield the maximal possible efforts when it is applied to the prototypical simple logit CSF. In this case it yields almost the highest valuation of the contested prize.

JEL-Code: D700, D720, D740, D780.

Keywords: contest design, balanced-budget-constraint, direct discrimination, structural discrimination, extreme dual discrimination, contest success function.

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1. Introduction

In the vast contest literature that has numerous applications (internal labor market tournaments, promotional competitions, R&D races, rent-seeking, political and public policy competitions, litigation and sports), the CSFs proposed by Tullock (1980) are most commonly assumed as the contest success function (CSF), see Konrad (2009) and references therein. In two-player contests, for $x_1 \geq 0$, $x_2 \geq 0$, $\alpha > 0$ and $\delta > 0$, these logit functions take the form:

(1) $$p_1(x_1, x_2) = \begin{cases} \dfrac{x_1^{\alpha}}{x_1^{\alpha} + (\delta x_2)^{\alpha}} & \text{if} \quad x_1 + x_2 > 0 \\ 0.5 & \text{if} \quad x_1 = x_2 = 0 \end{cases}$$

Usually, x_1 and x_2 are interpreted as the contestants' efforts. However, p_1 has two possible interpretations. It can be interpreted as contestant 1's winning probability of an indivisible prize or as his share in a divisible prize. In turn, the winning probability of contestant 2 or his share in the prize is equal to $p_2 = 1 - p_1$. Henceforth, we use the second interpretation, as in Corchon and Dahm (2010), Franke et al. (2013), Lee and Lee (2012), Warneryd (1998). Nevertheless, although under this interpretation there is no uncertainty in the model and the contestants compete on the certain shares of a divisible prize, we preserve the terms "contest" and "CSF". The exponent α is a parameter that represents the effect of a real unit of investment on the prize share of a contestant while the asymmetry between the impact of the contestants' efforts is captured by the parameter δ, $\delta > 0$. One reason for the popularity of this CSF is that it has appealing axiomatization, see Skaperdas (1996), Clark and Riis (1998), Jia (2008, 2010), Corchon and Dahm (2010), Hirshleifer and Riley (1992), Fullerton and McAfee (1999), Baye and Hoppe (2003)[1]. The special attention given to the simple logit CSF, where $\alpha = 1$ and $\delta = 1$, can be justified, as recently argued by Franke et al. (2013), on the grounds that it lends itself to a very appealing competitive-market interpretation.

In our setting, the exponent α is viewed as a given parameter and it is assumed that $0 < \alpha \leq 2$, which guarantees, as is well known, see Konrad (2009) and references therein, that the contest game has a unique pure-strategy equilibrium. However, we do enable the contest designer to control the parameter δ, as first

[1] Munster (2009) has recently generalized the axiomatic approach to group CSFs.

suggested in Lien (1986, 1990) and later by Clark and Riis (2000). This means that the designer can apply *structural discrimination* that affects the contestants' shares in the contested prize (the same efforts may yield different shares, depending on the value of this parameter). By (1), a reduction in δ increases the bias in favor of contestant 1, who is assumed to be, with no loss of generality, the more motivated contestant (the one with the higher prize valuation). Furthermore, $0 < \delta < 1$ implies a bias in favor of contestant 1. When $\delta = 1$ the contest is fair, there is no bias. When $\delta > 1$ the bias is in favor of contestant 2. The empirical relevance of such discrimination in contests with a logit CSF is thoroughly discussed in Epstein et al. (2011), Franke (2012) and Franke et al. (2013). Epstein et al. (2013) have recently shown that structural discrimination is effective; it is useful as a means of increasing the contestants' efforts when applied independently.

The contest designer can also carry out another type of discrimination that affects the contestants' incentives, not by controlling the parameter δ (in which case $\delta = 1$), but by directly changing the contestants' prize valuations (their rewards in case of winning the contest), thereby increasing or decreasing the gap between these valuations. In other words, the designer can manipulate the size of the divisible prize. Such a policy is usually based on a "give and take" mechanism in case of winning, which is henceforth referred to as *direct discrimination*. This form of discrimination has been recently introduced in Mealem and Nitzan (2013) and extensively discussed, focusing on its comparative application in an all-pay-auction relative to a logit CSF.

A crucial element in this second type of discrimination is the balanced-budget constraint faced by the contest designer. This constraint, which limits the design of the optimal tax schedule, implies that when one contestant's winning a share of the prize is subjected to a positive tax, the share of the prize won by the other contestant must be subjected to a negative tax, viz., the granting of a subsidy. The tax scheme consists then of two numbers (one negative and one positive) that are added to the contestants' initial valuations of the divisible prize. These numbers need not be equal in their absolute value, but they need to satisfy the requirement that in equilibrium the designer's net expenditures are equal to zero. Of course, whether the constraint is satisfied or not depends both on the applied structural and direct discrimination; the former determining the contestants' shares in the prize and the latter the actual modified values of the prize.

Our model and in particular, structural and direct discrimination subject to a balanced-budget constraint is of particular relevance in certain applications. In these applications asymmetric agents compete for a divisible prize; one agent (contestant 1) with a well established reputation competes against another agent (contestant 2) who is relatively unknown. Because of contestant 1's reputation, a larger significance is given to the contested prize (project) and therefore his prize valuation is larger than the prize value of his rival, contestant 2. Two such applications are presented below where the balanced-budget constraint is plausible. In the following example, the contest designer typically engages in a certain activity (some well defined task or project) restricted to a certain budget. Although the budget is earmarked only for this activity, it can be used to manipulate and affect the incentives of the contestants (the contractors) who compete for the outsourced project. But a designer who engages in such manipulations and in particular, in discrimination, must satisfy the contest balanced-budget constraint that we assume in order to ensure the overall budget constraint is satisfied:

1. *Municipal projects*. A municipal authority is conducting a tender for a divisible project such as urban development including development of a sewage system, roads, sidewalks and gardening. Two companies compete for a share in the project. The municipal authority is restricted to a budget allocated, for example, by the federal government. Although the budget is allocated only to urban development, it can be used to influence the incentives of the competing contestants by applying the two possible modes of discrimination. In order to satisfy the overall budget constraint, a designer who resorts to structural and direct discrimination must also satisfy the assumed balanced budget constraint.

The next application describes situations where the balanced-budget constraint is due to a different reason. The constraint is no longer related to a fixed budget which is at the disposal of the designer for the purpose of carrying out a particular project. It is due to the fact that the two competing contestants are (at least partly) controlled by a parent company. The parent company may not prevent competition between its two subsidiaries by custom or by the law. However, despite the existing competition, the parent company still has the ability to enforce some overall financial discipline as well as the power to ensure that the designer's strength in manipulating the companies is limited. The control of the parent company on its two subsidiaries and its power in dealing with the designer, given the conflict of interests between them, explains its success in enforcing the balanced-budget constraint:

2. *Marketing and distribution of a new product*. A reputable producer tries to introduce a new product to the market turning to two competing marketing companies that belong to the same conglomerate. The first marketing company (contestant 1) is large and well known. It has many branches located in areas with potentially high demand for the new product. The other company (contestant 2) is relatively small and unknown. Contestant 1 assigns a high value to obtaining the marketing and distribution of the new product because of the producer's reputation. In light of this, the producer can take advantage of contestant 1's desire to obtain the project and impose extra marketing and/or distribution costs thus enabling reduction of the costs of the less motivated marketing company (contestant 2). The producer has to satisfy the balanced – budget constraint because of the strength of the parent conglomerate that takes care of the combined interest of its subsidiaries.

Discrimination via contingent taxation of the prize won in a contest can be applied in various public-economic contexts. In particular, it can be used to explain the expected change in the existing income inequality between interest groups (e.g., the “poor” and the “rich”) that compete for the prize (gain or loss of income) associated with a proposed reform in the tax system. Such interest groups are typically represented by lobbyists who are the actual contestants. Mealem and Nitzan (2013) have shown that discrimination implemented by taxing and subsidizing the prize subject to a balanced budget constraint is also effective when applied independently. That is, when the designer resorts solely to this mode of discrimination, he can increase the contestants’ efforts. Note that Mealem and Nitzan (2013) focus on the application of direct discrimination disregarding structural discrimination. Their main purpose is to show that in this setting the all-pay-auction induces more efforts than any logit CSF with $0 < \alpha < 2$. In contrast, in the current study, we allow the two modes of discrimination focusing on the maximal efforts the designer can induce in a contest based on a logit CSF.

In light of the separate effectiveness of the above two modes of discrimination, the main objective of this study is to examine whether both of these modes of discrimination are needed when they can be applied simultaneously and to study their effectiveness in generating efforts. Given that the gap between the contestants can be closed by discrimination, either by modifying the contestants' prize valuations or by structurally changing the impact of their efforts, it seems that the designer can resort just to one type of discrimination. Interestingly, our preliminary claim establishes that, when $0 < \alpha < 2$, both types of discrimination are effective, not only when applied independently, but also when applied simultaneously. Furthermore, when $0 < \alpha < 2$, by our first preliminary result (Lemma 1), under any logit CSF exhibiting constant or increasing returns to scale and under some CSFs exhibiting decreasing returns to scale, the combined effects of the proposed dual discrimination increase the designer's revenue beyond the average value of the initial prize valuations, which is the maximal effort obtained by either mode of discrimination under *any* possible logit CSF.[2] In particular, when $\alpha = 1$, the combined effects of the two proposed modes of discrimination can yield efforts that are almost equal to the highest initial prize valuation. These efforts are exerted by the contestant who initially has the lower prize valuation. This contestant is offered an illusion of winning a very large prize. However, this attractively prize is almost always unattainable, because the designer allocates him a negligible share of the prize.

The extreme effectiveness of dual discrimination is robust to an increase in the number of the contestants. That is, if $\alpha = 1$, a designer who simultaneously applies the two modes of discrimination can induce the largest possible efforts in any *N*-player contest. Our main result reinforces the preliminary result by establishing that the extreme, twofold discrimination strategies presented in the preliminary result can yield efforts that are arbitrarily close to the maximal efforts. Surprisingly, this result implies that if the designer can control the two modes of discrimination as well as the exponent α of the CSF in (1), he can secure the largest possible efforts that are almost equal to the initial higher prize valuation by selecting the widely studied simple logit CSF where $\alpha = 1$.[3] The superiority of this constant-returns-to-scale-logit CSF is in marked contrast to its non-optimality when the designer is not allowed to

[2] See Epstein et al. (2013) and Mealem and Nitzan (2013).
[3] In the designer's problem (10), the exponent α is a given parameter, $0 < \alpha \le 2$. That is, the designer does not control α. Still, the solution of his problem for any $0 < \alpha \le 2$ implies that the (almost) maximal contestants' efforts are obtained for $\alpha = 1$. In other words, the indirect effort function is (almost) maximized at $\alpha = 1$.So this value of the exponent would be the designer's preferred value if he could select the parameter α.

apply any mode of discrimination between the contestants, or when he is allowed to apply just one of these modes of discrimination.

Alternative mechanisms based on the “take it or leave it” principle can yield maximal efforts that are equal to the highest prize valuation, see Proposition 2 part B in Nti (2004). In one simple version of this mechanism, the designer introduces a minimal effort requirement which is equal to the highest prize valuation. This enables participation of the individual with the highest valuation and precludes participation of the other individuals. In both cases, the attainment of maximal efforts is ensured by contestant exclusion. In contrast, in our setting, the convergence to the maximal efforts is rendered possible by a viable contest without exclusion (both contestants exert efforts) and without setting any limits on efforts.

Interestingly, in our setting, the individual with the lower prize valuation is offered the illusion of competing on a very large prize, albeit only a very small share of it can be won. The expected value of his prize is nevertheless positive and in fact, almost equal to the initial prize valuation of his rival, the individual with the higher prize valuation. The existence of effective incentives that induce participation in the contest together with the existence of an extreme illusion that results in efforts incurred by the individual with the lower prize valuation is a distinctive interesting feature of our contest. This feature is manifested in some realistic examples of the sort presented above.

Still, a natural question is what is the merit of convergence to the maximal effort by dual discrimination in our contest, given that effort maximization can be ensured by applying a "take it or leave it" setting.[4] The answer to this question can be based on three arguments. First, practically convergence to the maximal efforts might be satisfactory for the contest designer. Second, dual discrimination in a logit CSF contest might be appealing for a contest designer because it ensures active voluntary participation of the contestants without resorting to exclusionary means. Third, applying dual discrimination subject to a budget constraint, the designer offers an effective (extreme) illusion to the contestant with the lower prize valuation. This is consistent with the economic reality of contests, and lotteries in particular, which is

[4] Note that in order to provide effective participation incentives to the individual with the highest prize valuation in the first price all-pay auction with a reserve price, the designer may need to award him the prize when his effort is sufficiently close to his prize valuation, see Nti (2004, p.1063). In such a case, the outcome in our setting and in the “take it or leave it” setting is identical; the designer can come arbitrarily close to generating the maximal efforts.

often predicated on the existence of illusions. In any case, the preferred dual-discrimination policy in our theoretical setting actually occurs in some of the examples.

2. The setting

In our contest there are two contestants, the high and low benefit contestants, 1 and 2. The initial prize valuations of the contestants are denoted by n_1 and n_2 and the contest designer has full knowledge of the contestants' prize valuations. With no loss of generality, we assume that $n_1 \geq n_2$ or $k = \frac{n_1}{n_2} \geq 1$. Heterogeneity in the contestants' prize valuations is usually attributed to differences in preferences or to differences in the value of the awarded non-monetary privilege (monopoly permit). Given the contestants' fixed prize valuations and the CSF, the function that specifies the contestants' prize shares given their efforts, $p_i(x_1, x_2)$, the net payoff (surplus) of contestant *i* is:

$$u_i = p_i(x_1, x_2)n_i - x_i\,, \quad (i=1,2) \tag{2}$$

where $p_1(x_1, x_2)$ is the CSF given by (1) and $p_2 = 1 - p_1$. In the optimal contest design setting, the objective function of the contest designer is:

$$G = x_1 + x_2 \tag{3}$$

Resorting just to structural discrimination means that the contest designer maximizes his objective function (3) by selecting $\delta > 0$[5], given any α that satisfies $0 < \alpha \leq 2$.[6] Resorting solely to direct discrimination means that the designer changes the contestants' prize valuations from n_1 and n_2 to $(n_1 + \varepsilon_1)$ and $(n_2 + \varepsilon_2)$ by selecting the (positive or negative) amounts ε_1 and ε_2. A contest designer who applies such discrimination must ensure that the transformed prize valuations are positive, because

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[5] $\delta = 0$ implies that there is no competition because contestant 1 can win the whole prize by exerting a negligible effort. The assumption $\delta > 0$ is therefore consistent with the objective of the designer, viz., effort maximization.

[6] The analysis in this study is confined to CSFs with an exponent α such that $0 < \alpha \leq 2$. These functions include the constant and decreasing-returns-to-scale logit CSFs that are economically the most plausible ones. A discussion of equilibrium efforts in contests based on a logit CSF with $\alpha > 2$ appears in the last paragraph of the Conclusion.

otherwise the contestants will not voluntarily take part in the contest and the designer's revenue will be equal to zero. The contest designer also faces the balanced-budget constraint. That is, ε_1 and ε_2 must also satisfy the equality $p_1\varepsilon_1 + p_2\varepsilon_2 = 0$. This means that the sum of the positive and negative taxes applied by the discriminating designer is equal to zero. In this case, ε_1 (ε_2) is the tax (subsidy) levied on (given to) contestant 1 (2) if he wins the whole prize (his share in the prize is equal to 1).[7]

When the designer can apply both types of discrimination, he maximizes his objective function (3) by selecting δ, ε_1 and ε_2 (again, for any α that satisfies $0 < \alpha \le 2$), given the anticipated Nash equilibrium efforts of the contestants. The particular choice of his preferred discrimination policy together with the corresponding efforts of the contestants, constitute the equilibrium of the game. The contest game that we study has therefore a two-stage structure:

1. In the first stage the designer determines the discrimination policy, by selecting δ, ε_1 and ε_2 (for any α that satisfies $0 < \alpha \le 2$),
2. In the second stage the contestants simultaneously make decisions on their exerted efforts x_1 and x_2 taking as given the discrimination policy set by the designer.

The solution of this contest game is a sub-game-perfect Nash equilibrium.

Suppose that given a CSF of the logit form (1) where $0 < \alpha \le 2$, the designer can apply the two modes of discrimination, that is, select δ, ε_1 and ε_2. In this case the two contestants maximize their payoffs:

(4) $$u_1 = \frac{x_1^\alpha}{x_1^\alpha + (\delta x_2)^\alpha}(n_1 + \varepsilon_1) - x_1 \text{ and } u_2 = \frac{(\delta x_2)^\alpha}{x_1^\alpha + (\delta x_2)^\alpha}(n_2 + \varepsilon_2) - x_2$$

[7] Note that although the balanced-budget constraint limits the designer's budgetary means to be equal to zero in equilibrium, the contestants believe that his budget is unlimited because otherwise the designer would not be able to credibly apply direct discrimination and, in particular, promise arbitrarily large prize increments. The fact that the balanced-budget-constraint is satisfied in equilibrium clearly enhances the credibility of the designer because de-facto his promises are kept.

Let $a = \frac{n_1 + \varepsilon_1}{n_2 + \varepsilon_2}$ and $d = \left(\frac{a}{\delta}\right)^{\alpha}$. By the first order conditions,

(5) $$x_1^* = \frac{\alpha d(n_1 + \varepsilon_1)}{(d+1)^2} \text{ and } x_2^* = \frac{\alpha d(n_2 + \varepsilon_2)}{(d+1)^2}$$

and therefore,

(6) $$G = x_1^* + x_2^* = \frac{\alpha d(n_1 + \varepsilon_1 + n_2 + \varepsilon_2)}{(d+1)^2}$$

(7) $$p_1 = \frac{d}{d+1} \text{ and } p_2 = \frac{1}{d+1}$$

and the balanced-budget constraint takes the form

(8) $$p_1\varepsilon_1 + p_2\varepsilon_2 = \frac{d}{d+1}\varepsilon_1 + \frac{1}{d+1}\varepsilon_2 = 0$$

or

(9) $$d\varepsilon_1 + \varepsilon_2 = 0$$

The designer's problem is therefore to find the most effective dual discrimination strategy that yields the largest possible efforts subject to six constraints (see 1-6 below). Since the set of the feasible discrimination strategies is not compact (see constraints 3, 5 and 6), the designer objective is:

$$\sup_{\varepsilon_1,\varepsilon_2,\delta} \left(x_1^* + x_2^*\right) = \frac{\alpha d\left(n_1 + \varepsilon_1 + n_2 + \varepsilon_2\right)}{(d+1)^2}$$

(10)

$$\begin{aligned}
&s.t.\\
&1.\ 1 - \alpha + d \geq 0\\
&2.\ (1-\alpha)d + 1 \geq 0\\
&3.\ \delta > 0\\
&4.\ d\varepsilon_1 + \varepsilon_2 = 0\\
&5.\ n_1 + \varepsilon_1 > 0\\
&6.\ n_2 + \varepsilon_2 > 0
\end{aligned}$$

Recall that the term *d* like p_1 and p_2 is not a parameter but rather a function of all the choice variables $(\varepsilon_1, \varepsilon_2, \delta)$. In Appendix A we present the justification for constraints 1 and 2. It will be shown that these constraints guarantee that the contestants' utilities are not negative as well as the fulfillment of the second-order conditions in the contestants' maximization problems.[8]

3. Two-mode discrimination

3.A Results

Let us start by clarifying the effectiveness of discrimination when it can take the form of both direct and structural discrimination. For $k > 1$ and $0 < \alpha \leq 1$, dual discrimination yields efforts that are almost equal to αn_1.The proof of this claim (see the first part of Lemma 1 and the proof in Appendix B) uses the following idea: On the one hand, the designer applies direct discrimination in favor of contestant 2 by reducing the stake of contestant 1 (the contestant with the initially higher prize value) almost to zero ($\varepsilon_1 \to -n_1^+$) and increases the stake of contestant 2 (the contestant with the initially lower prize value) to a "very large" level ($\varepsilon_2 \to \infty$). On the other hand, in order to satisfy the balanced-budget constraint, the designer must create an appropriate bias in favor of contestant 1 by selecting δ, such that the balanced-budget constraint (9) is satisfied:[9]

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[7] The justification in Appendix A of constraints 1 and 2 in problem (10) is similar to that presented in Appendix B in Epstein et al. (2013).

[9] Given δ that satisfies (11), for $\left(\overline{\varepsilon_1}, \overline{\varepsilon_2}\right)$ satisfying $0 > \overline{\varepsilon_1} > -n_1^+$ and $0 < \overline{\varepsilon_2} < \infty$, the designer can always increase efforts by choosing a smaller ε_1 and a larger ε_2. That is, an equilibrium does not exist since the designer can induce total efforts that are arbitrarily close to αn_1 by choosing ε_1 and ε_2, such

(11) $$\delta = \left(\frac{n_1+\varepsilon_1}{n_2+\varepsilon_2}\right)\left(-\frac{\varepsilon_1}{\varepsilon_2}\right)^{\frac{1}{\alpha}}$$

In the first part of Lemma 1 it is shown that for $\alpha = 1$ this extreme dual discrimination, that is, $\varepsilon_1 \to -n_1^+$ and $\varepsilon_2 \to \infty$, is an optimal strategy yielding efforts that are almost equal to n_1, the initial higher prize valuation of contestant 1.

Undertaking the extreme dual discrimination, $\varepsilon_1 \to -n_1^+$ and $\varepsilon_2 \to \infty$, while choosing δ according to (11), such that the balanced-budget constraint (9) is satisfied, is possible for $0 < \alpha \le 1$, but it is not possible for $1 < \alpha \le 2$. The reason is that the designer's selection of $(\varepsilon_1, \varepsilon_2, \delta)$ must ensure that the utility of the contestants is not negative, to prevent their abandonment from the competition and, in turn, the decline of the contestants' efforts to zero. In other words, constraints 1 and 2 in problem (10) that ensure the existence of competition, as well as the second order conditions for utility maximization, must be satisfied. It can be verified that when $0 < \alpha \le 1$, for any positive value of *d*, these two constraints are satisfied. However, an increase of α beyond 1 does not enable any value of *d* in which contestant 1's stake is reduced and contestant 2's stake is increased, as implied by constraint 2 in problem (10) that ensures the participation of contestant 2 in the competition. In this case ($1 < \alpha \le 2$) the designer can set a maximal value for *d* which is equal to $d_{\max} = \frac{1}{\alpha - 1}$.[10] Combining this equality with the condition for the existence of the balanced-budget constraint, $d = -\frac{\varepsilon_2}{\varepsilon_1}$, gives the maximal value of ε_2 (given $\varepsilon_1 < 0$) that can be set by the designer, $\varepsilon_2 = -\frac{\varepsilon_1}{\alpha - 1}$. This means that the extreme direct discrimination in this case is obtained when $\varepsilon_1 \to -n_1^+$ and $\varepsilon_2 \to \frac{n_1}{\alpha - 1}$. Can this

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that $\overline{\varepsilon_1} > \varepsilon_1 > -n_1^+$ and $\overline{\varepsilon_2} < \varepsilon_2 < \infty$. A similar situation exists in the case dealt with in the sequel, where $1 < \alpha \le 2$. For this reason in problem (10) we look for the supremum and not the maximum of the contestants' efforts.

[10] Notice that if the designer chooses $d = d_{\max} = \frac{1}{\alpha - 1}$, then constraint 1 in problem (10) is also satisfied.

extreme direct discrimination together with extreme structural discrimination, the value of δ determined by (11), yield total efforts that converge to n_1? By the second part of Lemma 1, the answer is negative.

Lemma 1:[11]

1. For $k>1$ and $0<\alpha\le 1$, when $\varepsilon_1\to -n_1^+$, $\varepsilon_2\to\infty$ and δ is set according to (11), the prize share of contestant 1 converges to 1, but his effort converges to zero and the prize share of contestant 2 converges to zero, but his effort converges to αn_1. Total efforts therefore converge to αn_1, $u_1\to 0$ and $u_2\to(1-\alpha)n_1$.
2. For $k>1$ and $1<\alpha\le 2$, when $\varepsilon_1\to -n_1^+$, $\varepsilon_2=\dfrac{\varepsilon_1}{1-\alpha}$ and δ is set according to (11), the prize share of contestant 1 is $\dfrac{1}{\alpha}$, but his effort converges to zero and the prize share of contestant 2 is $\left(1-\dfrac{1}{\alpha}\right)$ and his effort converges to $\dfrac{1}{\alpha}n_1+\left(1-\dfrac{1}{\alpha}\right)n_2$. Total efforts therefore converge to a value that is smaller than n_1 and $u_1=u_2\to 0$.[12]

The special appeal of the dual discrimination strategies presented in Lemma 1 is highlighted by our main result.

Proposition 1: For any $0<\alpha\le 2$, the dual discrimination strategies applied in Lemma 1 yield total efforts that converge to the least upper bound of the possible equilibrium efforts of the contestants. The largest possible efforts are obtained under a simple logit CSF where $\alpha=1$. These efforts converge to n_1.

The relationship between the exponent α of a logit CSF and the maximal attainable efforts G is presented in Figure 1. By Lemma 1, under any logit CSF exhibiting

[11] The proofs of this and the next propositions appear in Appendix B.
[12] For $k=1$, that is, when $n_1=n_2=n$, in the range $0<\alpha<1$, we would get that $G\to\alpha n$ and in the range $1\le\alpha\le 2$, we would get that $G\to n$.

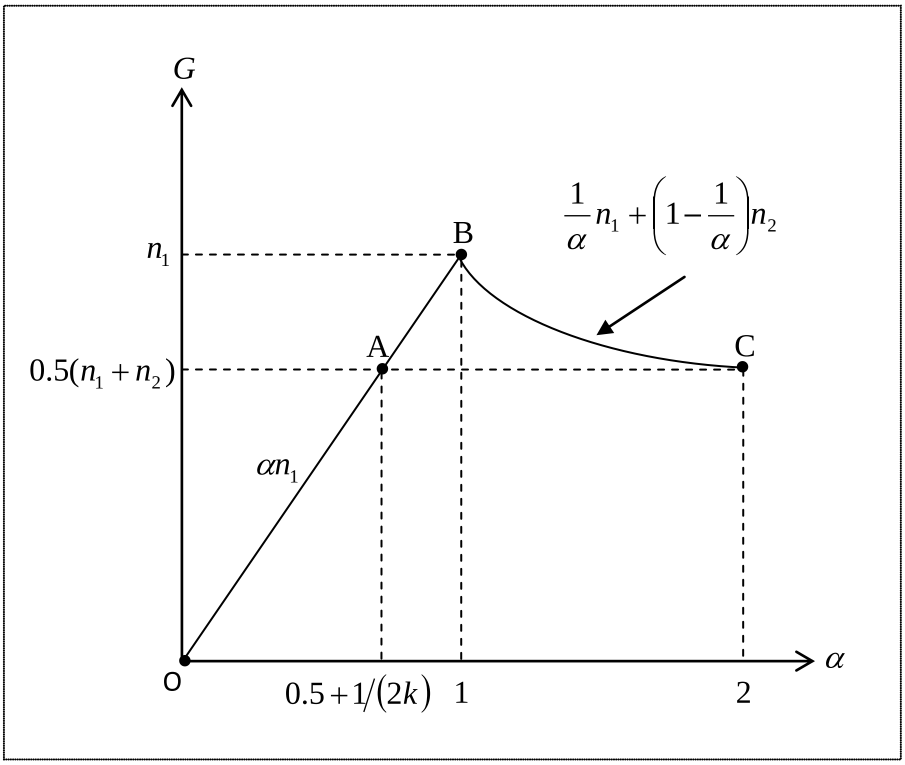
constant or increasing returns to scale, $1 \leq \alpha < 2$, and under logit CSFs exhibiting decreasing returns to scale, such that $0.5 + \frac{1}{2k} < \alpha < 1$, the combined effects of the extreme dual discrimination increase the designer's revenue beyond the average value of the initial prize valuations, $0.5(n_1 + n_2)$, which is the maximal effort obtained by either mode of discrimination under *any* possible logit CSF.

Proposition 1 implies that when the designer applies the two modes of discrimination, each type has a positive "added value" that enhances the exertion of efforts relative to the situation where the designer resorts to just one mode of discrimination. That is, the two modes of discrimination are supportive or "complementing" - their combination yields larger efforts than those obtained by separate application of one of these modes of discrimination for almost any given level of α ($0 < \alpha < 2$). Furthermore, under logit CSFs with increasing or constant returns to scale, as well as under some logit CSF with decreasing returns to scale, such dual discrimination yields efforts that are larger than the average prize valuation (see ABC in Figure 1), which is the largest possible total effort under separate application of these modes of discrimination. The advantage of combining these two types of discrimination relative to the use of a single mode of discrimination is due to the distinctive features of the contribution of each of these modes of discrimination to the exerted efforts as described below.

(i) Direct discrimination increases as much as possible the initially lower prize valuation while reducing the initially higher prize valuation almost to zero. This increases the sum of the contestants' prize valuations to infinity and makes the 'income effect' (associated with a scheme that increases the sum of the final stakes from $(n_1 + n_2)$ to $(n_1 + \varepsilon_1 + n_2 + \varepsilon_2)$) of this mode of discrimination the dominant effect.[13]

[13] For a clarification of the meaning of the income effect associated with direct discrimination, see the discussion following Proposition 2 in Mealem and Nitzan (2013).

***Figure 1*: The relationship between the exponent α and the maximal attainable efforts G**



(ii) The maximal possible increase in the sum of the contestants' prize valuations is not the result of direct discrimination alone. It is rendered possible by structural discrimination that makes sure that the balanced-budget constraint is satisfied. Specifically, structural discrimination counterbalances the above 'income effect' by almost completely favorably discriminating contestant 1, ensuring that his prize share converges to 1. The moderating effect described in (ii) is necessary to attain the maximal efforts. While structural discrimination has a 'second order' effect on efforts that moderates the income effect of direct discrimination, it also enables the dominance of this 'first order' income effect on efforts described in (**i**), namely, the increase in efforts due to the increase in the sum of the contestants' prize valuations. The dominance of the effect of direct discrimination means that the more extreme this mode of discrimination, the higher the total efforts and this requires the extremity of structural discrimination.

Proposition 1 also implies that if the designer can control δ, ε_1, ε_2 as well as α, he can secure almost the largest possible efforts n_1 by selecting $\alpha = 1$ (recall that we have already proved in part 2 of Lemma 1 that the efforts exerted when $1 < \alpha \leq 2$ converge to a value that is smaller than n_1). Any logit CSF with $\alpha \neq 1$ is therefore inferior to a simple CSF where $\alpha = 1$, when in both cases the designer applies the optimal discrimination strategy, viz, the dual discrimination strategy. Note that the combination of the two modes of discrimination results in an outcome which is practically reasonable; the amount transferred between the contestants is finite, such that the actual payment to contestant 2 (the tax taken from contestant 1) is equal to $p_2\varepsilon_2 \to n_1$. The superiority of $\alpha = 1$ is in marked contrast to its non-optimality when the designer is not allowed to discriminate between the contestants or when the designer is allowed to discriminate between the contestants, but apply just one mode of discrimination.

In the first example of a m*unicipal project* (see the examples discussed in the Introduction), despite the fact that both modes of discrimination are extreme, their combined use still results in a balanced effect. The designer promises the small company a very large value in case it receives the entire project. The designer, by structural discrimination, ensures that the small company's share in the project is sufficiently small, such that the large reputable company wins almost the entire project. In this extreme and most effective case from the designer's point of view, the large company transfers the small company a reasonable finite amount which is almost equal to its initial valuation of the entire project.

In the second example of *marketing and distribution of a new product*, in light of Proposition 1, it is expected that the producer assigns most of the project to the reputable marketing company (contestant 1) considerably increasing its cost while giving just a negligible share of the project to the second company (contestant 2) yet considerably reducing its costs. This dual discrimination strategy is the most effective one from the producer's point of view.

Two important conclusions can be drawn from the results. First, for $k > 1$, when the designer applies the (optimal) dual discrimination strategy (the strategy that maximizes the contestants' efforts), an increase in α from $\alpha = 1$ to $\alpha = 2$ reduces efforts. Second, Proposition 1 can be extended to the case of any number of contestants. In particular, under the simple logit CSF. where $\alpha = 1$ and any number of contestants *N*, the most effective dual discrimination strategy can secure total efforts that are equal almost to n_1. In the more general multi-player contest, the designer has to reduce the stakes of $N-2$ contestants to zero, making sure that contestant 1 with the highest stake is not included among them. That is,

Corollary 1: Given any number of contestants *N*, such that $n_1 \geq n_2, ..., \geq n_N$, if $\alpha = 1$ and the designer applies dual discrimination strategy, he can attain efforts that are equal almost to n_1.

Proof: The proof is based on the following simple three-stage strategy that the designer applies:

1. *Stage 1*: The designer selects a contestant *j* that satisfies $j \in (2, ..., N)$.
2. *Stage 2*: For any contestant i that satisfies $i \neq 1, j$, the designer chooses $\varepsilon_i = -n_i$. That is, he reduces the initial prize valuations of $N-2$ players to zero.
3. *Stage 3*: Applying the dual discrimination strategy with respect to the two contestants 1 and *j*, according to Proposition 1, the designer can induce efforts that are almost equal to n_1.

3. Conclusion

3.A A brief summary of the main contribution

Under common knowledge of the contestants' prize valuations and any Tullock-type CSF associated with a pure-strategy equilibrium, optimal contest design can be implemented by applying structural discrimination that biases the effect of the contestants' exerted efforts, Epstein et al. (2011, 2013). Alternatively, such design can be carried out by affecting the contestants' prize valuations via direct discrimination, subject to a balanced-budget constraint. Our results establish that:

(i) Both modes of discrimination are effective and therefore will be used by the designer, when they can be applied simultaneously; Furthermore, under CSFs exhibiting constant or increasing returns to scale (with the exception of the case $\alpha = 2$) and under CSFs exhibiting decreasing returns to scale, $0.5 + \frac{1}{2k} < \alpha < 1$, the combined effects of these modes of discrimination can increase the designer's revenue beyond the average value of the initial prize valuations, which is the maximal effort obtained by either mode of discrimination under *any* possible logit CSF, Epstein et al. (2013), Mealem and Nitzan (2013);

(ii) The dual discrimination strategies corresponding to CSFs where $0 < \alpha \leq 2$ are optimal;

(iii) When $\alpha = 1$, a variant of the prototypical simple and most commonly studied CSF that applies the extreme dual discrimination yields the largest possible efforts (efforts that are almost equal to the initially higher prize valuation);

3.B Generalization to *N*-player contests

A potential interesting extension of our study is the analysis of the multiple-player case. Only few studies dealt with *N*-player contests assuming logit CSFs with asymmetric contestants. Stein (2002), Franke (2012) and Franke et al. (2013, 2012) assumed, for *N*-player, that $\alpha = 1$, and Cornes and Hartley (2005) allowed any α. Stein (2002) extended the two-player contest to *N*-player contest and examined how changes in the contestants' prize valuations and in the measure of their prior relative chance of winning affect the equilibrium efforts. Franke (2012) compared these efforts under Affirmative Action (AA), where the designer affects the prize shares of the contestants (in our case, via the selection of δ) to the efforts obtained under Equal Treatment (ET). For two contestants, he extended his analysis to the case where $\alpha \leq 1$, but for *N* players he confined the analysis to $\alpha = 1$. For *N*-player contests, Franke et al. (2013, 2012) have recently allowed structural discrimination ($\delta \neq 1$), but still focusing on the simple CSF case ($\alpha = 1$). Franke et al. (2013) have shown that in this setting the designer will level the playing field by encouraging weak contestants, but he will not equalize the contestants' chances of winning the contest. Franke et al. (2012) have shown that the maximal efforts secured by the optimal APA are larger than those obtained by any logit CSF.

For *N* players, Cornes and Hartley proposed an elegant way to examine the existence of equilibrium for any α. Among other things, they have shown that, for $\alpha = 1$, there exists a unique equilibrium in pure strategies. But, for $\alpha > 1$, there is no explicit presentation of equilibrium and, in fact, multiple equilibria are possible, which precludes the possibility of conducting comparative statics, see footnote 24 in Franke (2012). This implies that, to attain consistency of the results, we can choose $\alpha > 1$, for two players or $\alpha = 1$, for any number of players. In our study the focus is on two-player contests and therefore, we can compare the two modes of discrimination also for logit CSFs with $\alpha \leq 2$, despite our inability to compute explicitly the equilibrium outcome under direct discrimination. This case is also more general than the one examined by Franke (2012), since he assumed for 2 players that $\alpha \leq 1$. The challenging question what happens when we move to an *N*-player contest for any α, $0 < \alpha \leq 2$, (note that in our study we have dealt only with the case $\alpha = 1$) seems an especially demanding challenge and is left for future research.

Let us finally discuss the possibility of extending our analysis to the case of $\alpha > 2$. Only few studies have dealt with this case and so far a characterization of the complete set of mixed-strategy equilibria is not available, even for the relatively tractable case of a simple logit CSF ($\alpha = 1$) and no discrimination of any form. Three relevant studies that allow $\alpha > 2$ are Baye et al. (1994), Alcalde and Dahm (2010) and Wang (2010). The former focuses on a special two-player contest with symmetric prize valuations allowing only discontinuous effort strategies. Even under these restrictions, the authors have left to future research the explicit solution under asymmetric prize valuations and no discrimination. Alcalde and Dahm (2010) have dealt with a more general fair contest with continuous efforts allowing $\alpha > 2$. Their main result is that there exists an equilibrium in mixed strategies that is equivalent to the equilibrium of the APA ($\alpha = \infty$). However, as already mentioned, they did not characterize the set of mixed-strategy equilibria.

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Appendix A

In problem (10), the designer controls the parameters δ, ε_1 and ε_2. The first two constraints in this problem are:

(A1) $$1-\alpha+d \geq 0 \text{ and } (1-\alpha)d+1 \geq 0$$

Let us show that, given the above two constraints, the unique equilibrium in pure strategies in the game between the two contestants is given by (5).[14]

From the payoff of the contestants, equations (4), we get the first order conditions:

$$\frac{\partial u_1}{\partial x_1} = \frac{\alpha x_1^{\alpha-1}(\delta x_2)^{\alpha}(n_1+\varepsilon_1)}{\left[x_1^{\alpha}+(\delta x_2)^{\alpha}\right]^2} - 1 = 0$$

(A2) and

[14] The proof is based on the reasoning proposed in Nti (1999), see his proof of Proposition 3, p. 423.

$$\frac{\partial u_2}{\partial x_2} = \frac{\alpha \delta^{\alpha} x_2^{\alpha-1} x_1^{\alpha} (n_2 + \varepsilon_2)}{\left[x_1^{\alpha} + (\delta x_2)^{\alpha}\right]^2} - 1 = 0$$

and after rearranging, we get the contestants' efforts, see (5). Substituting these efforts in the contestants' payoffs, see (4), we get:

(A3) $$u_1^* = \frac{d(n_1 + \varepsilon_1)(1 - \alpha + d)}{(d+1)^2} \text{ and } u_2^* = \frac{(n_2 + \varepsilon_2)[(1-\alpha)d + 1]}{(d+1)^2}$$

By (A3) one obtains the two constraints in (A1); The designer has to satisfy these constraints because in equilibrium of the game between the two contestants the contestants' payoffs must be non-negative, that is, $u_1^* \geq 0$ and $u_2^* \geq 0$. The second order conditions (SOC) of equilibrium in this game are:

$$\frac{\partial^2 u_1}{\partial x_1^2} = \frac{\alpha (n_1 + \varepsilon_1)(\delta x_2)^{2\alpha} x_1^{\alpha-2} \left[\alpha - 1 - (\alpha+1)\left(\frac{x_1/x_2}{\delta}\right)^{\alpha}\right]}{\left[x_1^{\alpha} + (\delta x_2)^{\alpha}\right]^3} \leq 0$$

and

$$\frac{\partial^2 u_2}{\partial x_2^2} = \frac{\alpha (n_2 + \varepsilon_2)\delta^{2\alpha} x_1^{\alpha} x_2^{2\alpha-2} \left[(\alpha - 1)\left(\frac{x_1/x_2}{\delta}\right)^{\alpha} - (\alpha+1)\right]}{\left[x_1^{\alpha} + (\delta x_2)^{\alpha}\right]^3} \leq 0$$

By the first order conditions, we obtain that $\frac{x_1^*}{x_2^*} = a$. Since $d = \left(\frac{a}{\delta}\right)^{\alpha}$, the SOC can be written as:

(A4) $$1 - \alpha + (\alpha+1)d \geq 0 \text{ and } (1-\alpha)d + \alpha + 1 \geq 0$$

The conditions in (A1) ensure that the SOC in (A4) are satisfied. Since the prize valuation of every contestant is positive, the equilibrium efforts in a pure-strategy equilibrium must be positive. By the FOC, there exists a unique solution to the contestants' problems which is given by (5), because given x_1^* contestants 2 maximizes his payoff by selecting x_2^* and vice versa. Hence, equation (5) provides the unique Nash equilibrium of the game between the contestants.

Q.E.D

Appendix B

Lemma 1:[15]

1. For $k>1$ and $0<\alpha\leq 1$, when $\varepsilon_1 \to -n_1^+$, $\varepsilon_2 \to \infty$ and δ is set according to (11), the prize share of contestant 1 converges to 1, but his effort converges to zero and the prize share of contestant 2 converges to zero, but his effort converges to αn_1. Total efforts therefore converge to αn_1, $u_1 \to 0$ and $u_2 \to (1-\alpha)n_1$.

2. For $k>1$ and $1<\alpha\leq 2$, when $\varepsilon_1 \to -n_1^+$, $\varepsilon_2 = \dfrac{\varepsilon_1}{1-\alpha}$ and δ is set according to (11), the prize share of contestant 1 is $\dfrac{1}{\alpha}$, but his effort converges to zero and the prize share of contestant 2 is $\left(1-\dfrac{1}{\alpha}\right)$ and his effort converges to $\dfrac{1}{\alpha}n_1 + \left(1-\dfrac{1}{\alpha}\right)n_2$. Total efforts therefore converge to a value that is smaller than n_1 and $u_1 = u_2 \to 0$.

Proof:

Part 1. By the balanced-budget constraint (9), $d=-\dfrac{\varepsilon_2}{\varepsilon_1}$ and therefore, when $0<\alpha\leq 1$, constraints 1 and 2 in problem (10) are always satisfied. Substituting $d=-\dfrac{\varepsilon_2}{\varepsilon_1}$ in (6) we get that $G=\dfrac{\alpha\left(-\dfrac{\varepsilon_2}{\varepsilon_1}\right)(n_1+\varepsilon_1+n_2+\varepsilon_2)}{\left(-\dfrac{\varepsilon_2}{\varepsilon_1}+1\right)^2}$. Multiplying the nominator and denominator of the above expression by $\left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2$ we get that

[15] Note that the proof is indirect, not using the Kuhn-Tucker conditions. The reason is that the constraints in problem (10) imply that the feasible set of the control variables is not compact. In particular, note that constraints 3, 5 and 6 have the form of strict inequalities. In addition, note that the objective function is not continuous at $\varepsilon_i = -n_i$. The standard Kuhn-Tucker conditions cannot therefore be used.

$$G=-\frac{\alpha\varepsilon_1\left(\frac{n_1}{\varepsilon_2}+\frac{\varepsilon_1}{\varepsilon_2}+\frac{n_2}{\varepsilon_2}+1\right)}{\left(-1+\frac{\varepsilon_1}{\varepsilon_2}\right)^2}$$

. Now, for $0<\alpha\le 1$, where $\varepsilon_1 \to -n_1^+$, $\varepsilon_2 \to \infty$ and δ is determined according to (11), we get that

$$G \to \frac{\alpha n_1\left(\frac{n_1}{\varepsilon_2}-\frac{n_1}{\varepsilon_2}+\frac{n_2}{\varepsilon_2}+1\right)}{\left(-1-\frac{n_1}{\varepsilon_2}\right)^2}=\frac{\alpha n_1(0+1)}{(-1-0)^2}=\alpha n_1$$

Since $d \to \infty$, the prize share of contestant 1 converges to 1, because

$$p_1=\frac{d}{d+1}=\frac{1}{1+\frac{1}{d}}\to 1$$

so $p_2 \to 0$. By (5), the exerted effort of contestant 1 is:

$$x_1^*=\frac{\alpha d(n_1+\varepsilon_1)}{(d+1)^2}=\alpha(n_1+\varepsilon_1)\frac{d}{(d+1)}\frac{1}{(d+1)}$$

Since $\varepsilon_1 \to -n_1^+$, $d \to \infty$ and $p_1=\frac{d}{d+1}\to 1$, $x_1^* \to 0$. The exerted effort of contestant 2 is equal to:

$$x_2^*=\frac{\alpha d(n_2+\varepsilon_2)}{(d+1)^2}=\alpha\left(\frac{n_2+\varepsilon_2}{d+1}\right)\frac{d}{(d+1)}$$

Substituting $d=-\frac{\varepsilon_2}{\varepsilon_1}$ in the second term of the above expression, we get that

$$x_2^*=\alpha\left(\frac{n_2+\varepsilon_2}{-\frac{\varepsilon_2}{\varepsilon_1}+1}\right)\frac{d}{(d+1)}$$

Multiplying the nominator and the denominator in the second term by $-\frac{\varepsilon_1}{\varepsilon_2}$, we get that

$$x_2^*=\alpha\left(\frac{-\frac{\varepsilon_1 n_2}{\varepsilon_2}-\varepsilon_1}{1-\frac{\varepsilon_1}{\varepsilon_2}}\right)\frac{d}{(d+1)}$$

Since $\varepsilon_1 \to -n_1^+$, $\frac{d}{d+1} \to 1$ and $\varepsilon_2 \to \infty$, we get that $x_2^* \to \alpha n_1$,

$$x_2^* = \alpha\left(\frac{-\frac{\varepsilon_1 n_2}{\varepsilon_2} - \varepsilon_1}{1 - \frac{\varepsilon_1}{\varepsilon_2}}\right)\frac{d}{(d+1)} \to \alpha\left(\frac{\frac{n_1 n_2}{\infty} + n_1}{1 - \frac{n_1}{\infty}}\right)\cdot 1 = \alpha n_1$$

The utility of contestant 1 is $u_1 = p_1(n_1 + \varepsilon_1) - x_1$. Since $p_1 \to 1$, $n_1 + \varepsilon_1 \to 0^+$ (because $\varepsilon_1 \to -n_1^+$) and $x_1^* \to 0$ hence, $u_1 \to 0$. The utility of contestant 2 is $u_2 = p_2(n_2 + \varepsilon_2) - x_2$. By the balanced-budget constraint (9), $\varepsilon_2 = -d\varepsilon_1 \to dn_1$ and therefore $n_2 + \varepsilon_2 \to n_2 + dn_1$. Since $n_2 + \varepsilon_2 \to n_2 + dn_1$, $p_2 = \frac{1}{d+1}$ and $x_2^* \to \alpha n_1$, $u_2 \to \frac{1}{d+1}(n_2 + dn_1) - \alpha n_1$. When $d \to \infty$, we get that $u_2 \to (1-\alpha)n_1$.

Part 2. As already noted in the discussion before Lemma 1, for $1 < \alpha \leq 2$, extreme dual discrimination requires that $\varepsilon_1 \to -n_1^+$ and $\varepsilon_2 \to \frac{n_1}{\alpha - 1}$, where δ is determined by (11). To find out the limit of the corresponding efforts, let us substitute in (6), $\varepsilon_1 = -n_1$, $\varepsilon_2 = \frac{n_1}{\alpha - 1}$ and $d = \frac{1}{\alpha - 1}$ to obtain:

$$G \to \frac{\alpha \frac{1}{\alpha - 1}\left(n_1 - n_1 + n_2 + \frac{n_1}{\alpha - 1}\right)}{\left(\frac{1}{\alpha - 1} + 1\right)^2} = \frac{1}{\alpha} n_1 + \left(1 - \frac{1}{\alpha}\right) n_2 < n_1$$

(it can be readily verified that the last inequality holds because, by assumption, $n_2 < n_1$). Since $d = \frac{1}{\alpha - 1}$, the prize share of contestant 1 is $p_1 = \frac{d}{d+1} = \frac{1}{\alpha}$ so $p_2 = 1 - \frac{1}{\alpha}$. By (5), the exerted effort of contestant 1 is equal to $x_1^* = \frac{\alpha d(n_1 + \varepsilon_1)}{(d+1)^2}$. Since $\varepsilon_1 \to -n_1^+$, $x_1^* \to 0$. The exerted effort of contestant 2 is equal to $x_2^* = \frac{\alpha d(n_2 + \varepsilon_2)}{(d+1)^2}$. Substituting $d = \frac{1}{\alpha - 1}$ and $\varepsilon_2 \to \frac{n_1}{\alpha - 1}$, we get that $x_2^* \to \frac{1}{\alpha} n_1 + \left(1 - \frac{1}{\alpha}\right) n_2$. Since $\varepsilon_1 \to -n_1^+$, $n_1 + \varepsilon_1 \to 0^+$ and therefore, $u_1 \to 0$. The utility of contestant 2 is:

$$u_2 = p_2(n_2+\varepsilon_2)-x_2 \to \left(1-\frac{1}{\alpha}\right)\left(n_2+\frac{n_1}{\alpha-1}\right)-\left[\frac{1}{\alpha}n_1+\left(1-\frac{1}{\alpha}\right)n_2\right]=0$$

Q.E.D.

Proposition 1: For any $0<\alpha\le 2$, the dual discrimination strategies applied in Lemma 1 yield total efforts that converge to the least upper bound of the possible equilibrium efforts of the contestants. The largest possible efforts are obtained under a simple logit CSF where $\alpha=1$. These efforts converge to n_1.

Proof: Let us divide the proof to two parts dealing with $0<\alpha\le 1$ and then with $1<\alpha\le 2$.

Part 1. If $0<\alpha\le 1$, then by part 1 of Lemma 1, the extreme dual discrimination strategy yields efforts that are equal to αn_1. We therefore have to show that the total efforts given by (6) do not exceed αn_1. That is, $\frac{\alpha d(n_1+\varepsilon_1+n_2+\varepsilon_2)}{(d+1)^2}\le \alpha n_1$ or, after some simplifications, $0\le d^2 n_1 - d\varepsilon_2 + dn_1 - dn_2 + n_1 - d\varepsilon_1$. By the balanced-budget constraint, $\varepsilon_2=-d\varepsilon_1$. Substituting this term (twice) in the last inequality and then adding and subtracting n_2, the inequality takes the form:

$$0\le d^2 n_1 - d(-d\varepsilon_1) + dn_1 - dn_2 + n_1 - n_2 + n_2 + \varepsilon_2,$$

which, after simplification becomes:

$$0\le d^2(n_1+\varepsilon_1)+(d+1)(n_1-n_2)+n_2+\varepsilon_2$$

Since, $d>0$, $n_1+\varepsilon_1>0$, $n_1\ge n_2$ and $n_2+\varepsilon_2>0$, the above inequality holds.

Part 2. If $1<\alpha\le 2$, then by part 2 of Lemma 1 the extreme dual discrimination strategy yields efforts that converge to $\frac{1}{\alpha}n_1+\left(1-\frac{1}{\alpha}\right)n_2$. We therefore have to show that the total efforts given by (6) do not exceed this level. Let us first show that the contestants' equilibrium efforts do not exceed $\frac{dn_1+n_2}{d+1}$. For that purpose, let us substitute the equilibrium efforts of (5) in (4), to obtain the equilibrium utilities:

$$u_1^* = \frac{(n_1+\varepsilon_1)d(1-\alpha+d)}{(d+1)^2} \text{ and } u_2^* = \frac{(n_2+\varepsilon_2)[(1-\alpha)d+1]}{(d+1)^2}$$

In equilibrium, the utility of a contestant is not negative so the sum of these utilities is not negative. That is,

$$u_1^* + u_2^* = \frac{(n_1 + \varepsilon_1)d(1 - \alpha + d)}{(d+1)^2} + \frac{(n_2 + \varepsilon_2)[(1-\alpha)d + 1]}{(d+1)^2} \geq 0$$

or, after some simplification,

$$dn_1 + n_2 + d\varepsilon_1 + \varepsilon_2 \geq \frac{\alpha d(n_1 + \varepsilon_1 + n_2 + \varepsilon_2)}{(d+1)}$$

Since, by the balanced-budget constraint $d\varepsilon_1 + \varepsilon_2 = 0$, the above inequality takes the form:

$$dn_1 + n_2 \geq \frac{\alpha d(n_1 + \varepsilon_1 + n_2 + \varepsilon_2)}{(d+1)}$$

or, dividing both sides of the inequality by $(d+1)$,

$$\frac{dn_1 + n_2}{d+1} \geq \frac{\alpha d(n_1 + \varepsilon_1 + n_2 + \varepsilon_2)}{(d+1)^2}$$

To complete the proof, let us show that

$$\frac{1}{\alpha} n_1 + \left(1 - \frac{1}{\alpha}\right) n_2 \geq \frac{dn_1 + n_2}{d+1}$$

or

$$(d+1)n_1 + (\alpha - 1)(d+1)n_2 \geq \alpha d n_1 + \alpha n_2$$

or

$$[(1-\alpha)d + 1]n_1 \geq [(1-\alpha)d + 1]n_2$$

Since the utility of contestant 2 is not negative, by constraint (2) in Problem (10), $(1-\alpha)d + 1 \geq 0$. Therefore, if $(1-\alpha)d + 1 = 0$, the latter condition is satisfied as equality and if $(1-\alpha)d + 1 > 0$, the latter condition takes the form $n_1 \geq n_2$, which is also satisfied.

Q.E.D.