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Credit Risk in General Equilibrium

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CESIFO WORKING PAPER NO. 4602
CATEGORY 12: EMPIRICAL AND THEORETICAL METHODS
JANUARY 2014

CESifo *Working Paper No. 4602*

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Abstract

This paper contributes to the literature on default in general equilibrium. Borrowing and lending takes place via a clearing house (bank) which monitors agents and enforces contracts. Our model develops a concept of bankruptcy equilibrium that is a direct generalization of the standard general equilibrium model with financial markets. Borrowers may default in equilibrium and returns on loans are determined endogenously. Restricted to a special form of mean variance preferences, we derive a version of the Capital Asset Pricing Model with bankruptcy. In this case we can characterize equilibrium prices and allocations and discuss implications for credit risk modeling.

JEL-Code: D530, G100.

Keywords: financial markets equilibrium, bankruptcy.

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October 7, 2013

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1 Introduction

This paper contributes to the literature about default in general equilibrium. It presents a framework that allows us to use the central ideas of this literature while at the same time keeping the model as close as possible to the standard general equilibrium model with financial markets. We also aim at providing a framework which allows a characterization of equilibrium prices and allocations going beyond abstract existence results and fully parametrized examples. We hope that this approach can mobilize the conceptual power of general equilibrium thinking for financial stability analysis.

Our paper makes the following new contributions: First, we model default penalties by extending the choices of agents to negative consumption plans. While the non-negative parts of the consumption plan is interpreted in the usual way the negative part of the consumption plan is identified with a default penalty. The utility function can simultaneously evaluate both consumption plans and default penalties. Second, we model default by restricting the default options of individuals to situations where they default only if all resources from endowments and other assets are insufficient to cover existing promises. Such a bankruptcy mechanism can be implemented while keeping anonymity of market exchange and price taking behavior. This is possible by introducing a clearing house that pools financial promises and has a monitoring and enforcement technology to verify bankruptcies and initiate the feasible payments. While building on the existing literature on default in general equilibrium our approach provides a structure that allows for a simple analysis of bankruptcy as an equilibrium phenomenon. It allows staying close to the standard model without default on financial promises. While non-linearities are introduced by driving a wedge between borrowing and lending rates, the single person decision problems remain linear. Looking at bankruptcies rather than arbitrary defaults, excludes equilibria of extreme pessimism, where financial markets break down. These potential pessimism equilibria led to relatively complicated equilibrium refinements in the previous literature. In our approach these refinements are not needed. We can thus give a fairly standard existence proof. Furthermore, making some more specific assumptions on preferences, our approach allows for an application that yields a bankruptcy version of the Capital Asset Pricing Model (CAPM). This bankruptcy-CAPM contains the standard CAPM as a special case. This allows for a more detailed study of the economics of bankruptcy and credit risk in general equilibrium by explicitly pinning down equilibrium prices and allocations.

Related Research The older literature on bankruptcy (Green [1973], Grandmont [1977] , Grandmont [1985]) is mostly conducted in a temporary equilibrium setting. It addresses already the main issues regarding existence of equilibrium. In particular it analyzed how to get a well-defined optimization problem of a borrower/lender. Issues that were already discussed in this literature are equilibrium existence with choice sets that are unbounded below, the modeling of penalties for choices which do not respect the feasibility of repayment in all states of the world ("planned default") as well as continuity of the budget and demand correspondence when the asset span depends on endogenous variables (Radner [1972]).

The literature on debt contracts during the eighties is focused primarily on information problems. An important topic in this literature was that costly state-verification can provide a justification for debt contracts with state-independent payoffs in states with no monitoring and where all assets of the debtor will be seized in states where no repayment occurs and monitoring takes place(Townsend [1979], Gale and Hellwig [1985]). This literature also discussed the idea that costly monitoring of debt contracts provides a rationale for a specialized institution that can be interpreted as a bank (Diamond [1984]).

The more recent literature most importantly Zame [1993] and Dubey et al. [2005] focuses on default rather than bankruptcy. This means that agents, regardless of their resources may decide to which degree they will fulfill financial promises, given a penalty proportional to the shortfall. The general equilibrium model with default and penalties was generalized to a continuum of states (Araujo et al. [1998]) and to infinite horizon models (Araujo et al. [1996]).

While our model is close to the models used in this literature our model has three main features in which it differs: First, we use a more general approach to model penalties. Instead of working with a separable function that adds utility of consumption and a penalty function for exceeding financial promises we use one function which simultaneously evaluates consumption and penalties. Second we study an "ability to pay" model (bankruptcy) rather than a "willingness to pay model" (default). In combination with our approach of modeling penalties, this feature allows an analysis that is closer to the standard model without default than the previous literature. Moreover, in special cases it also allows for a characterization of equilibrium prices and allocations. Finally the bankruptcy approach excludes no trade-equilibria due to extreme pessimism and therefore does not need equilibrium refinements as in Zame [1993] and Dubey et al. [2005].

We appeal to monitoring- and state-verification costs in order to justify the institution of a clearing house (bank) which acts as trading partner for debtors and lenders. It buys and sells bonds at differing rates for borrowing and lending and guarantees feasibility of contracts. It verifies the remaining assets of a debtor in case of bankruptcy and determines an aggregate recovery rate. In our model, individuals buy bonds from the clearing house at a guaranteed rate and lenders pay back to the clearing house either the contracted amount or their assets will be verified and seized.

Modeling the choice set of agents in a way that allows for feasible choices with negative components also arose in the finance literature discussing the notion of arbitrage. (see Werner [1987]. Dana et al. [1999] [1] In contrast to our paper this comes from introducing assets and securities directly in the preference relation. A negative component in a choice vector of an agent is in this case to be interpreted as a short position in a security. Negative consumption plans are not analyzed or allowed for in this literature.

Finally we would like to discuss our paper in the context of the large literature on

[1] In addition to providing new results the paper by Dana et al. [1999]) contains an overview of this literature and clarifies all the different arbitrage notions used there. Insead of enumerating the rather long list of papers on arbitrage and general equilibrium we refer the interested reader to this paper.

pecuniary default penalties in infinite horizon models, such as Kehoe and Levine [1993], Alvaraez and Jerman [2001], Kocherlakota [2008], Hellwig and Lorenzoni [2009], Bloise and Reichlin [2011] and Azariadis and Kaas [2013]. All of these papers consider in some way or another the option of temporary or partial exclusion of agents from financial markets trading as a consequence of defaulting on financial promises. This default option creates additional constraints in the agents decision problem and impacts on the welfare properties of equilibria. Actual default does not occur in equilibrium. In our model default (bankruptcy) does occur in equilibrium.

Structure of the paper We begin in section (2) with the analysis of a simple example of competitive borrowing and lending that illustrates the main concepts and idea of our analysis of credit risk in general equilibrium. In section (3) we describe and analyze the model, we define the concept of bankruptcy equilibrium. In section (4) we present the central results. We first prove existence of a bankruptcy equilibrium. In a next step, making more specific assumptions on preferences we derive a version of the Capital Asset Pricing Model with bankruptcy. This allows us to characterize equilibrium prices and quantities. Section (5) briefly discusses the implications of our analysis for credit risk modeling. Finally section(6) concludes. An appendix contains proofs of propositions stated in the main text.

2 Bankruptcy Equilibrium: An Example

We begin our analysis of bankruptcy equilibrium with an example that is simple enough to allow for a graphic exposition, yet rich enough to introduce the main concepts and ideas. The problem we would like to analyze is competitive markets for borrowing and lending with the possibility of bankruptcy. Building on the main ideas in the literature on default in general equilibrium, we aim at a conceptual framework that is both a simple and a natural extension of the traditional concept of financial market equilibrium as discussed for instance in Magill and Quinzii [1995].

In our example, two risk averse agents live for one period starting today ($t = 0$) and ending tomorrow ($t = 1$). They have endowments of a consumption good today and tomorrow. The endowments are described by the vectors $\omega^1 = (\omega^1_0, \omega^1_1)$ and $\omega^2 = (\omega^2_0, \omega^2_1)$ with all entries positive. They have standard preferences for the consumption good modeled by a utility function $u^i : X^i \mapsto \mathbb{R}$, where X^i denotes the agent's consumption set.

At $t = 0$, agents competitively trade a bond which promises one unit of income at $t = 1$. The bond trades at price q and the quantities of the bond chosen by the agents are denoted by z^i. The bond is in zero net supply. It is a financial instrument that allows borrowing and lending and, thus, for an inter-temporal transfer of the consumption good.

In the textbook model of financial market equilibrium, institutional arrangements of market exchange are such that, knowing only their own goals and endowments and observing prices, agents will always be able to stay within their budget constraints in each state of the world. We relax this assumption. Agents can make financial promises exceeding their resources tomorrow in some state of the world. Hence, bankruptcy may occur. We will consider uncertainty in the general form of our model described in the next section. In this example, however, for the sake of outlining our institutional setup in the simplest possible way, we restrict attention to a single state of the world at $t = 1$.

To allow for anonymous and competitive exchange the institutional and informational structure of financial market exchange is specified in the following way. There is a clearing house through which the agents can indirectly exchange financial promises. The clearing house is a passive intermediary. In $t = 0$, it collects payments from agents long in the bond. With these funds it provides resources to agents with a short position in the bond. In $t = 1$, the clearing house collects repayments. If there is a shortfall in contracted repayments the clearing house will confiscate the resources of the respective agents and use these proceeds to partially redeem financial claims of lenders. The distribution of proceeds follows a proportional rationing rule. The amount of the collected proceeds endogenously determines the rate of return for lenders. Moreover, if bankruptcy occurs, a utility penalty is applied to the agent who does not fulfill his promises. The penalty increases with the shortfall. Bankruptcy penalties are a modeling shortcut to describe costs of default for the agent without modeling them in detail.

We view this institutional arrangement as a costly state verification setup (Townsend [1979]), where the clearing house monitors, collects and distributes payments. It has the possibility to verify the state in case payments are not forthcoming and it has the authority to enforce payments and apply penalties.

A simple way to formalize these ideas is to extend the domain X^i of preferences: We assume that agents have a utility function with standard properties allowing to evaluate *both* positive consumption plans as well as penalties. Penalties are identified with negative consumption plans. Thus, with bankruptcy penalties, $X^i \subset \mathbb{R}_+ \times \mathbb{R}$, in contrast to the no bankruptcy case $X^i \subset \mathbb{R}_+ \times \mathbb{R}_+$. We will assume that a negative consumption value, $x^i < 0$, reflects the degree of bankruptcy, while a non-negative consumption value, $x^i \geq 0$, represents actual consumption. With this interpretation, our utility function embeds a *penalty function* which is strictly increasing in the value of the planned shortfall in a bankruptcy:[2] $x^{i-} := x^i \wedge 0$. Denoting the value of actual consumption by the non-negative value $x^{i+} = x^i \vee 0$ we have $x_1^i = x_1^{i+} + x_1^{i-}$. Thus, the consumer evaluates a consumption plan with respect to *both* the real consumption x_s^{i+} and the default penalty x_s^{i-} by a common utility index $u^i(x_0^i, x_1^i) = u^i(x_0^i, (x_1^{i+} + x_1^{i-}))$. To our knowledge, this is a novel approach to model utility penalties for bankruptcy that has not been used in the literature before.

The general idea to model costs of default by a utility penalty is not new and may be traced back to Dubey et al. [2005] and Zame [1993]. As most of the older literature, Dubey et al. [2005] and Zame [1993] model utility from real consumption $\widetilde{u}^i(x_0^i, x_1^{i+})$ and the penalty function $w^i(x_1^{i-})$ as additively separable preferences $\widetilde{u}^i(x_0^i, x_1^{i+}) + \lambda w^i(x_1^{i-})$. This utility function can be viewed a a special case of our utility function which evaluates real consumption and default penalty jointly: $u^i(x_0^i, x_1^i) := u^i(x_0^i, (x_1^{i+} + x_1^{i-}))$. Another

[2] We use the notation $\vee$ and $\wedge$ as the maximum and minimum operator. Applied to vectors the operators give the component-wise maximum of minimum.

example consistent with our modeling approach would be the quadratic utility function as observed by Magill and Quinzii [2000] [3].

There are thus two key elements in our bankruptcy model which in combination distinguish it from the older literature: First agents can not arbitrarily default on their promises. As long as their resources allow they have to pay. In our institutional framework this can be enforced because the clearing house is assumed to have the monitoring technology to enforce feasible payments. Second the utility function can rank bundles of consumption plans and bankruptcy penalties thought of as negative consumption plans. Since exchange of promises is intermediated by the clearing house this exchange is anonymous. Agents only observe security prices and the repayment rate on the bond.

This institutional framework can be viewed as a standard debt contract as analyzed in Gale and Hellwig [1985]: A standard debt contract is characterized by fixed repayment in all states where no bankruptcy occurs and full recovery (seizure of the entire endowment) in bankruptcy states.

For the simple example with two agents, one state and one bond, the equilibrium problem can be discussed geometrically in a net trade diagram as in [Magill and Quinzii, 1995, Figure 10.1.]. The diagram is drawn in the space of net income transfers defined by $\tau^i = x^i - \omega^i$, with net income transfers in $t = 0$ on the x-axes and net income transfers in $t = 1$ on the y-axes. With the possibility of bankruptcy, the payoff of a bond depends on whether the agent is a borrower or a lender. Hence, long and short positions have to be distinguished. At price q an agent can achieve all net income transfers along the vector $(-q, r)$ with a long position $z^i_+ = (0 \vee z^i)$. With a short position $z^i_- = -(0 \wedge z^i)$ he can achieve all net income transfers starting at the origin and extending along the ray $(q, -1)$. The difference in the return rate on bond holdings and on bond sales reflects the fact that in case of a bankruptcy the bond pays not the promise 1 but only the smaller return rate r

$$r = \frac{z^i_- \wedge \omega^i_1}{z^i_-}. \tag{1}$$

If we denote by $Z = \mathbb{R}_+ \times \mathbb{R}_+$ the set of feasible portfolios $z^i = (z^i_+, z^i_-)$ and by T the bond return matrix

$$T = \begin{bmatrix} -q & q \\ r & -1 \end{bmatrix},$$

the set of feasible income transfers is given by

$$\mathcal{C} = \{\tau \in \mathbb{R}^2 | \tau = Tz \quad z \in Z\}$$

where the index of agents has been suppressed.

Without bankruptcy (the case $r = 1$) this set is a linear space since the promise of the bond is always 1 no matter whether the position is long or short. This linear space becomes the cone $\mathcal{C}$ in the bankruptcy case.

The rays of the net transfer space $\mathcal{C}$ are drawn from the upper-left orthant to the lower right orthant reflect the fact that financial markets must be free of arbitrage

[3] Assume there are $S + 1$ states of the world, each state occurring with probability $\rho_s > 0$, this would be the function $u^i(x^i) = -\frac{1}{2}\sum_{s=0}^{S} \rho_s(\alpha^i - x^i_s)^2$.

opportunities in equilibrium. So every unit long in the bond yielding a positive payoff r tomorrow requires to give up $-q$ in terms of consumption good today. Every unit short in the bond pays q units of the consumption good today but carries an obligation of repayment tomorrow and eventually a utility penalty. The requirement of no arbitrage is equivalent to the requirement that there exist strictly positive state prices $\pi = (\pi_0, \pi_1)$ in the polar cone of the net transfer cone $\mathcal{C}$:

$$\mathcal{C}^* = \{\pi \in \mathbb{R}^2 | \pi\tau \leq 0 \quad \forall \tau \in \mathcal{C}\} \tag{2}$$

In the graph this is the shaded cone starting at the origin in the upper right orthant of the net transfer space. In a model without bankruptcy this cone is a linear space. Since long bonds have a different return than short bonds, however, the present value of an income stream is different depending on whether it implies a positive or negative transfer at $t = 1$. Hence, the net transfer space has a kink. Bankruptcy leads to a non-linear equilibrium valuation of contingent claims. In Figure 1 it is shown that in equilibrium contingent claims that amount to a long position in the bond are valued at $\bar{\pi}^l$ while contingent claims that amount to a short position are valued at $\bar{\pi}^s$.

The consumption spaces of agents 1 and 2 in the net trade diagram are half spaces through the endowment points containing all bundles x^i to their right. To mark the difference to the textbook model the shadings of the axes are such that both negative as well as positive consumption plans belong to the choice sets of agents.

In this example, agent 2 is the lender. He finds it optimal to choose a consumption bundle which requires an investment in the bond. At his optimal decision the gradient of his utility function is orthogonal to the ray $(-q, r)$. Since negative consumption, interpreted as utility penalties, is possible, budget sets are not bounded below because the choice sets are not bounded from below[4]. Agent 1 is the borrower. His optimal choice is a bundle with high consumption today at the cost of no consumption plus a penalty tomorrow. At the optimum the gradient of his utility function is orthogonal to the ray $(q, -1)$. Borrowing excessively today, he can achieve this choice within the prevailing financial structure.

We define a bankruptcy equilibrium as a situation where agents take optimal decisions, financial markets clear for these decisions and positive consumption plans are compatible with the available resources. The picture illustrates how the clearing house arrangement makes such an equilibrium allocation feasible. Contrary to the standard model in which q adjusts such that the net transfer space is rotated until all net trades balance, in a bankruptcy equilibrium two parameters, bond price q and the recovery rate r, must be adjusted.

Denoting the agents' net transfers for given $(\bar{q}, \bar{r})$ by $\tau^1(\bar{q}, \bar{r}))$ and $\tau^2(\bar{q}, \bar{r})$, the bond price and return rate $(\bar{q}, \bar{r})$ are a *bankruptcy equilibrium* when

$$(\tau^1(\bar{q}, \bar{r})) \vee -\omega^1) + (\tau^2(\bar{q}, \bar{r}) \vee -\omega^2) = 0.$$

[4] To obtain a bounded budget set additional assumptions are necessary. As explained in the next section, one can either introduce a lower bound bond trades through some kind of short selling restriction as in Radner [1972] or appeal to a stricter notion of no arbitrage as in Werner [1987] and Dana et al. [1999].

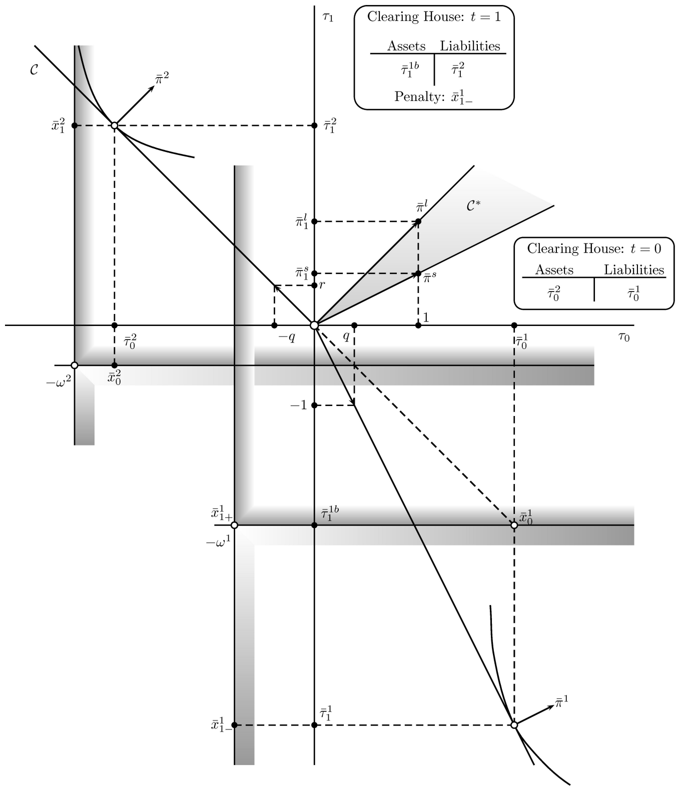

Figure 1: Bankruptcy equilibrium for the case with two agents and one state of the world. Agent 1 is the borrower and agent 2 is the lender. Agent 1 takes an optimal choice by going bankrupt at $t = 1$. In equilibrium $(\bar{q}, \bar{r})$ are such that the actual net trades balance. The bond is exchanged indirectly via a clearing house which has the opportunity to enforce payments and apply penalties and rationing of claims in a bankruptcy. The clearing house is like a passive intermediary. It's balance sheet at $t = 0$ and at $t = 1$ are shown in the picture. The balance sheet shows that the claim of agent 2, the lender is rationed to $\bar{\tau}_1^{1b}$ and the clearing house confiscates and distributes agent 1's, the borrower's, endowment and applies the utility penalty $\bar{x}_-^1$.

To interpret the equilibrium condition in terms of portfolio choices, let $z^1(\bar{q}, \bar{r})$ and $z^2(\bar{q}, \bar{r})$ be the optimal portfolio choices and $x^1(\bar{q}, \bar{r})$ and $x^2(\bar{q}, \bar{r})$ the optimal consumption choices at $(\bar{q}, \bar{r})$. Since agents must not choose a negative consumption in $t = 0$, security markets must clear [5],

$$z^1(\bar{q}, \bar{r}) + z^2(\bar{q}, \bar{r}) = 0,$$

and consumption in $t = 1$ must be feasible,

$$(x^1(\bar{q}, \bar{r}) \vee 0) + (x^2(\bar{q}, \bar{r}) \vee 0) = \omega_1^1 + \omega_1^2.$$

The condition that consumption at $t = 1$ must be feasible is equivalent to rational expectations about the return rate $\bar{r}$, i.e., the return rate enters consumers' decision

[5] This condition is equivalent to $\tau_0^1 + \tau_0^2 = 0$.

problems equals the return rate actually realized at $t = 1$. Hence, an alternative set of equilibrium conditions would require security markets to clear and agents correctly expecting the equilibrium return rate $\bar{r}$.[6]

The diagram shows how the bond market can equilibrate with bankruptcy. The transfer space of the lender has to be rotated by rationing his claim from the ray $(-q, 1)$ to $(-q, r)$ such that if this ray is prolonged to an imaginary linear space it passes through the zero consumption line of agent 1 at $\bar{x}_1^1 = 0$. In this position the market can clear because at this point the *actual* net income transfers sum to zero and thus $(\tau^1(\bar{q}, \bar{r})) \vee -\omega^1) + (\tau^2(\bar{q}, \bar{r}) \vee -\omega^2) = 0$. The picture also shows the balance sheets of the clearing house at $t = 0$ and at $t = 1$.

This example reveals in a nutshell the basic ideas and concepts of our notion of a bankruptcy equilibrium. Clearly, in a two-agent example, the clearing-house mechanism appears artificial. For many agents, however, trading financial promises solves a complicated information and coordination problem. Moreover, there are securitization markets which work exactly like asset pools financed by debt as in the clearing house construction.

Finally, how is credit risk entering the picture? This aspect of bankruptcy can only be seen in the general more complex model with many states of nature which we will develop in the next section. In the context of several states of nature, an endowment of zero in a single state would make loans impossible and, thus, eliminate all intertemporal trade, if no bankruptcy were possible. In such a situation, a bankruptcy equilibrium can be an improvement for all agents even in the face of bankruptcy penalties. This aspect of default was a central point of Zame [1993] as well as Dubey et al. [2005].

The general model considers also other financial instruments which may be used for risk sharing. Agents then take optimal portfolio decisions and financial instruments are priced according to their risk characteristics. Clearly the decision of some agents to choose a consumption plan that implies bankruptcy in some state is a credit risk from the viewpoint of the lenders. This is a risk, however, that arises endogenously as a consequence of the agents' decisions. Hence, parameters of credit risk such as the probability of default, the exposure at default and the recovery rate, which are assumed to follow an exogenous probability law in the usual credit risk models, will be endogenously determined in equilibrium.

3 The Model

3.1 A Bond Equity Economy

We consider a pure exchange economy with one commodity and a finite number I of agents. There are two dates, $t = 0$ and $t = 1$, and a finite number S of states of the world at date $t = 1$.

[6] This is analogous to the general equilibrium, multigood financial market model, where agents have to correctly anticipate equilibrium goods prices at $t = 1$ when making their plans today (see Radner [1972]).

Each agent is characterized by a closed and convex choice set $X^i \subset \mathbb{R}_+ \times \mathbb{R}^S$, a continuous, strongly monotone and concave utility function $u^i : X^i \to \mathbb{R}$ and an initial endowment $\omega^i \in X^i \cap \mathbb{R}^{S+1}_{++}$. We denote by ω^i_0 the endowment at $t = 0$ and by $\omega^i_{\mathbf{1}} = (\omega^i_1, \ldots, \omega^i_S)$ the endowment vector at $t = 1$. In a similar manner we denote by $x^i = (x^i_0, x^i_{\mathbf{1}}) \in X^i$ a consumption-bankruptcy plan of agent i.

The choice set is the same for all agents and consists of both consumption plans and potential bankruptcy penalties at $t = 1$. We assume that for all agents $X^i \subset \mathbb{R}_+ \times \mathbb{R}^S$, since we identify negative consumption plans with bankruptcy penalties. There are no outstanding claims in period 0. Hence, we assume that $x^i_0 \geq 0$. As explained above, the utility function u^i evaluates *both* consumption plans and bankruptcy penalties.

To achieve a consumption profile optimally adapted to their risk preferences agents can trade $J + 1$ securities. These securities are best thought of as a bond-equity security structure as in [Magill and Quinzii, 1995, p. 177]. First, there are J securities with payoff profile $y^j = (y^j_1, \ldots, y^j_S) \in \mathbb{R}^S_+$. These state-contingent payoffs can be though of as the exogenous output of J firms at $t = 1$. Equity represents claims to this output. Each agent owns $\delta^i \in \mathbb{R}^J_+$ claims initially. Equity can not be sold short. The $S \times J$ matrix of all equity payoffs y^j is denoted by Y. Each consumer chooses a portfolio $\theta^i \in \mathbb{R}^J_+$ of equity. The net-purchases (sales) of equities are denoted by $z^i_e = (\theta^i - \delta^i) \in \mathbb{R}^J_+ - \left\{\delta^i\right\}$. The no-short sale constraint for equities requires that or z^i_e must always be greater or equal to $-\delta^i$. Equity prices are denoted by $q_e \in \mathbb{R}^J$.

In addition to the equity markets there is also a market for a debt instrument. It can be though of as a bond, which allows the agents to make loans. For simplicity, we consider only one bond. The bond promises one unit of the consumption good in each state of the world. Agents can take on debts by trading the bond. A consumer may sell bonds even if the sale would require repayments exceeding the resources in some state. The consumer may well choose a bond sale leading to "negative consumption" in some state if the benefits from the loan in other states justify it. As discussed before, we interpret the disutility from "negative consumption" as a bankruptcy penalty which the agent suffers, even if the actual allocation will deliver a feasible consumption of zero in such a state.

Trade in bonds takes place between consumers and an agency which operates as a clearing house managing repayments. If there is a state where an agent's endowment does not suffice to cover the promised repayment from a bond sale, the agent will be bankrupt. A bankrupt agent will lose all resources in the respective state to the clearing house agency. The agency determines a return rate $r_{\mathbf{1}} \in (0, 1]^S$ which is paid to the bond holders. Hence, an agent who has invested in the bond has to take into account that the payoff profile of a bond purchase is perhaps only $r_{\mathbf{1}} \in (0, 1]^S$, falling short of the contracted repayment in some states.

We use a different notation for long and short positions in the bond. We define the positions of agent i long in the bond by z^i_{b+} and the positions short in the bond by z^i_{b-}. We assume that there is a short-selling constraint κ on the bond. The set of feasible portfolios is the set $Z := \mathbb{R}_+ \times [0, \kappa] \times \mathbb{R}^J_+ - \left\{\delta^i\right\}$. A portfolio is a tuple $z^i = (z^i_{b+}, z^i_{b-}, z^i_e) \in Z$. We assume that $J < S$. Hence, our analysis covers both the case of incomplete as well as complete markets. If $J = S - 1$ the availability of a bond completes the financial market system. When $J < S - 1$ we have incomplete markets.

Normalizing the price of the consumption good to 1, we can now define the budget set of agent i by

$$\mathbb{B}^i(q, r_{\mathbf{1}}) = \left\{ (x_0^i, x_{\mathbf{1}}^i) \in X^i \;\middle|\; \begin{array}{lll} x_0^i - \omega_0^i & \leq & -q_b z_{b+}^i + q_b z_{b-}^i - q_e z_e^i \\ x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i - Y\delta^i & \leq & r_{\mathbf{1}} z_{b+}^i - \mathbb{1} z_{b-}^i + Y z_e^i \\ & & (z_{b+}^i, z_{b-}^i, z_e^i) \in Z \end{array} \right\} \quad (3)$$

Note that the recovery rate r_s on the bond in each state is taken as a parameter by the consumer. Consumers are assumed to maximize their utility subject to this budget constraint. Recall also that consumption x_s^i may become negative in some state s, indicating that the consumer is bankrupt in this state and receives a bankruptcy penalty corresponding to this negative consumption value.

The bounds on short sales implicitly bound the choice set X^i from below [7]. This is a modeling assumption that ensures that we always have a well defined agent optimization problem when security prices fulfill a standard no arbitrage condition as for instance in Magill and Quinzii [1995]. An advantage of this lower bound is the fact, that we can consider various special cases depending on the size of this borrowing constraint κ. For $\kappa = 0$, we cover the case of an economy without credit. For $\kappa > 0$ but small enough, one can consider an economy without bankruptcy. With $\kappa > 0$ sufficiently large, bankruptcy is a possibility and returns on the bond will be determined endogenously.

3.2 Bankruptcy and the clearing house

Bankruptcy refers to a set of institutional arrangements specifying the reallocation of claims among economic agents. An agent is *bankrupt* , when the value of his debts exceeds the value of his assets. In the two period framework employed here this condition can be unambiguously defined. [8]

In case of a bankruptcy, all remaining assets of the debtor will be seized and distributed among the creditors. The remaining debt will be forgiven. Two institutional aspects are essential for the economic outcome: Firstly, how will the remaining assets be distributed among the claim holders? Secondly, what kind of penalty will be imposed on the bankrupt agent for not paying back the contracted amount of debt?

[7] There is a literature starting from Werner [1987] and analyzed in depth in Dana et al. [1999] where stronger no-arbitrage notions together with additional assumptions on utility functions endogenously bound the choice set, so that absence of arbitrage is necessary and sufficient for equilibrium existence when the choice set is unbounded. Since we want to focus attention on the bankruptcy clearing mechanism and the implications for the endogenous return on bonds we choose the short selling constraint approach.

[8] Such a formalization has been used in the literature in different versions by Modica et al. [1998], Sabarwal [2003], Araujo and Pascoa [2002]. It is also close to the framework of Eisenberg and Noe [2001], which shows how one can extend our bankruptcy rule to many loan instruments and non-anonymous bankruptcy in a pure balance sheet mechanics framework. A bankruptcy occurs if agents cannot repay their due liabilities. In contrast to Zame [1993] and Dubey et al. [2005], we do not allow agents to default on their loans. Agents will repay their debts as long as the value of their endowments and equity allows it. If liabilities exceed this value a bankruptcy occurs.

Bankruptcy laws specify these rules. A penalty is necessary in order to provide incentives for borrowers to repay their debts. Penalties for bankruptcies usually consist in excluding bankrupt individuals, at least temporarily, from further credit and constraining their consumption to a minimum for some period. These penalties depend also on the size of the losses which creditors suffer. Modeling the consequences of bankruptcy by a utility penalty is a simplification for not modeling these consequences in detail.

If a bankruptcy occurs existing nominal claims of the bond holders can no longer be satisfied. In order to model bankruptcy in perfect competition, where agents act as price takers, the bankruptcy mechanism must be anonymous. Anonymity of bankruptcy can be formalized by assuming that bond transactions are mediated through some central clearing institution that distributes shortfalls on promised payments among creditors. In this model, the clearing house collects the remaining assets of bankrupt agents $\omega_s^i + Y_s(\delta^i + z_e^i) < z_{b-}^i$ and distributes their value to the creditors. If repayments $\left(\sum_{i=1}^I z_{b-}^i \wedge (\omega_s^i + Y_s(\delta^i + z_e^i)\right)$ fall short of aggregate promises $\sum_{i=1}^I z_{b-}^i$ in some state s then these claims will be reduced proportionally.[9] Hence, one obtains the *return rate* [10] $r_s \in [0, 1]$,

$$r_s = \begin{cases} \frac{\sum_{i=1}^I z_{b-}^i \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\sum_{i=1}^I z_{b-}^i} & \text{if } \sum_{i=1}^I z_{b-}^i > 0 \\ 1 & \text{if } \sum_{i=1}^I z_{b-}^i = 0 \end{cases} \tag{4}$$

When planning their consumption and investments consumers will take this *recovery rate* r_s into account as an expected parameter which *will be determined in equilibrium.*

3.3 Bankruptcy Equilibrium

Let $u = (u^1, \ldots, u^I)$, $\omega = (\omega^1, \ldots, \omega^I)$ and $\delta = (\delta^1, \ldots, \delta^I)$ be the vectors of individual utility functions and individual endowments of goods and equity, respectively, and denote by $V = [-\mathbb{1}, Y]$ the exogenously given equity payoffs. We denote by $\mathcal{E} = (u, \omega, \delta, V)$ the corresponding economy.

[9] Define for any two vectors $x, y \in \mathbb{R}^n$ the lattice operations $x \wedge y := (min(x_1, y_1), \cdots, min(x_n, y_n))$ and $x \vee y := (max(x_1, y_1), \cdots, max(x_n, y_n))$. By Y_s we denote the s-th row of the matrix Y

[10] Since the recovery rate is only defined when there is some trade in the bond we define the recovery rate in cases where there is no bond trade as 1 by a continuous extension.

Definition 1 (Equilibrium)**.** *A* financial market equilibrium with bankruptcy *of the economy* $\mathcal{E} = (u, \omega, \delta, V)$ *is a tuple of consumption plans, portfolio choices, security prices and recovery rates* $(\bar{x}, \bar{z}_{b+}, \bar{z}_{b-}, \bar{z}_e, \bar{q}, \bar{r}_{\mathbf{1}}) \in X^I \times Z^I \times \mathbb{R}_+^{J+1} \times (0, 1]^S$ *such that for all* $i = 1, \ldots, I$

(i) $\bar{x}^i \in \arg\max\{u^i(x^i) \mid x^i \in \mathbb{B}^i(\bar{q}, \bar{r}_{\mathbf{1}})\}$, (optimal behavior)

(ii) $\sum_{i=1}^I \bar{z}_{b+}^i - \sum_{i=1}^I \bar{z}_{b-}^i = 0$ *and* $\sum_{i=1}^I \bar{z}_e^i = 0$, (asset market clearing)

(iii) $\sum_{i=1}^I \bar{x}^{i+} = \sum_{i=1}^I (\omega^i + Y\delta^i)$, (feasible allocation)

(iv) $\bar{r}_s = \begin{cases} \frac{\sum_{i=1}^I (\bar{z}_{b-}^i \wedge (\omega_s^i + Y_s(\delta^i + \bar{z}_e^i))}{\sum_{i=1}^I \bar{z}_{b-}^i} & \text{if } \sum_{i=1}^I \bar{z}_{b-}^i > 0 \\ 1 & \text{if } \sum_{i=1}^I \bar{z}_{b-}^i = 0 \end{cases}$ (clearing mechanism)

The definition of equilibrium has some redundancy, since conditions (ii), (iii) and (iv) are not independent. Indeed, in the appendix we use this fact in the existence proof. The credit market clearing mechanism is a central feature of our economy. Including it explicitly in the definition of an equilibrium makes the interrelation between the clearing mechanism, the feasibility of consumption and the endogeneity of the return rate more transparent.

In equilibrium feasibility of the consumption allocation is guaranteed by condition (iii). This does not preclude that consumers choose an amount of debt which leads to a negative value of the consumption plan in some state. Hence, $\bar{x}_s^i$ can be negative in some states, representing the bankruptcy penalty experienced by this consumer. Creditors hold rational expectations about the recovery rate in states where bankruptcy occurs (iv). Only in this case security market clearing (*ii*) and good market clearing (*iii*) can be fulfilled simultaneously. Bankruptcy is factored into the asset price system $\bar{q}$. Note that the standard general equilibrium concept without bankruptcy is a special case of the financial market equilibrium with bankruptcy when $r_s = 1$ for all states s. This special case can be obtained by choosing the credit constraint κ small enough.

4 Bankruptcy Equilibrium: Results

4.1 Bankruptcy Equilibrium: Existence

We first show that our equilibrium concept is well defined. We show that, given the assumptions on preferences, endowments and securities in Proposition 1, a bankruptcy equilibrium will always exist.

Proposition 1 (Existence of Bankruptcy Equilibrium)**.** *Let $\mathcal{E} = (u, \omega, \delta, V)$ be a finance economy. If*

A1	(consumption sets)	X^i *is a non-empty, closed and convex subset of* $\mathbb{R}_+ \times \mathbb{R}^S$,
A2	(preferences)	$u^i : X^i \to \mathbb{R}$ *is continuous, strongly monotone, and concave,*
A3	(endowments)	$\omega^i = (\omega^i_0, \omega^i_{\mathbf{1}}) \in X^i \cap \mathbb{R}^{S+1}_{++}$ *and* $\sum_{i=1}^I \delta^i \in \mathbb{R}^J_{++}$,
A4	(asset returns)	*the matrix* $V = [-\mathbb{1}\ Y]$ *has full column rank,*
A5	(asset trades)	*there is* $\kappa \geq 0$ *such that* $z^i_{b-} \leq \kappa$ *for all* $i = 1, \ldots, I$.

then a bankruptcy equilibrium exists.

Proof: The proof is given in the appendix. □

In the appendix, we prove existence of a bankruptcy equilibrium. Mostly, we could use standard arguments. We had to modify the market equilibrium lemma of Grandmont [1988], however, in order to guarantee the existence of a consistent bankruptcy scheme operated by a clearing house.

Existence of a bankruptcy equilibrium has been proved in slightly different settings by Sabarwal [2003] and for a slightly different version of the model by Modica et al. [1998] and Araujo and Pascoa [2002].[11] Zame [1993] and Dubey et al. [2005] prove existence for a similar model with default. In these models consumers can deliberately decide what fraction of their promise they are going to repay. Hence, returns on a loan may become zero and there can be equilibria where expectations on bond recoveries are so pessimistic that there is no trade in the bond. In contrast, in our model where bankruptcy of an agent implies the seizure of all remaining assets by the clearing house, there is always some positive return on a loan from the endowments of the debtor. Hence, no trade in the bond due to overly pessimistic expectations cannot occur in an equilibrium under bankruptcy.

4.2 Bankruptcy Equilibrium and the CAPM

To make the bankruptcy model more useful for economic analysis, we want to go beyond the abstract discussion of the previous sections and add enough structure to the model such that we are able to study equilibrium prices and allocations explicitly. The aim is thus to arrive at a formulation of the model that is more specific than the abstract discussion yet more general than a fully parametrized example.

The formulation we are going to suggest and analyze now is to specify the bankruptcy model along the lines of a CAPM model, widely used in finance and economics.

[11] Araujo and Pascoa [2002] have no utility penalties but short selling constraints on the debt instruments, Sabarwal [2003] has T periods and no penalties but short selling constraints on the debt instruments. Modica et al. [1998] study a model where agents can become bankrupt without penalty in states of the world of which they are ex ante unaware of. Obviously all these models are closely related.

Instead of general preferences we now assume that preferences are defined by an additively separable, linear quadratic utility function

$$u^i(x^i) = \alpha_0^i x_0^i - \frac{1}{2}\sum_{s=1}^{S} \rho_s(\alpha_1^i - x_s^i)^2, \quad i = 1, \ldots, I, \tag{5}$$

where we have also time 1 state probabilities are given by the ρ_s for $s = 1, \ldots, S$. Note that this utility function provides a natural example of the class of preferences we studied in the previous section. It jointly evaluates the utility from consumption and utility penalties identified with negative consumption plans. This is an observation that has been made by Magill and Quinzii [2000] to give an interpretation to negative consumption plans usually admitted in the CAPM.[12]

Define $\alpha_0 = \sum_i \alpha_0^i$, $\alpha_{\mathbf{1}} = \sum_i \alpha_1^i$ and $\omega_{\mathbf{1}} = \sum_i \omega_{\mathbf{1}}^i$ and $\delta = \sum_i \delta^i$. By assuming that all preference parameters $(\alpha_0^i, \alpha_1^i) \in \mathbb{R}_{++}^2$ are chosen such that for each agent $\alpha_1^i \mathbb{1} - \omega_{\mathbf{1}} \in \mathbb{R}_{++}^S$ where $\mathbb{1}$ is the S-dimensional vector consisting of components equal to 1 we can exclude specific technical problems associated with potential satiation. Such an assumption is used for instance in Nielsen [1989] to ensure existence of equilibrium with preferences allowing satiation. Basically the assumption ensures monotonicity of utility on those regions of the choice set X that correspond to a feasible allocation.

Our specification of preferences assumes agents who care about the mean and the variance of their consumption plans. The advantage of this specific restriction is that it allows a characterization of equilibrium prices, portfolios and consumption.

We are now going to discuss equilibrium pricing and allocations in this CAPM specification of the model with bankruptcy. We discuss these results along the lines of the exposition in [Magill and Quinzii, 1995, chapter 3, 17] to highlight the similarity as well as the differences to the standard CAPM model.

4.3 Equilibrium Security Pricing: Adjusting the Market Portfolio for Credit Risk

Our first result refers to security pricing with linear quadratic preferences. To simplify the exposition we assume that the short selling constraints on equity will not be binding in equilibrium. We can always achieve this by an appropriate choice of security endowments δ^i.

For our discussion we use the probability induced inner products

$$\langle x_{\mathbf{1}}, y_{\mathbf{1}} \rangle = \sum_{s=1}^{S} \rho_s x_s y_s \quad \forall\, x_{\mathbf{1}}, y_{\mathbf{1}} \in \mathbb{R}^S \text{ and} \tag{6}$$

$$\langle x, y \rangle = \sum_{s=0}^{S} \rho_s x_s y_s \quad \forall\, x, y \in \mathbb{R}^{S+1}, \tag{7}$$

[12] Unlike in the general case discussed before these preferences have a satiation point and thus the utility function is not monotone on its whole domain. It is well-known that one can obtain monotonicity on the relevant compact and convex set of state-contingent consumption by choosing for instance for all consumers satiation points outside the set of feasible allocations.

where $\rho_0 := 1$.

Define the vector

$$\bar{\gamma} := \sum_{i=1}^{I} \nabla u^i(\bar{x}^i) = (\alpha_0, \alpha_1 \mathbb{1} - ((\omega_{\mathbf{1}} + Y\delta) - d_{\mathbf{1}}))^T,$$

where $d_{\mathbf{1}} = \sum_{i=1}^{I} d_{\mathbf{1}}^i := \sum_{i=1}^{I} (\mathbb{1} - r_{\ \mathbf{1}}) z_{b-}^i$ is the aggregate shortfall from promises on the bond. This vector $\bar{\gamma}$ expresses the equilibrium marginal evaluation of income streams that can be generated within the given financial structure.

Proposition 2. *If $(\bar{x}, \bar{z}, \bar{q}, \bar{r}_{\mathbf{1}})$ is a bankruptcy equilibrium of the economy $\mathcal{E}(u, \omega, \delta, V)$ with non-binding short selling constraints on equity and strictly positive date zero consumption $\bar{x}_0^i$ then*

(i) there exist strictly positive constants a and b such that $\bar{\gamma}$, fulfills

$$\bar{\gamma}_{\mathbf{1}} = a\,\mathbb{1} - b\,\tilde{\omega}_{\mathbf{1}},$$

(ii) denote the equilibrium market value of any income stream m that can be generated by a linear combination of the existing securities by $c(m)$. It fulfills the weak inequality

$$c(m) \geq \langle \bar{\gamma}_{\mathbf{1}}, m \rangle = E(\bar{\gamma}_{\mathbf{1}})E(m) - b\,cov(\tilde{\omega}_{\mathbf{1}}, m),$$

where $\tilde{\omega}_{\mathbf{1}} = \big((\omega_{\mathbf{1}} + Y\delta) - (\mathbb{1} - r_{\mathbf{1}}) \sum_i \bar{z}_-^i\big)$ is the aggregate endowment $\omega_{\mathbf{1}} + Y\delta$ reduced by the aggregate shortfall in promises on the bond market.

Proof: The proof is given in the appendix. □

From Proposition 2 we see that security prices in a bankruptcy equilibrium look similar to security prices in a financial market equilibrium without bankruptcy and quadratic preferences (see for instance [Magill and Quinzii, 1997, Proposition 1]. The most important change is that in a bankruptcy equilibrium the role taken by the aggregate endowment $\omega_{\mathbf{1}}$ is now replaced by the aggregate endowment corrected for the aggregate shortfalls from bankruptcy $\tilde{\omega}_{\ \mathbf{1}}$.

The pricing formula that results from the CAPM without bankruptcy shows that the price of a security is a decreasing linear function of the covariance of the income stream provided by this security with aggregate income risk in the economy as a whole. If for instance $\omega_{\mathbf{1}} \in \text{span}(Y)$, the aggregate income risk could be interpreted as a benchmark portfolio, called the market portfolio in the finance literature. If a security is positively (negatively) correlated with aggregate income it's covariance value is negative (positive). Bankruptcy changes this insight in the sense that aggregate income risk can only be described in equilibrium. Aggregate income risk or the "market portfolio" has to be adjusted by the planned shortfall in financial promises. Aggregate income risk that is relevant for determining the covariance value of securities is endogenous.

Since this quantity determines the marginal value of income that can be achieved by trading in financial securities, the prices of *all* securities whether or not they are affected by credit risk are influenced by the trading of a defaultable security. This is of course a typical general equilibrium effect. The specific structure of our model allows however to say much more. The bankruptcy CAPM says that in the valuation of any financial security in a market containing at least one defaultable financial instrument, we can make a CAPM like valuation by correcting the market portfolio by the aggregate shortfall in promises on the credit instruments.

Since in a bankruptcy equilibrium agents can't go short in the security promising $r_{\mathbf{1}}$ and also can't go long in the security $\mathbb{1}$ income streams m that can be generated from the existing securities can not anymore be valued by a linear function. However Proposition 2 shows that we can give valuation bounds for income streams that can be replicated, similar as in the literature on portfolio constraints (see Luttmer [1996]).

4.4 Allocations: A Two Fund Separation Result

We can also characterize equilibrium allocations such that the relationship to two fund separation theorems characteristic for the CAPM can be seen. Again for simplifying notation and to make the analogy to the CAPM more visible, let us assume that short selling constraints on equity are not binding. The structure of bankruptcy equilibrium requires that we characterize the consumption-default plans of agents depending on whether they are long or short in the bond in a bankruptcy equilibrium. As in the case with pricing the role of the aggregate endowment $\omega_{\ \mathbf{1}} + Y\delta^i$ is now taken by the aggregate endowment corrected for aggregate shortfalls in promises $\tilde{\omega}_{\mathbf{1}}$. Since constraints on the possible bond positions (z_b^{i+}, z_b^{i-}) may bind for some agents, we get additional terms that depend on the Lagrangian multipliers of the respective constraints.

Proposition 3. *Let* $(\bar{x}, \bar{z}, \bar{q}, \bar{r}_{\mathbf{1}})$ *is a bankruptcy equilibrium of the economy* $\mathcal{E}(u, \omega, \delta, V)$ *with non-binding short selling constraints on equity and strictly positive date zero consumption* $\bar{x}_0^i$*: If in a bankruptcy equilibrium an agent i trades long in the bond, her equilibrium consumption plan is given by*

$$\bar{x}_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + P_{Y_{b+}}\left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - (\omega_{\mathbf{1}}^i + Y\delta^i - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}})\right) - \sigma_{b+}\frac{\alpha_0^i}{\alpha_0}r_{\mathbf{1}e},$$

where $\sigma_{b+} := \sum_{i=1}^I \sigma_{b+}^i$ *is the sum of all agents' Lagrange multipliers corresponding to the constraints* $z_{b+}^i \geq 0$, $P_{Y_{b+}}$ *is the projection on the span of the matrix* $(r_{\mathbf{1}}, Y)$ *and*

$$r_{\mathbf{1}e} := \frac{r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})}{||r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})||^2}.$$

If agent i trades short in the bond, her equilibrium allocation is given by

$$\bar{x}_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + P_{Y_{b-}}\left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - (\omega_{\mathbf{1}}^i + Y\delta^i - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}})\right) - \sigma_{b-}\frac{\alpha_0^i}{\alpha_0}\mathbb{1}_e,$$

where $\sigma_{b-} := \sum_{i=1}^I \sigma_{b-}^i$ *is the sum of all agents' Lagrange multipliers corresponding to the constraints* $z_{b-}^i \geq 0$, $P_{Y_{b+}}$ *is the projection on the span of the matrix* $(-\mathbb{1}, Y)$ *and*

$$\mathbb{1}_e := \frac{\mathbb{1} - P_Y(\mathbb{1})}{||\mathbb{1} - P_Y(\mathbb{1})||^2}.$$

If in equilibrium agent i does not trade in the bond, her equilibrium allocation is given by

$$\bar{x}_{\mathbf{1}}^{i} = \omega_{\mathbf{1}}^{i} + P_Y \left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - (\omega_{\mathbf{1}}^i + Y\delta^i - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \right).$$

Proof: The proof is given in the appendix. □

From Proposition 3 we see that in the bankruptcy equilibrium consumption is characterized by an approximate linear risk sharing rule both for agents long and short in the bond with the aggregate endowment corrected for shortfalls in the promises made by bond trading. (compare to the no bankruptcy case in Magill and Quinzii [1997]).

In the standard CAPM an investor's date 1 consumption is obtained from aggregate income via a linear sharing rule. In this model therefore investor's hold fully diversified portfolios and each investor's portfolios are a proportion of the "market portfolio". In a bankruptcy equilibrium this property is changed in two ways. First, and most importantly even, if the aggregate endowment is in the span of the available equities as can thus be interpreted as an aggregate output, the relevant benchmark portfolio is endogenous. This is because aggregate output has to be corrected by the planned shortfall in financial promises. Second, the optimal date one consumption allocations and thus portfolio decisions differ between agents who are short in the bond and agents who are long in the bond. It is as if the agents who are going long in the bond and agents who are going short in the bond face a different asset structure. For the first class of agents the bond pays $r_{\mathbf{1}}$, for the others the relevant promise is $\mathbb{1}$.

While Proposition 2 and 3 show the analogy between the standard CAPM and the bankruptcy CAPM, they do not allow to explicitly calculate equilibrium values from the exogenous parameters, since the short positions of the bond traders have to be known to determine $\tilde{\omega}_{\mathbf{1}}$, the shortfall corrected market portfolio. Still the modeling approach to bankruptcy suggested here, does allow - in contrast to other models in the literature - to arrive at characterizations of equilibrium prices and allocations which are slightly more general than fully parameterized numerical examples. We hope that these results will turn out useful for the further study of the economics of bankruptcy and default in a general equilibrium context.

One issue where the explicit equilibrium characterization presented here might add new insights to the existing literature is the question of welfare benefits of default in incomplete markets as initiated in examples by Zame [1993] and Dubey et al. [2005]. Our characterization in combination with the analysis in Magill and Quinzii [1997] might help to investigate these issues more systematically. We leave this for future research.

5 Endogenous Risk

Our model provides a framework to reflect some recent developments in credit risk modeling (see McNeil et al. [2005] for an overview). Most of the models emerging from this literature and applied in risk management at banks and financial institutions try to capture default risk and credit losses from debt exposures by deriving a loss distribution of a portfolio of debt instruments by taking the probability of default, the recovery rate and the exposure at default as stemming from an exogenous source of randomness. In this viewpoint risk management decisions are conceptually treated like a game against nature or a single person decision problem. While our model has also an abstract notion of exogenous randomness, modeled as different states of the world, the parameters of credit risk are really endogenous variables that are determined by the self interested interactions of many different individuals. In the perspective of our model, credit risk is conceptually more like an equilibrium problem rather than a decision problem. It would be an interesting problem for future research to systematically derive the consequences for credit risk modeling for risk management purposes. Our paper could provide a useful starting point for such a project. Clearly, a discussion of this kind needs to build on a more specific model such as the linear quadratic utility case studied in section 4.2.

The perspective on credit risk as an endogenous risk also has consequences for analyzing liquidity. For instance in Figure 1 used in our introductory example, it becomes immediately clear that the liquidity of the bond market is directly related to the credit risk of the bond. If credit risk is high, return spreads between long and short bond positions are high. If we look at liquidity from this perspective it immediately becomes clear that it makes no sense to think of liquidity as an exogenous property of particular asset classes, as it frequently seems to be done in policy discussions. Liquidity, like credit risk, is endogenous and is an equilibrium phenomenon.

6 Conclusions

In this paper we have studied a model of bankruptcy in general equilibrium that develops the bankruptcy model as a simple generalization of the standard general equilibrium model with financial markets. The key idea to achieve this is to extend the choices of agents to negative consumption plans and to interpret the utility of a negative consumption plan as a utility penalty. This idea together with the requirement that individuals can only be in default when all their resources are exhausted (bankruptcy) makes the model particularly tractable at the level of individual decisions. With the assumption of strictly positive endowments it also rules out extreme pessimism equilibria where financial market trading breaks down. We therefore can dispense with equilibrium refinements used in the literature on default in general equilibrium. The simplicity of the model also allows together with the assumption of mean variance preferences the formulation of a bankruptcy version of the Capital Asset Pricing Model (CAPM). This allows for this case also the characterization of equilibrium prices and allocations.

We hope that this paper helps to make the literature on default in general equilibrium more useful for economic applications. In particular the CAPM version of the model could be an interesting starting point to study some issues concerning default in general equilibrium beyond specific examples. Two questions, which we hope to tackle in future research concern the welfare effects of default and the question of coherent approaches to credit risk management once it is acknowledged that credit risk has conceptually more the nature of an equilibrium problem rather than a single person decision problem.

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7 Appendix

7.1 Proof of Proposition 1

Our proof consists of eight steps. In Step (i), we show that the set of arbitrage-free security prices for a recovery rate of 1, i.e., without bankruptcies, remains arbitrage-free for all return rates with bankruptcies, $r_{\mathbf{1}} \leq 1$. Appealing to strong monotonicity of preferences (A2), we show in Step (ii) that no consumer will go long and short in the bond simultaneously and that the budget constraints in all states in period 1 must be binding as long as the recovery rates are strictly positive in every state $r_{\mathbf{1}} > 0$. Step (ii) shows that utility as a function of a portfolio and consumption in period 0 is continuous and strongly monotone. Given assumptions A3 and A5, standard arguments show that the budget correspondence in period 0 must be compact-, convex-valued and continuous (Step (iii)). Using the concavity of the utility function (A2), in Step (iv), we prove that demand for assets and consumption in period 0 is a compact- and convex-valued u.h.c. correspondence of prices and recovery rates. In Step (v) we study excess demand correspondences for assets and consumption in period 0. In particular, we derive boundary conditions and show that Walras' law holds. Step (vi) shows that the recovery rates are continuous functions of asset trades and, for strictly positive endowments (A3) and a borrowing constraint κ (A5) are bounded away from zero, i.e., $r_{\mathbf{1}} > 0$. Step (vii) adapts the market equilibrium lemma of Grandmont [1988] in order to prove that there is an asset price system and consistent recovery rates for all states such that asset markets and the market for consumption in period 0 clear. Finally, in step (viii), we prove that consistent return rates and asset market clearing ensures that consumption in all states of period 1 is feasible, i.e., that also condition (iii) of Definition 1 is satisfied.

(i) Arbitrage-free prices: We begin with a characterization of the security prices which admit no arbitrage.

By assumptions A3 and A5, aggregate bond supply $\sum_{i\in I} z^i_{b-}$ is bounded above by $|I| \cdot \kappa$ and aggregate equity holdings are $\delta := \sum_{i\in I} \delta^i \in \mathbb{R}^J_{++}$. since there are no short sales in equity, the set of feasible security trades is $Z = \mathbb{R}_+ \times [0,\kappa] \times (\mathbb{R}^J_+ - \delta)$. Recall with the payoff matrix

$$T = \begin{bmatrix} -q_b & q_b & -q_e \\ r_{\mathbf{1}} & -\mathbb{1} & Y \end{bmatrix},$$

the set of net income transfers that can be generated by security trades is

$$\mathcal{C} = \{\tau \in \mathbb{R}^{S+1} | \tau = Tz \quad z \in Z\}.$$

Given T the financial markets admit no arbitrage opportunities if there is no $(z_{b+}, z_{b-}, z_e) \in Z$ such that $T\,(z_{b+}, z_{b-}, z_e) \geq 0$. This is the same as saying that $\mathcal{C} \cap (\mathbb{R}^{S+1}_+ \setminus \{0\}) = \emptyset$, i.e., it is impossible to find a portfolio which generates a non-negative state-contingent consumption plan.

Since Z is bounded below, we need to modify the fundamental theorem of arbitrage-free pricing appropriately. Denote by $R := (0,1]^S$ the set of state-contingent return rates.

Lemma 1. *For given $r_{\mathbf{1}} \in R$, there are no arbitrage opportunities if and only if there is a vector of positive state prices $\pi \in \mathbb{R}^S_{++}$ such that $\pi T \leq 0$.*

Proof: Let $\Delta = \{\tau \in \mathbb{R}^{S+1}_{+} | \sum_{s=0}^{S} \tau_s = 1\}$ denote the non-negative simplex in $\mathbb{R}^{S+1}$. Then $\mathcal{C} \cap (\mathbb{R}^{S+1}_{+} \setminus \{0\}) = \emptyset$ implies that $\mathcal{C} \cap \Delta = \emptyset$. Obviously, both $\mathcal{C}$ and Δ are convex sets. Since $\mathcal{C}$ is the image of the closed polyhedral set $Z \subset \mathbb{R}^{J+2}$ under the linear map T, it is closed. Moreover, the simplex Δ is a compact set. Hence, we can apply a strict separation theorem (e.g., Magill and Quinzii [1995], Theorem 9.4, p. 73) in order to conclude that there is a linear functional $0 \neq \pi \in \mathbb{R}^{S+1}$ such that

$$\sup_{\tau \in \mathcal{C}} \pi\tau < \inf_{\tau \in \Delta} \pi\tau.$$

Applying the same arguments as in the proof of Magill and Quinzii [1995] Theorem 9.3. (iii), one obtains the inequality $\pi\tau \leq 0$ for all $\tau \in \mathcal{C}$. This inequality cannot be turned into an equality, however, because $\tau \in \mathcal{C}$ does not imply that $-\tau \in \mathcal{C}$ because of the lower bound on Z. $\square$

For incomplete markets, there will be many π fulfilling the no arbitrage inequality [13]. For a given recovery rate $r_{\mathbf{1}} \in R := (0,1]^S$, let us denote by

$$\widetilde{Q}(r_{\mathbf{1}}) := \left\{ q \in \mathbb{R}^{J+1} | \pi T \leq 0,\ \pi \in \mathbb{R}^{S+1}_{++} \right\}$$

the set of all arbitrage free security prices. This can be written equivalently as

$$\widetilde{Q}(r_{\mathbf{1}}) := \left\{ (q_b, q_e) \in \mathbb{R}^{J+1} |\ \pi_{\mathbf{1}} r_{\mathbf{1}} \leq \pi_0 q_b \leq \pi_{\mathbf{1}} \mathbb{1},\ \pi_{\mathbf{1}} Y - \pi_0 q_e \leq 0, \quad \text{for some} \quad (\pi_0, \pi_{\mathbf{1}}) \in \mathbb{R}^{S+1}_{++} \right\}$$

Since $(\pi_0, \pi_{\mathbf{1}}) > 0$, w.l.o.g., one can normalize $\pi_0 = 1$. For any $j \in J$, denote by

$$\underline{y}^j = \min \left\{ y^j_s |\ s \in S \right\}, \quad \underline{r} = \min \left\{ r_s |\ s \in S \right\}$$

and let

$$\underline{y} = \min \left\{ \underline{y}^j |\ j \in J \right\}.$$

Lemma 2.

$$\widetilde{Q}(r_{\mathbf{1}}) = \left\{ (q_e, q_b) \in \mathbb{R}^{J+1} | \quad q_b > 0,\ \ q^j_e > \underline{y} q_b \quad \textit{for all} \quad j \in J. \right\}.$$

Proof: *(i) Necessity*: Suppose there exists $\pi \in \mathbb{R}^{S+1}_{++}$ such that

$$\text{for all} \quad j \in J \quad \sum_{s \in S} y^j_s \pi_s \leq q^j_e, \quad \sum_{s \in S} r_s \pi_s \leq q_b \leq \sum_{s \in S} \pi_s.$$

By Assumption A4 and the premise $r_{\mathbf{1}} \in R := (0,1]^S$, $y^j \neq 0$ for all $j \in J$ and $r_{\mathbf{1}} \neq 0$. Hence,

$$\begin{aligned} q^j_e &\geq \sum_{s \in S} y^j_s \pi_s > 0, \\ q_b &\geq \sum_{s \in S} r_s \pi_s > 0. \end{aligned}$$

[13] For general convex constraint sets the characterization is more involved (see Elsinger and Summer [2001]).

Moreover, by Assumption A4, y^j cannot be constant. Hence, for all $j \in J$,

$$q_e^j \geq \sum_{s \in S} y_s^j \pi_s > \underline{y}^j \sum_{s \in S} \pi_s \geq \underline{y} q_b.$$

(ii) Sufficiency: Suppose $(q_e, q_b) \in \mathbb{R}^{J+1}$ are such that

$$q_b > 0, \;\; q_e^j > \underline{y} q_b \quad \text{for all} \quad j \in J.$$

Since $q_e^j > 0$ for all $j \in J$ and $q_b > 0$, there exists ε such that, for all $j \in J$,

$$\begin{aligned} 0 &< \varepsilon \sum_{s \in S} y_s^j < q_e^j, \\ 0 &< \varepsilon \sum_{s \in S} r_s < q_b. \end{aligned}$$

a) If there exists $s \in S$ such that $y_s^j = 0$ for all $j \in J$, then one can choose any π with $\pi_s \geq \varepsilon$ for all $s \in S$ and $\sum_{s \in S} \pi_s \geq q_b$.
b) If for all $s \in S$ there exists some $j \in J$ such that $y_s^j > 0$, then consider the following optimization problem

$$\begin{array}{lll} & \max_{\pi} \sum_{s \in S} \pi_s & \\ \text{subject to} & q_e^j - \sum_{s \in S} y_s^j \pi_s \geq 0 & \text{for all } j \in J \\ & \pi_s - \varepsilon \geq 0 & \text{for all } s \in S. \end{array}$$

For $J < S$,[14] the constraint set is compact and convex with a non-empty interior. The objective function is continuous. Hence, a maximizer exists and it is characterized by the FOC of the appropriate Lagrangian function.
Given non-negative multipliers λ^j, $j \in J$, and μ_s, $s \in S$, the FOCs of the Lagrangian

$$\sum_{s \in S} \pi_s + \sum_{j \in J} \lambda_j \left[q_e^j - \sum_{s \in S} y_s^j \pi_s \right] + \sum_{s \in S} \mu_s \left[\pi_s - \varepsilon \right]$$

are

$$\begin{array}{lll} \text{(i)} & 1 - \sum_{j \in J} \lambda_j y_s^j + \mu_s = 0 & \text{for } s \in S, \\ \text{(ii)} & \lambda^j \left[q_e^j - \sum_{s \in S} y_s^j \pi_s \right] = 0, & \text{for all } j \in J, \\ \text{(iii)} & \mu_s \left[\pi_s - \varepsilon \right] = 0 & \text{for } s \in S. \end{array}$$

By (i) and (ii), $\sum_{j \in J} \lambda_j y_s^j = 1 + \mu_s > 0$. Hence, for some $j \in J$, $\lambda^j > 0$ and

$$q_e^j - \sum_{s \in S} y_s^j \pi_s = 0.$$

[14] Note that we write $J < S$ to be consistent with our assumption (A4). The proof of Lemma 2 would also work with $J \leq S$. In the motivating example we have $J = S = 1$. In this case there are no issues stemming from potential linear dependencies in the asset span, since there is only the bond.

Also from (i), we can conclude for all $s \in S$,

$$\mu_s = \sum_{j \in J} \lambda_j y_s^j - 1 \geq 0.$$

It is impossible that $\mu_s > 0$ for all s, since $\pi_s = \varepsilon$ for all s implies

$$\sum_{s \in S} y_s^j \pi_s = \varepsilon \sum_{s \in S} \pi_s < q_e^j$$

for all $j \in J$, which contradicts the fact that, for some $j \in J$, we have $q_e^j - \sum_{s \in S} y_s^j \pi_s = 0$. Hence, $\mu_s = 0$ must hold for some $s \in S$. It follows from (iii) that

$$\begin{aligned} \mu_s &= \sum_{j \in J} \lambda_j y_s^j - 1 = 0 \quad \text{for all } s \text{ with } y_s^j = \underline{y} \text{ for some } j \in J \\ \mu_s &= \sum_{j \in J} \lambda_j y_s^j - 1 > 0 \quad \text{for all } s \text{ with } y_s^j > \underline{y} \text{ for all } j \in J. \end{aligned}$$

Hence

$$\begin{aligned} \mu_s &= \sum_{j \in J} \lambda_j y_s^j - 1 = 0 \quad \text{for all } s \text{ with } y_s^j = \underline{y} \text{ for some } j \in J \\ \mu_s &= \sum_{j \in J} \lambda_j y_s^j - 1 > 0 \quad \text{for all } s \text{ with} y_s^j > \underline{y} \text{ for all} j \in J. \end{aligned}$$

For an optimal (π, λ, μ), complementary slackness implies

$$\begin{aligned} 0 &= \sum_{j \in J} \lambda_j \left[q_e^j - \sum_{s \in S} y_s^j \pi_s \right] + \sum_{s \in S} \mu_s \left[\pi_s - \varepsilon \right] \\ &= \sum_{j \in J} \lambda_j q_e^j - \sum_{j \in J} \lambda_j \sum_{s \in S} y_s^j \pi_s + \sum_{s \in S} \mu_s \pi_s - \sum_{s \in S} \mu_s \varepsilon \\ &= \sum_{j \in J} \lambda_j q_e^j - \varepsilon \sum_{s \in S} \mu_s - \sum_{s \in S} \underbrace{\left[\sum_{j \in J} \lambda_j y_s^j - \mu_s \right]}_{=1} \pi_s \\ &= \sum_{j \in J} \lambda_j q_e^j - \varepsilon \sum_{s \in S} \mu_s - \sum_{s \in S} \pi_s. \end{aligned}$$

Hence, the optimal π satisfies:

$$\begin{aligned}
\sum_{s\in S}\pi_s &= \sum_{j\in J}\lambda_j q_e^j - \varepsilon\underbrace{\sum_{s\in S}\mu_s}_{:=A} \\
&> \sum_{j\in J}\lambda_j \underline{y} q_b - \varepsilon A \\
&= \underbrace{q_b\left[\sum_{j\in J}\lambda_j \underline{y}\right]}_{=1} - \varepsilon A \\
&= q_b - \varepsilon A
\end{aligned}$$

where the strict inequality follows from the premise $\underline{y}q_b < q_e^j$ for all $j \in J$. From $\mu_s = \sum_{j\in J}\lambda_j y_s^j - 1 > \sum_{j\in J}\lambda_j \underline{y} - 1 = 0$ for some $s \in S$, one concludes $A > 0$. Hence, for ε small enough, we have

$$\sum_{s\in S}\pi_s \geq q_b.$$

□

The set

$$\widetilde{Q}(r_{\mathbf{1}}) = \left\{(q_e, q_b) \in \mathbb{R}^{J+1} |\ \ q_b > 0,\ \ q_e^j > \underline{y}q_b \quad \text{for all} \quad j \in J. \right\}$$

is clearly an open cone in $\mathbb{R}^{J+1}$.

Note that, for any $r_{\mathbf{1}}$, we have $\underline{r}q_b \leq q_b$ for all s. Hence, the set of arbitrage-free asset prices for any $r_{\mathbf{1}} \in R$, is given by

$$\widetilde{Q} := \widetilde{Q}(r_{\mathbf{1}}).$$

The set $\widetilde{Q}$ is also an open cone.

(ii) Induced utility function: For given $r_{\mathbf{1}} \in R$ and $q \in \widetilde{Q}$, each consumer $i \in I$ maximizes $u^i(x_0^i, x_{\mathbf{1}}^i)$ by choosing asset trades $(z_{b+}^i, z_{b-}^i, z_e^i) \in Z$ and a consumption allocation $(x_0^i, x_{\mathbf{1}}^i) \in X^i$ subject to

$$\begin{array}{lll}
(a) & x_0^i - \omega_0^i & \leq -q_b z_{b+}^i + q_b z_{b-}^i - q_e z_e^i, \\
(b) & x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i - Y\delta^i & \leq r_{\mathbf{1}} z_{b+}^i - \mathbb{1} z_{b-}^i + Y z_e^i, \\
(c) & z_{b-}^i & \leq \kappa, \\
(d) & z_e^i & \geq -\delta^i,
\end{array}$$

where the constraint (c) follows from A5 in Proposition 1.

Strong monotonicity of u^i(Proposition 1, A2) implies for an optimal choice $(x_0^i, x_{\mathbf{1}}^i, z_{b+}^i, z_{b-}^i, z_e^i)$ that condition (b) is binding:

$$x_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + Y\left(\delta^i + z_e^i\right) + r_{\mathbf{1}} z_{b+}^i - \mathbb{1} z_{b-}^i. \tag{8}$$

Moreover, for $r_{\mathbf{1}} \neq \mathbb{1}$, the consumer will never take a short position $z^i_{b-} > 0$ and a long position $z^i_{b+} > 0$ simultaneously. These observations allow us to restrict attention to net trades in bonds z^i_b and to simplify notation by restricting asset trades $\left(z^i_b, z^i_e\right)$ to the space $\mathbb{R}^{J+1}$.

Noting that $z^i_b := \left(z^i_{b+} - z^i_{b-}\right) = \left(z^i_b \vee 0\right) + \left(z^i_b \wedge 0\right) \in \mathbb{R}$, Equation 8 allows us to define the following *induced utility function* $v^i : \mathbb{R}_+ \times \mathbb{R}^{J+1} \times R \to \mathbb{R}$

$$v^i(x^i_0, z^i_b, z^i_e; r_{\mathbf{1}}) := u^i(x^i_0, \omega^i_{\mathbf{1}} + Y\left(\delta^i + z^i_e\right) + r_{\mathbf{1}}\left(z^i_b \vee 0\right) + \mathbb{1}\left(z^i_b \wedge 0\right))$$

By Assumption A2, the induced utility function v^i is a *continuous* function. Moreover, v^i is *strongly monotone* in $\left(x^i_0, z^i_b, z^i_e\right)$.

(iii) Budget correspondence in period $\mathbf{t = 0}$: Using the induced utility function, we can now study the simplified consumer problem of maximizing $v^i(x^i_0, z^i_b, z^i_e; r_{\mathbf{1}})$ subject to

$$\begin{array}{llcl}
(a) & x^i_0 - \omega^i_0 & \leq & -q_b z^i_b - q_e z^i_e, \\
(b') & x^i_0 & \geq & 0, \\
(c') & z^i_b & \geq & -\kappa, \\
(d) & z^i_e & \geq & -\delta^i.
\end{array}$$

Without loss of generality, we give up the normalization of commodity prices in period $t = 0$, $p_0 = 1$, which has been used in the text so far. Denote by

$$\mathbb{B}^i_0(p_0, q_b, q_e) := \left\{(x^i_0, z^i_b, z^i_e) \in \mathbb{R}_+ \times \mathbb{R}^{J+1} |\ p_0\left(x^i_0 - \omega^i_0\right) + q_b z^i_b + q_e z^i_e \leq 0,\ z^i_b \geq -\kappa,\ z^i_e \geq -\delta^i\right\}$$

the *budget correspondence* for $t = 0$.

Lemma 3. $\mathbb{B}^i_0 : \mathbb{R}_{++} \times \widetilde{Q} \to \mathbb{R}_+ \times \mathbb{R}^{J+1}$, *is a compact- , convex-valued, and continuous correspondence.*

Proof: Since prices $(p_0, q_b, q_e) \in \mathbb{R}_{++} \times \widetilde{Q}$ are strictly positive and consumption and asset trades are bounded below (Assumptions A3 and A5), the set $\mathbb{B}^i_0(p_0, q_b, q_e)$ is compact. It is obviously convex. Continuity follows by standard arguments, e.g., Debreu [1956], p. 63. $\square$

(iv) Demand correspondences for consumption in period $\mathbf{t = 0}$ and asset trades: Since $\mathbb{B}^i_0(\lambda p_0, \lambda q_b, \lambda q_e) = \mathbb{B}^i_0(p_0, q_b, q_e)$ holds for any $\lambda > 0$, one can normalize prices to the unit simplex. Consider the price simplex in $\mathbb{R}^{J+2}_+$,

$$\overline{Q} := \left\{(p_0, q_b, q_e) \in \mathbb{R}^{J+2}_+ |\ p_0 + q_b + \sum q^j_e = 1\right\}.$$

$\overline{Q}$ is a compact and convex subset of $\mathbb{R}^{J+2}_+$. Let $\widehat{Q} := \overline{Q} \cap \left(\mathbb{R}_{++} \times \widetilde{Q}\right)$. It is easily checked that $\widehat{Q}$ is an open and convex subset of $\overline{Q}$.

For $r_{\mathbf{1}} \in R$ and $q \in \widehat{Q}$, define the demand correspondence $f^i : \widehat{Q} \times R \to \mathbb{R}^{J+2}$ by

$$f^i(p_0, q; r_{\mathbf{1}}) = \arg\max\left\{v^i(x^i_0, z^i_b, z^i_e; r_{\mathbf{1}}) |\ \left(x^i_0, z^i_b, z^i_e\right) \in \mathbb{B}^i_0(p_0, q_b, q_e)\right\}.$$

Lemma 4. *$f^i(p_0, q; r_{\mathbf{1}})$ is a non-empty, compact- and convex-valued, u.h.c. correspondence from $\widehat{Q} \times R$ into $\mathbb{R}^{J+2}$.*

Proof: The indirect utility v^i is a continuous function on $\mathbb{R}_+ \times \mathbb{R}^{J+1} \times R$. By Lemma 3, $\mathbb{B}_0^i(p_0, q_b, q_e)$ is a compact-, convex-valued and continuous correspondence on $\widehat{Q}$. Hence, by the maximum theorem, the demand correspondence $f^i(p_0, q; r_{\mathbf{1}})$ is non-empty, compact-valued, and u.h.c. for all $(p_0, q; r_{\mathbf{1}}) \in \widehat{Q} \times R$. It remains to show that $f^i(p_0, q; r_{\mathbf{1}})$ is convex-valued. Suppose there are two maximizers $\left(\widehat{x}_0^i, \widehat{z}_b^i, \widehat{z}_e^i\right)$ and $\left(\widetilde{x}_0^i, \widetilde{z}_b^i, \widetilde{z}_e^i\right)$ both in $f^i(p_0, q; r_{\mathbf{1}})$. To simplify notation, let $R(z_b) := r_{\mathbf{1}} \left(z_b^i \vee 0\right) + \mathbb{1} \left(z_b^i \wedge 0\right)$. It is easy to check that $R(z_b)$ is a concave function. Consider the state-contingent consumption $\widehat{x}_{\mathbf{1}}^i$ and $\widetilde{x}_{\mathbf{1}}^i$ corresponding to the portfolios $\left(\widehat{z}_b^i, \widehat{z}_e^i\right)$ and $\left(\widetilde{z}_b^i, \widetilde{z}_e^i\right)$,

$$\widehat{x}_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + Y\left(\delta^i + \widehat{z}_e^i\right) + R(\widehat{z}_b^i) \quad \text{and} \quad \widetilde{x}_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + Y\left(\delta^i + \widetilde{z}_e^i\right) + R(\widetilde{z}_b^i),$$

and note that, for any $\lambda \in [0,1]$, the convex combination $x_{\lambda\mathbf{1}}^i$ of $\widehat{x}_{\mathbf{1}}^i$ and $\widetilde{x}_{\mathbf{1}}^i$ satisfies

$$\begin{aligned} & x_{\lambda\mathbf{1}}^i := \lambda \widehat{x}_{\mathbf{1}}^i + (1-\lambda)\widetilde{x}^i{}_{\mathbf{1}} \\ = \;& \lambda\left[\omega_{\mathbf{1}}^i + Y\left(\delta^i + \widehat{z}_e^i\right) + R(\widehat{z}_b^i)\right] + (1-\lambda)\left[\omega_{\mathbf{1}}^i + Y\left(\delta^i + \widetilde{z}_e^i\right) + R(\widetilde{z}_b^i)\right] \\ \leq \;& \omega_{\mathbf{1}}^i + Y\left(\delta^i + \lambda\widehat{z}_e^i + (1-\lambda)\widetilde{z}_e^i\right) + R(\lambda\widehat{z}_b^i + (1-\lambda)\widetilde{z}_b^i), \end{aligned}$$

where the weak inequality follows since $R(z_b)$ is a concave function. Hence, $x_{\lambda\mathbf{1}}^i$ is affordable with the portfolio $\left(z_{\lambda b}^i, z_{\lambda e}^i\right) := \left(\lambda\widehat{z}_b^i + (1-\lambda)\widetilde{z}_b^i, \lambda\widehat{z}_e^i + (1-\lambda)\widetilde{z}_e^i\right)$. Moreover, since the budget correspondence is convex-valued

$$\left(x_{\lambda 0}^i, z_{\lambda b}^i, z_{\lambda e}^i\right) := \left(\lambda\widehat{x}_0^i + (1-\lambda)\widetilde{x}_0^i, \lambda\widehat{z}_b^i + (1-\lambda)\widetilde{z}_b^i, \lambda\widehat{z}_e^i + (1-\lambda)\widetilde{z}_e^i\right) \in \mathbb{B}_0^i(p_0, q_b, q_e).$$

Case (i): If $\widehat{x}_{\mathbf{1}}^i = \widetilde{x}_{\mathbf{1}}^i$, then $x_{\lambda\mathbf{1}}^i = \widehat{x}_{\mathbf{1}}^i = \widetilde{x}_{\mathbf{1}}^i$ and $u^i(x_{\lambda 0}^i, x_{\lambda\mathbf{1}}^i) = u^i(\widehat{x}_0^i, \widehat{x}_{\mathbf{1}}^i) = u^i(\widetilde{x}_0^i, \widetilde{x}_{\mathbf{1}}^i)$. Hence,

$$v^i\left(x_{\lambda 0}^i, z_{\lambda b}^i, z_{\lambda e}^i; r_{\mathbf{1}}\right) = v^i\left(\widehat{x}_0^i, \widehat{z}_b^i, \widehat{z}_e^i; r_{\mathbf{1}}\right) = v^i\left(\widetilde{x}_0^i, \widetilde{z}_b^i, \widetilde{z}_e^i; r_{\mathbf{1}}\right)$$

and $(x_{\lambda 0}^i, z_{\lambda b}^i, z_{\lambda e}^i) \in f^i(p_0, q; r_{\mathbf{1}})$.
Case (ii): If $\widehat{x}_{\mathbf{1}}^i \neq \widetilde{x}_{\mathbf{1}}^i$, then

$$u^i(x_{\lambda 0}^i, x_{\lambda\mathbf{1}}^i) \geq \lambda u^i(\widehat{x}_0^i, \widehat{x}_{\mathbf{1}}^i) + (1-\lambda)u^i(\widetilde{x}_0^i, \widetilde{x}_{\mathbf{1}}^i)$$

since u^i is concave by Assumption A2. Hence,

$$\begin{aligned} v^i\left(x_{\lambda 0}^i, z_{\lambda b}^i, z_{\lambda e}^i; r_{\mathbf{1}}\right) &\geq u^i(x_{\lambda 0}^i, x_{\lambda\mathbf{1}}^i) \\ &\geq \lambda u^i(\widehat{x}_0^i, \widehat{x}_{\mathbf{1}}^i) + (1-\lambda)u^i(\widetilde{x}_0^i, \widetilde{x}_{\mathbf{1}}^i) \\ &= \lambda v^i(\widehat{x}_0^i, \widehat{z}_b^i, \widehat{z}_e^i; r_{\mathbf{1}}) + (1-\lambda)v^i(\widetilde{x}_0^i, \widetilde{z}_b^i, \widetilde{z}_e^i; r_{\mathbf{1}}), \end{aligned}$$

where the first inequality follows from the fact that the utility of any $x_{\mathbf{1}}^{*i}$ which maximizes $u^i(x_{\lambda 0}^i, x_{\mathbf{1}}^i)$ subject to $x_{\mathbf{1}}^i \leq \omega_{\mathbf{1}}^i + Y\left(\delta^i + z_{\lambda e}^i\right) + r_{\mathbf{1}}\left(z_{\lambda b}^i \vee 0\right) + \mathbb{1}\left(z_{\lambda b}^i \wedge 0\right)$, yielding indirect utility $v^i\left(x_{\lambda 0}^i, z_{\lambda b}^i, z_{\lambda e}^i; r_{\mathbf{1}}\right) = u^i(x_{\lambda 0}^i, x_{\mathbf{1}}^{*i})$, must be at least as high as the utility of the feasible consumption vector $(x_{\lambda 0}^i, x_{\lambda\mathbf{1}}^i)$. Hence, $(x_{\lambda 0}^i, z_{\lambda b}^i, z_{\lambda e}^i) \in f^i(p_0, q; r_{\mathbf{1}})$.
This proves that $f^i(p_0, q; r_{\mathbf{1}})$ is convex-valued. □

(v) Excess demand correspondences Define the *individual excess demand correspondence* $\zeta^i : \widehat{Q} \times R \to \mathbb{R}^{J+2}$ as

$$\zeta^i(p_0, q; r_{\mathbf{1}}) := f^i(p_0, q; r_{\mathbf{1}}) - \left\{(\omega_0^i, 0, 0)\right\}.$$

Obviously, ζ^i inherits all relevant properties of f^i, in particular, ζ^i is a non-empty, compact- and convex-valued u.h.c. correspondence. Since $f^i(p_0, q; r_{\mathbf{1}}) \subseteq \mathbb{B}_0^i(p_0, q)$, individual excess demand correspondences ζ^i are bounded below by $(-\omega_0^i, -\kappa, -\delta^i)$. Consequently, the aggregate excess demand correspondence $\sum_{i\in I} \zeta^i(p_0, q; r_{\mathbf{1}})$ is bounded below by $\left(-\sum_{i\in I}\omega_0^i, -|I|\kappa, -\sum_{i\in I}\delta^i\right)$. Feasibility, i.e., $0 \in \sum_{i\in I}\zeta^i(p_0, q; r_{\mathbf{1}})$, implies that, in any equilibrium, the aggregate allocation must be also bounded above by

$$\left(\sum_{i\in I}\omega_0^i, |I|\kappa, \sum_{i\in I}\delta^i\right).$$

Denote by $\overline{K} := \left[\left(-\sum_{i\in I}\omega_0^i, -|I|\cdot\kappa, -\sum_{i\in I}\delta^i\right), \left(\sum_{i\in I}\omega_0^i, |I|\kappa, \sum_{i\in I}\delta^i\right)\right] \subset \mathbb{R}^{J+2}$ the compact and convex cube of feasible allocations in $\mathbb{R}^{J+2}$. By assumption A3 and A5, $\overline{K}$ is not empty.

Define the *bounded excess demand correspondence* $\overline{\zeta}^i : \widehat{Q} \times R \to \overline{K}$ as

$$\overline{\zeta}^i(p_0, q; r_{\mathbf{1}}) = \arg\max\left\{v^i(z_0^i + \omega_0^i, z_b^i, z_e^i; r_{\mathbf{1}}) \mid \left(z_0^i + \omega_0^i, z_b^i, z_e^i\right) \in \mathbb{B}_0^i(p_0, q_b, q_e) \cap \overline{K}\right\}.$$

Noting that $\mathbb{B}_0^i(p_0, q_b, q_e) \cap \overline{K}$ is a compact-, convex-valued, and continuous correspondence[15], it is obvious from the proof of the Lemma 4 that $\overline{\zeta}^i(p_0, q; r_{\mathbf{1}})$ is a non-empty, compact- and convex-valued u.h.c. correspondence.

Finally, let $\Xi(p_0, q; r_{\mathbf{1}}) := \sum_{i\in I}\overline{\zeta}^i(p_0, q; r_{\mathbf{1}})$ be the bounded aggregate excess demand correspondence. We will denote aggregate excess demand by $(sz_0, sz_b, sz_e) := \left(\sum_{i\in I} z_0^i, \sum_{i\in I} z_b^i, \sum_{i\in I} z_e^i\right)$, in order to distinguish it from an excess demand allocation $(z_0, z_b, z_e) := \left((z_0^1, z_b^1, z_e^1), ..., (z_0^I, z_b^I, z_e^I)\right)$.

It remains to check Walras law and to investigate the boundary behavior of $\Xi(p_0, q; r_{\mathbf{1}})$.

Lemma 5. *(i) Let $(p_0^n, q^n; r_{\mathbf{1}}^n)$ be a sequence in $\widehat{Q}\times R$ such that $(p_0^n, q^n; r_{\mathbf{1}}^n) \to (\overline{p}_0, \overline{q}; \overline{r}_{\mathbf{1}}) \in \partial\widehat{Q} \times R$. Then for every sequence $(sz_0^n, sz_b^n, sz_e^n) \in \Xi(p_0^n, q^n; {r^n}_{\mathbf{1}})$ there exists $(\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \in \widehat{Q}$ such that*

$$(\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e)\,(sz_0^n, sz_b^n, sz_e^n) > 0$$

holds for infinitely many n.

[15] If one views $\overline{K}$ as a constant correspondence on $\widehat{Q} \times R$, it is clearly a non-empty, compact- and convex-valued correspondence. Hence, $\mathbb{B}_0^i(p_0, q_b, q_e) \cap \overline{Z}$ is a non-empty, compact- and convex-valued correspondence.
The intersection of two closed-valued u.h.c. correspondences is always a u.h.c. correspondence (Green and Heller [1981], u.h.c.-Property (8), p. 48). The intersection of two *convex-valued* l.h.c. correspondences is l.h.c. if the intersection of their interiors is not empty (Green and Heller [1981], l.h.c.-Property (8), p. 48). The correspondences $\mathbb{B}_0^i(p_0, q_b, q_e)$ and $\overline{K}$ satisfy these properties. Thus, we can conclude that $\mathbb{B}_0^i(p_0, q_b, q_e) \cap \overline{K}$ is a continuous correspondence.

(ii) $\Xi(p_0, q; r_{\mathbf{1}})$ *satisfies Walras law: for all* $(sz_0, sz_b, sz_e) \in \Xi(p_0, q; r_{\mathbf{1}})$,

$$(p_0, q_b, q_e)(z_0, z_b, z_e) = 0.$$

Proof:

(i) Consider a sequence of prices $(p_0^n, q^n; r_{\mathbf{1}}^n) \to (\overline{p}_0, \overline{q}; \overline{r}_{\mathbf{1}}) \in \partial \widehat{Q} \times R$ and suppose the claim is not true. Then there exists a sequence

$$(sz_0^n, sz_b^n, sz_e^n) = \left(\sum_{i \in I} z_0^{in}, \sum_{i \in I} z_b^{in}, \sum_{i \in I} z_e^{in} \right) \in \Xi(p_0^n, q^n; r_{\mathbf{1}}^n)$$

with $\left(z_0^{in}, z^{in}\right) \in \overline{\zeta}^i(p_0^n, q^n; r_{\mathbf{1}}^n)$ for all $i \in I$ such that for all $(p_0, q_b, q_e) \in \widehat{Q}$

$$(p_0, q_b, q_e)(sz_0^n, sz_b^n, sz_e^n) \leq 0 \tag{9}$$

holds for all but a finite number of n.

For all $i \in I$, the sequence $\left(z_0^{in}, z^{in}\right)$ is contained in the compact set $\overline{K}$. Thus, there exists a converging subsequence $\left(z_0^{i\nu}, z^{i\nu}\right) \to \left(\overline{z}_0^i, \overline{z}^i\right)$ with $\left(z_0^{i\nu}, z^{i\nu}\right) \in \overline{\zeta}^i(p_0^\nu, q^\nu; r_{\mathbf{1}}^\nu)$. By strong monotonicity (Assumption A2), $(p_0^\nu, q_b^\nu, q_e^\nu)\left(z_0^{i\nu}, z_b^{i\nu}, z_e^{i\nu}\right) = 0$ and, by continuity,

$$(\overline{p}_0, \overline{q}_b, \overline{q}_e)\left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) = 0.$$

We show that $(\overline{z}_0^i, \overline{z}^i)$ maximizes utility $v^i(z_0^i + \omega_0^i, z^i; \overline{r}_{\mathbf{1}})$ at $(\overline{p}_0, \overline{q}; \overline{r}_{\mathbf{1}})$ for all consumers $i \in I$ for whom $\left(\overline{p}_0 \omega_0^i + \overline{q}_b \kappa + \overline{q}_e \delta^i\right) > 0$ holds. By assumptions A3 and A5 and the fact that $(\overline{p}_0, \overline{q}) \neq 0$, there exist consumers $i \in I$ with $\left(\overline{p}_0 \omega_0^i + \overline{q}_b \kappa + \overline{q}_e \delta^i\right) > 0$.
Consider a consumer $i \in I$ with $\left(\overline{p}_0 \omega_0^i + \overline{q}_b \kappa + \overline{q}_e \delta^i\right) > 0$ and assume that $(\overline{z}_0^i, \overline{z}^i)$ were not optimal at $(\overline{p}_0, \overline{q}; \overline{r}_{\mathbf{1}})$, then there would exists $(\tilde{z}_0^i, \tilde{z}^i)$ such that $(\tilde{z}_0^i + \omega_0^i, \tilde{z}^i) \in \mathbb{B}_0^i(\overline{p}_0, \overline{q})$ with $v^i(\tilde{z}_0^i + \omega_0^i, \tilde{z}^i; \overline{r}_{\mathbf{1}}) > v^i(\overline{z}_0^i + \omega_0^i, \overline{z}^i; \overline{r}_{\mathbf{1}})$. Since $\left(\overline{p}_0 \omega_0^i + \overline{q}_b \kappa + \overline{q}_e \delta^i\right) > 0$ holds, there exists a sequence $\left(\tilde{\overline{z}}_0^{i\nu}, \tilde{\overline{z}}^{i\nu}\right) \to (\tilde{z}_0^i, \tilde{z}^i)$ with $\left(\tilde{\overline{z}}_0^{i\nu} + \omega_0^i, \tilde{\overline{z}}^{i\nu}\right) \in \mathbb{B}_0^i(p_0^\nu, q^\nu)$ and, by continuity of v^i, $v^i(\tilde{\overline{z}}_0^{i\nu} + \omega_0^i, \tilde{\overline{z}}^{i\nu}; r_{\mathbf{1}}^\nu) > v^i(\overline{z}_0^{i\nu} + \omega_0^i, \overline{z}^{i\nu}; r_{\mathbf{1}}^\nu)$ for all ν sufficiently large. This contradicts the claim $\left(\overline{z}_0^{i\nu}, \overline{z}^{i\nu}\right) \in \overline{\zeta}^i(p_0^\nu, q^\nu; r_{\mathbf{1}}^\nu)$ for all ν. Thus $(\overline{z}_0^i, \overline{z}^i)$ must be optimal at $(\overline{p}_0, \overline{q}; \overline{r}_{\mathbf{1}})$.
For $(\overline{p}_0, \overline{q}) \in \partial \widehat{Q}$, however, the budget set is unbounded. This is obvious if some price $(\overline{p}_0, \overline{q})$ equals zero. If, for some $j \in J$, $q_e^j = \underline{y} q_b$ holds, then the portfolio $\left(z_b^i, z_e^i\right) = \left(-\underline{y}, e_j\right) a$, where e_j denotes the j-th unit vector in $\mathbb{R}^J$, is self-financing, i.e., $\left(q_e^j - q_b \underline{y}\right) a = 0$ for any $a > 0$, and yields a return $\left(y_s^j - \underline{y}\right) a \geq 0$, which is strictly positive in at least one state $s \in S$, by Assumption A4. Strong monotonicity of u^i (Assumption A2) implies that $(\overline{z}_0^i, \overline{z}^i)$ is an element of the upper boundary of $\overline{K}$. Since $(\overline{z}_0^i, \overline{z}^i)$ is in the upper boundary of $\overline{K}$, we have $\overline{z}_0^i > 0$ if $\overline{p}_0 = 0$, $\overline{z}_b^i > 0$ if $\overline{q}_b = 0$ and $\overline{z}_e^{ji} > 0$ if $\overline{q}_e^j = 0$. Moreover, if $\overline{q}_e^j = \overline{q}_b \underline{y}$, we have $\overline{z}_e^{ji} = \frac{\kappa}{\underline{y}} > 0$ and $\overline{z}_b^i = -\kappa$ with $\left(\overline{q}_e^j \overline{z}_e^i + \overline{q}_b \overline{z}_b^i\right) = \left(\overline{q}_e^j - \overline{q}_b \underline{y}\right) \frac{\kappa}{\underline{y}} = 0$. Choose (p_0^o, q_b^o, q_e^o) such that $p_0^o = \varepsilon_0 > 0$ if $\overline{p}_0 = 0$,

$q_b^o = \varepsilon_b > 0$ if $\overline{q}_b = 0$, $q_e^{jo} = \varepsilon_e^j + \overline{q}_b \underline{y}$ with $\varepsilon_e^j > 0$ if $\overline{q}_e^j = \overline{q}_b \underline{y}$, and $\varepsilon_0 + \varepsilon_b + \sum_{j \in J} \varepsilon_e^j = 1$. By construction, $(p_0^o, q_b^o, q_e^o) \in \widehat{Q}$ and $(p_0^o, q_b^o, q_e^o) \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) > 0$. Let $I' \subseteq I$ be the non-empty set of consumers with $\left(\overline{p}_0 \omega_0^i + \overline{q}_b \kappa + \overline{q}_e \delta^i\right) > 0$. Then, for $\alpha > 0$

$$(\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) := \alpha \left(p_0^o, q_b^o, q_e^o\right) + (1 - \alpha) \left(\overline{p}_0, \overline{q}_b, \overline{q}_e\right) \in \widehat{Q}$$

and $(\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) > 0$ for all $i \in I'$.

Hence, $(\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \sum_{i \in I'} \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) > 0$ and, since $\left(z_0^{i\nu}, z_b^{i\nu}, z_e^{i\nu}\right) \to \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right)$, it follows that there exists N such that

$$(\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \sum_{i \in I'} \left(z_0^{i\nu}, z_b^{i\nu}, z_e^{i\nu}\right) > 0 \text{ for all } \nu > N.$$

Moreover, $(\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \sum_{i \in I \setminus I'} \left(z_0^{i\nu}, z_b^{i\nu}, z_e^{i\nu}\right)$ converges to $\alpha \left(p_0^o, q_b^o, q_e^o\right) \sum_{i \in I \setminus I'} \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right)$ since, for all $i \in I$, $\left(\overline{p}_0, \overline{q}_b, \overline{q}_e\right) \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) = 0$. Hence, there is α small enough such that

$$\begin{aligned} & (\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \sum_{i \in I} \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) \\ = \quad & (\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \sum_{i \in I'} \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) + \alpha \left(p_0^o, q_b^o, q_e^o\right) \sum_{i \in I \setminus I'} \left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) > 0. \end{aligned}$$

This implies that there is N such that, for all $\nu > N$,

$$(\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \left(sz_0^\nu, sz_b^\nu, sz_e^\nu\right) = (\widetilde{p}_0, \widetilde{q}_b, \widetilde{q}_e) \sum_{i \in I} \left(z_0^{i\nu}, z_b^{i\nu}, z_e^{i\nu}\right) > 0$$

in contradiction to Equation 9.

(ii) For every $(sz_0, sz_b, sz_e) \in \Xi(p_0, q; r_{\mathbf{1}})$, strong monotonicity of v^i implies

$$p_0 z_0^i + q_b z_b^i + q_e z_e^i = 0$$

for all $\left(z_0^i, z_b^i, z_e^i\right) \in \overline{\zeta}^i(p_0, q; r_{\mathbf{1}})$. Summing over $i \in I$, yields the result. □

Finally, we denote by $\overline{\zeta} : \widehat{Q} \times R \to \overline{K}^I$,

$$\overline{\zeta}(p_0, q; r_{\mathbf{1}}) := \overline{\zeta}^1(p_0, q; r_{\mathbf{1}}) \times ... \times \overline{\zeta}^I(p_0, q; r_{\mathbf{1}}),$$

the Cartesian product of individual excess demand correspondences. By Lemma 4, $\overline{\zeta}(p_0, q; r_{\mathbf{1}})$ is a non-empty, compact- and convex-valued u.h.c. correspondences on $\widehat{Q} \times R$. as a product of correspondences with these properties.

(vi) The bankruptcy clearing mechanism: Let $\overline{K} \subset \mathbb{R}^{J+2}$ be the set of individually feasible excess demands. A vector of individual excess demands $z = (z_0, z_b, z_e) := \left((z_0^1, z_b^1, z_e^1), ..., (z_0^I, z_b^I, z_e^I)\right)$ is an element of $\overline{K}^{|I|}$.

The *bankruptcy clearing mechanism* (compare Equation 4) is a function $\rho : \overline{K}^{|I|} \to R := (0,1]^S$ defined by its component functions

$$\rho_s(z_0, z_b, z_e) = \begin{cases} \frac{\sum_{i=1}^I \left(-z_b^i \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\sum_{i=1}^I \left(-z_b^i \vee 0\right)}. & \text{for } \sum_{i=1}^I \left(-z_b^i \vee 0\right) > 0 \\ 1 & \text{for } \sum_{i=1}^I \left(-z_b^i \vee 0\right) = 0 \end{cases}.$$

$\rho_s(z_0, z_b, z_e)$ is a continuous function on $\overline{K}^{|I|}$. Notice that, by Assumption A3, $\omega_s^i > 0$ for all $s \in S$ and all $i \in I$. Hence, $\left(-z_b^i \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^i)) = -z_b^i$ for $0 < -z_b^i < \omega_s^i$. This implies that $\rho_s(z_0, z_b, z_e) = 1$ in a neighborhood of $z_b = 0$. Thus, $\rho_s(z_0, z_b, z_e)$ is well-defined and continuous at $z_b = 0$. Moreover, $\frac{\left(-z_b^i \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\left(-z_b^i \vee 0\right)}$ is an increasing function of z_b^i with a minimum $\frac{\kappa \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\kappa} \geq \frac{\kappa \wedge \omega_s^i}{\kappa}$ at $z_b^i = -\kappa$. Therefore, we have

$$\begin{aligned}
\rho_s(z_0, z_b, z_e) &= \frac{\sum_{i=1}^I \left(-z_b^i \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\sum_{i=1}^I \left(-z_b^i \vee 0\right)} \\
&= \sum_{i=1}^I \left[\frac{\left(-z_b^i \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\left(-z_b^i \vee 0\right)}\right] \frac{\left(-z_b^i \vee 0\right)}{\sum_{i=1}^I \left(-z_b^i \vee 0\right)} \\
&\geq \sum_{i=1}^I \left[\frac{\kappa \wedge \omega_s^i}{\kappa}\right] \frac{\left(-z_b^i \vee 0\right)}{\sum_{i=1}^I \left(-z_b^i \vee 0\right)} \\
&\geq \min_{i \in I} \left[\frac{\kappa \wedge \omega_s^i}{\kappa}\right] \underbrace{\sum_{i=1}^I \frac{\left(-z_b^i \vee 0\right)}{\sum_{i=1}^I \left(-z_b^i \vee 0\right)}}_{=1} \\
&= \min_{i \in I} \left[\frac{\kappa \wedge \omega_s^i}{\kappa}\right] =: \underline{\rho}_s > 0.
\end{aligned}$$

Hence, for all $s \in S$, $\rho_s(z_0, z_b, z_e)$ is bounded below by a strictly positive value $\underline{\rho}_s$. Let $\widehat{R} := \prod_{s \in S} \left[\underline{\rho}_s, 1\right] \subset R$. Then ρ is a continuous function $\overline{K}^{|I|} \to \widehat{R}$.

(vii) Existence of Equilibrium in period t = 0: In order to prove existence of an equilibrium in $t = 0$, we will adapt the existence proof of Grandmont [1988] (pp. 11-12).

Consider an increasing sequence of compact and convex subsets $Q^k$of $\widehat{Q}$ such that $\widehat{Q}$ is contained in the union of all Q^k.

For each k and every $(p_0, q_b, q_e, r_1) \in Q^k \times \widehat{R}$, the Cartesian product of individual excess demand correspondences $\overline{\zeta}(p_0, q_b, q_e; r_\mathbf{1})$ is a non-empty, compact- and convex-valued and u.h.c. correspondence.

For any excess demand vector $z := \left(z^1, ..., z^I\right) = \left(\left(z_0^1, z_b^1, z_e^1\right), ..., \left(z_0^I, z_b^I, z_e^I\right)\right) \in \overline{K}^{|I|}$, let

$$\mu^k(z) := \arg\max \left\{ p_0 \sum_{i=1}^{I} z_0^i + q_b \sum_{i=1}^{I} z_b^i + q_e \sum_{i=1}^{I} z_e^i | \ (p_0, q_b, q_e) \in Q^k \right\}$$

be the set of prices in Q^k that maximize the value of the excess demand . Since Q^k is compact and $p_0 \sum_{i=1}^{I} z_0^i + q_b \sum_{i=1}^{I} z_b^i + q_e \sum_{i=1}^{I} z_e^i$ is a continuous function on Q^k, $\mu^k(z)$ is not empty. By the maximum theorem, $\mu^k : \overline{K}^{|I|} \to Q^k$ is a compact, convex-valued, and u.h.c. correspondence on $\overline{K}^{|I|}$.

For each $(p_0, q_b, q_e, r_{\mathbf{1}}, z) \in Q^k \times \widehat{R} \times \overline{Z}^{|I|}$, define the correspondence

$$\Phi^k(p_0, q_b, q_e, r_{\mathbf{1}}, z) := \mu^k(z) \times \{\rho(z)\} \times \overline{\zeta}(p_0, q_b, q_e, r_{\mathbf{1}}).$$

The correspondence Φ^k is a mapping $Q^k \times \widehat{R} \times \overline{K}^{|I|} \to Q^k \times \widehat{R} \times \overline{K}^{|I|}$. The correspondence Φ^k is non-empty, compact, convex-valued, and u.h.c., as a Cartesian product of correspondences with these properties. Hence, by Kakutani's fixed point theorem, there is $(p_0^k, q_b^k, q_e^k, r_{\mathbf{1}}^k, z^k) \in \Phi^k(p_0^k, q_b^k, q_e^k, r_{\mathbf{1}}^k, z^k)$.

By construction of μ^k,

$$0 = p_0^k \sum_{i=1}^{I} {z_0^i}^k + q_b^k \sum_{i=1}^{I} z_b^{ik} + q_e^k \sum_{i=1}^{I} z_e^{ik} \geq p_0 \sum_{i=1}^{I} z_0^{ik} + q_b \sum_{i=1}^{I} z_b^{ik} + q_e \sum_{i=1}^{I} z_e^{ik} \tag{10}$$

for all $(p_0, q_b, q_e) \in Q^k$. The sequence $\left(p_0^k, q_b^k, q_e^k\right) \in Q^k$ must be bounded away from $\partial\widehat{Q}$ or one would contradict Lemma 5 (i).

Since $Q^k \times \widehat{R} \times \overline{K}^{|I|} \subset \overline{Q} \times \widehat{R} \times \overline{Z}^{|I|}$, there exists a converging subsequence (same notation) $(p_0^k, q_b^k, q_e^k, r_{\mathbf{1}}^k, z^k) \to (\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}}, \overline{z}) \in \overline{Q} \times \widehat{R} \times \overline{K}^{|I|}$ such that

(i) $\overline{r}_{\mathbf{1}} = \rho(\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}})$,

(ii) $\overline{z} \in \overline{\zeta}(\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}})$,

(iii) $(\overline{p}_0, \overline{q}_b, \overline{q}_e) \in \mu^k(z^k)$ for all k.

By continuity, Equation 10 implies

$$0 = \overline{p}_0 \sum_{i=1}^{I} \overline{z}_0^i + \overline{q}_b \sum_{i=1}^{I} \overline{z}_b^i + \overline{q}_e \sum_{i=1}^{I} \overline{z}_e^i \geq p_0 \sum_{i=1}^{I} \overline{z}_0^i + q_b \sum_{i=1}^{I} \overline{z}_b^i + q_e \sum_{i=1}^{I} \overline{z}_e^i$$

for all $(p_0, q_b, q_e) \in \widehat{Q}$. Since $(\overline{p}_0, \overline{q}_b, \overline{q}_e) \in \widehat{Q}$ implies $(\overline{p}_0, \overline{q}_b, \overline{q}_e) > 0$, one can conclude that $\sum_{i \in I} \overline{z}^i = 0$.

In summary, we have a consistent return rate $\overline{r}_{\mathbf{1}}$ for the bond and a price system $(\overline{p}_0, \overline{q}_b, \overline{q}_e)$ for which the asset markets and the market for commodities in period $t = 0$ clear:

(i) $\overline{r}_{\mathbf{1}} = \widehat{\rho}(\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}})$,

(ii) $\overline{z} \in \overline{\zeta}(\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}})$, with $\sum_{i \in I} \overline{z}^i = 0$.

(viii) Existence of Equilibrium in period $\mathbf{t = 1}$: It remains to show that in a bankruptcy equilibrium

(i) $\overline{r}_{\mathbf{1}} = \widehat{\rho}(\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}})$,

(ii) $\overline{z} \in \overline{\zeta}(\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}})$, with $\sum_{i \in I} \overline{z}^i = 0$,

consumption will be non-negative in every state $s \in S$, i.e., condition (iii) of the definition of a bankruptcy equilibrium

$$\sum_{i=1}^{I} \left(x_s^i \vee 0\right) = \sum_{i=1}^{I} (\omega_s^i + Y_s \delta^i)$$

for all $s \in S$ is also satisfied.

Recall that, for an equilibrium price system $(\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}})$ and an equilibrium allocation $\left(\overline{z}_0^i, \overline{z}_b^i, \overline{z}_e^i\right) \in \overline{\zeta}^i(\overline{p}_0, \overline{q}_b, \overline{q}_e, \overline{r}_{\mathbf{1}})$, by Equation 8,

$$x_s^i = \omega_s^i + Y_s \left(\delta^i + \overline{z}_e^i\right) + r_s(\overline{z}_b^i \vee 0) + (\overline{z}_b^i \wedge 0)$$

and, by Equation 4,

$$\overline{r_s} = \frac{\sum_{i=1}^{I} \left(-z_b^i \vee 0\right) \wedge \left(\omega_s^i + Y_s(\delta^i + z_e^i)\right)}{\sum_{i=1}^{I} (-\overline{z}_b^i \vee 0)}$$

for all $s \in S$ hold. These conditions imply non-negative consumption in all states.

Lemma 6. *If for all $i \in I$ and all $s \in S$ equations 8 and 4 hold, then*

$$\textstyle\sum_{i=1}^{I} z_b^i = 0 \quad \textit{and} \quad \sum_{i=1}^{I} z_e^i = 0$$

imply

$$\sum_{i=1}^{I} \left(x_s^i \vee 0\right) = \sum_{i=1}^{I} (\omega_s^i + Y_s \delta^i)$$

for all $s \in S$.

Proof: Consider an arbitrary $s \in S$. By Equation 8, summing over all consumers $i \in I$, one obtains

$$\sum_{i=1}^{I} \left(x_s^i - \omega_s^i - Y_s \delta^i\right) = \sum_{i=1}^{I} \left[r_s(z_b^i \vee 0) + (z_b^i \wedge 0) + Y_s z_e^i\right],$$

which is equivalent to

$$\sum_{i=1}^{I} \left[(x_s^i \vee 0) - \omega_s^i - Y_s \delta^i\right] = \sum_{i=1}^{I} \left[r_s(z_b^i \vee 0) + (z_b^i \wedge 0) + Y_s z_e^i - (0 \wedge x_s^i)\right].$$

The asset market equilibrium conditions

$$\sum_{i=1}^{I} z_b^i = 0 \quad \text{and} \quad \sum_{i=1}^{I} z_e^i = 0$$

imply

$$\begin{aligned}
&\sum_{i=1}^{I} \left[r_s(z_b^i \vee 0) + (z_b^i \wedge 0) + Y_s z_e^i - (x_s^i \wedge 0) \right] \\
= \;& r_s \underbrace{\sum_{i=1}^{I} (z_b^i \vee 0)}_{= -\sum_{i=1}^{I}(z_b^i \wedge 0)} + \sum_{i=1}^{I} (z_b^i \wedge 0) - \sum_{i=1}^{I} (x_s^i \wedge 0) + Y_s \underbrace{\sum_{i=1}^{I} z_e^i}_{=0} \\
= \;& r_s \sum_{i=1}^{I} (-z_b^i \vee 0) + \sum_{i=1}^{I} (z_b^i \wedge 0) - \sum_{i=1}^{I} (x_s^i \wedge 0).
\end{aligned}$$

Denote by $I^+ := \{i \in I | \ z_b^i \geq 0\}$ the set of consumers who are not borrowing, by $I_s^- := \{i \in I | \ z_b^i < 0, \ x_s^i \geq 0\}$ the set of loan takers who are solvent, and by $I_i^- : \{i \in I | \ z_b^i < 0, \ x_s^i < 0\}$ consumers who are insolvent, then substituting Equation 4 for r_s yields

$$\begin{aligned}
& r_s \sum_{i=1}^{I} (-z_b^i \vee 0) + \sum_{i=1}^{I} (z_b^i \wedge 0) - \sum_{i=1}^{I} (x_s^i \wedge 0) \\
= \;& r_s \sum_{i \in I^-} (-z_b^i \vee 0) + \sum_{i \in I^-} (z_b^i \wedge 0) - \sum_{i \in I^-} (x_s^i \wedge 0) \\
= \;& \sum_{i \in I^-} \left[\left(-z_b^i\right) \wedge \left(\omega_s^i + Y_s(\delta^i + z_e^i)\right) \right] - \sum_{i \in I^-} (-z_b^i) - \sum_{i \in I^-} (x_s^i \wedge 0) \\
= \;& \sum_{i \in I_i^-} \left[\omega_s^i + Y_s(\delta^i + z_e^i)) \right] - \sum_{i \in I_i^-} (-z_b^i) - \sum_{i \in I_i^-} (x_s^i) \\
= \;& \sum_{i \in I_i^-} \left[\left(\omega_s^i + Y_s \left(\delta^i + z_e^i \right) + z_b^i \right) - x_s^i) \right] = 0,
\end{aligned}$$

since $x_s^i < 0$ implies $x_s^i = \omega_s^i + Y_s \left(\delta^i + z_e^i \right) + z_b^i < 0$. Hence, $\sum_{i=1}^{I} \left(x_s^i \vee 0 \right) = \sum_{i=1}^{I} (\omega_s^i + Y_s \delta^i)$. □

7.2 Proof of Proposition 2

We partition the matrix

$$T = \begin{bmatrix} -q_b & q_b & -q_e \\ r_{\mathbf{1}} & -\mathbb{1} & Y \end{bmatrix}$$

and the portfolio vector $z^i \in Z^i$ into long-bond, short-bond, and equity trades by $T = (T_{b+}, T_{b-}, T_e)$ and $z^i = (z_{b+}^i, z_{b-}^i, z_e^i)^T$, respectively. Using Lagrange multipliers $\pi^i \in$

$\mathbb{R}^{S+1}$, $\sigma^i_{b+} \geq 0$, and $\sigma^i_{b-} \geq 0$, and assuming we are at an interior solution (δ^i are such that short sales constraints on equity are not binding at equilibrium, and parameters are such that $x^i_0 > 0$ at equilibrium), the KKT conditions for the minimization of the Lagrange function

$$L^i(x^i, z^i, \pi^i, \sigma^i_{b+}, \sigma^i_{b-}) = -u^i(x^i) + \langle \pi^i, x^i - \omega^i - e^i - Tz^i \rangle - \sigma^i_{b+} z^i_{b+} - \sigma^i_{b-} z^i_{b-}$$

imply that

$$\begin{aligned} \langle \nabla u^i(\bar{x}^i), T_e \rangle &= (0, \ldots, 0), \\ \langle \nabla u^i(\bar{x}^i), T_{b+} \rangle &= -\sigma^i_{b+} \leq 0, \text{ and} \\ \langle \nabla u^i(\bar{x}^i), T_{b-} \rangle &= -\sigma^i_{b-} \leq 0, \end{aligned}$$

Since the optimization problem is convex, the KKT conditions are also sufficient for an optimal solution to the agent's problem.

The gradient of the linear quadratic utility function fulfills in the equilibrium allocation $\bar{x}^i$

$$\langle \nabla u^i(\bar{x}^i), \tau \rangle \leq 0 \quad \forall \tau \in \mathcal{C}.$$

Summing up all agent's equilibrium gradients we define the vector

$$\bar{\gamma} := \sum_{i=1}^{I} \nabla u^i(\bar{x}^i) = (\alpha_0, \alpha_1 \mathbb{1} - ((\omega_{\mathbf{1}} + Y\delta) - d_{\mathbf{1}}))^T,$$

where $d_{\mathbf{1}} = \sum_{i=1}^I d^i_{\mathbf{1}} := \sum_{i=1}^I (\mathbb{1} - r_{\ \mathbf{1}}) z^i_{b-}$ is the aggregate shortfall from promises on the bond. Still we have $\langle \bar{\gamma}, \tau \rangle \leq 0 \quad \forall \tau \in \mathcal{C}$. Since any trade $\tau \in \mathcal{C}$ can be decomposed as $\tau = (-c(m), m)$ we get for $\frac{1}{\alpha_0}\bar{\gamma}$ that $\gamma_{\mathbf{1}} = \frac{\alpha_1}{\alpha_0}\mathbb{1} - \frac{1}{\alpha_0}\tilde{\omega}_{\mathbf{1}}$ and $c(m) \geq \langle \bar{\gamma}_{\mathbf{1}}, m \rangle$, which proves the lemma. □

7.3 Proof of Proposition 3

Suppose agent i goes long in the bond. Define the matrix by $T_{\text{long}} = \begin{pmatrix} -q_b & q_e \\ r_{\mathbf{1}} & Y \end{pmatrix}$. We know from proposition 2 that

$$\begin{aligned} \langle T^T_{\text{long}}, \nabla u^i(\bar{x}^i) \rangle &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and} && (11) \\ \langle T^T_{\text{long}}, \bar{\gamma} \rangle &= \begin{pmatrix} -\sigma_{b+} \\ 0 \end{pmatrix}, && (12) \end{aligned}$$

where $\nabla u^i(x^i) = (\alpha^i_0, \alpha^i_1 - x^i_{\mathbf{1}})^T$ and $\bar{\gamma} = (\alpha_0, \alpha_1 \mathbb{1} - \tilde{\omega}_{\mathbf{1}})^T$. Divide equation (11) by α^i_0 and equation (12) by α_0 , subtract the equations and multiply the result by α^i_0 again. With $\bar{\tau}^i_{\mathbf{1}} := \bar{x}^i_{\mathbf{1}} - \omega^i_{\mathbf{1}} - Y\delta^i$ this gives

$$\langle Y^T_{b+}, \bar{\tau}^i_{\mathbf{1}} \rangle = \langle Y^T_{b+}, (\alpha^i_1 - \frac{\alpha^i_0}{\alpha_0}\alpha_1)\mathbb{1} - ((\omega^i_{\mathbf{1}} + Y\delta^i) - \frac{\alpha^i_0}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \rangle - \frac{\alpha^i_0}{\alpha_0}\begin{pmatrix} \sigma_{b+} \\ 0 \end{pmatrix}.$$

We can write

$$\begin{pmatrix} \sigma_{b+} \\ 0 \end{pmatrix} = \langle Y_{b+}^T, \sigma_{b+} \frac{r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})}{||r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})||^2} \rangle = \langle Y_{b+}^T, \sigma_{b+} r_{\mathbf{1}e} \rangle.$$

Now, since $\langle Y_{b+}^T, v_{\mathbf{1}} \rangle = \langle Y_{b+}^T, P_{Y_{b+}}(v_{\mathbf{1}}) \rangle$ for any vector $v_{\mathbf{1}} \in \mathbb{R}^S$, it follows that

$$\bar{\tau}_{\mathbf{1}}^i = P_{Y_{b+}} \left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0} \alpha_1) \mathbb{1} - ((\omega_{\mathbf{1}}^i + Y\delta^i) - \frac{\alpha_0^i}{\alpha_0} \tilde{\omega}_{\mathbf{1}}) - \sigma_{b+} \frac{\alpha_0^i}{\alpha_0} r_{\mathbf{1}e} \right).$$

Finally, since $r_{\mathbf{1}} - P_Y(r_{\mathbf{1}}) \in \mathrm{span}(Y_{b+})$, the result follows.

The results for agents going short and for agents that do not trade in the bond are proved similarly. □