The international platform of Ludwig-Maximilians University's Center for Economic Studies and the Ifo Institute

CESifo Working Papers

www.cesifo.org/wp

Markov Perfect Equilibria in Differential Games with Regime Switching Strategies

Ngo Van Long
Fabien Prieur
Klarizze Puzon
Mabel Tidball

CESifo Working Paper No. 4662
Category 12: Empirical and Theoretical Methods
February 2014

CESifo
Center for Economic Studies & Ifo Institute

CESifo *Working Paper No. 4662*

Markov Perfect Equilibria in Differential Games with Regime Switching Strategies

Abstract

We propose a new methodology exploring Markov perfect equilibrium strategies in differential games with regime switching. Specifically, we develop a general game with two players having two kinds of strategies. Players choose an action that influences the evolution of a state variable, and decide on the switching time from one regime to another. Compared to the optimal control problem with regime switching, necessary optimality conditions are modified for the first-mover. When choosing her optimal switching strategy, this player considers the impact of her choice on the other player's actions and payoffs. In order to determine the equilibrium timing of regime changes, the notion of wrong timing is introduced and necessary conditions for a particular timing to be wrong are derived. We then apply this new methodology to an exhaustible resource extraction game. Sufficient conditions for the existence of an interior solution are compared to those characterizing a wrong timing. The impact of feedback strategies for the equilibrium adoption time depends on the balance between two conflicting effects: the first mover incurs an indirect cost due to the future switching of her rival (incentive to delay the switch). But she is able to affect the other player's switching decision (incentive to switch more rapidly). In a particular case without direct switching cost, the interplay between the two ensures that the first-mover adopts the new technology in finite time. Interestingly, this result differs from what is obtained in a non-game theoretic framework, i.e. immediate adoption.

JEL-Code: C610, C730, Q320.

Keywords: differential games, regime switching, technology adoption, non-renewable resources.

Ngo Van Long
Department of Economics
McGill University
855 Sherbrooke Street West,
Canada – Montreal, QC H3A 2T7
ngo.long@mcgill.ca

Fabien Prieur
LAMETA, University of
Montpellier I & INRA
2 place Viala
France – 34060 Montpellier
prieur@supagro.inra.fr

Klarizze Puzon
LAMETA, University of Montpellier I
Montpellier / France
puzon@lameta.univ-montp1.fr

Mabel Tidball
INRA-LAMETA
Montpellier / France
tidball@supagro.inra.fr

February 1, 2014
We thank Raouf Boucekkine, Larry Karp, Santanu Roy, Amos Zemel and participants in conferences and seminars in Annecy, Baton Rouge, Lisbon, Los Angeles and Toulouse.

1 Introduction

Several decision making problems in economics concern the timing of switching between alternative and consecutive regimes. Regimes may refer to technological and/or institutional states of the world. For instance, a firm with an initial level of technology may find it optimal to either adopt a new technology or to stick with the old one (Boucekkine, Saglam and Vallée, 2004). Another example is the decision to phase out existing capital controls in a given economy (Makris, 2001). In all non-trivial problems, the switching decision involves a trade-off, since adopting a new regime brings with it immediate costs as well as potential future benefits. Given these considerations, multi-stage optimization is generally used for the analysis of regime switching (Tomiyama, 1985), which endogenously determines switching times.

In this article, we consider regime switching strategies in differential games. The game theoretic literature involving regime switching choice is sparse. Early papers on dynamic games of regime change do not involve a stock variable. In these models, the only relevant state of the system is the identity of the players which have adopted the new technology. An example is Reinganum (1981)'s model of technological adoption decisions of two ex ante identical firms. She assumed that firms adopt pre-commitment (open-loop) strategies. That is, it is as if a firm enters a binding commitment on its date of technology switch, knowing the adoption date of the other firm. Reinganum's primary finding is that, with two ex-ante identical firms using open-loop strategies, the equilibrium features diffusion: One firm will innovate first and the other will innovate at a later date. The first mover earns higher profits. Fudenberg and Tirole (1985) revisited Reinganum's study by using the concept of pre-emption equilibrium. Focusing on Markov perfect equilibrium as the solution concept, they noted that the second-mover may try to preempt its rival and become the first-mover. At the preemption equilibrium, the first mover advantage vanishes.[1]

A second strand of literature pertains to the strategic interaction of agents in relation to the dynamics of a given stock. For instance, Tornell (1997) presented a model relating economic growth and institutional change. Infinitely-lived agents solve a differential game over the choice of property-rights regimes, e.g. common or private property, defined over a capital stock. It was shown that a potential equilibrium of the game involves multiple switching between regimes. However, only the symmetric equilibrium was considered, such that the players always choose to switch at the same instant. Consequently, the question of the timing between switching points was not addressed. In addition, even though Tornell explicitly defined the Markov perfect equilibrium for the class of differential games with regime switching, a rigorous modelling of these strategies, for switching time, is missing in his analysis. A more recent example is the analysis by Boucekkine, Krawczyk and Vall (2011). They analyzed the trade-off between environmental quality and economic performance using a two-player differential game. Assuming that pollution results from the sum of consumption levels and

[1] See Long (2010) for a survey of this literature.

there is no decay, they proved the existence of an open-loop Nash equilibrium. They found that each player chooses the technology without considering the choice made by the other player. There was no interior switching instant. At the open-loop Nash equilibrium, either a player adopts technology immediately, or he sticks to the old one.

To our knowledge, there seems to be no existing study which formally defines Markov perfect equilibrium in differential games with regime switching strategies. This is where the first theoretical contribution of this paper lies. We develop a general differential game with two players having two kinds of strategies. First, players have to choose at each point in time an action that influences the evolution of a state variable. Second, they may decide on the timing of switching between alternative and consecutive regimes that differ both in terms of the payoff function and the state equation. For simplicity, we assume that each player can affect a regime change only once. Focusing on Markov perfect equilibrium, we define the switching or timing strategy as a function of the *state of the system*, which is described by the level of the stock variable, and the regime that is in current operation. The relevant level of the stock on which the strategy is based is the one corresponding to the instant when the switching problem arises.

For any of the two possible general timings, we characterize the necessary optimality conditions for switching times, both for interior and corner solutions. One interesting finding is that, compared to the standard optimal control problem with regime switching, necessary optimality conditions are modified only for the player who finds it optimal to move first. Indeed, when choosing the optimal date and level of the state variable for switching, this player must take into account that *(i)* her decision will influence the other player's switching strategy and *(ii)* the other player's switch will impact on her own welfare. Therefore, player's strategy must take these factors into consideration. Depending on the particular economic problem at hand, the interaction through switching times may be an incentive to either postpone or expedite regime switching. Another important issue is how to determine the optimal timing at the Markov perfect equilibrium. This issue is solved by providing necessary conditions for a particular timing to be *wrong*.[2]

The second contribution of this paper is the application of this new game theoretic material to study a model of the tragedy of the commons in the present of switching strategies. A game of exhaustible resource extraction is considered. By incurring a lumpy cost, players can make use of a more efficient extraction technology. Not only do players choose their consumption levels, they also decide whether to adopt the new technology and when. To date, there are only a few papers that have studied the relationship between natural resource exploitation and the timing of technology adoption. Using a finite horizon two-stage optimal control problem, Amit (1986) explored the case of a petroleum producer who considers switching from primary to secondary recovery process. He observed that a technological switch occurs if the desired extraction rate is larger than

[2]By wrong we mean that at least one player would prefer the alternative timing.

can be obtained by the natural drive, or when the desired final output is more than can be obtained using the primary process. In a more recent paper, Valente (2011) analyzed a two-phase endogenous growth model which concerns a switch from an exhaustible resource input into a backstop technology. He showed that adoption of new technology implies a sudden fall in consumption, but an increase in the growth rate. Finally, Boucekkine, Pommeret, and Prieur (2013) explored a general control problem with both technological and ecological regime (induced by the crossing of a critical threshold) switches. They applied it to address the issue of optimal resource extraction under ecological irreversibility, and with the possibility to adopt backstop technology. It was observed that the opportunity to switch to a backstop technology may lead to an irreversible ecological regime.

Overall, while the above-mentioned studies have explored resource management and regime switching, they only do so using single-agent optimization programs. None have conducted an analysis using a differential game approach. Indeed, our section 4 tries to fill this gap in the resource extraction literature. It is assumed that heterogenous players start with a less efficient extraction technology and have to decide: *(i)* whether to switch to a more efficient technology, and *(ii)* when, given that switching involves a direct cost that depends on both the switching date and the level of the state variable.

Our main findings can be summarized as follows. We first eliminate candidate equilibriums by identifying a meaningful necessary condition for a proposed timing to be erroneous. This condition involves on the one hand the difference of players' switching costs, and on the other hand the difference in technological gains from switching. Indeed, it is possible that both players find a given timing wrong. This happens when the player who is supposed to be the first to adopt has a relative disadvantage in adoption costs that is not compensated by any relative technological advantage. This notably encompasses the obvious situation in which the first mover incurs the higher switching cost and, at the same time, is the one who benefits the less from adoption at any level of the resource stock. When this condition is not met, we deduce that the proposed timing is not wrong. We then provide sufficient conditions for the existence of an interior solution where both players adopt the new technology in finite time and investigate the impact of feedback strategies regarding switching time on the first-mover switching strategy (compared to the single-agent case). We emphasize the interplay between two opposite effects. First, in our application, the switch made by the second mover is costly for the first mover because it implies a drop in her consumption of the resource. The switching cost of the latter is thus augmented by this term, which gives her an incentive, other things equal, to delay the switch. On the other hand, however, it turns out that the length of time between the two switches is increasing in the level of the stock at the time of the first switch. From the point of view of the first mover, who controls this level, switching at a relatively abundant stock of resource is a means to postpone the switch of the other. Because of discounting, delaying the switch of the other player will allow the first-mover to incur a lower cost. This is an incentive to switch at an earlier date. In the particular case where the first player does not bear a direct switching cost, we show that she finds it worthwhile to adopt the new technology at finite date, but not immediately at the beginning of the game. This result differs from what one would obtain in the absence of interaction between players, i.e. immediate adoption.

The plan of the paper is as follows. Section 2 describes the main assumptions of the general differential game with regime switching. Section 3 analyzes the optimality conditions that characterize a Markov perfect equilibrium. Section 4 applies these theoretical findings to a game of exhaustible resource extraction. Section 5 concludes.

2 The general problem

We consider a two-player differential game in which the instantaneous payoff of each player and the differential equation describing the stock dynamics depend on what regime the system is in. There are a finite number of regimes, indexed by s, and we assume that under certain conditions, the players are able to take action (at some cost) to affect a change of regime. Let $\mathcal{S}$ be the set of regimes. For simplicity, we assume that each player can make a regime switch only once. This implies that regime changes are irreversible, i.e. switching back is not allowed. In this case, there are four possible regimes and the set $\mathcal{S}$ is simply

$$\mathcal{S} \equiv \{11, 12, 21, 22\}$$

We assume that the system is initially in regime 11. Player 1 can take a "regime switching – or regime change – action" to switch the system from regime 11 to regime 21, if player 2 has not taken her regime change action before him.[3] Once the system is in regime 21, only player 2 can take a regime switching action, and this leads the system to regime 22. From regime 11, player 2 can switch to regime 12 (if player 1 has not made his regime change before her). From regime 12, only player 1 can make a regime change, and this switches the system to regime 22. If the system is in regime 11 and players 1 and 2 take regime change action simultaneously, the regime will be switched to 22. Finally, the system may remain in 11 forever if neither agent takes a regime change action. Let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ in which player i can make a regime change. Then $\mathcal{S}_1 = \{11, 12\}$ and $\mathcal{S}_2 = \{11, 21\}$.

The state variable x could be in any space $\mathbb{R}_+^m$, $1 \leq m \leq M$. To simplify the exposition, we set $m = 1$. At each instant, each player chooses an action u_i, with $u_i \in \mathbb{R}^n$, $1 \leq n \leq N < \infty$, that affects the evolution of x. The instantaneous payoff to player i at time t when the system is in regime s is

$$F_i^s(u_i(t), u_{-i}(t), x(t)).$$

If player i, $i = 1, 2$, takes a regime change action at time $t_i \in \mathbb{R}_+$, he/she incurs a lumpy cost $\Omega_i(x(t_i), t_i)$. Then, if for example $0 < t_1 < t_2 < \infty$, the total

[3] The first number in any regime index indicates player 1's moves. The second refers to player 2.

payoff for player 1 is

$$\int_0^{t_1} F_1^{11}(u_1,u_2,x)e^{-\rho t}dt + \int_{t_1}^{t_2} F_1^{21}(u_1,u_2,x)e^{-\rho t}dt$$
$$+\int_{t_2}^{\infty} F_1^{22}(u_1,u_2,x)e^{-\rho t}dt - \Omega_1(x(t_1),t_1)$$

with ρ the discount rate.

The differential equation describing the evolution of the state variable x in regime s is

$$\dot{x} = f^s(u_1,u_2,x)$$

In the subsequent analysis, we use Markov perfect equilibrium (MPE) as the solution concept. As illustrated by the decomposition above, if the equilibrium timing is such that $0 \leqq t_1 \leqq t_2 \leqq \infty$, there are three sub-games to be considered, each being associated with a particular regime. Indeed, for the timing considered, the sequence of regimes is: 11, 21 and 22. A natural way to proceed, for determining a MPE of this game, is to solve the problem recursively, starting from the regime arising after the final regime switching, here 22. This is a natural extension of the method originally developed by Tomiyama (1985) and Amit (1986) to solve their two-stage optimal control problems.

The next assumption ensures that our problem, seen as a sequence of three sub-games, is well-behaved.

Assumption 1

- *The functions $F_i^s(.)$ and $f^s(.)$, for any $s \in \mathcal{S}$, belong to the class C^1.*
- *The sub-game obtained by restricting the general problem to any regime s, satisfies the Arrow-Kurz's sufficiency conditions.*

These conditions will allow us to use some envelope properties that require the differentiability of the value function. (See Boucekkine, Pommeret and Prieur, 2013, for a detailed discussion).

Let us now define what is a MPE strategy in our model. Each player has two types of controls, the set of controls being given by $\mathcal{C}_i = \{u_i, t_i\}$. A MPE strategy consists of an action policy and a switching rule describing the actions undertaken by each player at every possible state of the system, $(x,s) \in \mathbb{R}_+ \times S$.[4]

The *action strategy* of player i is a mapping Φ_i from the state space $\mathbb{R}_+ \times \mathcal{S}$ to the set $\mathbb{R}^n$.

The *switching rule* can be defined as follows: Suppose player 1 thinks that if player 2 finds herself in regime 21 at date t, with $x(t)$ (which implies that he switched at an earlier date $t_1 < t$), she will make a switch at a date $t_2 \geq t$. Then player 1 should think that the interval of time between the current period and the switching date, $t_2 - t$, is a function of the state of the system. More

[4]We restrict attention to those strategies that are not time-dependent. This requires that the function $\Omega_i(x(t_i),t_i)$ takes the form $e^{-\rho t_i}\omega_i(x(t_i))$.

generally, we define the *time-to-go strategy (before switching)* of player i, given that $s \in \mathcal{S}_i$, as a mapping θ_i from $\mathbb{R}_+ \times \mathcal{S}$ to $\mathbb{R}_+ \cup \{\infty\}$. For instance, from the state $(x, 21)$, $\theta_2(x, 21)$ is the length of time that must elapse before player 2 takes her regime switching action. If $\theta_2(x, 21) = \infty$ for all x, it means she does not want to switch at all from regime 21.

Then we say that

Definition 1

- *A* ***strategy vector*** *of player i (as guessed by player $-i$) is a pair $\psi_i \equiv (\Phi_i, \theta_i)$, $i = 1, 2$.*
- *A* ***strategy profile*** *is a pair of strategy vectors, (ψ_1, ψ_2).*
- *A strategy profile (ψ_1^*, ψ_2^*) is called a* ***Markov-perfect Nash equilibrium****, if given that player i uses the strategy vector ψ_i^*, the payoff of player j, starting from any state $(x, s) \in \mathbb{R}_+ \times \mathcal{S}$, is maximized by using the strategy vector ψ_j^*, where $i, j = 1, 2$.*

The next section presents the set of necessary optimality conditions that characterize a MPE of our differential game with regime switching strategies.

3 Necessary Conditions for switching strategies

We first analyze and interpret optimality conditions for an interior solution, which allows us to emphasize the impact of the interaction through switching strategies on the solution. Next, we will pay attention to corner solutions and introduce the concept of *wrong timing.*

3.1 Interior solution

In the following analysis, player i's present value Hamiltonian and co-state variable in any regime s are denoted respectively by H_i^s and λ_i^s. The results are presented for a particular timing: $0 < t_1 < t_2 < \infty$.[5] Moreover, note that in the theorem below, attention is paid only to the necessary optimality conditions related to the switching problems. That is, we consider a path $(u_1^*(t), u_2^*(t), x^*(t))$ that satisfies the other standard Pontryagin conditions (see the Appendix A).

Theorem 1 *Let x_i^* be player i's switching point, i.e. the value of the state variable such that $x^*(t_i^*) = x_i^*$ for $i = 1, 2$. The necessary optimality conditions for the existence of a MPE featuring the timing $0 < t_1^* < t_2^* < \infty$ are:*

- *For player 2:*

$$\begin{aligned} H_2^{21*}(t_2^*) - \frac{\partial \Omega_2(x_2^*, t_2^*)}{\partial t_2} &= H_2^{22*}(t_2^*) \\ \lambda_2^{21*}(t_2^*) + \frac{\partial \Omega_2(x_2^*, t_2^*)}{\partial x_2} &= \lambda_2^{22*}(t_2^*). \end{aligned} \tag{1}$$

[5] Necessary optimality conditions for the other general timing, $0 < t_2 < t_1 < \infty$, can easily be derived by symmetry.

- *For player 1:*[6]

$$\begin{array}{l} H_1^{11*}(t_1^*) - \frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial t_1} = H_1^{21^*}(t_1^*) - [H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)] \\ \lambda_1^{11*}(t_1^*) + \frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial x_1} = \theta_2^{*\prime}(x_1^*, 21)[H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)] + \lambda_1^{21*}(t_1^*), \end{array} \tag{2}$$

Proof. See the Appendix A. ■

To understand these switching conditions for an interior solution, let us focus on the difference between the optimality conditions of the first-mover (player 1) and the second-mover (player 2) for the particular timing considered. Player 2's conditions (1) are similar to the ones derived in multi-stage optimal control literature. The first condition states that it is optimal to switch from the penultimate to the final regime when the marginal gain of delaying the switch, given by the difference $H_2^{21*}(.) - H_2^{22*}(.)$, is equal to the marginal cost of switching, $\frac{\partial \Omega_2(x_2^*, t_2^*)}{\partial t_2}$. The second condition equalizes the marginal benefit from an extra unit of the state variable x_2 with the corresponding marginal cost. It basically says that the value of the co-state, when approached from the intermediate regime, plus the incremental switching cost must just equal the value of the co-state, approached from the final regime. Hence, as long as a player finds it optimal to be the second mover, her optimality conditions are similar to the standard switching conditions of an optimal control problem.

The novel part of Theorem 1 stems from the problem faced by the player who opts to adopt first. Indeed, player 1's optimality conditions are modified (compared to the single agent framework). The first condition in (2) implies that player 1 takes into account how his situation changes as a consequence of a switch of player 2. Player 1 decides on his optimal switching time by equalizing the marginal gain of delaying the switch, which is given by the difference $H_1^{11*}(.) - H_1^{21*}(.)$ to the marginal switching cost, $\frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial t_1} - [H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)]$. The extra-term $[H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)]$ is the marginal impact of player 2's switch on player 1. So, player 1 anticipates the impact of player 2's switch on his payoff. Depending on the nature of the problem, the additional term can either be positive or negative. The second optimality condition is also modified. The cost of a marginal increase in x_1 now includes an extra-term: $\theta_2^{*\prime}(x_1^*, 21)[H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)]$. This term reflects the fact that player 1 takes into account the impact of the choice of his switching level x_1^* on player 2's timing strategy. Put differently, player 1 knows that modifying x_1^* is a means to delay or accelerate player 2's regime switching. In sum, the modified switching conditions of player 1 illustrate the existence of a two-way interaction through switching strategies.

A couple of comments are in order here:

• First, when deriving the conditions of Theorem 1, we implicitly assume that players follow their MPE strategies for the action policy, i.e. that the triplet $(u_1^*(t), u_2^*(t), x^*(t))$ is the path followed by each player's action policy and the state variable at a MPE. This boils down to considering that optimal switching conditions are conditional on the optimal action policies. Then, the question

[6] The "prime"in θ_2' refers to the derivative w.r.t to the state variable x.

is: Is the switching rule robust to deviations in the action policy? Consider the problem of player 1, once he is already in regime 21. Player 1's problem is to choose the time path $\{u_1\}$ that maximizes

$$\int_{t_1}^{t_1+\theta_2(x_1,21)} e^{-\rho t} F_1^{21}(u_1, \Phi_2(x,21), x)dt + V_1^{22*}(x_2, t_1 + \theta_2(x_2, 21))$$

with $V_1^{22*}(.)$ the continuation payoff (resulting from the play of the MPE actions in the final regime) and subject to,

$$\dot{x} = f^{21}(u_1, \Phi_2(x,21), x)$$

$$x(t_1) = x_1,\ x(t_1 + \theta_2(x_1, 21)) = x_2$$

where he takes as given x_1, x_2, $\Phi_2(x,21)$ and $\theta_2(x_1,21)$. If he deviates from the equilibrium from time t_1 to some time $t_1+\epsilon$, with $\epsilon > 0$, what would be his optimization problem at time $t_1 + \epsilon$? The point is that he should expect that player 2 still continues to use the strategy $(\Phi_2(x,21), \theta_2(x,21))$, with the switching point x_2, because he knows that $\Phi_1(x,21)$ will be played by him from time $t_1+\epsilon$ onward. The deviation will be reflected in the value of the state variable at $t_1+\epsilon$, $x(t_1+\epsilon) \neq x^*(t_1+\epsilon)$. This will in turn affect the length of time before the next switch by Player 2, $\theta_2(x(t_1+\epsilon), 21)$.

• Second, in condition (2), the term $\theta_2^{*\prime}(x_1, 21)$ may look like a kind of Stackelberg-leadership consideration: Player 1 knows the function $\theta_2^{*\prime}(x,21)$, and hence he knows that when he chooses x_1 he is indirectly influencing t_2. But this is not really Stackelberg leadership in a global sense. The situation is just like any standard game tree with sequential moves. If a player moves first, he knows how the second mover will move at each of the subgame that follows, and therefore he will take that into account in choosing which subgame he is going to induce.

3.2 Corner solutions

We now turn to the necessary conditions for the corner solutions, still for the same timing.[7]

Theorem 2 *1. Suppose player 1's switching problem has an interior solution* (t_1^*, x_1^*).

• *A necessary condition for player 2 to choose a corner solution with immediate switching, i.e.* $t_2^* = t_1^*$ *(instead of* $t_2^* > t_1^*$*) is*

$$H_2^{21*}(t_2^*) - \frac{\partial \Omega_2(x_2^*, t_2^*)}{\partial t_2} \leq H_2^{22*}(t_2^*) \text{ if } t_1^* = t_2^* < \infty \tag{3}$$

[7] For the sake of brevity, we don't review all the cases. Conditions for having $t_1 = t_2$ are briefly discussed at the end of this section.

- *A necessary condition for player 2 to choose a corner solution of the never switching type* $t_2^* = \infty$ *is*

$$H_2^{21*}(t_2^*) - \frac{\partial \Omega_2(x_2^*, t_2^*)}{\partial t_2} \geq H_2^{22*}(t_2^*) \text{ for all } t_2^* \geq t_1^* \tag{4}$$

2. *Suppose player 2's switching problem has an interior solution* (t_2^*, x_2^*).

- *A necessary condition for player 1 to choose a corner solution with immediate switching* $0 = t_1^*$ *is*

$$H_1^{11*}(t_1^*) - \frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial t_1} \leq H_1^{21^*}(t_1^*) - [H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)] \text{ if } 0 = t_1^* < t_2^* \tag{5}$$

- *A necessary condition for player 1 to choose a corner solution of the never switching type* $t_1^* = t_2^*$ *is*

$$H_1^{11*}(t_1^*) - \frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial t_1} \geq H_1^{21^*}(t_1^*) - [H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)] \text{ if } 0 < t_1^* = t_2^* \tag{6}$$

Proof. See the appendix A. ■

When examining the corner solutions, a distinction should be made between different cases. The situations $t_1^* = 0$ and $t_2^* = \infty$ have been studied in literature, for open-loop strategies (Boucekkine, Krawczyk and Vallée, 2011). If $t_1^* = 0$, it must be because player 1 wants to escape from regime 11 as soon as possible. This is the case if a delay in switching yields a marginal gain that is not greater than the marginal loss of foregoing for an instant the benefit of the new regime. Similarly, if player 2 were to adopt a never switching strategy then it would mean that for all $t_2 \geq t_1^*$, the marginal gain from delaying a switch away from regime 21 is greater than the marginal cost.

Of further interest is the interpretation of players' "corner" solution $t_1^* = t_2^*$. The switching conditions are of the same meaning as before. Consider for instance player 1's problem. The inequality in (6) is the optimality condition for player 1: if he chooses the corner $t_1^* = t_2^*$, it must be true that at t_2^* a delay in switching yields a marginal gain that is at least as high as the marginal loss of foregoing for an instant the benefit of new regime. To analyze player 1's choice of switching time t_1, we have proceeded as if that player were subject to the constraint $t_1 \leq t_2^*$, with t_2^* having been determined. Then, using the tools originally developed by Tomiyama (1985) and Amit (1986), the corresponding finite horizon switching problem was solved. However, because the current analysis pertains to a differential game, Condition (6) cannot simply be interpreted as a necessary condition for having a corner solution $t_1^* = t_2^*$. Rather, this condition is necessary for the timing $0 \leqq t_1 \leqq t_2 \leqq \infty$ to be wrong. Indeed, under (6), Player 1 would prefer switching at a later date than t_2^*. This is feasible because t_2^* is not fixed.[8]

[8] The same reasoning applies to the case where player 2 finds the timing non-optimal.

In other words, till now we have derived optimality conditions for each player, under a particular given timing. But the timing is not fixed and an important task is to determine what will be the timing at the equilibrium, or under which conditions a particular timing will occur. The analysis of these wrong timing situations is of crucial importance to address this non-trivial issue. Indeed, it should allow us to gain information about the optimal timing by pointing out the conditions under which one player will optimally accept to make the first switch, while the other will choose to be the second-mover. This would allow the elimination of some MPE candidates. In our problem, there are a priori fifteen possible timings corresponding to the set of possible combinations between t_1 and t_2. But, it is highly unlikely that heterogeneous players decide on the same switching time. So, in the case of heterogeneous players, the timing $0 < t_1 = t_2 < \infty$ should not give MPE candidates.[9] Logically, one expects that several cases are mutually exclusive. Analyzing the wrong timing conditions, (5)-(6) and the ones obtained when analyzing the other timing, should be a means to understand, once the benefit and cost functions are specified, which timing, between $0 \leqq t_1 < t_2 \leqq \infty$ and $0 \leqq t_2 < t_1 \leqq \infty$, is consistent with MPE requirement. Therefore, in any particular application, an analysis of wrong timing situations should then be conducted in order to reduce the set of potential candidates for MPEs, before having a look at other interior or corner solutions.

The next section is devoted to an application of the theory to an exhaustible resource problem. Our purpose is to illustrate how the reasoning above works in a simple example from which we can extract (partial) analytical results.

4 Application: A resource extraction game

We consider a differential game of extraction of a non-renewable resource. In the related literature,[10] it is generally argued that the presence of rivalry among multiple agents tends to result to inefficient outcomes, e.g. overextraction of natural resources. Another common feature of the frameworks developed in this literature is the assumption that players cannot adopt new technology that will improve their extraction efficiency. It is usually assumed that consumption is a fixed fraction of the extraction level. In this section, we relax this assumption and consider the possibility of technological adoption among players. That is, players not only choose their consumption. They also decide when to adopt

[9] However, it's quite easy to derive the optimality conditions is this case. Suppose that it is optimal for the two players to switch at the same date $t^* = t_1 = t_2 \in (0, \infty)$, for the same level of the state $x^* = x_1^* = x_2^*$, then the following conditions must hold, for $i = 1, 2$:

$$\begin{aligned} H_i^{11*}(t^*) - \tfrac{\partial \Omega_i(x^*, t^*)}{\partial t_i} &= H_i^{22*}(t^*) \\ \lambda_i^{11*}(t^*) + \tfrac{\partial \Omega_i(x^*, t^*)}{\partial x_i} &= \lambda_i^{22*}(t^*). \end{aligned} \tag{7}$$

Finally, conditions corresponding to the cases $t_1 = t_2 = 0$ or $t_1 = t_2 = \infty$, that are more relevant to our analysis, are basically the same as in the single-agent framework. They are presented in Appendix A.

[10] For extensive surveys on dynamic games in resource economics, refer to Long (2010, 2011).

the more efficient extraction technology. This puts forth another innovative contribution of this paper.[11]

Our resource extraction game comprises $I = 2$ players. Let $u_i(t)$ denote the consumption level of player i, $i = 1, 2$, at time $t \geq 0$. Meanwhile, let $e_i(t)$ be player i's extraction rate from the resource at time $t \geq 0$. Extraction is converted into consumption according to the following technology: $\gamma_i u_i(t) = e_i(t)$, where γ_i^{-1} is a positive number that reflects a player's degree of efficiency in transforming the extracted natural resource into a consumption good.

Two production technologies, described only by the parameter γ_i, are available to player i from $t = 0$. Because players' technological menus may differ, one needs to introduce a specific index for the player's actual technology. It is assumed that player 1 starts with technology $l = 1$ and has to decide: *(i)* whether she switches to technology $l = 2$, and *(ii)* when. The state of technology of the other player, 2, is labelled as k and a technological regime is represented by $s = lk$, with $l, k = 1, 2$. For each player i, the ranking between the parameters satisfies: $\gamma_i^1 > \gamma_i^2$, which means that the second new technology is more efficient than the old one. A possible indicator of technological gain for player i from adoption is the ratio $\frac{\gamma_i^2}{\gamma_i^1} \in (0, 1)$, such that the smaller is the ratio, the higher is the gain.

Let $x(t)$ be the stock of the exhaustible resource, with the initial stock x_0 given. As in section 2, t_1 and t_2 are the switching times. Suppose $0 < t_1 < t_2$, then the evolution of the stock is given by the following differential equation:

$$\dot{x} = \begin{cases} -\gamma_1^1 u_1 - \gamma_2^1 u_2 \text{ if } t \in [0, t_1) \\ -\gamma_1^2 u_1 - \gamma_2^1 u_2 \text{ if } t \in [t_1, t_2) \\ -\gamma_1^2 u_1 - \gamma_2^2 u_2 \text{ if } t \in [t_2, \infty) \end{cases}$$

At the switching time, if any, player i incurs a cost that is defined in terms of the level of the state variable at which the adoption occurs, $x(t_i) = x_i$. Let $\omega_i(x_i)$ be this cost, with $\omega_i'(.) \geq 0$. The direct switching cost is discounted at rate ρ. As seen from the initial period, if a switch occurs at t_i, the discounted cost amounts to $e^{-\rho t_i}\omega_i(x_i)$ (this is our $\Omega(x_i, t_i)$ of Section 2). It takes the following form: $\omega_i(x_i) = \chi_i + \beta_i x_i, \chi_i > 0$ and $\beta_i \geq 0$. χ_i is the fixed cost related to technology investment. These may include initial outlay for machinery, etc. On the other hand, β_i represents the sensitivity of adoption cost to the level of the exhaustible resource at the instant of switch. Our assumption implies that the cost of adopting new technology is increasing in x_i. This assumption conveys the idea that the lower the level of the (remaining) stock of resource, the lower the cost of adopting the new technology. It could reflect the fact that scientific progress on installation of resource-saving technology is continually made as the scarcity becomes more acute. Finally, each player's gross utility function depends on her consumption only and takes the logarithmic form: $F(u_i, u_{-i}, x) = \ln(u_i)$.

[11] As mentioned in the Introduction section, technology adoption issues have been studied in the multi-stage optimal control literature but there are very few references in dynamic game theory.

In the next subsections, attention is paid first to the corner solutions, which allows us to address the issue of the equilibrium timing. Then an analysis of the interior solution – for the correct timing – is conducted with the aim to discuss the features of the equilibrium with regime switching. From now on, the timing considered is $0 \leqq t_1 \leqq t_2$. Results for the other timing are obtained by symmetry.

4.1 Equilibrium timing

The first part of the subsequent analysis examines the conditions under which the timing under scrutiny is wrong, i.e. at least one player would choose the opposite timing. This will be followed by identifying necessary conditions for corner solutions. Once these tasks are done, we can solve for interior solutions.

4.1.1 Wrong timing

A player may find the timing $0 \leqq t_1 \leqq t_2 \leqq \infty$ *non-optimal*. For instance, guessing that player 1 will switch at $t_1 (< \infty)$, player 2 may prefer switching at a date *no later* than t_1. In general, it can be shown that[12]

Proposition 1 *If player i, $i = 1, 2$, finds the above timing non-optimal then it must hold that*

$$\rho\omega_2(x_{-i}^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \leq \rho\omega_1(x_{-i}^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right), \tag{8}$$

where x_{-i} is the switching point of the other player that corresponds to this hypothetical scenario.

Proof. See the appendix B.3. ∎

As mentioned in Section 3, following Theorem 2, condition (8) characterizes a situation that is more than a simple corner solution. As an illustration, let's consider player 2's situation. Let (t_1^*, x_1^*) be the interior solution of player 1's switching problem when he anticipates that player 2 will stick to his timing strategy, with $0 < t_1^* < \infty$ given. In order to obtain (8), we have determined under which conditions it is "optimal" for player 2, who maximizes the discounted value between t_1^* and ∞, to switch immediately. This means that the necessary conditions are similar to the usual conditions of the multi-stage optimal control theory for immediate switching. However, this particular situation cannot be interpreted as a corner solution precisely because the framework under scrutiny is a differential game. This implies that in fact t_1^* (the beginning of the planning period for player 2) is not given. So, we should interpret this degenerate corner solution as a situation where it is not optimal for player 2 to adopt after player 1. Player 1, who is supposed to be the first mover, may find the timing

[12] Even though the analysis of wrong timing comes logically before considering any other solution, it should be noted that in the Appendix B, the proof of proposition 1 cannot be read independently of the other parts. This is also true for the proofs devoted to corner solutions.

non-optimal as well. What is worth noting is that the necessary condition for a wrong timing is the same for the two players.[13] Therefore, as long as condition (8) holds for at least one of the player, the correct timing, at the MPE, if a MPE exists, should be $0 \leqq t_2 \leqq t_1 \leqq \infty$.

Next, we can exploit the result that necessary conditions for a wrong timing are similar to establish that:

Corollary 1 *A sufficient condition for the equilibrium timing to be* $0 \leqq t_1 \leqq t_2 \leqq \infty$ *is:*

$$\rho[\omega_1(x) - \omega_2(x)] < -\left[\ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) - \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] \text{ for all } x \in [0, x_0] \tag{9}$$

If condition (9) holds, then the timing $t_2 < t_1$ cannot occur in equilibrium. This condition can easily be interpreted in economic terms. First note that $-\ln\left(\frac{\gamma_i^2}{\gamma_i^1}\right)$, for $i = 1, 2$, is a measure of the gain from switching. Then, this condition basically states that for player 1, the relative advantage of adoption (RHS), measured in terms of the differential of gains, is greater than the relative disadvantage in terms of adoption costs (LHS). Put differently, player 1 has a relative disadvantage in adoption costs that is compensated by a relative technological advantage. Of course, this inequality is satisfied when player 1 incurs a lower direct switching cost and, at the same time, derives a higher benefit of adoption. But, it might also hold in intermediate situations where player **1**'s adoption cost is higher provided that the differential in technological gains is largely favorable to player 1.

From now on, let's assume that (9) holds. In the next section, we briefly review the corner solutions associated with the timing $0 \leqq t_1 \leqq t_2 \leqq \infty$.

4.1.2 Corner solutions

First, we emphasize the conditions under which the MPE may feature a corner solution. Next, we tackle the issue of the occurrence of a simultaneous switch.

Proposition 2

- *Assume player 1's switching problem has an interior solution t_1^*. A necessary condition for player 2 to choose the "never switching strategy," so that $0 < t_1^* < t_2^* = \infty$ is that*

$$\rho\omega_2(0) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \geq 0. \tag{10}$$

- *Assume player 2's switching problem has an interior solution t_2^*. A necessary condition for player 1 to switch immediately at the beginning, so that $0 = t_1^* < t_2^* < \infty$ is*

$$\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq e^{-\rho\theta_2^*(x_0,21)} \ln(1 - \beta_2\rho x_2^*). \tag{11}$$

[13] There is no single condition however because the reference point in (8), that is given by the switching point of the other player, matters.

- *For a combination of immediate and never switching* $0 = t_1^* < t_1^* = \infty$ *to arise at the MPE, it is necessary that (10) and the condition below hold:*

$$\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq 0. \tag{12}$$

Proof. See respectively Appendix B.1.2, B.2.2 and B.4. ■

Necessary conditions for being at a corner solution have very simple interpretations. For instance, according to condition (10), a player never finds it worthwhile to adopt the new technology when the fixed cost of adoption, weighted by the rate of time preference, is larger than the gain from switching. In the same vein, a player is willing to adopt the new technology immediately when the switching cost at the initial resource level is lower than the gain from adoption. In the latter case, the particular tradeoff is influenced by the other player's switching decision to switch in finite time (11) or keep the old technology forever (12).

Beyond corner solutions, there are three remaining cases: (a) Players might wish to adopt their new technology at the same date and for the same stock of resource. Or, (b) they might both prefer switching instantaneously; or (c) on the contrary adopting never switching strategies. If there is heterogeneity in switching costs, case (a) cannot be an equilibrium outcome. The conditions for having the two other possibilities can easily be derived from proposition 2 (see the Appendix B.4).

Finally, it is useful to establish that:

Corollary 2 *The following (double) inequality*

$$\rho\omega_i(0) < -\ln\left(\frac{\gamma_i^2}{\gamma_i^1}\right) < \rho\omega_i(x_0) \text{ for } i = 1, 2, \tag{13}$$

is sufficient for not to be in a corner solution.

Thus, if the switching cost is low enough (compared to the gain from switching) when the resource gets close to exhaustion, then it is worthwhile to adopt the new technology in finite time. Moreover, this cost at the beginning of the game should be high for making players unwilling to adopt immediately.

With this information in mind, we now turn to the analysis of the interior solution.

4.2 Impact of the interaction through switching strategies on the equilibrium

At the interior solution ($0 < t_1 < t_2 < \infty$), our differential game can be divided into three subgames. We proceed backward by examining first the solution to the last period problem, i.e. to the subgame arising after player 2's regime switch. This is a standard infinite horizon differential game. Recalling that we restrict our attention to linear feedback strategies, we find that the consumption strategies $\Phi_i(x, 22)$ $(i = 1, 2)$ at the MPE satisfy[14]

$$\gamma_1^2 \Phi_1^*(x, 22) = \gamma_2^2 \Phi_2^*(x, 22) = \rho x. \tag{14}$$

From these strategies, we can easily compute players' present values corresponding to the last period problem, which are used as scrap value functions for the preceding problem. Indeed, the next step is to examine the second period problem in which player 2 has now to take her regime switching action. The resolution consists in determining not only consumption strategies valid in regime 21 but also the switching time and switching point of player 2 at the MPE. Results are summarized in the proposition below.[15]

Proposition 3

- *In regime 21, consumptions strategies are given by*

$$\gamma_1^2 \Phi_1^*(x, 21) = \gamma_2^1 \Phi_2^*(x, 21) = \Gamma + \rho x \text{ with } \Gamma = \frac{\rho^2 \beta_2 (x_2^*)^2}{1 - \beta_2 \rho x_2^*}. \tag{15}$$

- *The optimal switching point, x_2^*, is defined by*

$$\rho \omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) = \ln(1 - \beta_2 \rho x_2^*). \tag{16}$$

Sufficient conditions for the existence of a unique x_2^* are:

$$\begin{aligned} &-\ln\left(\tfrac{\gamma_2^2}{\gamma_2^1}\right) > \rho\omega_2(0), \\ &\rho\omega_2(x_1) + \ln\left(\tfrac{\gamma_2^2}{\gamma_2^1}\right) > \ln(1 - \beta_2 \rho x_1). \end{aligned} \tag{17}$$

- *The time-to-go (before switching) strategy is $t_2^* - t_1 = \theta_2^*(x_1, 21)$ with*

$$\theta_2^*(x_1, 21) = \frac{1}{2\rho} \ln\left[(1 - \rho\beta_2 x_2^*)\frac{x_1}{x_2^*} + \rho\beta_2 x_2^*\right] = \frac{1}{2\rho} \ln\left[\frac{\Phi_i^*(x_1, 21)}{\Phi_i^*(x_2^*, 21)}\right]. \tag{18}$$

where (t_1, x_1) can be any solution to the switching problem of player 1.

Proof. See the appendix B.1.1. ■

Several comments are in order here. First, from equations (14) and (15), one can observe that $\gamma_2^1 u_2^{21*}(t_2^*) = \gamma_2^2 u_2^{22*}(t_2^*)$ iff $\beta_2 = 0$. Thus, if $\beta_2 > 0$, players' resource extraction experiences a jump at the switching date of player 2. This results from the dependence of the direct switching cost on the level of the state variable at the switching date.[16] Second, the first sufficient condition (for the

[14] Players share the same extraction rate regardless of the regime. This is due to the symmetric structure of the extraction game. Of course the common extraction rate is regime-dependent.

[15] For simplicity, we assume that: $x_0 > (\rho\beta_2)^{-1}$.

[16] A similar pattern is observed by Valente (2011) and Prieur, Tidball and Withagen (2013), in different frameworks.

existence of a unique switching point, x_2^*) in (17) is satisfied whenever we assume that the condition (13) of Corollary 2 holds. This condition basically states that player 2 will switch in finite time as long as the technology differential – gain from switching – is large enough compared to the fixed cost of switching, when the stock of resource approaches zero. Third, the time-to-go before switching (18) is defined in terms of player 1's switching point, x_1, the discount rate and some parameters characterizing regime 21, that players leave, and regime 22, that players reach. Hence, player 1 is able to affect player 2's switching strategy and will take this influence into account in the first period problem. Note also that the optimal switching date of player 2 is increasing in x_1. The larger the resource stock at which player 1 decides to switch, the later the adoption of player 2. In other words, switching rapidly for player 1 tends to delay the adoption time of player 2.

Adopting the same methodology as before (notably by computing player 1's present value from regime 21 on), we can finally have a look at the first period problem. In regime 11, player 1 guesses that *(i)* player 2 has in mind an optimal switching point x_2 (not necessarily the same as x_2^* that we found above), and that *(ii)* player 2 has a time-to-go strategy $\theta_2(x_1, 21)$ (not necessarily $\theta_2^*(x_1, 21)$ defined by (18)). He also guesses that player 2's consumption strategy in regime 11 takes the form $\Phi_2(x, 11)$. Now we characterize the MPE in consumption strategies in regime 11 and provide sufficient conditions for having a unique solution to the first player's switching problem. Note that at the MPE, player 1's guesses have to be consistent with player 2's actual strategies.

Proposition 4

- *In regime 11 consumption strategies satisfy*

$$\gamma_1^1 \Phi_1^*(x, 11) = \gamma_2^1 \Phi_2^*(x, 11) = \Lambda + \rho x, \text{ with } \Lambda = \frac{\Gamma + \rho x_1^*[1 - \zeta(x_1^*; x_2^*)]}{\zeta(x_1^*; x_2^*)},$$

and

$$\zeta(x_1^*; x_2^*) = 1 - \frac{e^{-\rho\theta_2^*(x_1^*, 21)}}{2} \ln(1 - \rho\beta_2 x_2^*) - \beta_1(\Gamma + \rho x_1^*),$$

where Γ *is given in Proposition 3.*

- *The optimal level of the stock for switching* x_1^* *solves*

$$\rho\omega_1(x_1^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = e^{-\rho\theta_2^*(x_1^*, 21)} \left[\rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] + \ln[\zeta(x_1^*; x_2^*)] \tag{19}$$

- *If the following conditions hold:* $\zeta(x_2^*; x_2^*) \geq 1$, $\zeta(x_0; x_2^*) \in (0, 1]$ *and*

$$\begin{aligned} &\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) > e^{-\rho\theta_2^*(x_0, 21)} \left[\rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right], \\ &\rho\omega_1(x_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) < \rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right), \end{aligned} \tag{20}$$

then there exists a unique $x_1^* \in (x_2^*, x_0)$.

- *With the pair* (x_1^*, x_2^*) *being determined above, the optimal switching time* $t_1^* = \theta_1^*(x_0, 11)$ *is*

$$\theta_1^*(x_0, 11) = \frac{1}{2\rho} \ln \left(\frac{x_0 + \frac{\Lambda}{\rho}}{x_1^* + \frac{\Lambda}{\rho}} \right) = \frac{1}{2\rho} \ln \left[\frac{\Phi_i^*(x_0, 11)}{\Phi_i^*(x_1^*, 11)} \right].$$

Proof. See the Appendix B.2.2. ■

Recalling that $\gamma_1^2 \Phi_1^*(x_1^*, 21) = \Gamma + \rho x_1$ (see 15), we observe that $\zeta(.)$ defined above provides information on the direction ($\lesseqgtr 1$) and magnitude of the jump in the extraction rate at the switching time t_1. The third item of Proposition 4 basically enumerates the sufficient conditions for the existence of an interior solution to player 1's switching problem. At first glance, these boundary conditions may seem difficult to interpret. But, referring once again to Corollaries 1 and 2, it is clear that conditions in (20) are necessarily satisfied under the sufficient conditions we discussed previously for not being in a wrong timing and not having corner solutions.

In the remainder of this section, we further address the impact of MPE strategies for switching times on the equilibrium. Indeed, given that (player 2) switching strategy is based on the state of the system and player 1 is able to affect this state, it is crucial to understand how player 1 adapts his strategy to player 2's switching decision. This also requires the solution to the following related issue: what is the impact of player 2's future switch on player 1? For the sake of interpretation, player 1's switching conditions are rewritten as:[17]

$$\begin{aligned} &\ln \left[\frac{u_1^{11*}(t_1^*)}{u_1^{21*}(t_1^*)} \right] = -\rho\omega_1(x_1^*) + e^{-\rho\theta_2^*(x_1^*,21)} \ln \left[\frac{u_1^{22*}(t_2^*)}{u_1^{21*}(t_2^*)} \right] \\ &\left[\gamma_1^2 u_1^{21*}(t_1^*) \right]^{-1} - \left[\gamma_1^1 u_1^{11*}(t_1^*) \right]^{-1} = \omega_1'(x_1^*) + \theta_2^{*\prime}(x_1^*, 21) e^{-\rho\theta_2^*(x_1^*,21)} \ln \left[\frac{u_1^{22*}(t_2^*)}{u_1^{21*}(t_2^*)} \right] \end{aligned} \tag{21}$$

Compared to the single-agent problem, both conditions are modified. The LHS of the *first equation* in (21) reflects the marginal gain from extending the horizon of the first regime. If there exists $0 < t_1^* < t_2^*$ then this marginal gain must be equal to the marginal cost of switching at t_1^*. Now, the marginal switching cost (RHS) is augmented (in absolute magnitude) by the extra-term $e^{-\rho\theta_2^*(x_1^*,21)} \ln[\frac{u_1^{22*}(t_2^*)}{u_1^{21*}(t_2^*)}]$. Player 1 anticipates that his switching decision will be followed by the switch (in finite time too) of the second player and that this switch will be costly. Why is it so? Adopting a new technology translates into a decrease in the extraction rate: $\gamma_2^1 u_2^{21*}(t_2^*) > \gamma_2^2 u_2^{22*}(t_2^*)$. Intuitively, with the new technology, one needs less resource to produce a given amount of the consumption good. The impact of player 2's adoption on her own consumption is unclear because it depends on the size of the productivity differential $\frac{\gamma_2^2}{\gamma_2^1}$. However, it is clear that player 1 is worse off after player 2's switch because he bears the decrease in extraction (as both players have the same extraction rate in regime 22) and is not able to compensate this loss by an adaptation of his

[17] At the MPE, the guess of player 1 must be consistent with the switching strategy actually adopted by player 2.

technology. So, it means that the marginal cost of switching is higher than it would be in the absence of player 2. Other things equal (x_1 constant), it implies that the switch should occur at a later date, *i.e.* player 1, when interacting with player 2, has an incentive to postpone adoption.

The *second equation* in (21) equalizes the marginal benefit from an extra unit of the state variable x_1 (LHS) with the corresponding marginal cost (RHS). The marginal cost is lower in the game than in the control problem because $\theta_2^{*\prime}(x_1^*, 21)e^{-\rho\theta_2^*(x_1^*,21)} \ln[\frac{u_1^{22*}(t_2^*)}{u_1^{21*}(t_2^*)}] < 0$. Indeed, other things equal (t_1 constant), by increasing x_1, player 1 induces player 2 (because player 1 controls x_1 on which is based player 2's decision) to delay the instant of her switch that is, the instant when player 1 will incur the additional indirect (marginal) cost. The impact of player 2's switch will then be felt less acutely because of discounting. This in turn implies that player 1's adoption should occur at a higher x_1^*. This second effect makes it worthwhile for player 1 to adopt at an earlier date (because the trajectory of x is monotone non increasing).

In summary, as a result of the interaction with player 2, player 1 will delay the adoption of the new technology (first-order effect corresponding to the first condition in (21)). It does not mean however that he will not adopt before player 2. According to the second condition in (21), the sooner the adoption of player 1, the lower the negative impact of player 2's adoption on his welfare (second-order effect).

To conclude this analysis, let us highlight a striking result that can be obtained by focusing on the special case where $\omega_1(x_1) = 0$ (the switching cost is independent of the stock of resource). In this case, player 1 does not bear any (direct) cost when he switches. Then, we know that the solution of the optimal control problem (single-agent problem) is $t_1^* = 0$: One adopts instantaneously because the new technology is more efficient than the old one. But, it is clear that if the equations in (21) have a solution,[18] then conclusions will be very different in the switching game. Player 1 incurs a indirect (marginal) cost when player 2 adopts. Then, it is optimal for player 1 to switch at a date t_1^* such that $0 < t_1^* < t_2^*$ because it allows him to compensate for the loss by increasing extraction (which implies that consumption increases too) at the switching time, i.e. one must have $\gamma_1^1 u_1^{11*}(t_1^*) < \gamma_1^2 u_1^{21*}(t_1^*)$. Interestingly enough, the interaction through switching time offsets the previous effect of adoption (identified for player 2): switching to the new, more efficient, technology translates into an increase in the extraction rate.

5 Conclusion

In this paper, we have developed a general two-player differential game with regime switching strategies. The interaction between players is assumed to be governed by two kinds of strategies. At each point in time, they have to

[18]This can be guaranteed by deriving the existence conditions for this special case from the analysis of Appendix B.2.2.

choose an action that influences the evolution of a state variable. In addition, they may decide on the switching time between alternative and consecutive regimes. At a feedback Nash equilibrium, the switching strategy is defined as a function of the state of the system. Compared to the standard optimal control problem with regime switching, necessarily optimality conditions are modified only for the first-mover. When choosing the optimal date and the level of state variable for switching, this player must take into account that *(i)* his decision will influence the other player's switching strategy, and *(ii)* the other player's switch will affect his welfare. Furthermore, we have exhibited the necessary conditions characterizing the timing at the Markov perfect equilibria. Wrong timing situations were also analyzed.

In the second part of this paper, we applied this new theoretical framework to solve a game of exhaustible resource extraction with technological regime switching. It was assumed that, at a given cost, players have the option to adopt a more efficient extraction technology. We then obtained sufficient conditions guaranteeing that both players switch in finite time. Moreover, we investigated the impact of feedback strategies for switching time on the first-mover technology adoption strategy. There is an interplay between two conflicting effects. First, the switch of the second mover is costly for the first-mover because it implies a drop in his consumption. Thus, the first-mover may opt to delay adoption. Meanwhile, because of discounting, delaying the switch of the other play will allow the first-mover to incur a lower indirect cost. This is an incentive for the first-mover to adopt at an earlier date.

Overall, the methodology presented in this paper may pave the way to handle a wider class of problems in economics. Potential extensions include the analysis of technology adoption in a climate change game, the consideration of the interaction between the elites and the citizens in a game of institutional regime changes (Acemoglu and Robinson, 2006, 2008), and the like. These issues will be addressed in the authors' future research endeavors.

References

[1] Acemoglu, D. and Robinson, J.: Economic Origins of Dictatorship and Democracy, Cambridge Books, Cambridge University Press (2006).

[2] Acemoglu, D. and Robinson, J.: Persistence of Power, Elites, and Institutions, American Economic Review **98**(1), 267-293 (2008).

[3] Amit, R.: Petroleum reservoir exploitation: Switching from primary to secondary recovery. Operations Research **34**, 534-549 (1986).

[4] Boucekkine, R., Saglam, C. and Vallée T.: Technology adoption under embodiment: a two-stage optimal control approach. Macroeconomic Dynamics **8**, 250-271 (2004).

[5] Boucekkine, R., Krawczyk, J. and Vallée T.: Environmental quality versus economic performance: A dynamic game approach. Optimal Control Applications and Methods **32**, 29-46 (2011).

[6] Boucekkine, R., Pommeret, A. and Prieur F. Optimal regime switching and threshold effects. Journal of Economic Dynamic and Control **37**(12), 2979-2997. (2013).

[7] Fudenberg, D Tirole, J.: Preemption and rent equalization in the adoption of new technology. Review of Economic Studies, **52**(3), 383-401 (1985).

[8] Long, NV.: A Survey Of Dynamic Games In Economics. World Scientific Publishing Co. Pte. Ltd. (2010).

[9] Long, NV.: Dynamic Games in the Economics of Natural Resources: A Survey. Dynamic Games Applications **1**(1), 115-148 (2011).

[10] Makris, M.: Necessary conditions for infinite horizon discounted two-stage optimal control problems. Journal of Economic Dynamics and Control **25**,1935-1950 (2001).

[11] Reinganum, JF.: On the diffusion of new technology: A game theoretic approach. Review of Economic Studies **48**, 385-405 (1981).

[12] Prieur, F., Tidball, M. and Withagen, C.: Optimal emission-extraction policy in a world of scarcity and irreversibility. Resource and Energy Economics **35**(4), 637-658 (2013).

[13] Tomiyama, K.: Two-stage optimal control problems and optimality conditions. Journal of Economic Dynamics and Control **9**, 317-337 (1985).

[14] Tornell, A.: Economic growth and decline with endogenous property rights. Journal of Economic Growth, **2**(3), 219-50, (1997).

[15] Valente, S.: Endogenous growth, backstop technology adoption, and optimal jumps. Macroeconomic Dynamics **15**, 293-325 (2011)

Appendix

A Proof of Theorems 1&2

Let the triplet $(u_1^*(t), u_2^*(t), x^*(t))$ be the path followed by each player's strategy and the stock variable at a Markov perfect equilibrium (MPE), for every $t \in [0, +\infty)$. A restriction of this path to $[t_{j-1}, t_j]$, which corresponds to a particular regime say s, $j = 1, 2$ with $t_0 = 0$, continues to characterize the solution of the subgame with $x^*(t_j) = x_j^*$, t_{j-1} and t_j fixed and with the maximization of $\int_{t_{j-1}}^{t_j} F_i^s(u_1, u_2, x)e^{-\rho t}dt$ as player i's objective, $i = 1, 2$. In addition, a restriction of $(u_1^*(t), u_2^*(t), x^*(t))$ to $[t_2, +\infty)$ is a MPE of the infinite horizon game with $x^*(t_2) = x_2^*$, t_2 fixed and with the maximization of $\int_{t_2}^{\infty} F_i^{22}(u_1, u_2, x)e^{-\rho t}dt$ as player i's objective.

The proof uses standard calculus of variations techniques in a sequence of three subgames as explained in the main text. The problem is solved recursively, starting from the game arising after the last switch. The proof focuses on the timing $0 \leqq t_1 \leqq t_2 \leqq \infty$, i.e. on the case where player 1 is the first to switch, followed by player 2.[19]

In each subgame, player's optimization problems are solved using the Pontryagin method.[20] This implies that when solving player i's problem, in any regime, we have to introduce a guess about the other player's strategy, $u_{-i}(t) = \Phi_{-i}(x(t), s)$. Moreover, attention is mainly paid to the problem faced by the player who undertakes the switching decision. When required, we also present the optimality conditions of the other player.

- *Last regime* 22*, for* $t \geq t_2$: Player i solves:

$$\max_{u_i} \int_{t_2}^{\infty} F_i^{22}(u_i, \Phi_{-i}(x, 22), x)e^{-\rho t}dt, \tag{22}$$

subject to,

$$\dot{x} = f^{22}(u_i, \Phi_{-i}(x, 22), x). \tag{23}$$

where t_2 and the initial condition $x(t_2) = x_2$ are fixed. $x(t_2)$ will be made free (end-point) in the next stage. The present value Hamiltonian of the problem, H_i^{22}, is given by $H_i^{22} = F_i^{22}(u_i, \Phi_{-i}(x, 22), x)e^{-\rho t} + \lambda_i^{22} f^{22}(u_i, \Phi_{-i}(x, 22), x)$, where λ_i^{22} is player i co-state variable associated with x in regime 22. This problem does not deserve further attention since it yields straightforward first-order necessary conditions. Let us denote by superscript $*$ the paths identified by these conditions (abstracting from the issues of existence and uniqueness). Let $V_i^{22*}(x_2, t_2)$ be the value function, we have the usual envelope conditions:

$$\begin{aligned} \frac{\partial V_i^{22*}}{\partial t_2} &= -H_i^{22}(t_2), \\ \frac{\partial V_i^{22*}}{\partial x_2} &= \lambda_i^{22}(t_2) \text{ for } i = 1, 2. \end{aligned} \tag{24}$$

[19]The necessary optimality conditions for the other timing $0 \leqq t_2 \leqq t_1 \leqq \infty$ can be obtained by symmetry.

[20]We may alternatively adopt the HJB approach and obtain the same results that will take the form of the well-known continuity and smooth-pasting conditions.

• *Second regime* 21*, for* $t \in [t_1, t_2]$: In this regime, player 2 (the one who decides on the switching time) solves:

$$\max_{u_2, t_2} \int_{t_1}^{t_2} F_2^{21}(u_2, \Phi_1(x, 21), x)e^{-\rho t}dt - \Omega_2(x_2, t_2) + V_2^{22*}(x_2, t_2)$$

subject to,

$$\dot{x} = f^{21}(u_2, \Phi_1(x, 21), x),$$

where t_1 and the initial condition $x(t_1) = x_1$ are fixed. But, $x(t_2) = x_2$ is a free end-point and t_2 is a control variable. After some calculations, one obtains:

$$V_2^{21} = \int_{t_1}^{t_2} (H_2^{21} + \dot{\lambda}_2^{21} x)dt - \left[\lambda_2^{21}(t_2)x_2 - \lambda_2^{21}(t_1)x_1\right] - \Omega_2(x_2, t_2) + V_2^{22*}(x_2, t_2).$$

To find the necessary optimality conditions, we derive the first-order variations of V_2^{21} with respect to the state and control variables' paths, for fixed t_1, $x(t_1) = x_1$ and free t_2 and x_2. This yields, after rearranging terms:[21]

$$\begin{array}{l} \delta V_2^{21} = \int\limits_{t_1}^{t_2} [(\frac{\partial H_2^{21}}{\partial x} + \frac{\partial H_2^{21}}{\partial u_1}\Phi_1'(x, 21) + \dot{\lambda}_2^{21})\delta x + \frac{\partial H_2^{21}}{\partial u_2}\delta u_2]dt \\ +(H_2^{21}(t_2) - \frac{\partial \Omega_2(x_2,t_2)}{\partial t_2} + \frac{\partial V_2^{22*}}{\partial t_2})\delta t_2 - (\lambda_2^{21}(t_2) + \frac{\partial \Omega_2(x_2,t_2)}{\partial x_2} - \frac{\partial V_2^{22*}}{\partial x_2})\delta x_2. \end{array} \tag{25}$$

A trajectory is optimal if any small departure from it decreases the value function, that is $\delta V_2^{21} \leq 0$ for any $\delta x(t)$, $t \in (t_1, t_2)$, for any $\delta u_2(t)$, $t \in [t_1, t_2]$, and for any δt_2 and δx_2. This gives the following necessary conditions for an interior maximizer, $t_1 < t_2 < \infty$:

$$\left\{ \begin{array}{l} \frac{\partial H_2^{21}}{\partial u_2} = 0, \ \frac{\partial H_2^{21}}{\partial x} + \frac{\partial H_2^{21}}{\partial u_1}\Phi_1'(x, 21) + \dot{\lambda}_2^{21} = 0, \\ H_2^{21}(t_2) - \frac{\partial \Omega_2(x_2,t_2)}{\partial t_2} + \frac{\partial V_2^{22*}}{\partial t_2} = 0, \ \lambda_2^{21}(t_2) + \frac{\partial \Omega_2(x_2,t_2)}{\partial x_2} - \frac{\partial V_2^{22*}}{\partial x_2} = 0. \end{array} \right. \tag{26}$$

The first two equations are the standard Pontryagin conditions, the last two are optimality conditions with respect to the switching time, t_2, and the free state value, x_2. Together with conditions in (24) obtained from the third sub-problem, one gets conditions (1) of Theorem 1, that is:

$$\begin{array}{l} H_2^{21*}(t_2^*) - \frac{\partial \Omega_2(x_2^*,t_2^*)}{\partial t_2} = H_2^{22*}(t_2^*) \\ \lambda_2^{21*}(t_2^*) + \frac{\partial \Omega_2(x_2^*,t_2^*)}{\partial x_2} = \lambda_2^{22*}(t_2^*). \end{array} \tag{27}$$

Optimality conditions for corner solutions can easily be deduced from the analysis above:

Suppose $t_1 = t_2^*$, with t_1 the switching time of player 1 taken as given. Then the only possible variations of t_2^* are such that $\delta t_2 \geq 0$. From (24) and (25), $\delta V_2^{21} \leq 0$ only if (condition 3 of Theorem 2)

$$H_2^{21*}(t_2^*) - \frac{\partial \Omega_2(x_2^*, t_2^*)}{\partial t_2} \leq H_2^{22*}(t_2^*) \tag{28}$$

[21]The derivative of the action strategy, $\Phi_i(.)$ is necessarily taken w.r.t the state variable x.

Similarly, for the last case, $t_2^* = \infty$, to occur, it must hold that:

$$H_2^{21*}(t_2^*) - \frac{\partial \Omega_2(x_2^*, t_2^*)}{\partial t_2} \geq H_2^{22*}(t_2^*) \text{ (for any } t_2^* > t_1) \tag{29}$$

Let us now have a look at player 1's problem. By definition, player 2's switching strategy is defined in terms of the state of the system at any $t \geq t_1$. Given that her switching problem starts at t_1 for a given x_1, the time-to-go strategy (before switching) at this date is: $t_2 - t_1 = \theta_2(x_1, 21)$. Player 1's does not take any switching decision in this regime but he has to make a guess about player 2's switching decision.[22] Following the same steps as before, player 1's value function in regime 21, V_1^{21}, can be written as

$$\int_{t_1}^{t_1+\theta_2(x_1,21)} (H_1^{21}+\dot{\lambda}_1^{21}x)dt - \left\{\lambda_1^{21}[t_1+\theta_2(x_1,21)]x_2 - \lambda_1^{21}(t_1)x_1\right\} + V_1^{22*}[x_2, t_1+\theta_2(x_1,21)],$$

and one can obtain the Pontryagin conditions:

$$\frac{\partial H_1^{21}}{\partial u_1} = 0,\ \frac{\partial H_1^{21}}{\partial x} + \frac{\partial H_1^{21}}{\partial u_2}\Phi_2'(x,21) + \dot{\lambda}_1^{21} = 0. \tag{30}$$

For the last step of the proof, we need to compute the partial derivatives of the value function with respect to t_1 and x_1, which yields:

$$\begin{aligned} \frac{\partial V_1^{21*}}{\partial x_1} &= \theta_2'(x_1,21)[H_1^{21}(t_2) - H_1^{22}(t_2)] + \lambda_1^{21}(t_1) \\ \frac{\partial V_1^{21*}}{\partial t_1} &= H_1^{21}(t_2) - H_1^{21}(t_1) - H_1^{22}(t_2). \end{aligned} \tag{31}$$

These are some envelope conditions.

• *First regime* 11*, for* $t \in [0, t_1]$: In the initial regime, player 1 has now to choose whether he switches and when. The optimization program is:

$$\max_{u_1,t_1} \int_0^{t_1} F_1^{11}(u_1, \Phi_2(x,11), x)dt - \Omega_1(x_1,t_1) + V_1^{21*}(x_1,t_1)$$

subject to,

$$\dot{x} = f^{11}(u_1, \Phi_2(x,11), x),$$

where x_0 is given and $x(t_1) = x_1$ and t_1 are free.

Using the same techniques, the value function is:

$$V_1^{11} = \int_0^{t_1} (H_1^{11} + \dot{\lambda}_1^{11}x)dt - \left[\lambda_1^{11}(t_1)x_1 - \lambda_1^{11}(0)x_0\right] - \Omega_1(x_1) + V_1^{21*}(x_1,t_1).$$

Computing the first-order variation of V_1^{11} with respect to the state and control variables' paths, for free t_1 and x_1, one obtains:

$$\begin{aligned} \delta V_1^{11} = &\int_0^{t_1}[(\tfrac{\partial H_1^{11}}{\partial x} + \tfrac{\partial H_1^{11}}{\partial u_2}\Phi_2'(x,11) + \dot{\lambda}_1^{11})\delta x + \tfrac{\partial H_1^{11}}{\partial u_1}\delta u_1]dt \\ &+(H_1^{11}(t_1) - \tfrac{\partial \Omega_1(x_1,t_1)}{\partial t_1} + \tfrac{\partial V_1^{21*}}{\partial t_1})\delta t_1 - (\lambda_1^{11}(t_1) + \tfrac{\partial \Omega_1(x_1,t_1)}{\partial x_1} - \tfrac{\partial V_1^{21*}}{\partial x_1})\delta x_1. \end{aligned} \tag{32}$$

[22] At the MPE, this guess will be consistent with the actual switching strategy of player 2.

Again, we say that a trajectory is optimal if any small departure from it decreases the value function, that is $\delta V_1^{11} \le 0$ for any $\delta x(t)$, $t \in (0, t_1)$, for any $\delta u_1(t)$, $t \in [0, t_1]$, and for any δt_1 and δx_1. Hence, the necessary conditions for an interior maximizer, $0 < t_1 < t_2$ are:

$$\begin{cases} \frac{\partial H_1^{11}}{\partial u_1} = 0, \; \frac{\partial H_1^{11}}{\partial x} + \frac{\partial H_1^{11}}{\partial u_2}\Phi_2'(x, 11) + \dot{\lambda}_1^{11} = 0, \\ H_1^{11}(t_1) - \frac{\partial \Omega_1(x_1, t_1)}{\partial t_1} + \frac{\partial V_1^{21*}}{\partial t_1} = 0, \; \lambda_1^{11}(t_1) + \frac{\partial \Omega_1(x_1, t_1)}{\partial x_1} - \frac{\partial V_1^{21*}}{\partial x_1} = 0. \end{cases} \tag{33}$$

The (last) two switching conditions can be rewritten, using (31), as:

$$\begin{array}{l} H_1^{11*}(t_1^*) - \frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial t_1} = H_1^{21^*}(t_1^*) - [H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)] \\ \lambda_1^{11*}(t_1^*) + \frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial x_1} = \theta_2'(x_1^*, 11)[H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)] + \lambda_1^{21*}(t_1^*), \end{array} \tag{34}$$

they correspond to conditions (2) of Theorem 1. Thus, conditions in (27) and (34) give the necessary conditions for the optimal timing to be $0 < t_1 < t_2 < \infty$.

Regarding the necessary conditions for corner solutions, we have:

Suppose $0 = t_1^*$, then the only possible variations of t_1^* are such that $\delta t_1 \ge 0$. For $\delta V_1^{11} \le 0$ it must be true that

$$H_1^{11*}(t_1^*) - \frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial t_1} \le H_1^{21^*}(t_1^*) - [H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)] \text{ if } 0 = t_1^* < t_2^* \tag{35}$$

In the opposite situation, $t_1^* = t_2$, the variations t_1^* are non positive: $\delta t_1 \le 0$. For $\delta V_1^{11} \le 0$, we must have

$$H_1^{11*}(t_1^*) - \frac{\partial \Omega_1(x_1^*, t_1^*)}{\partial t_1} \ge H_1^{21^*}(t_1^*) - [H_1^{21*}(t_2^*) - H_1^{22*}(t_2^*)] \text{ if } t_1^* = t_2^*. \tag{36}$$

These conditions are similar to the ones stated in Amit (1986), Theorem 1 p. 537, for corner solutions.

There exist other less interesting cases that we discuss briefly here. Another eventuality is that both players find it optimal to switch at the same instant $t_1 = t_2 = t \in \mathbb{R}_+^*$, and for the same resource stock, $x_1 = x_2 = x$. In that case, the set of necessary conditions reduce to:[23]

$$\begin{array}{l} H_i^{11*}(t^*) - \frac{\partial \Omega_i(x^*, t^*)}{\partial t_i} = H_i^{22*}(t^*), \\ \lambda_i^{11*}(t^*) + \frac{\partial \Omega_i(x^*, t^*)}{\partial x_i} = \lambda_i^{22*}(t^*), \end{array} \tag{37}$$

Finally, there may exist double corner solutions: $t = 0$ and $t = \infty$. The optimality conditions can easily be derived from what we already have. In particular, for the case $t^* = 0$ to occur, it must hold that $H_i^{11*}(t^*) - \frac{\partial \Omega_i(x^*, t^*)}{\partial t_i} \le H_i^{22*}(t^*)$ if $0 = t^* < \infty$ and for any $i = 1, 2$.

[23] This is of course a knife-edge situation, which is highly unlikely at least when one assumes a sufficient degree of heterogeneity among players.

B Application

We restrict attention to linear feedback strategies: $\Phi_j(x,s) = a_j^s + b_j^s x$. In any regime s, player i's present value Hamiltonian is given by:

$$H_i^s = \ln(u_i^s)e^{-\rho t} - \lambda_i^s(\gamma_i^l u_i^s + \gamma_j^k(a_j^s + b_j^s x))$$

The FOCs are:

$$\begin{aligned} &(u_i^s)^{-1}e^{-\rho t} = \gamma_i^l \lambda_i^s \\ &\dot{\lambda}_i^s = b_j^s \lambda_i^s \\ &\dot{x} = -\gamma_i^l u_i^s - \gamma_j^k(a_j^s + b_j^s x) \end{aligned} \tag{38}$$

and have to be combined with the appropriate transversality condition, which depends on whether the regime is terminal, or not. Whatever the regime, it can easily be checked that players' extraction strategies satisfy:

$$\gamma_i^l u_i^s = \gamma_j^k u_j^s \Leftrightarrow a_i^s = a_j^s \text{ and } b_i^s = b_j^s. \tag{39}$$

When regime s is terminal, we obtain: $\gamma_1^l \Phi_1^*(x,s) = \gamma_2^k \Phi_2^*(x,s) = \rho x$. If $s = 22$ is the terminal regime (both players have switched), then the value function is

$$V_i^{22*}(x_2,t_2) = \frac{e^{-\rho t_2}}{\rho}\left[\ln(x_2) + \ln(\rho) - \ln(\gamma_i^2) - 2\right] = e^{-\rho t_2} v_i^{22*}(x_2), \tag{40}$$

where $v_i^{22*}(x_2)$ is the continuation payoff, in current value.

B.1 Player 2's switching problem

B.1.1 Interior solution (proof of Proposition 3)

Switching conditions: Assume player 1 has switched at some $t_1 \in (0,\infty)$, for a switching point x_1. For an interior solution (t_2, x_2), conditions (1) of Theorem 1 are (derivative of the value function w.r.t. x_2)

$$u_2^{21}(t_2) = \frac{\rho x_2}{\gamma_2^1(1-\beta_2\rho x_2)}, \tag{41}$$

and (derivative of the value function w.r.t. t_2)

$$\ln(u_2^{21}(t_2)) - \lambda_2^{21}(t_2)e^{\rho t_2}\left[\gamma_2^1 u_2^{21}(t_2) + \gamma_1^2 u_1^{21}(t_2)\right] + \rho\omega_2(x_2) - \rho v_2^{22*}(x_2,t_2) = 0. \tag{42}$$

Using (39) and (41), the consumption strategies satisfy:

$$\gamma_1^2 \Phi_1^*(x,21) = \gamma_2^1 \Phi_2^*(x,21) = \frac{\rho^2\beta_2(x_2)^2}{1-\beta_2\rho x_2} + \rho x = \Gamma + \rho x. \tag{43}$$

From (38)-(41) and (43), (42) simplifies to

$$\rho\omega_2(x_2) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) = \ln(1-\beta_2\rho x_2), \tag{44}$$

which defines the optimal level for switching, x_2^*. Conditions (41) and (44) are the necessary conditions for an interior solution.

Characterization of the solution: The LHS of (44) is defined for all $x_2 \in \big[0, (\rho\beta_2)^{-1}\big)$, increasing in x_2 and varying from zero to ∞ as x goes from zero to $(\rho\beta_2)^{-1}$. The RHS is strictly positive at $x_2 = 0$ iff $\ln\left(\frac{\gamma_2^1}{\gamma_2^2}\right) > \rho\omega_2(0)$. Since $\beta_2 > 0$, the RHS is strictly decreasing in x_2. Thus, if $x_1 \geq (\rho\beta_2)^{-1}$ and

$$\rho\omega_2(0) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) < 0 \tag{45}$$

then, there exists a unique solution x_2^* in $\big[0, (\rho\beta_2)^{-1}\big)$. Otherwise $(x_1 < (\rho\beta_2)^{-1})$, another boundary condition is

$$\rho\omega_2(x_1) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) > \ln(1 - \rho\beta_2 x_1).$$

Replacing consumptions with the expressions given by (43) in the state equation, and solving the resulting differential equation (with the boundary condition $x(t_1) = x_1$) yield the expression of the state variable for any $t \in [t_1, t_2^*)$:

$$x^{21*}(t) = \left[x_1 + \frac{\rho\beta_2 (x_2^*)^2}{1 - \beta_2\rho x_2^*}\right] e^{-2\rho(t-t_1)} - \frac{\rho\beta_2 (x_2^*)^2}{1 - \beta_2\rho x_2^*}.$$

Evaluating this equation at t_2^* and solving for $\theta_2 = t_2^* - t_1$, one obtains

$$\theta_2^*(x_1, 21) = \frac{1}{2\rho}\ln\left[(1 - \rho\beta_2 x_2^*)\frac{x_1}{x_2^*} + \rho\beta_2 x_2^*\right] = \frac{1}{2\rho}\ln\left[\frac{\Phi_i(x_1, 21)}{\Phi_i(x_2^*, 21)}\right], \tag{46}$$

which gives the time-to-go (before switching) strategy of player 2 for any switching point (and more generally, any level of the state variable) x_1.

B.1.2 Never switching condition (proof of Proposition 2, first item)

Still assuming that there exists $t_1 \in (0, \infty)$, the condition for a never switching solution ($t_2^* = \infty$) is given by:

$$\ln[u_2^{21}(t_2)] + \rho\omega_2(x_2) \geq \ln[u_2^{22}(t_2)], \tag{47}$$

for all $t_2 \in (t_1^*, \infty) \cup \{\infty\}$. This corresponds to condition (4) of Theorem 2. When $t_2 \to \infty$ (and $x_2 \to 0$ because the stock of resource is exhausted asymptotically), we use the feature that regime 21 becomes the final regime and $\gamma_1^2 u_1^{21}(t) = \gamma_2^1 u_2^{21}(t) = \rho x$, and take the limit of (47) to obtain

$$\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho\omega_2(0) \geq 0. \tag{48}$$

This necessary condition for a never switching solution is also sufficient to have

$$\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho\omega_2(x_2) \geq \ln(1 - \beta_2\rho x_2) \text{ for all } t_1 < t_2 \leqq \infty.$$

The analysis of the last case (wrong timing: $t_1 = t_2$) is postponed to Appendix B.3 because it requires player 1's switching problem be examined first.

B.2 Player 1's switching problem

B.2.1 Interior solution (proof of Proposition 4)

Switching conditions: Suppose that player 2's regime switching takes place at some $t_2 \in (0, \infty)$. Using (40) and (43), the value function of player 1 (in current value), from regime 21 on, for any x_1 and any guess $(x_2, \theta_2(x_1, 21))$, is:[24]

$$v_1^{21*}(x_1, \theta_2(x_1, 21)) = \frac{1}{\rho}\left\{\ln[\Phi_1^*(x_1, 21)] - 2 + e^{-\rho\theta_2(x_1,21)} \ln(1 - \rho\beta_2 x_2)\right\}. \tag{49}$$

This yields the continuation payoff as seen from regime 11, the one in which player 1 takes his regime switching decision. Let $\zeta(x_1; x_2^*)$ be defined as follows:

$$\zeta(x_1; x_2) = 1 - \frac{e^{-\rho\theta_2(x_1,21)}}{2} \ln(1 - \rho\beta_2 x_2) - \beta_1(\Gamma + \rho x_1). \tag{50}$$

Then the switching conditions (2) of Theorem 1 are, for our example

$$\gamma_1^1 u_1^{11}(t_1) = \frac{\Gamma + \rho x_1}{\zeta(x_1; x_2)}, \tag{51}$$

with Γ defined in (43), and (using the relationship in 39 and 38):

$$\ln\left[u_1^{11}(t_1)\right] - 2 + \rho\omega_1(x_1) - \rho v_1^{21*}[x_1, \theta_2(x_1, 21)] = 0. \tag{52}$$

Solving for the MPE in consumption strategies in regime 11, one finds

$$\gamma_1^1 \Phi_1^*(x, 11) = \gamma_2^1 \Phi_2^*(x, 11) = \Lambda + \rho x \text{ with } \Lambda = \frac{\Gamma + \rho x_1[1 - \zeta(x_1; x_2)]}{\zeta(x_1; x_2)}.$$

Substituting $u_1^{11}(t_1)$ with the expression in (51), using (44) and $\gamma_1^2 u_1^{21}(t_1) = \Gamma + \rho x_1$, the optimality condition (52) can be rewritten as:

$$\rho\omega_1(x_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = e^{-\rho\theta_2(x_1,21)}\left[\rho\omega_2(x_2) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] + \ln[\zeta(x_1; x_2)]. \tag{53}$$

At the MPE, player 1's guess must be consistent with player 2's actual strategy, which implies that x_2 and θ_2 are given by x_2^* and θ_2^*, defined by (44) and (46). Thus, player 1's optimality conditions for switching (51) and (53) have to be evaluated at this particular point and for this particular strategy.

Characterization of the solution: $\zeta(x_1; x_2^*)$ is defined over (x_2^*, x_0) with $\zeta'(x_1; x_2^*) < 0$. Let us assume that $\zeta(x_0; x_2^*) > 0$, which requires x_0 be high enough. The LHS of (53) increases with x_1 on the interval $[x_2^*, x_0]$ whereas the

[24] Note that the third term is exactly the difference between the present value Hamiltonians evaluated at t_2, $H_1^{22}(t_2) - H_1^{21}(t_2)$, discounted from t_1.

RHS is non monotone because the time-to-go (before switching), θ_2^*, is increasing in x_1. Therefore, imposing $\zeta(x_2^*; x_2^*) \geq 1$, with

$$\zeta(x_2^*; x_2^*) = \frac{1 - \rho(\beta_1 + \beta_2)x_2^*}{1 - \rho\beta_2 x_2^*} - \frac{\ln(1 - \rho\beta_2 x_2^*)}{2},$$

$\zeta(x_0; x_2^*) \leq 1$, and

$$\begin{aligned} \rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) &> \rho\omega_2(x_0) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right), \\ \rho\omega_1(x_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) &< \rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right), \end{aligned}$$

guarantees the existence of a unique $x_1^* \in (x_2^*, x_0)$ that satisfies (53).[25]

Using all the material above, the resource stock is given by: $x^{11}(t) = \left(x_0 + \frac{\Lambda}{\rho}\right) e^{-2\rho t} - \frac{\Lambda}{\rho}$. Evaluating this expression at t_1^*, one obtains:

$$t_1^* = \theta_1^*(x_0, 11) = \frac{1}{2\rho} \ln\left(\frac{x_0 + \frac{\Lambda}{\rho}}{x_1^* + \frac{\Lambda}{\rho}}\right).$$

Remark. There is no reason for player 1's switching point to be the same when $t_2^* < \infty$ than when $t_2^* = \infty$. Indeed, when $t_2^* = \infty$, it can easily be shown that x_1^* solves: $\ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) + \rho\omega_1(x_1) = \ln(1 - \beta_1\rho x_1)$.

B.2.2 Immediate switching (proof of Proposition 2, second item)

Still assuming that player 2's switching problem has a solution t_2^* (with x_2^* that solves 44), if player 1 finds it optimal to switch instantaneously then, according to Theorem 2 condition (5) must hold. In our application, it is given by:

$$\ln[u_1^{11}(t_1^*)] + \rho\omega_1(x_1^*) \leq \ln[u_1^{21}(t_1^*)] + e^{-\rho\theta_2^*(x_1, 21)} \ln(1 - \beta_2\rho x_2^*) \qquad (54)$$

if $0 = t_1^* < t_2^*$. A the particular date $0 = t_1^*$ (implying that $x_1^* = x_0$), exploiting the fact that regime 11 vanishes in regime 21, which implies that $\gamma_1^1 u_1^{11}(0) = \gamma_2^1 u_1^{21}(0) = \Gamma + \rho x_0$, condition (54) reduces to:

$$\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq e^{-\rho\theta_2^*(x_0, 21)} \ln(1 - \beta_2\rho x_2^*) \qquad (55)$$

B.3 Wrong timing (proof of Proposition 1)

B.3.1 For player 2

The wrong timing situation, for player 2, corresponds to the limit case where $t_1^* = t_2^*$. Condition (3) of Theorem 2 reduces to:

$$\ln[u_2^{21}(t_2^*)] \leq \ln[u_2^{22}(t_2^*)] - \rho\omega_2(x_2^*) \text{ if } t_1^* = t_2^* \qquad (56)$$

[25] Note that $\zeta(x_2^*; x_2^*) > 1$ holds under specific assumptions. Assuming that $\beta_2 > 2\beta_1$, it is pretty easy to show that $\exists! \bar{x}_2^* \in (0, (\rho\beta_2)^{-1})$ such that $\zeta(x_2^*; x_2^*) > 1$ for all $x_2^* < \bar{x}_2^*$. From now on, we will assume that this technical condition holds.

with $u_2^{22}(t_2^*) = \rho x_2^*$. Next, we use the relationship (39) which, combined with the fact that regime 21 actually vanishes into regime 11, i.e. $t_1^* = t_2^*$, implies that: $\gamma_2^1 u_2^{21}(t_1^*) = \gamma_2^1 u_2^{11}(t_1^*)$. From the resolution of player 1's switching problem in this hypothetical case, we first obtain $\gamma_1^1 u_1^{11}(t_1) = \frac{\rho x_1^*}{\gamma_1^1(1-\beta_2\rho x_1^*)} \frac{1}{\zeta(x_1^*;x_1^*)}$. Thus, (56) simplifies to:

$$\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho\omega_2(x_1^*) \leq \ln(1-\beta_2\rho x_1^*) + \ln[\zeta(x_1^*;x_1^*)]. \tag{57}$$

Moreover, player 1's second switching condition is $\rho\omega_1(x_1^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = \ln(1 - \rho\beta_2 x_1^*) + \ln\left[\zeta(x_1^*;x_1^*)\right]$, which implies that (57) can be rewritten as:

$$\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho\omega_2(x_1^*) \leq \rho\omega_1(x_1^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right). \tag{58}$$

B.3.2 For player 1

Assume now that player 2's switching problem has an interior solution, (t_2^*, x_2^*). Applying the condition of Theorem 2 to our example, the timing is wrong for player 1 only if:

$$\ln[u_1^{11}(t_1^*)] + \rho\omega_1(x_1^*) \geq \ln[u_1^{21}(t_1^*)] + e^{-\rho\theta_2^*(x_1^*,21)} \ln(1-\beta_2\rho x_2^*) \tag{59}$$

if $t_1^* = t_2^*$. Making use of $x_1^* = x_2^*$ (and $\theta_2^*(x_2^*, 21) = 0$), $\gamma_1^1 u_1^{11}(t_2^*) = \gamma_2^1 u_2^{21}(t_2^*) = \Gamma + \rho x_2^*$, condition (59) is equivalent to: $\rho\omega_1(x_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \ln(1-\beta_2\rho x_2^*)$, which, from (44), can be rewritten as:

$$\rho\omega_1(x_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \tag{60}$$

B.4 Remaining cases

• **Immediate and never switching:** $0 = t_1 < t_2 = \infty$. From Appendix B.1.2, we know that (48) is a necessary condition for player 2 to be at the corner $t_2 = \infty$. In this case, player 1 compares the (marginal) value he would obtain under the permanent regime 11 with the corresponding value he would get by switching directly to 21. Given that $\gamma_1^l u_1(0) = \rho x_0$ for $l = 1, 2$, the condition for an immediate switching is: $\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq 0$.

• **Simultaneous interior switches:** $0 < t_1 = t_2 = t < \infty$. From (38) and (39), we have $\lambda_1^s = \lambda_2^s$ in any regime s. It is clear that the last switching condition in (37) cannot be simultaneously satisfied for the two players whenever $\omega_1'(x) \neq \omega_2'(x)$ for all x (recall that $\omega_i'(x)e^{-\rho t} = \frac{\partial\Omega_i(x,t)}{\partial x_i}$ for $i = 1, 2$).

• **Simultaneous instantaneous switches:** $t_1 = t_2 = 0$: In this case, it must be true that $\rho\omega_i(x_0) + \ln\left(\frac{\gamma_i^2}{\gamma_i^1}\right) \leq 0$ for $i = 1, 2$.

• **Never switching for both players:** $t_1 = t_2 = \infty$. In the same vein, here condition (48) must hold for the two players.