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Abstract

Interbank claims are a concern to regulators as they might facilitate the dissemination of defaults and generate spill-over effects. Building on a simple model, this paper introduces a measure of the spill-over effects that a bank generates when it defaults. The measure is based on an explicit criterion, the aggregate debt repayments, and is bank's specific, affected by the bank's characteristics and links to other banks. Such measure can be useful to a regulator to determine in which banks cash should be injected during a default episode or to evaluate the impact of raising capital before the occurrence of default. The approach applies more generally to a system of entities that are linked through financial claims. This is illustrated to evaluate the consolidated foreign claims of 10 EU countries.

JEL-Code: G010, G210, G280.

Keywords: default, contagion, systemic risk, financial linkages, intervention policy.

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1 Introduction

An intricate web of claims and obligations ties together the balance sheets of a wide variety of financial institutions, banks, hedge funds, and various intermediaries. Some argue that these ties have played a large role in the dissemination of the financial crisis of 2007-2008. As such, interbank claims are an important concern for both bankers and regulators and there is a general call for addressing their role in the risk of the system, the so-called 'systemic' risk.[1] This paper proposes to measure the spill-over effects that interbank liabilities generate on the propagation of default. The measure is based on an explicit criterion: the aggregate debt repayments (equivalently, in the model, the aggregate assets' size), and is bank's specific, affected by the bank's characteristics and links to other banks. Such measure can be useful to a regulator to determine in which banks cash should be injected during a default episode or to evaluate the impact of raising capital before the occurrence of default. The approach applies more generally to a system of entities that are linked through financial claims. This is illustrated to evaluate the consolidated foreign claims of 10 EU countries.

The analysis builds on a simplified description of a banking system due to Eisenberg and Noe (2001) (hereafter EN). Banks have liabilities to each other and the result of the activities of each bank is summarized by a single value, its *cash-flow*. Under limited liability, the capacity of a bank to repay its liabilities depends not only on its realized cash-flow but also on the reimbursements of its loans, calling for a *joint* determination of the repayments. The clearing mechanism defined by EN solves this: at clearing repayment ratios, each bank in default reimburses as much as it can under limited liability and given others' repayments. The aggregate repayments at the clearing ratios thus measure the reimbursement capacity of the system as a function of the cash-flows of all banks.

The variation in the aggregate repayments due to a decrease in a bank's cash-flow is our measure of the bank's impact on the system. When the bank defaults, the variation includes not only the decrease in the bank's repayment following the decrease in its cash-flow but also the decrease caused by the propagation of defaults, the spill-over effects. When the bank does not default, the variation is null. The variation is simple to compute if the bank's cash-flow decrease is moderate enough to leave unchanged the set of defaulting banks. In

[1] The new framework proposed by the Basel committee (Basel III) identifies some 'systemically important financial institutions' (SIFI) from which higher standards are required. The SIFI are mainly determined by their size.

that case, the variation is proportional to a term called hereafter threat index. The index of a bank depends not only on its own liabilities but also on the whole set of defaulting banks and the liabilities between them. The threat index and default ratio of a bank (the complement to 1 of its repayment ratio) are both measures of its weakness. However, they typically differ, and are, in some precise sense, dual to each other. A bank's default ratio is determined by the capacity of its debtors to repay their debts whereas its threat index is determined by the capacity of its creditors to repay their debt.

The analysis has several implications from a regulation perspective. First, it is helpful to determine a policy of cash injection into banks during a default episode. As an injection is similar to an increase in banks' cash-flows, the policy must account for the spill-over effects across banks so as to improve aggregate repayments as much as possible. When the amount of cash is moderate, the optimal policy is characterized by the threat indices: Cash should be injected into the defaulting banks with the largest threat index. As a result, due to the discrepancy between threat indices and default levels, injecting cash into the banks that appear the weakest, those with the largest default ratios, may be sub-optimal.

Second, the lack of information on the bilateral liabilities between financial institutions is a concern to regulators. How valuable is this information? To address this question, we compare the optimal injection strategy, which assumes complete information, with the benchmark in which the regulator knows the total liabilities and total loans of each bank only, but not the bilateral ones. The value of information is defined as the improvement in the aggregate repayments reached by the optimal policy under complete information over the benchmark. The value is driven by the heterogeneity and asymmetry in liabilities structure. When the injected amount is moderate, the value is proportional to the difference between the maximal index and the average index over the defaulting banks. This difference is determined partly by the heterogeneity in the proportions of their liabilities that defaulting banks owe towards defaulting banks. It also depends on the heterogeneity in intricate spill-over effects that account for liabilities cycles between defaulting banks, as described by the dominant eigenvector of a certain matrix.

Third, the impact of increasing the capital level of a bank can be assessed by considering an ex ante stage before the cash-flows are realized and the clearing ratios are determined. The value of raising capital at the margin is simply the expectation of the threat index. Since the threat index depends on the liabilities structure within the defaulting banks, its ex ante value depends both on the whole liabilities structure and the distribution of the defaulting set, which, in turn, is determined by the cash-flows' distribution. In particular, the value is increasing in the probability that the defaulting set is large when the bank defaults. This probability is affected by other factors than the liabilities, such as the correlation between the cash-flows or the size of the bank.

Finally, the analysis can be applied more generally to a system of entities that are linked through financial claims. The approach is illustrated on the consolidated foreign claims for 10 EU countries and uncovers interesting features not easy to see from the liabilities structure, due to its asymmetry and heterogeneity.

The literature on financial contagion is growing. Empirical studies have examined the potential for contagion in real banking systems. Often models use a binary variable that represents either failure or safeness. The contagion risk of a bank is defined as the expected number of subsequent failures (possibly weighted by their size) following its initial failure. The first simulations were calibrated on real payment systems (see e.g. Furfine 2003 on Fedwire) or on interbank networks (e.g. Upper and Worms 2004, Elsinger et al. 2004, Degryse and Nguyen 2004 for Germany, Austria, Belgium respectively). These studies concluded that systemic risk was extremely limited, in the sense that the probability of a large number of failures triggered by the single initial failure of a bank was almost null, though the possibilities of bail-out were not taken into account. One explanation for these very low effects is that data on bilateral exposures is limited and the technics to 'fill' the missing data is likely to lead to underestimate contagion. This point is related to the analysis of the value of information on the bilateral liabilities and their heterogeneity (Section 4).

Interbank liabilities have two opposing effects on contagion: they increase the opportunities for sharing liquidity shocks among counter-parties but also facilitate the channels through which default spreads. Recent studies aim to assess the impact of the liabilities structure on this trade-off[2] (Gai and Kapadia 2008, Elliott, Golub, and Jackson 2013, Glasserman and Young 2015, Acemoglu, Ozdaglar and Tahbaz-Salehi 2015). For example, Gai and Kapadia (2008) use a random graph in which the expected number of links (i.e. number of creditors) of a bank is given by a parameter, identical for all banks. They find that financial systems exhibit a robust-yet-fragile tendency: while greater connectivity reduces the likelihood of widespread default, the impact on the financial system, should

[2] A more economic approach has been initiated by Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000).

problems occur, can be on a significantly larger scale than hitherto.[3] The analysis however relies on (a priori) symmetric networks and measures systematic risk by the expected contagion size given an expected bank picked at random. Given the observed heterogeneity of financial institutions, both in their size and connections, an important question is to assess the spill-over effects initiated by a *given* bank, as is performed by the measure introduced in this paper.

Regulation is typically defined at the unit level, determined by the balance sheet of the bank under consideration. A prominent example is the Value at Risk indicator (VaR), which is based on a statistical assessment of a bank's payoffs independently from what is happening to other banks. Various measures of systemic risks have been recently introduced, such as CoVaR (Adrian and Brunnermeier 2008) or Marginal Expected shortfall (Acharya *et al.* 2008, Brownee and Engle 2010). CoVaR for example is a measure similar to VaR but conditional on systemic events, defined, say, as those in which the stock market portfolio falls below a threshold. These measures are based on a reduced form and cannot distinguish between the initiation and contagion effects. Gouriéroux, Héam, and Monfort (2012) propose an approach related to ours, also based on the equilibrium EN model. They decompose for each bank the impact of shocks on the balance sheets into a direct and a contagion effect, and illustrates it on a French banking system. The analysis differs from here as they do not base the analysis on the aggregate asset size nor evaluate targeting strategies.

Finally, the paper also relates to the large literature that studies the interactions and externalities channeled through a network of connections. Firstly, the threat index provides an assessment of a position in a network, and, as such, is related to the power or centrality indices introduced in the sociological literature by Katz (1953) or Bonacich (1987). These indices depend on an 'attenuation' parameter, which captures the importance of indirect links. Our approach however differs in an important way as it is based on an explicit objective; as a result, the relevant network is endogenous, restricted to the links between defaulting banks and the threat index does not depend on an exogenous attenuation parameter because the importance of indirect links is endogenously determined as well. Secondly, injection policies have been investigated in alternative network models in which individ-

[3] This result is driven by the assumption that a bank's total amount of assets and liabilities is kept fixed, irrespective of the network structure. In practice this amount is related positively with the number of links, which is likely to hamper the risk-sharing benefit of forming links.

ual actions generate externalities channelled through a network. In a criminal network for example, the 'key player' to remove, the one whose arrest triggers the largest decrease in global criminal activity, may not be the one the more active (see Ballester, Calvó-Armengol, Zenou 2005). Similar insights hold in our model since the most threatening banks are not necessarily those with the lowest repayment ratios.

The paper is organized as follows. Section 2 presents the model and the clearing mechanism, and introduces the aggregate repayments value. Section 3.1 analyzes how this value varies with the cash-flows and introduces the threat indices. Section 3.2 examines the optimal injection policies and relates them to the threat indices. Section 4 computes the value of the information on the bilateral liabilities to a regulator who decides on a cash injection policy. Section 5 takes an ex ante point of view and introduces the value of raising capital. Section 6 illustrates the approach on the consolidated foreign claims for 10 EU countries and Section 7 concludes. Section 8 gathers the proofs.

2 A liquidation model with default

There are n financial institutions, called banks for simplicity. Denote $N = \{1, \cdots, n\}$. Banks draw some revenues from their activities and are linked through claims on each other. We are at an ex-post stage, or liquidation stage. The revenues are realized, summarized by a single value for each bank, called its *cash-flow*, z_i for bank i, which is positive.[4] The interbank assets and liabilities are described by the nominal debt obligations of each bank i towards any other bank j, ℓ_{ij}. Liabilities have the same priority. The vector of cash-flows $\boldsymbol{z} = (z_i)$ and the $n \times n$ matrix $\boldsymbol{\ell} = (\ell_{ij})$ where ℓ_{ii} is null for each i summarizes the relevant data.

The liabilities structure depends on the situation under consideration, in particular on the horizon of the debts. In payment systems, liabilities are often both ways, reflecting common clienteles for example. Not only both ℓ_{ij} and ℓ_{ji} can be simultaneously positive but they are likely to be both positive or both null. In long term arrangements however, some patterns are more directed, such as the ones described in the Austrian banking system, with almost a pyramidal structure (see for example Upper and Worms 2004).

[4] z_i may also be interpreted as the market value of the bank's activities Shin (2008). As long as these values are not affected by the clearing mechanism, the analysis goes through.

Clearing repayment ratio vectors The capacity for banks to repay their debts depends on the cash-flow levels $\boldsymbol{z}$, the mutual liabilities $\boldsymbol{\ell}$, and the repayments they effectively receive. Some banks may be unable to repay fully their debts and default possibly propagates due to the mutual liabilities. The clearing mechanism determines the outcome of propagation. It bears on the proportion of the debt reimbursed -as liabilities have the same priority, a bank reimburses the same fraction of its liability to each creditor- and is based on two simple rules, limited liability and creditors' priority over stockholders. It turns out that these two rules pin down the outcome and furthermore maximize the aggregate repayments under the limited liability constraint. This will serve as a basis for defining spill-over effects.

Let us denote by ℓ_i^* i's total nominal liabilities

$$\ell_i^* = \sum_j \ell_{ij}. \tag{1}$$

Start by assuming that all banks fully repay their debts to i. In that case, i's asset is equal to $z_i + \sum_j \ell_{ji}$. Due to the limited liability of stockholders, bank i will fully repay its debts only if its asset is larger than ℓ_i^*. Otherwise i defaults; i is said to 'initiate' default, since it defaults even under full repayments of its loans.

Default can be partial, meaning that a bank in difficulty pays a fraction of its liability. This fraction, between 0 and 1, is called *repayment ratio* or simply ratio, denoted by θ_i. A ratio (vector) $\boldsymbol{\theta} = (\theta_i)$ specifies the repayment ratio of each bank, where θ_i is between 0 and 1. The *default ratio* is then defined as $1 - \theta_i$. Under limited liability, i's repayments are constrained by its *asset value*, which is the sum of its cash-flow and the loans repayments by other banks. Formally, given the ratio vector $\boldsymbol{\theta}$, bank i's asset value denoted by $a_i(\boldsymbol{\theta})$ is

$$a_i(\boldsymbol{\theta}) = z_i + \sum_j \theta_j \ell_{ji}. \tag{2}$$

i's repayments, $\theta_i \ell_i^*$, are constrained to be less than $a_i(\boldsymbol{\theta})$ since stockholders cannot be forced to add cash. Equivalently, i's equity defined as

$$e_i(\boldsymbol{\theta}) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* \tag{3}$$

must be non-negative. In particular, bank i initiates default when $e_i(\mathbb{1}) < 0$ where $\mathbb{1}$ denotes the n-vector of 1.

The clearing mechanism basically requires debtors to reimburse as much as they can under limited liability.[5]

Definition 1 *Given* $(\boldsymbol{z}, \boldsymbol{\ell})$*, a vector* $\boldsymbol{\theta} = (\theta_i)$ *in* $[0,1]^n$ *is said to be a clearing ratio if it satisfies for each* i

(limited liability): $a_i(\boldsymbol{\theta}) \geq \theta_i \ell_i^*$ *(equivalently* $e_i(\boldsymbol{\theta}) \geq 0$*)*

(priority of creditors over stockholders): either $\theta_i = 1$ *or* $a_i(\boldsymbol{\theta}) = \theta_i \ell_i^*$ *i.e.* $e_i(\boldsymbol{\theta}) = 0$*.*

One checks that $\mathbb{1}$ is a clearing ratio vector when no bank initiates defaults, since then $e_i(\mathbb{1}) \geq 0$ for each i. The existence of a clearing ratio vector follows from the complementarities between banks' ratios, as shown by EN: The capacity of a bank to repay its debts is increasing in its asset, hence in other banks' repayment ratios. Complementarities imply that there is a greatest (in each component) ratio vector under which limited liability is satisfied for each bank. Formally, let us say that $\boldsymbol{\theta}$ in $[0,1]^n$ is feasible if $e_i(\boldsymbol{\theta}) \geq 0$ for each i; the set of feasible ratios has a greatest element (the set is nonempty: the null ratio $\mathbf{0}$ is feasible). This greatest feasible ratio is a clearing ratio. To see this, assume, by contradiction, that creditors' priority is not satisfied: for some i, both $\theta_i < 1$ and $e_i(\boldsymbol{\theta}) > 0$. Then all equities are still non-negative under an increase in θ_i that is small enough to keep i's equity positive (since, by complementarity, all others' equities can only improve). This implies that the greatest feasible vector is a clearing vector; it is surely the greatest clearing vector. Furthermore, it turns out that this is the unique clearing vector when the cash-flows are all positive, as we have assumed.[6]

Creditors' priority can thus be seen as forcing the clearing ratio vector to maximize the payment of *each* bank within the system under the limited liability condition. As a result, the clearing vector maximizes any function that is increasing in the ratios, in particular it maximizes the aggregate repayments.

[5] The ratio for a bank with no liabilities towards other banks is defined, though this has no true meaning, and is equal to 1 according to the definition. This convention allows us not to distinguish these banks.

[6] The assumption of positive cash-flows is weak. It simplifies the presentation and furthermore does not change substantially the results. In case of multiple clearing vectors, equities are unaffected by the chosen clearing vector and the greatest clearing vector is the solution to the program $\mathcal{P}$.

The aggregate repayments As we have just seen, the clearing vector maximizes the aggregate debt repayments over the feasible set. Formally it solves

$$\begin{aligned}\mathcal{P}(\boldsymbol{z}) \quad &: \quad \max_{\boldsymbol{\theta}, \mathbf{0} \leq \boldsymbol{\theta} \leq \mathbf{1}} \sum_i \theta_i \ell_i^* \\ \theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} &\leq z_i \text{ for each } i. \end{aligned} \tag{4}$$

Let $V(\boldsymbol{z})$ denote the value of the program $\mathcal{P}(\boldsymbol{z})$. The aggregate payments V, will be taken as the objective of a regulator or of the payment system. In particular, a cash injection into banks or a decrease in banks' cash-flows are evaluated through this system's perspective by considering their impact on V.[7]

This objective coincides with that of maximizing the aggregate size of the balance sheets: Summing asset values (2) over i, the aggregate asset size, $\sum_{i \in N} a_i(\boldsymbol{\theta})$, writes as $\sum_{i \in N} z_i + \sum_{i \in N} \theta_i \ell_i^*$, which is the sum of the aggregate cash-flows and the debt repayments.

Discussion Our approach extends to alternative objectives, provided they are increasing in the ratios (since the clearing ratio vector maximizes such objectives under limited liability, as we have seen). It does not extend to an objective pertaining to aggregate equity because aggregate equity is constant over the ratios: Summing equity values (2) over i, $\sum_{i \in N} e_i(\boldsymbol{\theta})$ is equal to the aggregate cash-flow, $\sum_{i \in N} z_i$, whatever $\boldsymbol{\theta}$ because the payments within N cancel out. In case of default, the clearing mechanism thus performs transfers between the stockholders. The banks that initiate default, those for which the value $e_i(\mathbb{1})$ is negative, finally end up with a null equity value without the need for their stockholders to add cash. All the other banks end up with an equity that is smaller than $e_i(\mathbb{1})$, the level would be no default.[8] Partial default on liabilities hence plays the role of a 'buffer' to stockholders.

In what follows, a bank is said to be *safe* if its equity is positive at the clearing ratio. Since all equities are non-negative at the clearing ratio and their sum is equal to the positive aggregate cash-flow, surely one bank is safe. A bank is said to be *defaulting* if its ratio is strictly lower than 1. To state some results, we need to exclude the situations where a bank is at the *boundary* of being safe and defaulting: Bank i is at the boundary if $e_i(\boldsymbol{\theta}) = 0$ and $\theta_i = 1$. Typically there is no boundary bank: If there are, a small perturbation in $\boldsymbol{z}$ (or $\boldsymbol{\ell}$) makes either their ratios strictly smaller than 1 or their equities strictly positive.

[7] The liabilities structure also influences payments. The impact of a change in the liabilities is investigated in Section 3.2.5.

[8] This is obvious for those that do not default as their asset is decreased and repayments unchanged. For those that end up defaulting, their equity is null hence smaller than $e_i(\mathbb{1})$.

Notation. $\mathbb{I}$ denotes the $n \times n$ identity matrix. Given a n-vector $\boldsymbol{x}$ and S a subset of indices, $\boldsymbol{x}_S$ denotes the vector obtained from $\boldsymbol{x}$ by keeping the components indexed by S. If S is reduced to a singleton $\{i\}$, $\boldsymbol{x}_{-i}$ denotes $\boldsymbol{x}_{N-\{i\}}$ and $\boldsymbol{x} = (x_i, \boldsymbol{x}_{-i})$. Similarly, $A_{S\times T}$ denotes the matrix obtained from a matrix A by keeping the rows indexed by i in S and the columns indexed by j in T. A^t denotes the transpose of A.

3 Increasing the aggregate debt repayments

This section first analyzes how the objective V varies with the banks' cash-flows, define banks threat indices and relates them to optimal cash injection policies.

3.1 Properties of V and threat indices

Observe that a defaulting bank is surely indebted, $\ell_i^* > 0$. Let us define i's *liabilities shares* by $\pi_{ij} = \frac{\ell_{ij}}{\ell_i^*}$ for each j.

Proposition 1 *The function V is piece-wise linear and concave. V is differentiable at each $\boldsymbol{z}$ for which no bank is at the boundary. Given the set of defaulting banks D, the derivative vector $(\frac{\partial V}{\partial z_i})$ is null outside D, i.e., of the form $(\mathbf{0}_{N-D}, \boldsymbol{\mu}_D)$, and $\boldsymbol{\mu}_D$ is the unique solution to*

$$\mu_i = 1 + \sum_{j\in D} \pi_{ij}\mu_j \text{ for each } i \text{ in } D. \tag{5}$$

μ_i is called i's threat index. The clearing ratio is non-decreasing and convex in $\boldsymbol{z}$, the default set and the threat indices are non-increasing in $\boldsymbol{z}$. Furthermore V is sub-modular:

$$\text{For every } i, z_i' \geq z_i \text{ and } \boldsymbol{z}'_{-i} \geq \boldsymbol{z}_{-i} \text{ imply } V(z_i', \boldsymbol{z}'_{-i}) - V(z_i, \boldsymbol{z}'_{-i}) \leq V(z_i', \boldsymbol{z}_{-i}) - V(z_i, \boldsymbol{z}_{-i}). \tag{6}$$

The function V is well-behaved since it is piece-wise linear and concave. Assuming no bank at the boundary, its derivative (the threat index) depends on $\boldsymbol{z}$ through the set of defaulting banks D only, as can be seen from (5). Therefore V is linear over the set of cash-flows $\boldsymbol{z}$ that lead to the same set D and the kinks arise at cash-flows for which there is a bank at the boundary.

The derivative of V with respect to i's cash-flow is equal to the multiplier μ_i associated to the equity constraint (4) at points where V is differentiable (according to the envelope theorem). Thus, a marginal decrease of one unit in i's cash-flow decreases the payments by μ_i units, hence the term 'threat' index for μ_i (considering an increase instead, the index can be interpreted as a credit multiplier as well). For a safe bank, its index is null since its repayments are full and unchanged by a small variation of its cash-flow, hence no change occurs. For the defaulting banks, the indices follow expression (5) by applying standard complementarity relationships between the repayment ratios (the solutions to $\mathcal{P}$) and the threat indices (the solutions to its dual).[9] The threat indices of defaulting banks are jointly determined: the index of defaulting i is the sum of 1 -one unit in i's cash-flow induces i to repay one unit of its debts- and a term that depends on the indices of i's creditors that are themselves defaulting. This additional term thus represents the *spill-over effects* due to liabilities. It will be investigated more closely in Section 3.2.2, which also considers the case where a bank is at the boundary.

The monotony properties for the ratios and the threat indices relate to the complementarity between ratios: increasing the cash-flow of one bank not only (weakly) increases its ratio but also those of the other banks. As a result, the set of defaulting banks, hence the threat indices, decrease. Sub-modularity extends this to non-marginal variations of the cash-flows. Equation (6) compares the incremental values in V due to an increase in i's cash-flow (from z_i to z'_i) when the other banks' cash-flows are either $\boldsymbol{z}_{-i}$ (the right-hand side) or the largest values $\boldsymbol{z}'_{-i}$ (the left-hand side). When the defaulting set is the same set D for the cash-flows $\boldsymbol{z}$ and $\boldsymbol{z}'$, V is linear for the cash-flows larger than $\boldsymbol{z}$ and smaller than $\boldsymbol{z}'$; hence the two incremental values are equal to $\mu_i(z'_i - z_i)$ where μ_i is the index of i given D. When the defaulting set changes, sub-modularity says that the incremental value in V due to an increase in i's cash-flow is larger the lower other banks' cash-flows are. As shown in the proof, this is due to the fact that the decreases in default and in spill-over effects due to an increase in i's cash-flow are larger the weaker other banks are.

A pyramidal structure Consider the pyramidal structure described in Figure 1. An arrow from i to j represents a positive liability of i towards j. Thus each intermediary lends to its unique 'superior' and collects funds from its 'subordinates'. This corresponds to a situation in which chains of intermediaries collect funds for the bank at the top, bank 1. Let the level of an intermediary be defined as the number of links between it and top bank

[9] The fact that they are unique -a necessary condition for the differentiability of V- is not straightforward without further assumptions on $\boldsymbol{\ell}$. See Section 3.2.2.

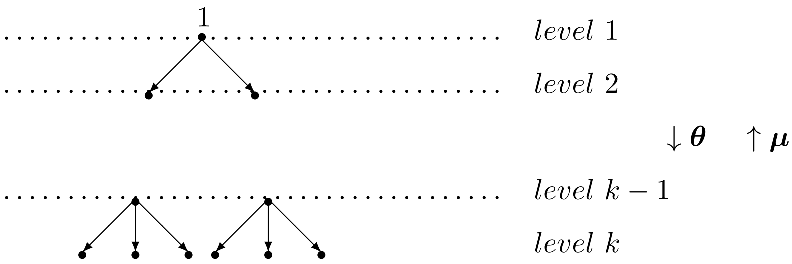

Figure 1: Pyramidal network

1; k denotes the maximum level. Clearing ratios are computed recursively from the top. Once the default set is known, indices are also computed recursively from the bottom.

Clearing ratio vector. Bank 1 has no claims on other banks so its repayment ratio is determined by its cash-flow as the minimum of 1 and z_1/ℓ_1^*. The repayment ratios of the banks at level 2 can now be computed since they receive payments from bank 1 only and these are known. The computation proceeds: Once the ratios are determined for the banks at some level l, the repayments to a bank at level $l+1$, hence its asset value, are known; the repayment ratio is thus determined (equal to the minimum of 1 and the ratio of the asset value to total liabilities). After k steps, the clearing vector is obtained.

Threat index vector. Knowing which banks are defaulting, the threat indices are computed recursively, starting from the bottom. Assume no boundary bank.[10] A bank at level k has no creditors, so its threat index is either equal to 0 (the bank does not default) or 1 (the bank defaults). The computation proceeds: At level l, the index of a non-defaulting bank is set to zero; that of a defaulting bank is computed using expression (5) since the threat indices of its creditors (which are a level $l+1$) have been determined at the previous step.

When the pyramidal structure reduces to a single debt chain, one easily checks that the threat index of a defaulting bank is simply equal to 1 plus the number of its consecutive direct or indirect creditors that are defaulting. Clearly, the orders given by the default ratios and the threat indices may differ.

Similar recursive computations can be performed in the reverse situation in which all the liabilities point toward the top, as in the case of a 'conglomerate' in which each institution

[10]If there are boundary banks, the computation can be performed by either considering these boundary banks as defaulting banks, so as to obtain the maximal threat indices, or by considering them as safe banks, so as to obtain the minimal indices, as explained in next section.

lends funds to its direct subordinates. The top bank has a null threat index, and the structure can be qualified as less prone to contagion than the previous one because a single default cannot affect all banks. ■

In a general network, a recursive computation is not possible because of the cycles. There are various ways to compute the clearing vector, for instance through linear programming by solving the program $\mathcal{P}$ or through an algorithm that exploits the complementarities structure (see Section 3.2.4). Once the set D is known, the threat indices are computed by solving the linear system (5), which entails considering the cycles between the defaulting banks, as explained in section 3.2.2. The backward-forward relationships between clearing ratios and threat indices that arise in pyramidal structures are still present in the form of dual relationship, as we examine now.

Comparing clearing ratios and threat indices Let us write down the conditions satisfied by the clearing ratio and threat index vectors, assuming no boundary bank. They are respectively of the form $(\mathbb{1}_{N-D}, \boldsymbol{\theta}_D)$ and $(\mathbf{0}_{N-D}, \boldsymbol{\mu}_D)$ where D denotes the default set. The clearing ratio satisfies the system of linear inequalities that says that equity is positive for banks that are not in D, i.e., $\ell_i^* - \sum_{j \in D} \theta_j \ell_{ji} - \sum_{j \in S} \ell_{ji} \leq z_i$, and equity is null for those in D, that is, $\theta_i \ell_i^* - \sum_{j \in D} \theta_j \ell_{ji} - \sum_{j \in S} \ell_{ji} = z_i$. Dividing by ℓ_i^* the equations on D, $\boldsymbol{\theta}_D$ and $\boldsymbol{\mu}_D$ respectively solve the linear systems

$$\begin{aligned} \text{for each } i \text{ in } D \quad &: \quad \theta_i = \sum_{j \notin D} \pi_{ji} + \sum_{j \in D} \theta_j \pi_{ji} \frac{z_i}{\ell_i^*}, \\ \text{for each } i \text{ in } D \quad &: \quad \mu_i = 1 + \sum_{j \in D} \pi_{ij} \mu_j. \end{aligned}$$

Whereas the distress of a bank i as measured by its repayment ratio depends on the distress of its debtors (through the π_{ji}), the threat the bank imposes on the payment system depends on the threat of its creditors (through the π_{ij}). Also, the repayment ratios are affected by the precise values taken by the cash-flow whereas the indices depend on these values only through the default set (see Section 3.2.2 for an explanation). This explains why the ratios and threat indices of the defaulting banks are not necessarily aligned.

3.2 Threat indices and moderate injection policies

This section examines policies of cash injection into banks during a default episode when the objective is to improve aggregate repayments. It first considers a regulator who injects cash into banks. As the threat indices play a major role in determine the optimal policies, we investigate more closely their determinants and compared them with the default ratios and the order of defaults in EN algorithm. An alternative policy that relies on banks and forces some banks to write-off part of their claims is examined.

3.2.1 Cash injection policies

Consider a planner whose objective is to maximize aggregate payments (or aggregate asset values), by injecting an amount of cash denoted by m. A *feasible injection policy* is described by a non-negative n-vector $\boldsymbol{x} = (x_i)$ whose total $\sum_i x_i$ less than or equal to m (budget equation). x_i represents the amount received by i, which changes z_i into $z_i + x_i$. As a result, aggregate payments are changed from $V(\boldsymbol{z})$ to $V(\boldsymbol{z} + \boldsymbol{x})$.

A injection policy $\boldsymbol{x}$ is said to be *optimal* is it maximizes $V(\boldsymbol{z} + \boldsymbol{x})$ over the set of feasible strategies. A 'target' is a bank that receives a positive amount. The optimal injection policies are characterized by the threat indices when m is small enough.

Proposition 2 *A marginal injection of cash is optimal if it targets the defaulting banks with the largest threat index. The same strategy is optimal for larger amounts that are moderate enough to keep the set of defaulting banks unchanged. The increase in V is equal to $\mu^{max}m$ where μ^{max} denotes the maximal threat index.*

The proposition follows from Proposition 1. The policy is especially simple when the injected amount is moderate since there is no need to modify the targets. These targets may not be the banks with the largest default ratios, since ratios and threat indices can be in different order. The targets may not be the banks with a large 'size' either (the size can be measured in different ways, for instance by the liabilities total or the ratio of this total to the cash-flow). As clear from the expression (5), the threat index of a defaulting bank is determined by its liabilities' shares towards its creditors that are in default (this property is interpreted in Section 3.2.2). Hence the defaulting banks with the largest index are not necessarily those with the largest size. This should not be interpreted, however, that size does not matter. The threat index is computed at the liquidation stage, given the realized default set. One may suspect that a large bank, more precisely a bank with large liabilities, generates more default among its creditors when it defaults. If true, its index is likely to be large when its default. This issue is addressed in Section 5 and the illustration of Section 6.

3.2.2 Explaining the link between cash injection and threat indices

Let us consider the process of reimbursements triggered by the injection of one unit of cash into a bank, say i. Assume first no bank at the boundary and the unit be small enough so that the banks' status do not change. If i is safe, i keeps the unit as it already fulfills its obligations: The process stops ($\mu_i = 0$). If i is defaulting, i uses the additional unit to increase its reimbursements. i's creditors that are themselves in default, in turn, use the additional cash to repay their debts, and so on. The initial unit thus triggers a sequence of additional reimbursements along chains of defaulting banks; their sum is precisely measured by μ_i. The explicit expression for the sequence can be obtained as follows. The relationships (5) between the threat indices of defaulting banks writes in matrix form[11]

$$(\mathbb{I} - \boldsymbol{\pi})_{D\times D}\boldsymbol{\mu}_D = \mathbb{1}_D. \tag{7}$$

The matrix $(\mathbb{I} - \boldsymbol{\pi})_{D\times D}$ is invertible and the inverse expressed as an infinite sum, as stated by the following Lemma.

Lemma 1 *Let D be the default set at a clearing ratio vector. Any subset of D has a creditor outside D and the dominant eigenvalue ρ_D of $\boldsymbol{\pi}_{D\times D}$ is smaller than 1. Thus the matrix $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}$ is invertible, with a positive inverse given by the converging infinite sum: $(\mathbb{I} - \boldsymbol{\pi})^{-1}_{D\times D} = \mathbb{I}_{D\times D} + \boldsymbol{\pi}_{D\times D} + \boldsymbol{\pi}^{(2)}_{D\times D} + ... + \boldsymbol{\pi}^{(p)}_{D\times D} + ...$ where $\boldsymbol{\pi}^{(p)}$ denotes the product of $\boldsymbol{\pi}$ by itself p times.*

The proof relies on the fact that any subset of D has creditors outside the subset: Since the z_is are positive and each defaulting bank has null equity, some payments must go out from the subset. This implies that, for some integer p, the total of each row of $\boldsymbol{\pi}^{(p)}_{D\times D}$ is strictly less than 1, hence the dominant eigenvalue of $\boldsymbol{\pi}_{D\times D}$ is smaller than 1.[12] Well known results

[11]There is a slight abuse of notation because the matrix $\boldsymbol{\pi}$ is not defined on $N \times N$ if some banks are not indebted. But these banks are surely not defaulting.

[12]The property is obvious for a complete liability structure ($\ell_{ij} > 0$ for each distinct i and j): In that case the total of each row of $\boldsymbol{\pi}_{D\times D}$ is strictly smaller than 1 (D is a strict subset of N). For an incomplete liability structure, the property is true for a set of defaulting banks but not necessarily for any subset of N. Without an outside creditor for each subset, invertibility may fail, as in the following example. Let subset T be made of indebted banks and

$$\boldsymbol{\pi}_{T\times T} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

on positive matrices then imply that $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}$ is invertible with an inverse given by the infinite sum of the powers of $\boldsymbol{\pi}_{D\times D}$.

Applying Lemma 1, (7) yields

$$\boldsymbol{\mu}_D = \mathbb{1}_D + \boldsymbol{\pi}_{D\times D}\mathbb{1}_D + \boldsymbol{\pi}^{(2)}_{D\times D}\mathbb{1}_D + ... + \boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D + ... \tag{8}$$

which also writes for each i in D as

$$\mu_i - 1 = \sum_{j\in D}\pi_{ij} + \sum_{j\in D, k\in D}\pi_{ij}\pi_{jk} + \cdots + \sum_{j_1,j_2,\cdots,j_p \text{ all in } D}\pi_{ij_1}\pi_{j_1j_2}\pi_{j_2j_3}..\pi_{j_{p-1}j_p} + \cdots \tag{9}$$

Recall that $\mu_i - 1$ measures the spill-over effects generated by the unit of cash injected in defaulting i, that is, the reimbursement flows in addition to the first unit initiated by i. The spill-over effects are decomposed into a sequence of triggered additional flows, corresponding to the terms on the right hand side of (9). First, each i's creditor j receives the share π_{ij} of the unit reimbursed by i, entirely used for reimbursement by those in default. This generates a first additional payment equal to i's *cumulated liabilities share* toward D, $\sum_{j\in D}\pi_{ij}$, the first term on the right hand side of (9). By the same argument, each of the π_{ij} units received by defaulting i's creditor j generates $\sum_{k\in D}\pi_{jk}$ extra units of payments. Summing over all defaulting creditors of i, the second indirect additional payment equals $\sum_{j\in D}\pi_{ij}(\sum_{k\in D}\pi_{jk})$, or $\sum_{k\in D}(\sum_{j\in D}\pi_{ij}\pi_{jk})$. Iterating, the additional indirect impact along a path of p banks, each one defaulting and indebted to its successor, is given by the p-th term in (9).

The above process explains why the indices are determined by the liability *shares* (not the absolute liabilities) within the set D and furthermore do not depend upon cash-flows' levels. The priority of creditors triggers automatic payments that are entirely determined by the liability shares of the recipient defaulting banks.

The process when some banks are at the boundary When a bank, say i, is at the boundary, an increase in the cash flow of has to be distinguished from that of a decrease. An injection of one unit in i's cash-flow has no impact on payments because i already repays its debt: $\frac{\partial V}{\partial z_i^+} = 0$. A decrease in i's cash-flow instead triggers its default; thus surely the

Rows totals are 1 for banks 1 and 2 and 0 for bank 3. This implies that bank 3 has a creditor outside T, hence T has a creditor outside T. But banks 1 and 2 have no creditor outside $\{1,2\}$. The matrix $(\mathbb{I}-\boldsymbol{\pi})_{T\times T}$ is not invertible because the vector $\boldsymbol{x}=(1,1,0)$ satisfies $\boldsymbol{x}=\boldsymbol{x}\boldsymbol{\pi}_{T\times T}$. One checks that T cannot be a default set: since 1 and 2 are isolated, surely θ_1 or θ_2 is equal to 1 at a clearing ratio.

left derivative $\frac{\partial V}{\partial z_i^-}$ is at least equal to 1, which explains why the value function V is not differentiable. Using the same argument as above, the exact value of the left derivative can be computed by taking for D the whole set of banks that are defaulting or at the boundary.[13]

Finally, note that an alternative interpretation of expression (7) is in stochastic term. Interpret $\boldsymbol{\pi}$ as a transition matrix in which element π_{ij} is the probability of reaching j from i (by definition the sum $\sum_j \pi_{ij}$ is equal to 1). Start with a bank in D and stops the process as soon as a safe bank is reached. Since $\rho_D < 1$, the process reaches a safe bank for sure. In that interpretation, the element i, j of the matrix $\boldsymbol{\pi}^{(p)}_{D\times D}$ is the probability of reaching j from i in p steps while staying all along in D and μ_i is the average number of times where the process stays in D when it starts at i. Such an interpretation allows one to use standard probability techniques.

3.2.3 Two main determinants of the μ_i: Cumulated shares and dominant eigenvector

The cumulated liability shares to D are a primary determinant of the indices but they are not the only ones, as we have seen: They are only the first term in (9). In the following example, indices are in a different order than the cumulated shares.

Example 1 *There are 6 defaulting banks. Figure 2 represents the network within D. An arrow from i to j represents a liability from i to j. Banks 5, 6 and 7 are 1's creditors but are not indebted to defaulting banks. Their indices are thus equal to 1. For each other bank, let us assume that its liability total is shared equally among its creditors and that it has a creditor outside D. The positive shares within D are thus :* $\pi_{1i} = 1/5$ *for* $i = 2, 5, 6, 7$, $\pi_{2i} = 1/4$ *for* $i = 1, 3, 4$, $\pi_{3i} = 1/3$ *for* $i = 2, 4$, *and* $\pi_{4i} = 1/3$ *for* $i = 2, 3$. *Indices are approximately:* $\mu_1 = 2.2$, $\mu_2 = 3.09$, $\mu_3 = \mu_4 = 3.03$. *The cumulated shares to D are not in the same order since bank 1 has the largest share,* $4/5$, *(compared to* $3/4$ *and 2/3) but the smallest index. This is explained by the indirect liabilities as follows. Consider the process following an injection into a bank as described above. Bank 1 is the one that transfers the largest proportion,* $4/5$, *to the defaulting banks but the proportion 3/5 is transferred to 6,5,7,*

[13] V is not differentiable with respect to the cash-flow of a defaulting bank j either if there is a chain of defaulting creditors from j to a boundary bank i: An increase in j's cash-flow makes i safe whereas a decrease makes it default, generating a further decrease in payments.

Figure 2: Index versus cumulated shares

which then transfer the received amount entirely to safe banks. Thus, after two 'rounds', only the banks $1, 2, 3, 4$ *matter with a network depicted on the right of Figure2.*

In this example, the impact of 'long run' effects overturn the order of the cumulated shares. To analyze these effects, let us write the threat indices, by using the dominant eigenvalue ρ_D of the matrix $\boldsymbol{\pi}_{D\times D}$. Write (8) as

$$\boldsymbol{\mu}_D - \mathbb{1}_D = \rho_D(\frac{1}{\rho_D}\boldsymbol{\pi}_{D\times D}\mathbb{1}_D) + \rho_D^2(\frac{1}{\rho_D^2}\boldsymbol{\pi}_{D\times D}^{(2)}\mathbb{1}_D) + ... + \rho_D^p(\frac{1}{\rho_D^p}\boldsymbol{\pi}_{D\times D}^{(p)}\mathbb{1}_D) + ... \tag{10}$$

We know that $\rho_D < 1$ (Lemma 1). The vectors $(\frac{1}{\rho_D^p}\boldsymbol{\pi}_{D\times D}^{(p)}\mathbb{1}_D)$ are all in the same simplex: choosing for $\boldsymbol{u}$ the left dominant eigenvector of $\boldsymbol{\pi}_{D\times D}$ that satisfies $\sum_i u_i = 1$, the vectors all satisfy $\sum_i u_i y_i = 1$. Thus the infinite sum (10) converges geometrically with a rate equal to ρ_D. It follows that when ρ_D is small, the first terms in (9) drive the value of the indices: the spill-over effects are close to the cumulated shares within D. When ρ_D is not too small, the indirect payments matter. One can say more if the matrix is primitive (its p-th power is positive for some p). In that case, by an extension of Perron-Frobenius theorem, the sequence of matrices $\boldsymbol{\pi}_{D\times D}^{(p)}/\rho_D^p$ converges to a matrix whose columns are all proportional to a dominant eigenvector $\boldsymbol{v}$ of $\boldsymbol{\pi}_{D\times D}$ and rows are all proportional to $\boldsymbol{u}$. It follows that, whatever moderate strategy $\boldsymbol{x}$, the additional flow received by banks at step p becomes proportional to $\boldsymbol{u}$ with a proportionality factor equal to $\rho_D^p \sum_i x_i v_i$. Hence ρ_D measures the persistence of the flow in D; furthermore, injecting a unit into a defaulting bank i whose component v_i is maximal in $\boldsymbol{v}$ generates the maximal long run spill-over effects (i must be involved in cycles since otherwise v_i is null). This explains why the eigenvector plays an important role when ρ_D is large.

In example 1, ρ_D is equal to 0.634 associated to the eigenvector $(0.1672, 0.5298, 0.5879, 0.5879, 0, 0, 0)$, whose components are in a different order from the cumulated shares.

3.2.4 Discussion

Relationships with centrality indices Threat indices have a flavor of the centrality indices introduced in sociology by Katz (1953) and Bonacich (1987). In the specific case described in the next example, the threat index coincides, up to a linear transformation, with a Katz-Bonacich index of the sub-network linking defaulting banks.

Recall that, given a network with incidence matrix $\boldsymbol{g}$, the Katz-Bonacich index is defined as $\boldsymbol{\beta} = (\mathbb{I} - a\boldsymbol{g})^{-1} \boldsymbol{g}\mathbb{1}$ where a is an 'attenuation' parameter, which captures the importance of indirect links. Observe[14] that $a\boldsymbol{\beta} = (\mathbb{I} - a\boldsymbol{g})^{-1} \mathbb{1} = (\mathbb{I} - a\boldsymbol{g})^{-1} \boldsymbol{g}\mathbb{1} - \mathbb{1}$.

Example 2 *Let each positive liability have an identical level and each bank have the same number of creditors, say p, hence the same total liabilities. The matrix $\boldsymbol{\pi}$ is thus proportional to the incidence matrix of the liabilities network : $\boldsymbol{\pi} = \frac{1}{p}\boldsymbol{g}$ where $\boldsymbol{g}$ has 1 if ℓ_{ij} is positive and 0 otherwise. Given a defaulting set D, $\boldsymbol{\mu}_D$ is equal to $\left((\mathbb{I} - \frac{1}{p}\boldsymbol{g})_{D\times D}\right)^{-1} \mathbb{1}_D$, which is a linear transformation of the Bonacich index associated to the network $\boldsymbol{g}_{D\times D}$ with attenuation parameter $1/p$. Therefore, the more numerous creditors each bank has, the smaller the attenuation factor and the more dissipated the impact of default is along a chain of creditors.*

This example is very specific. Furthermore, our approach differs in an important way: The threat index is based on an objective. This explains why both the subset of relevant nodes and the attenuation parameter are endogenous, respectively determined by the set of defaulting banks and the number of total creditors of a bank.

Threat indices and the order of default in EN algorithm Let us compare the order given by the threat indices with the order suggested by EN algorithm. The EN algorithm starts by setting all ratios equal to 1 and computes equities. If all equities are non-negative, $\mathbb{1}$ is the clearing ratio. Otherwise, $e_i(\mathbb{1}) < 0$ for at least one i. In that case, the algorithm adjusts sequentially the ratios downward (with cycles, several adjustments may be necessary for a given bank contrary to the case of a pyramidal network). EN suggest that the step at which a bank first fails in the algorithm indicates its fragility. These levels are not necessarily in the same order as the threat indices, as illustrated by the following example described in Figure 3.

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[14] Write $\mathbb{I} = (\mathbb{I} - a\boldsymbol{g}) + a\boldsymbol{g}$; multiplying this equation by $(\mathbb{I} - a\boldsymbol{g})^{-1}$ yields $(\mathbb{I} - a\boldsymbol{g})^{-1} \mathbb{1} = \mathbb{1} + a (\mathbb{I} - a\boldsymbol{g})^{-1} \boldsymbol{g}\mathbb{1}$.

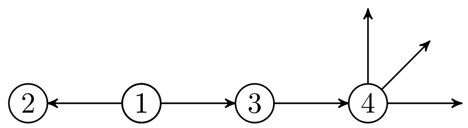

Figure 3: Index versus order of default in EN algorithm

Example 3 *Let 1 be equally indebted to 2 and 3. 4 is indebted to banks that are not indebted and not represented in the figure. Assume that 1 is the only one to initiate default:* $e_1(\mathbb{1}) < 0$ *and* $e_i(\mathbb{1}) > 0$ $i = 2, 3, 4$*, but that 3 and 4 also end up defaulting (this is possible for some values of the cash-flows, see below). In the algorithm, bank 1 defaults first, 3 second and 4 third. Since* $\mu_4 = 1$*, we have* $\mu_3 = 2$ *and* $\mu_1 = 3/2$*, it is more beneficial to inject a moderate amount of cash into bank 3 rather than 1. This is due to the fact that part of the cash injected in 1 ends up in the safe bank 2.*

Non–monotony of the optimal policy The targeting policy can be non-monotone in the amount m. Take Example 3 and the following values: $z_1 = 2$, $\ell_{12} = \ell_{13} = 2$, $z_3 = 2.2$, $\ell_{34} = 4$, $z_4 = 1.2$, $\ell_4^* = 6$, which are due to other banks. Other values do not matter. We have $e_1(\mathbb{1}) = -2$, $e_3(\mathbb{1}) = 0.2$ and $e_4(\mathbb{1}) = 0.2$. The clearing ratios for 1 and 3 are $\theta_1 = 0.5$ and $\theta_3 = 0.8$. The optimal strategy as a function of m is as follows:

for $m \leq 0.8$, inject m into 3 only,

for $0.8 \leq m \leq 1.6$, inject $2(m - 0.8)$ into bank 1 and $1.6 - m$ into bank 3,

for $1.6 < m < 1.8$, inject any amount between 1.6 and 1.8 into bank 1 and the remaining into bank 4 with a maximum of 1.

The explanation is as follows: As long as $m < 0.8$, the three banks 1, 3 and 4 are surely defaulting as in Example 3. It is thus optimal to target 3 only and the ratios for 1 and 3 are $\theta_1 = 0.5$ and $\theta_3 = (3.2 + m)/4$ (4's ratio does not matter as long as it is defaulting). At $m = 0.8$, bank 3 does not default any longer and becomes at the boundary, with a left index of $3/2$ (because 4 is still defaulting) and a right index equal to 0 (see section 3.2.2). Thus, when m is larger than 0.8, it becomes optimal to target 1 and 3 and to allocate to 3 the minimal amount that keeps it non-defaulting, i.e. at the boundary. Since 3 receives back half of the amount received by 1, this explains why 3's allocation decreases. When m reaches 1.6, bank 3 is no longer targeted because it receives indirectly the 0.8 units that make its equity non-negative. Then, it becomes optimal to target both 1 and 4 with a minimum of 1.6 allocated to bank 1 so as to keep 3 non-defaulting.

The non-monotony of the optimal policy implies that a 'greedy' policy, in which the amount is gradually allocated to the banks whose threat index is maximal, might not be optimal when the default set changes. It would be interesting to know whether the greedy algorithm nevertheless provides a good approximation of the optimal policy.

3.2.5 Alternative policy: Transfers across banks

Let us examine an alternative policy based on transfers across banks. Creditors to a bank in difficulty may be asked to write off part of their claims. (Alternative policies could be analyzed such as in 2007 for LTCM when creditors where asked to replace part of their claims into stocks.) A writing off of part of the claim from a safe bank j to i amounts to decrease ℓ_{ij}. The outcome is independent of the identity of the safe banks involved, which is thus not made explicit in the following proposition.

Proposition 3 *A marginal writing off by safe banks on one unit of a claim on a defaulting bank i leads to increase V by $\theta_i(\mu_i - 1)$. Thus it is optimal first to write off part of the claims of the banks with the largest $\theta_i(\mu_i - 1)$.*

According to the proposition, writing off part of the claims of safe banks on a defaulting bank never results in a decrease in the overall payments; furthermore the result is an increase if the defaulting bank is also indebted to another defaulting bank ($\mu_i > 1$). This is a priori not obvious because i reimburses less to safe banks. However, since i defaults, the total reimbursements of i are unchanged, and recomposed, with a relative increase to defaulting creditors, which trigger further payments. The larger θ_i, the more important the amount redistributed; the larger μ_i, the more beneficial the impact on subsequent creditors.

4 The value of information

The optimal targeting strategy uses information on the liabilities structure. In the absence of such information, a natural strategy is a *uniform strategy*, which injects the same amount in each defaulting bank. Such a uniform strategy is indeed optimal when there is a lack of data on bilateral interbank exposures, as we show at the end of this section. Next proposition computes the improvement in the aggregate repayments reached by the uniform policy and compares it with the optimal policy under complete information.

Proposition 4 *Let the injected amount m be allocated uniformly to the defaulting banks. If D does not change, the increase in V is equal to $(\frac{1}{d}\sum_{i\in D}\mu_i)m$ where d is the number of banks in D. Thus, the loss with respect to the optimal strategy is proportional to the difference $(\mu^{max}-\frac{1}{d}\sum_{i\in D}\mu_i)$, which is called the value of information.*

According to Proposition 4, the improvement in payments due to the knowledge of the structure is, for moderate amounts, proportional to the difference between the maximal index and the average index over the defaulting banks. In Example 1, the maximal index is equal to 3.09 and the average one to 2.05, so that the improvement is quite large, roughly equal to m. We investigate here how both quantities μ^{max} and $(\mu^{max}-\frac{1}{d}\sum_{i\in D}\mu_i)$ vary with the liabilities structure, keeping the set D fixed. These quantities differ, as is illustrated by the following example.

Example 4 *Let the cumulative liability shares be equal within D: the sums $\sum_{j\in D}\pi_{ij}$ are equal to the same value σ for each i in D. This also writes $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}\mathbb{1}_D=\sigma\mathbb{1}_D$, which yields $\boldsymbol{\mu}_D=\frac{1}{1-\sigma}\mathbb{1}_D$ Thus, the benefit of an injection is large when the common cumulated value σ is large, close to 1, or, equivalently, when the defaulting banks have little liabilities outside D (since, by definition, the overall cumulated share is 1). The indices in D, however, are all equal to $\frac{1}{1-\sigma}$ and the value of information is null: any moderate injection into the defaulting banks is optimal.*

Cumulative liability shares are however unlikely to be equal within D (this can be true for each possible D only if all the liabilities shares are equal). Whenever they differ, the indices differ as well[15] and the value of information is positive. As we saw in section 3.2.3, the cumulative liability shares in D and the dominant eigenvector of the matrix $\boldsymbol{\pi}_{D\times D}$ are two main determinants of the indices; furthermore, the larger the dominant eigenvalue ρ_D, the more important the eigenvector. When the cumulated shares differ across the banks, the differences between the threat indices are increased if the eigenvector components are in the same order as the cumulated shares. This occurs, for instance, when there is a subset of banks in D whose cumulated shares to D are large and furthermore concentrated between themselves. In that case, knowing that 'core' of defaulting banks is valuable.

[15] The indices are equal when $\boldsymbol{\mu}_D$ is equal to $\mu^{max}\mathbb{1}_D$. From (7), this is equivalent to $\mu^{max}(\mathbb{I}-\boldsymbol{\pi})_{D\times D}\mathbb{1}_D=\mathbb{1}_D$, which writes as $(\mu^{max}-1)\mathbb{1}_D=\mu^{max}\boldsymbol{\pi}_{D\times D}\mathbb{1}_D$, hence the cumulated shares to D are equal across defaulting banks.

In example 1, ρ_D is equal to 0.634 associated to the eigenvector $(0.1672, 0.5298, 0.5879, 0.5879, 0, 0, 0)$. Its components are in a different order than the cumulated shares.

Log-fitting model When there is a lack of data on bilateral interbank exposures, the log-fitting method is used to estimate the missing data given the available information (see for example Upper and Worms 2004 and Elsinger, Lehar, and Summer 2004). Let us consider the situation in which the total amount of liabilities and total amount of loans are known but there is no specific information on bilateral exposures. In that case, the estimated i's liabilities shares are independent of i, equal to the overall proportions of the loans (i.e., j's proportion is $\frac{\sum_{i \in N} \ell_{ij}}{\sum_{ik, i \in N} \ell_{ik}}$). This implies that, for the estimated matrix, the cumulated shares to any default set D are equal. Hence the threat indices are equal as well (see example 4): A uniform strategy is optimal. Presumably, this index on the estimated matrix is lower than the 'true' maximal index.

5 Ex ante value of raising capital

So far, the analysis has taken place at a 'liquidation' stage. This section evaluates the impact of an increase in the requirement on capital at an ex ante stage. Specifically, it considers the situation in which the increase in capital has to be invested in a risk-free asset, keeping risky investments and liabilities unchanged.

Initially, banks expect cash-flows from their investments, $\widetilde{z}_i$ for i where $\widetilde{}$ means that the cash-flow is random. Let V be the payment value associated to the cross-liabilities. The aggregate payments will be $V(\boldsymbol{z})$ if the realized value for the cash-flows is $\boldsymbol{z}$, hence the expected payments are equal to $E[V(\widetilde{\boldsymbol{z}})]$.

Consider an increase in capital, k_i for i, invested in a risk-free asset with null return (to simplify notation); i's cash-flow is changed into $\widetilde{z}_i + k_i$ and the expected payments into $E[V(\widetilde{\boldsymbol{z}} + \boldsymbol{k})]$. It follows that raising i's capital by 1 (marginal) unit raises expected repayment by $E[\mu_i(\widetilde{\boldsymbol{z}})]$. This gives the marginal value of raising the capital. To compute it, recall that the indices of the safe banks are null so that the expression (5) writes as

$$\mu_i(\boldsymbol{z})\mathbb{1}_{D(\boldsymbol{z})}(i) = [1 + \sum_j \pi_{ij}\mu_j \mathbb{1}_{D(\boldsymbol{z})}(j)]\mathbb{1}_{D(\boldsymbol{z})}(i),$$

where $D(\boldsymbol{z})$ denotes the set of defaulting banks given $\boldsymbol{z}$. Taking the expectation over $\widetilde{\boldsymbol{z}}$, we obtain

$$E[\mu_i(\widetilde{\boldsymbol{z}})] = Proba(i \in D(\boldsymbol{z}))[1+\alpha_i] \text{ where } \alpha_i = E[\mu_i(\widetilde{\boldsymbol{z}}) - 1 | i \in D(\boldsymbol{z})]. \tag{11}$$

The marginal value of increasing i's capital is thus the product of two terms: the probability of i's default, $Proba(i \in D(\boldsymbol{z}))$, and the expectation of the spill-over effects conditional on i defaulting, α_i. We call hereafter α_i the *spill-over index.* Using (9), α_i an be written in developed form as

$$\begin{aligned} \alpha_i &= \sum_j \pi_{i,j} Proba[j \in D(\boldsymbol{z}) | i \in D(\boldsymbol{z})] \\ &+ \sum_{j,k} \pi_{i,j}\pi_{j,k} Proba[j \text{ and } k \in D(\boldsymbol{z}) | i \in D(\boldsymbol{z})] + . \\ &+ \ ... \end{aligned}$$

i's spill-over index is increasing in the distribution of i's creditors that are defaulting (through the liabilities shares) conditional on i defaulting and increasing indirectly in the whole distribution of the defaulting set, still conditional on i defaulting. This is reminiscent of the distinction put forward by Tarashev, Borio, and Tsatsaronis (2010) as the 'participation' to default. A bank may have large conditional spill-over effects if it is likely to default under systemic events when numerous others default as well. This correlation is increased by various factors: positive correlation in the cash-flows or a pure size effect as we examine now.

Impact of the size Let us say that two banks differ in size if the distribution of their cash-flows are equal up to a scale factor and their liabilities and loans are proportional with the same scale factor. Formally bank 1's size is λ times that of bank 2 if the distribution of 1's cash-flow $\widetilde{z}_1$ is the same distribution as that of $\lambda\widetilde{z}_2$, and 1's liabilities and loans toward banks other than 2 are λ times those of 2', and 1's liability toward 2 is λ times 2's liability toward 1. Thus 1's and 2's liabilities' shares towards other banks are equal and those towards each other as well.

No assumption has been made on the joint law of the cash-flows. It is easy to check that if the realized cash-flows of the two banks are in the same proportion as their size, $z_1 = \lambda z_2$, their ratios and threat indices are equal (in particular they default simultaneously). It follows that, if the banks' cash-flows are perfectly correlated, the marginal values of raising capital in the two banks are equal, independently of their sizes. This is no longer true if the banks' cash-flows are not perfectly correlated. To go further, consider the reasonable assumption of conditional proportional cash-flows. Given (z_1, z_2), denote $(z_1^\sigma, z_2^\sigma) = (\lambda z_2, z_1/\lambda)$ in which 1 and 2 cash-flows exchanged, accounting for their size. The distributions of $\widetilde{z}_i$ and $\widetilde{z}_i^\sigma$ for $i = 1, 2$ are identical by assumption. The following condition extends this property to the joint cash-flows conditional on other banks' cash-flows: The cash-flows of banks 1 and 2 are *conditionally proportional* if the distributions of $(\widetilde{z}_1, \widetilde{z}_2)$ and $(\widetilde{z}_1^\sigma, \widetilde{z}_2^\sigma)$ conditionally on $\boldsymbol{z}_{-12}$ are identical. Under conditional independence, events that affect the banks other than 1 and 2 are independent on the fact that bank 1 has a low cash-flow relative to bank 2 or the reverse. Of course, if 1 and 2 cash-flows are independent between each other and with $\boldsymbol{z}_{-12}$, then they are conditionally proportional.

Proposition 5 *Let banks 1 and 2 differ in size, with 1's size λ times 2's size for $\lambda > 1$. Assume their cash-flows to be conditionally proportional. Then*

$$E[\mu_1(\widetilde{\boldsymbol{z}})|z_1 \leq \lambda z_2] \geq E[\mu_2(\widetilde{\boldsymbol{z}})|z_1 \geq \lambda z_2] \text{ and } E[\mu_2(\widetilde{\boldsymbol{z}})|z_1 \leq \lambda z_2] \geq E[\mu_1(\widetilde{\boldsymbol{z}})|z_1 \geq \lambda z_2]. \quad (12)$$

Under conditional independence, though the distributions of (z_1, z_2) and (z_1^σ, z_2^σ) are identical, conditional on events on banks other than 1 and 2, the distributions of the clearing ratios and threat indices, and the values of raising capitals, depend on whether (z_1, z_2) or (z_1^σ, z_2^σ) is realized. The larger bank has a larger impact on default and this works in two directions. The impact is harmful when 1 is weaker than 2, i.e., for (z_1, z_2) with $z_1 \leq \lambda z_2$ because the ratios can only be lower and default set larger than in the symmetric situation (z_1^σ, z_2^σ) where 2 is weaker than 1 (as shown in the proof): This gives the left-hand-side inequality of (12). Conversely the impact of the size is beneficial for (z_1, z_2) with $z_1 \geq \lambda z_2$ relative to (z_1^σ, z_2^σ): This gives the right-hand-side inequality of (12). Since the value of raising capital is equal to the mean of the two conditional expected values, (half of $E[\mu_i(\widetilde{\boldsymbol{z}})|z_1 \leq \lambda z_2] + E[\mu_i(\widetilde{\boldsymbol{z}})|z_1 \geq \lambda z_2]$ for i) one cannot conclude that the bank with the larger size should have a relatively larger capital. This is however likely to be the case when 1 and 2 have few chances to default together: This follows from the left-hand side inequality of (12), since, in that case, 1's threat index is positive mostly when $z_1 < \lambda z_2$ and 2's threat index when $z_1 \geq \lambda z_2$.

6 European cross claims

This section illustrates the method on the consolidated foreign claims for 10 EU countries: Austria, Belgium, France, Germany, Greece, Ireland, Italy, the Netherlands, Portugal, Spain. These claims are those of reporting banks in one country on debt obligations of another country.[16] The exercise can be viewed as a thought experiment in which all these claims are liquidated. Given that a country allocates a fraction of its GDP to the liquidation of foreign claims, how much will it reimburse? (Note that the default of a country, in particular the default on its governmental bonds, is internalized among the residents of a country, since the claims are consolidated. This suppresses the corresponding spill-over effects.)

Liquidation is based on a fraction of the GDPs at the time of liquidation. Specifically, i's 'cash-flow' is given by

$$z_i = k_i GDP_i$$

where k_i is the fraction of i's GDP allocated to liquidation. Ex ante, this fraction is perceived as random, reflecting the uncertainty on growth rate or on political factors such as the resistance within the country. In the reported simulations, the simplest assumptions are taken: The $\widetilde{k}_i$s are log-normally and independently distributed, with the same mean m and standard error v for each country with $m = 1/2$ and $v = 1/8$. GDP_i is the level of GDP in country i at the year of reference, taken to be 2008. Simulations with 1000 draws are run for three years, 2009, 2010 and 2011. To concentrate on the impact of cross-liabilities only, the same reference year is used for the three years so that $\widetilde{z}_i$ follows the same distribution: The changes linked to default are only due to changes in the cross-liabilities.[17]

Let us first consider the situation in which all the z_is are equal to their expected value, here half of GDP_i. Belgium and Ireland initiate default, since their 'equities' are have negative as can be seen from Table 1. Their default does not propagate, i.e., no other country defaults at the clearing ratios. Table 2 reports the ratios and threat indices for Belgium and Ireland. The threat index of Belgium is much smaller than that of Ireland due

[16] The data is available from the Bank of International Settlements. The same data (except for Austria, Belgium, Ireland and The Netherlands) is used by Elliott, Golub, Jackson 2013 for a different exercise, in which there is a cost to default.

[17] It is not difficult to make simulations based on other assumptions. For example the reference year could be adjusted overtime, taken as 2009 for 2010 and so on. Given the huge drop in GDP for Greece, this increases its number of defaults.

to the asymmetry in their liabilities: Belgium's liabilities share to Ireland is much smaller than that of Ireland to Belgium. More generally, liabilities are very asymmetric, and this plays an important role in the results.

Table 1: Net equities at the mean values. Unit: 100 millions of dollars

	Aust	*Belg*	*France*	*Germany*	*Greece*	*Ireland*	*Italy*	*Netherlands*	*Port*	*Spain*
2009	335	−905	23278	18028	153	−1346	6001	5727	142	3424
2010	543	−43	19257	16733	668	−417	7776	4956	526	4838
2011	598	−318	18043	16764	866	−1150	8751	4847	594	5842

Table 2: The clearing ratio and threat index at the mean values for the defaulting countries

	Belg	*Ireland*	*Belg*	*Ireland*
	θ	θ	μ	μ
2009	0.776	0.664	1.014	1.162
2010	0.97	0.845	1.015	1.171
2011	0.868	0.494	1.001	1.155

Let us consider now the effect of the shocks on k_i. A country initiates default for the values of $\widetilde{k}_i$ such that $\widetilde{k}_i < \rho_i$ where ρ_i is equal to i's total net liabilities (the difference between total liabilities and loans) to GDP. The values of ρ_is are reported in Table 3.

Table 3: Net liabilities to GDP ratio ρ_i

	Aust	*Belg*	*Fra*	*Germ*	*Greece*	*Ireland*	*Italy*	*Neth*	*Port*	*Spain*
2009	0.3989	0.7282	−0.5623	−0.0915	0.4541	1.2104	0.1995	−0.3114	0.4465	0.2733
2010	0.3363	0.5110	−0.3788	−0.0490	0.2991	0.7200	0.1106	−0.2022	0.3015	0.1797
2011	0.3198	0.5803	−0.3234	−0.0500	0.2397	1.1071	0.0617	−0.1867	0.2757	0.1132

A country with a negative ratio never initiates default. This is the case for France, Germany, and the Netherlands. We will see that they never default through contagion as well. A country with a positive ratio initiates default with a positive probability as computed in Table 4 (this is the probability that the lognormal distribution with mean 1/2 and standard error 1/8 is less than ρ_i). There has been a general decrease in the net liabilities per GDP for the vulnerable countries in 2010, which translates into a decrease in the probability.

Table 4: Probability to initiate default

	Aust	*Belg*	*Greece*	*Ireland*	*Italy*	*Port*	*Spain*
2009	0.4856	0.8183	0.5663	0.9561	0.1303	0.5559	0.2643
2010	0.3805	0.6378	0.3126	0.8135	0.0201	0.3170	0.0988
2011	0.3507	0.7096	0.2016	0.9414	0.0015	0.2688	0.0220

Table 5 gives the expectation of the clearing ratios and threat indices, the expected clearing ratios and spill-over effects conditional on the default of the country, i.e., the spill-over index α_i defined in (11), and the number of defaults. France, Germany, and the Netherlands never default so they are excluded from the Table. Some general features are as follows. The clearing ratios at the mean are smaller than the expected ratios given in Table 3 (this follows from the convexity of the clearing ratio, established in Proposition 1). Comparing the frequency of default ($N/1000$) with the probability to initiate default, we see that Greece and Ireland do not suffer from contagion, but this is not true for the other countries. The net liabilities over the years change substantially, and mostly decrease. This explains the decrease both in defaults and in the spill-over index α_i. A fortiori the expected threat indices decrease.

Let us consider the countries. Ireland is by all accounts the weakest country: it has the largest probability of default, smallest expected repayment ratio, largest expected threat index and largest spill-over index. It cuts its liabilities in 2010 (from 404249 to 270618) and cuts its loans in 2011 (from 134 266 to 17 710). This explains why its frequency of default decreases in 2010 and increases in 2011; the payment ratio moves in the opposite direction. Belgium's default is also substantial but most of its liabilities are towards France, Germany and the Netherlands, which never default. This explains why it has the smallest spill-over index. Portugal in 2009 has the largest spill-over index. Inspecting the structure of its liabilities explains this: Portugal's liabilities towards Spain are more than half of its total and Spain is fragile since its frequency of default is almost 30 %. Spain's spill-over index on the other hand is rather small, partly due to the fact that its liabilities towards Portugal are less than 7 %. A similar analysis holds for Austria. Almost half of Austria's liabilities are due to Italy, which is also fragile. Thus a simultaneous default of both Austria and Italy (resp. Portugal and Spain) makes Austria's (resp. Portugal) threat index large; furthermore, the occurrence of a simultaneous default is larger than it would be without such a concentration in liabilities. This structure also explains the decrease in Portugal's and Austria's spill-over effects in 2010 and 2011 though their defaults are still substantial. These decreases are due to Spain and Italy, whose default sharply decrease thanks to cut in their liabilities (their net liabilities per GDP is divided by a third for Italy and a half for Spain between 2009 and 2010, see Table 3).

The analysis thus uncovers interesting features that are not easily seen from the 10 by 10 net liabilities matrices nor from aggregate net liabilities because these matrices are highly asymmetrical and change overtime in a non-uniform way.

Table 5 : Expected ratios, indices, and numbers of default.

Year 2009

$$\begin{pmatrix} & Aust & Belg & Greece & Ireland & Italy & Port & Spain \\ \mathbb{E}\theta & 0.901 & 0.717 & 0.804 & 0.641 & 0.982 & 0.852 & 0.947 \\ \mathbb{E}\mu & 0.594 & 0.881 & 0.658 & 1.168 & 0.181 & 0.759 & 0.339 \\ \mathbb{E}_{|def}\theta & 0.806 & 0.668 & 0.655 & 0.625 & 0.882 & 0.757 & 0.813 \\ \alpha & 0.136 & 0.031 & 0.149 & 0.217 & 0.128 & 0.218 & 0.144 \\ N & 509 & 854 & 568 & 958 & 151 & 612 & 284 \end{pmatrix}$$

Year 2010

$$\begin{pmatrix} & Aust & Belg & Greece & Ireland & Italy & Port & Spain \\ \mathbb{E}\theta & 0.934 & 0.84 & 0.915 & 0.799 & 0.999 & 0.942 & 0.987 \\ \mathbb{E}\mu & 0.421 & 0.686 & 0.329 & 0.954 & 0.026 & 0.393 & 0.11 \\ \mathbb{E}_{|def}\theta & 0.831 & 0.759 & 0.72 & 0.752 & 0.923 & 0.831 & 0.861 \\ \alpha & 0.045 & 0.018 & 0.064 & 0.164 & 0.077 & 0.107 & 0.104 \\ N & 390 & 663 & 303 & 812 & 19 & 341 & 94 \end{pmatrix}$$

Year 2011

$$\begin{pmatrix} & Aust & Belg & Greece & Ireland & Italy & Port & Spain \\ \mathbb{E}\theta & 0.938 & 0.767 & 0.948 & 0.474 & 1.0 & 0.94 & 0.998 \\ \mathbb{E}\mu & 0.381 & 0.77 & 0.203 & 1.089 & 0.003 & 0.346 & 0.027 \\ \mathbb{E}_{|def}\theta & 0.828 & 0.694 & 0.724 & 0.443 & 0.93 & 0.813 & 0.919 \\ \alpha & 0.018 & 0.003 & 0.046 & 0.152 & 0.055 & 0.032 & 0.07 \\ N & 362 & 761 & 187 & 944 & 3 & 323 & 24 \end{pmatrix}$$

7 Concluding remarks

This works represents a contribution to our understanding of the impact of cross-liabilities on a system, building on a simple description of financial interactions described by EN model. The analysis can be extended into several directions. To simplify, I have considered debts with equal priority but the analysis extends to the case of several levels of debts, say junior and senior. In that case, the clearing mechanism has to account for the differences of priority; similar expressions for the threat indices obtain by distinguishing the type of default. Another direction is to allow for active asset management as in Cifuentes *et al* (2005), Shin (2008), or Greenwood, Landier, and Thesmar (2015). Let defaulting banks sell their asset holdings to lower the extent of their default or to satisfy solvability ratio constraints. When markets are not perfectly liquid, these 'fire' sales trigger a decrease in stocks price, which in turn decreases the asset value of the balance sheets of *all* banks according to their positions. Thus, asset management amplifies the original impact of a default. The threat index of a bank would incorporate not only its liabilities to other defaulting banks but also the sensitivity of their stock holdings to prices. The extent of similarity in holdings and the sensitivity of prices to sales would determine the strength of this additional effect.

The analysis however may not extend easily to the case where portfolios contain derivatives on other banks such as CDS. In that case, a bank may benefit from the default of another one and multiple clearing ratio vectors might be possible due to the lack of complementarities.

Finally, the threat index reflects an externality imposed by a defaulting bank on the debt repayments of all other banks. While the default level of a bank depends on its assets and the safety of its debtors, its threat index depends on its liabilities and the safety of its creditors. A bank thus may not assess properly the externality it will impose on the system when it decides on its interbank relationships, since it is concerned with the safety of its debtors and not with that of its creditors. This raises the issue of which regulatory tools could help in improving incentives. Such an issue should be addressed by taking an ex ante perspective, in a model in which the liabilities and the investment decisions, which generate future cash-flow levels, are chosen.

8 Proofs

Proof of Lemma 1 Let D be the set of defaulting banks at a clearing ratio. To show that $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}$ is invertible with an inverse given by the infinite sum $\mathbb{I}_{D\times D}+\boldsymbol{\pi}_{D\times D}+\boldsymbol{\pi}^{(2)}_{D\times D}+...+\boldsymbol{\pi}^{(p)}_{D\times D}+....$, we prove that an iterate of the matrix $\boldsymbol{\pi}_{D\times D}$ has all its rows totals smaller than 1: $\boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D \ll \mathbb{1}_D$. The result then follows from standard results on productive matrices: the spectral radius of $\boldsymbol{\pi}^{(p)}_{D\times D}$ is strictly smaller than 1 hence also that of $\boldsymbol{\pi}_{D\times D}$.

Each bank in D has null equity, so, from Lemma 2 below, each subset of D has an outside creditor. Interpret $\boldsymbol{\pi}$ as a transition matrix in which element π_{ij} is the probability of reaching j from i. The (i,j) element of the matrix $\boldsymbol{\pi}^{(q)}_{D\times D}$ gives the probability of reaching j from i in q steps along a path included in D. Hence the sum $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)$ is the probability of the paths of length q that start from i and are included in D. Such a sum is non-increasing in q since a path included in D of length $q+1$ has necessarily its first q elements included in D.[18] So, once the inequality $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)<1$ holds for q it holds for all larger values than q. Thus, $\boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D \ll \mathbb{1}_D$ holds if for each i in D there is q for which $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)<1$.

By contradiction, assume that for some i in D we have $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)=1$ for each q. All the paths starting from i are included in D. Let C be composed with all the elements that can be reached from i. By construction, C has no outside creditor and is included in D, hence all its elements have null equity. Applying Lemma 2 gives the desired contradiction. ■

Lemma 2 *Let T be a non-empty subset of N such that each i in T has null equity and positive cash-flow: $e_i(\boldsymbol{\theta})=0$ and $z_i>0$. Then T is not the whole set N and has a creditor in $N-T$.*

Proof of Lemma 2 We first prove that for any ratio $\boldsymbol{\theta}$ and non-empty subset T of N, the following equality holds

$$\sum_{i\in T}e_i(\boldsymbol{\theta})=\sum_{i\in T}z_i+\sum_{i\in T,j\notin T}\theta_j\ell_{ji}-\sum_{i\in T,j\notin T}\theta_i\ell_{ij}. \tag{13}$$

The proof is straightforward by summing net equities values over T since the payments within T cancel out. Formula (13) says that the aggregate net equity of the banks in T

[18] Formally $\sum_{j\in D}\pi^{(q+1)}_{D\times D}(i,j) = \sum_{j\in D}\pi^{(q)}_{D\times D}(i,k)\sum_{k\in D}\pi(k,j)$ which is equal, exchanging sums, to $\sum_{k\in D}\pi^{(q)}_{D\times D}(i,k)(\sum_{j\in D}\pi(k,j))$ Since the term in bracket $\sum_{j\in D}\pi(k,j)$ is not larger than 1 for each k, we finally obtain $\sum_{j\in D}\pi^{(q+1)}_{D\times D}(i,j)\le\sum_{k\in D}\pi^{(q)}_{D\times D}(i,k)$, the desired inequality.

is equal to their aggregate cash-flow plus the net payment from banks outside T, i.e., the difference between the payments received by T from $N-T$ and those made by T to $N-T$.

The lemma follows. By contradiction, let T have no outside financial creditor: $\ell_{ij}=0$ for each i in T and j not in T. (13) applied to T at $\boldsymbol{\theta}$ implies $\sum_{i\in T} e_i(\boldsymbol{\theta}) = \sum_{i\in T} z_i + \sum_{i\in T, j\notin T} \theta_j \ell_{ji} = 0$, in contradiction with each z_i strictly positive. ∎

Proof of Proposition 1

1- *The properties of* V. First assume all banks to be indebted. Writing the constraint $\theta_i \le 1$ as $\theta_i \ell_i^* \le \ell_i^*$, the program $\mathcal{P}$ is equivalent to $\mathcal{P}'$:

$$\mathcal{P}'(\boldsymbol{z},\boldsymbol{\ell}) \quad : \quad \max_{\boldsymbol{\theta}} \sum_i \ell_i^* \theta_i$$

$$0 \le \theta_i \ell_i^* \le \ell_i^* \text{ for each } i \tag{14}$$

$$\theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} \le z_i \text{ for each } i \tag{15}$$

The values of program $\mathcal{P}$ and $\mathcal{P}'$ are identical. The derivative of V with respect to z_i is given by the multiplier associated to the i-th constraint (15) when the multiplier is unique. The multiplier to i's debt constraint (14) will be denoted by λ_i.

The program $\mathcal{P}'(\boldsymbol{z},\boldsymbol{\ell})$ has a finite solution: the feasible set is non-empty (it contains $\boldsymbol{\theta}=\mathbf{0}$ since $\boldsymbol{z}$ is positive) and is compact. From well known results on linear programming, the multipliers associated to the constraints are the solutions to the dual program of $\mathcal{P}'(\boldsymbol{z},\boldsymbol{\ell})$, and furthermore, the values of the primal and dual coincide. The dual program of $\mathcal{P}'(\boldsymbol{z},\boldsymbol{\ell})$ is

$$\mathcal{D} \quad : \min_{(\lambda,\mu)\ge 0} \sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^*$$

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j \ge 0 \text{ for each } i. \tag{16}$$

and furthermore that the constraints of the dual (16) are binding. To show this, recall that the dual of $\max \ell^* \cdot \boldsymbol{\theta}$ under $A\boldsymbol{\theta} \le b$, $\boldsymbol{\theta} \ge 0$ is $\min b \cdot \boldsymbol{\gamma}$ under $A^t \boldsymbol{\gamma} \ge \ell^*$, and $\boldsymbol{\gamma} \ge 0$. Apply this to $\mathcal{P}'(\boldsymbol{z},\boldsymbol{\ell})$ where A is the $2n \times n$ matrix and b is the $2n$-vector given by

$$A = \begin{pmatrix} dg(\ell^*) - \ell^t \\ dg(\ell^*) \end{pmatrix},\ b = \begin{pmatrix} \boldsymbol{z} \\ \ell* \end{pmatrix}$$

Writing the $2n$-vector $\boldsymbol{\gamma}$ as $\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix}$, the objective of the dual is to minimize $\sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^*$

under the constraints

$$\begin{pmatrix} dg(\ell^*) - \ell & dg(\ell^*) \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} \geq \ell^*.$$

Spelling out the i-th constraint yields $\ell_i^* \mu_i - \sum_j \ell_{ij}\mu_j + \ell_i^* \lambda_i \geq \ell_i^*$ which is (16).

We now show that these constraints are binding:

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j = 0 \text{ for each } i. \tag{17}$$

By contradiction, suppose $(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j > 0$ for some i.

If $\lambda_i > 0$, λ_i can be decreased without affecting the other constraints and the objective is decreased, a contradiction. If $\lambda_i = 0$, then μ_i must be strictly positive. A small decrease in μ_i is feasible: by the assumption, i-th constraint is not binding for i and furthermore a decrease in μ_i relaxes the constraints (16) for the j distinct from i. A decrease in μ_i results in a decrease in the objective, a contradiction again.

Now, let S be the set of safe banks, for which (15) is strict. By the slackness conditions, $\mu_i = 0$ for i in S. Let us assume that there are no boundary banks. All banks that are not in S have a repayment ratio strictly smaller than 1. By the slackness conditions, their debt multipliers λ_i are null. This pores that $\mu_i = 0$ for i in S and $\lambda_i = 0$ for i in D.

Plugging these values into equations (17) for each i in D, we obtain $\mu_i \ell_i^* - \sum_{j \in D} \ell_{ij}\mu_j = \ell_i^*$: this proves (5). The fact that the system (5) has a unique solution, which is furthermore positive, follows from Lemma 1.

Now, assume that some banks are not indebted. They are surely safe and the program $\mathcal{P}$ is equivalent to $\mathcal{P}'$ defined by considering the indebted banks only, which gives the values of their threat indices. It then suffices to set the threat indices of non-indebted banks equal to 0.

2. *The clearing ratio is non-decreasing and convex in* $\boldsymbol{z}$. Let $\boldsymbol{\theta}(\boldsymbol{z})$ denote the clearing ratio given $\boldsymbol{z}$. Recall that $\boldsymbol{\theta}(\boldsymbol{z})$ is the greatest element of the feasible set associated to $\boldsymbol{z}$. Let $\boldsymbol{z}'$ with $\boldsymbol{z}' \geq \boldsymbol{z}$. $\boldsymbol{\theta}(\boldsymbol{z})$ is feasible for $\boldsymbol{z}'$, hence is surely less than the greatest element of the feasible set associated to $\boldsymbol{z}'$, which is $\boldsymbol{\theta}(\boldsymbol{z}')$: This proves $\boldsymbol{\theta}(\boldsymbol{z}) \leq \boldsymbol{\theta}(\boldsymbol{z}')$. The convexity of $\boldsymbol{\theta}$ follows the same type of argument: Given two vectors $\boldsymbol{z}$ and $\boldsymbol{z}'$ and λ in $[0,1]$, the ratio $\lambda\boldsymbol{\theta}(\boldsymbol{z}) + (1-\lambda)\boldsymbol{\theta}(\boldsymbol{z}')$ is feasible for $\lambda\boldsymbol{z} + (1-\lambda)\boldsymbol{z}'$, thus lower than $\boldsymbol{\theta}(\lambda\boldsymbol{z} + (1-\lambda)\boldsymbol{z}')$.

The default set and the threat index are non-increasing in $\boldsymbol{z}$. The monotony of the default set follows from that of the ratio: Let $\boldsymbol{z}' \geq \boldsymbol{z}$. i in the default set D' associated to

$\boldsymbol{z}'$ satisfies $\theta_i(\boldsymbol{z}') < 1$, which implies $\theta_i(\boldsymbol{z}) < 1$, i.e. i belongs to the default set given $\boldsymbol{z}$: D' is a subset of D.

Consider $\boldsymbol{\mu}_D$ and $\boldsymbol{\mu}_{D'}$; they satisfy (8) for D and D'. Thus it suffices to show that $\boldsymbol{\pi}^{(p)}_{D'\times D'}(ij) \leq \boldsymbol{\pi}^{(p)}_{D\times D}(ij)$ for each i and j both in D'. $\boldsymbol{\pi}^{(p)}_{D\times D}(ij)$ is determined as follows: for each path of p elements in D linking i to j , compute the product of the π entries over the edges of the path; $\boldsymbol{\pi}^{(p)}_{D\times D}(ij)$ is the sum of these products over all the paths. $\boldsymbol{\pi}^{(p)}_{D'\times D'}(ij)$ is determined in the same way by considering the paths included in D' instead of D. There are less such paths since D' is a subset of D: the inequality $\boldsymbol{\pi}^{(p)}_{D\times D}(ij) \leq \boldsymbol{\pi}^{(p)}_{D\times D}(ij)$ follows.

3. *The sub-modularity of* V. Sub-modularity can be proved by considering vectors that differ in a single component j : Changing successively each component of $\boldsymbol{z}_{-i}$ to that of $\boldsymbol{z}'_{-i}$ produces the required inequality. The sub-modularity for two vectors $\boldsymbol{z}_i$ and $\boldsymbol{z}'_i$ with identical components $\boldsymbol{z}_{-ij}$ except for j writes:

$$V(z'_i, z'_j, \boldsymbol{z}_{-ij}) - V(z_i, z'_j, \boldsymbol{z}_{-ij}) \leq V(\boldsymbol{z}'_i, z_j, \boldsymbol{z}_{-ij}) - V(z_i, z_j, \boldsymbol{z}_{-ij}) \text{ for } z'_i \geq z_i, z'_j \geq z_j.$$

For a differentiable function V, the sub-modularity is satisfied if the partial derivative $\frac{\partial V}{\partial z_i}$ is non-increasing in z_j, as can be seen from the following expression:

$$V(z'_i, z_j, \boldsymbol{z}_{-ij}) - V(z_i, z_j, \boldsymbol{z}_{-ij}) = \int_{z_i}^{z'_i} \frac{\partial V}{\partial z_i}(t, z_j, \boldsymbol{z}_{-ij})dt. \tag{18}$$

Here V is not differentiable at all points but the partial derivative of V exists almost everywhere: $\frac{\partial V}{\partial z_i}(t, z_j, \boldsymbol{z}_{-ij})$ is given by the unique multiplier μ_i at $(t, z_j, \boldsymbol{z}_{-ij})$ when there is no boundary bank, which is the case but for a finite number of points as t runs in the interval (z_i, z'_i). Since V is continuous, the integral expression (18) is still valid. V is thus sub-modular if for each i the multiplier μ_i at $(t, z_j, \boldsymbol{z}_{-ij})$ is a non-decreasing function of z_j, for j distinct from i, given t and $\boldsymbol{z}_{-ij}$. This follows from point 2 since the vector $\boldsymbol{z}(t) = (t, z_j, \boldsymbol{z}_{-ij})$ increases as t increases. ∎

Proof of Proposition 2. The function V is concave and $\frac{\partial V}{\partial z_i} = \mu_i$ for each i at each $\boldsymbol{z}$ for which no bank is at the boundary. Since the marginal impact of cash injection towards i increases the value V of the payments within the system by μ_i units, cash should be allocated to the banks with the largest μ_i. As for the last statements, they follow from the fact that the threat index is constant and V is linear for all the cash values above $\boldsymbol{z}$ that lead to the same set D. ∎

Proof of Proposition 2 The proof relies on the computation of $\frac{\partial V}{\partial \ell_{ij}}$. Thanks to the envelope theorem, and assuming no boundary bank, the payment function V is differentiable with respect to ℓ_{ij} with a derivative given by

$$\frac{\partial V}{\partial \ell_{ij}} = \theta_i(1 - \mu_i + \mu_j). \tag{19}$$

The result follows. ∎

Proof of Proposition 5 There is nothing to prove if 1 and 2 are not indebted since then their indices are null. Let us assume that 1 and 2 are indebted.

Fix $\boldsymbol{z}_{-12}$. The proof uses two steps.

Step 1. We have $\theta_1(\boldsymbol{z}) \leq \theta_2(\boldsymbol{z})$ and $\mu_1(\boldsymbol{z}) \geq \mu_2(\boldsymbol{z})$ for $z_1 \leq \lambda z_2$; conversely, $\theta_1 \geq \theta_2$ and $\mu_1(\boldsymbol{z}) \leq \mu_2(\boldsymbol{z})$ for $z_1 \geq \lambda z_2$.

Let $e_i(\boldsymbol{\theta}, \boldsymbol{z})$ be i's equity at ratios $\boldsymbol{\theta}$ given $\boldsymbol{z}$. Using the proportionality of liabilities and loans, 1's equity value writes

$$\begin{aligned} e_1(\boldsymbol{z}, \boldsymbol{\theta}) &= z_1 + \lambda \sum_{j \geq 2} \theta_j \ell_{j2} + \lambda \theta_2 \pi_{1,2} - \lambda \theta_1 \ell_2^* & (20) \\ &= \lambda e_2(\boldsymbol{\theta}, \boldsymbol{z}) + z_1 - \lambda z_2 - \lambda(\theta_1 - \theta_2)\ell_2^*. & (21) \end{aligned}$$

Let $z_1 \leq \lambda z_2$. By contradiction, assume $\theta_1(\boldsymbol{z}) > \theta_2(\boldsymbol{z})$. The above equation implies $e_1(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z})) < \lambda e_2(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z}))$. However, since surely $\theta_2(\boldsymbol{z}) < 1$, 2 defaults and $e_2(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z})) = 0$. We thus obtain $e_1(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z})) < 0$, a contradiction. As for the indices, if 2 does not default, its index is null, so that surely $\mu_1(\boldsymbol{z}) \geq \mu_2(\boldsymbol{z})$. If 2 defaults, we have just shown that 1 surely defaults. Given that the liability shares of the two banks are equal, their indices are equal, hence again $\mu_1(\boldsymbol{z}) \geq \mu_2(\boldsymbol{z})$.

The proof is similar in the opposite case where $z_1 \geq \lambda z_2$ since we have not ued the fact that λ is larger than 1.

Step 2. Consider $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$ obtained from the clearing ratio $\boldsymbol{\theta}(\boldsymbol{z})$ for $\boldsymbol{z}$ by exchanging θ_1 and θ_2. For $z_1 \leq \lambda z_2$, $\boldsymbol{\theta}(\boldsymbol{z}^\sigma) \geq \boldsymbol{\theta}^\sigma(\boldsymbol{z})$ and $\boldsymbol{\mu}(\boldsymbol{z}^\sigma) \leq \boldsymbol{\mu}^\sigma(\boldsymbol{z})$, that is, the clearing ratio at $\boldsymbol{z}^\sigma$ is larger than the permuted ratio and the index is less than the permuted index.

Let $z_1 \leq \lambda z_2$. To show that the clearing ratio for $\boldsymbol{z}^\sigma$ is at least equal to $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$, it suffices to prove that the ratio $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$ is feasible given $\boldsymbol{z}^\sigma$, i.e., that the equities of all banks are non-negative. To simplify notation, let e_i denote i's equity given $\boldsymbol{z}$ and $\boldsymbol{\theta}$: $e_i = e_i(\boldsymbol{z}, \boldsymbol{\theta})$ and e_i' i's equity given $\boldsymbol{z}^\sigma$ and $\boldsymbol{\theta}^\sigma$: $e_i' = e_i(\boldsymbol{z}^\sigma, \boldsymbol{\theta}^\sigma)$. We have $e_i \geq 0$ for each i and want to prove $e_i\prime \geq 0$ for each i.

For the banks other than 1 or 2, their cash-flows are equal under $\boldsymbol{z}$ and $\boldsymbol{z}^\sigma$. Since their repayment ratios are equal under $\boldsymbol{\theta}(\boldsymbol{z})$ and $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$, equities e_i and e_i' only differ though 1 and 2's reimbursements. These reimbursements are equal to $(\lambda\theta_1(\boldsymbol{z})+\theta_2(\boldsymbol{z}))\ell_{21}$ at $\boldsymbol{\theta}(\boldsymbol{z})$ and to $(\lambda\theta_2(\boldsymbol{z})+\theta_1(\boldsymbol{z}))\ell_{2i}$ at $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$. Since we kow that $\theta_1(\boldsymbol{z}) \leq \theta_2(\boldsymbol{z})$ from Step 1 and $\lambda > 1$ by assumption, the latter are not smaller than the former, and the equities are in the same order: $e'_i \geq e_i$; this proves $e'_i \geq 0$ for each $i > 2$.

For banks 1 and 2, bank 1's equity at $\boldsymbol{z}^\sigma$ and $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$ and is λ times that of bank 2 at $\boldsymbol{z}$ and $\boldsymbol{\theta}(\boldsymbol{z})$ as can be checked from (20): $e'_1 = \lambda e_2$ and similarly bank 2's equity is $1/\lambda$ times that of bank 1, $e'_2 = e_1/\lambda$. Thus both e'_1 and e'_2 are non-negative.

We can now conclude. From Step 2, $(\mu_1(\boldsymbol{z}) - \mu_2(\boldsymbol{z}^\sigma))\mathbb{1}_{z_1 \leq \lambda z_2} \geq 0$ for each realization $\boldsymbol{z}_{-12}$, which implies $E[(\mu_1(\widetilde{\boldsymbol{z}}) - \mu_2(\widetilde{\boldsymbol{z}}^\sigma))\mathbb{1}_{\widetilde{z}_1 \leq \lambda \widetilde{z}_2}] \geq 0$. By the conditional independence assumption, $E[\mu_2(\widetilde{\boldsymbol{z}}^\sigma)\mathbb{1}_{\widetilde{z}_1 \leq \lambda \widetilde{z}_2}] = E[\mu_2(\widetilde{\boldsymbol{z}})\mathbb{1}_{\widetilde{z}_1 \geq \lambda \widetilde{z}_2}]$ We thus obtain the first inequality in (12) Similarly $E[(\mu_2(\widetilde{\boldsymbol{z}}) - \mu_1(\widetilde{\boldsymbol{z}}^\sigma))\mathbb{1}_{\widetilde{z}_1 \leq \lambda \widetilde{z}_2}] \geq 0$, and $E[\mu_1(\widetilde{\boldsymbol{z}}^\sigma)\mathbb{1}_{\widetilde{z}_1 \leq \lambda \widetilde{z}_2}] = E[\mu_1(\widetilde{\boldsymbol{z}})\mathbb{1}_{\widetilde{z}_1 \geq \lambda \widetilde{z}_2}]$ proves the second inequality.. ∎

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