CESifo Working Papers
www.cesifo.org/wp

Endogenous Capital- and Labor-Augmenting Technical Change in the Neoclassical Growth Model

Andreas Irmen
Amer Tabaković

CESifo Working Paper No. 5643
Category 6: Fiscal Policy, Macroeconomics and Growth
December 2015

ISSN 2364-1428

CESifo
Center for Economic Studies & Ifo Institute

CESifo *Working Paper No. 5643*

Endogenous Capital- and Labor-Augmenting Technical Change in the Neoclassical Growth Model

Abstract

The determinants of the direction of technical change and their implications for economic growth and economic policy are studied in the one-sector neoclassical growth model of Ramsey, Cass, and Koopmans extended to allow for endogenous capital- and labor-augmenting technical change. We develop a novel micro-foundation for the competitive production sector that rests on the idea that the fabrication of output requires tasks to be performed by capital and labor. Firms may engage in innovation investments that increase the productivity of capital and labor in the performance of their respective tasks. These investments are associated with new technological knowledge that accumulates over time and sustains long-run growth. We show that the equilibrium allocation is not Pareto-efficient since both forms of technical change give rise to an inter-temporal knowledge externality. An appropriate policy of investment subsidies may implement the efficient allocation.

JEL-Codes: O310, O330, O410.

Keywords: endogenous technical change, induced innovation, capital- and labor-augmenting technical change, neoclassical growth model.

*Andreas Irmen**
University of Luxembourg
CREA / Faculty of Law, Economics and Finance
162a, avenue de la Faïencerie
Luxembourg – 1511 Luxembourg
airmen@uni.lu

Amer Tabaković
University of Luxembourg
CREA / Faculty of Law, Economics and Finance
162a, avenue de la Faïencerie
Luxembourg – 1511 Luxembourg
amer.tabakovic@uni.lu

*corresponding author

This Version: November 30, 2015
We would like to thank Burkhard Heer, Martin Hellwig, Christos Koulovatianos, Thomas Seegmuller, Henri Sneessens, Gautam Tripathi, and Benteng Zou for useful comments and suggestions. Both authors gratefully acknowledge financial support from the University of Luxembourg under the program "Agecon C - Population Aging: An Exploration of its Effect on Economic Performance and Culture."

1 Introduction

Since its inception in the late 1980s modern growth theory has strongly emphasized the importance of endogenous technical change for our understanding of the differential growth performance of actual economies (see, e. g., Acemoglu (2009)). However, this theory almost always neglects the possibility of capital-augmenting technical change and, by design, focusses on the causes and the consequences of labor-augmenting technical change. Is this neglect benign? Do the positive implications and the policy recommendations of these models still hold in the presence of capital-augmenting technical change? Can policy recommendations still be justified on normative grounds? Moreover, what determines the direction of technical change? To address all these questions we introduce endogenous capital- and labor-augmenting technical change into the workhorse model of Dynamic Macroeconomics, namely, the competitive one-sector neoclassical growth model of Ramsey (1928), Cass (1965), and Koopmans (1965).

To render this possible, we devise a novel micro-foundation for a competitive production sector that builds on and complements concepts developed in the competitive endogenous growth models of Hellwig and Irmen (2001) and Irmen (2013a). It rests on the idea that the fabrication of output requires tasks to be performed. Some tasks are carried out by capital, others by labor. Innovation investments increase the productivity of capital and labor in the performance of their respective tasks. Identical, price-taking firms undertake these innovation investments in an attempt to maximize infra-marginal rents. In equilibrium, these rents cover factor costs as well as all investment outlays. Innovation investments are associated with new technological knowledge that accumulates over time. Inter-temporal knowledge spill-overs support sustained economic growth.

Our main new findings include the following. First, the key determinant of the direction of technical change is the relative scarcity of "efficient capital" with respect to "efficient labor" measured by the ratio of these two variables at the beginning of each period. This ratio determines relative factor prices and the relative profitability of innovation investments. For instance, if this ratio falls then efficient capital becomes relatively scarcer and the price of capital increases relative to the price of labor. Accordingly, an innovation investment enhancing the productivity of capital is more advantageous and the direction of technical change shifts towards capital. It is in this sense that our analysis provides a formal interpretation of Hicks' famous assertion according to which technical change is directed to economizing the use of a factor that has become relatively more expensive (Hicks (1932), pp. 124-125).

Second, along the transition towards the steady state, the growth rate of the economy reflects both capital- and labor-augmenting technical progress. However, in steady state capital-augmenting technical progress vanishes. Hence, in the long run, the growth rate of per-capita variables reflects only labor-augmenting technical change. The reason for this finding is closely related to the generalization of Uzawa's steady-state growth theorem formulated in Irmen (2013b). Roughly speaking, in steady state technical progress must be labor-augmenting since capital accumulates and the economy's net output function, which cannot be written in Cobb-Douglas form, exhibits constant returns to scale in capital and labor.

The third set of results relates to the positive properties of the steady state. We show that the steady state is a "balanced growth path" that satisfies all of Kaldor's famous stylized facts (Kaldor (1961)). Moreover, the steady-state growth rate of all per-capita variables is predicted to increase in parameters capturing the positive effect of institutions, technical infrastructure, or geography on the efficiency of the production sector. However, other parameters that often bring about steady-state growth effects in models that feature only labor-augmenting technical change such as the discount factor of the representative household, the size and the growth rate of the population have no impact on the steady-state growth rate. The mere feasibility of capital-augmenting technical change is shown to be the reason for this. Due to its presence, the steady-state growth rates of capital- and labor-augmenting technical change are determined by the properties of the production sector alone.

Fourth, we analyze the local stability properties of the steady state and establish saddle-path stability in the state space. Interestingly, this finding does not hinge on the elasticity of substitution between efficient capital and efficient labor. The relative scarcity measured by the ratio of efficient capital to efficient labor is a key stabilizing force. In steady state, this ratio and, therewith, the direction of technical change are constant. A small, one-time shock that lowers this ratio renders efficient capital relatively scarcer and shifts the direction of technical change towards more capital-augmenting and less labor-augmenting technical progress. This adjustment and the concomitant effect of capital accumulation tend to reduce the relative scarcity of efficient capital and move the economy back towards its steady state.

Fifth, we study the effect of different fiscal policies on the steady-state growth rate. First, we consider a linear tax on capital. We find that the steady-state growth rate of the economy is unaffected by the tax. This reflects the fact that due to the presence of capital-augmenting technical change this growth rate is determined within the production sector. Second, we study the consequences of a policy that pays a subsidy to either capital- or labor-augmenting innovation investments. We find that both kinds of subsidy increase the steady-state growth rate of the economy. However, while the subsidy to labor-augmenting innovation investments induces a direct effect on investment incentives, the subsidy to capital-augmenting innovation investments increases the steady-state growth rate through a general equilibrium effect.

Finally, we conduct a welfare analysis. We find that the equilibrium allocation is not Pareto efficient. Innovation investments in a given period increase the stock of knowledge that is available for subsequent innovative activity and create inter-temporal knowledge spill-overs which are not taken into account by private firms. A benevolent social planner may choose a policy of investment subsidies that implements the Pareto efficient allocation. We fully characterize the policy that implements the efficient steady state. It involves a subsidy to both kinds of innovation investments, however, at different rates.

The present paper contributes to at least three strands of the modern literature linking technical change to economic growth. First, it is related to the endogenous technical change literature initiated by Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992). These contributions and those that emanated from them focus on endogenous labor-augmenting technical change.[1] The present paper shows that the determination of the steady-state growth rate is very different in the presence of capital-augmenting technical change. In particular, it no longer hinges on the interplay between the household side and the production side. We clarify this point in Section 3.4. As a consequence, parameters, e. g., the discount factor, or policies, e. g., a tax on capital, that appear in the household's Euler equation will no longer affect the steady-state growth rate.

Second, this paper makes a contribution to the modern literature on endogenous capital- and labor-augmenting technical change (see, e. g., Funk (2002), Acemoglu (2003b), Irmen (2011), and Irmen (2013a)). This research has its roots in the so-called induced innovations literature of the 1960s (see, e. g., von Weizsäcker (1962), Kennedy (1964), Samuelson (1965), or Drandakis and Phelps (1966)).

In contrast to the literature of the 1960s, the present paper depicts capital- and labor-augmenting technical change as resulting from a well-defined profit-maximization problem solved by price-taking firms. The solution to this problem gives rise to a (possibly convex) *equilibrium* innovation possibility frontier that replaces the exogenous and concave innovation possibility frontier stipulated in the old literature. Moreover, in our

[1] Observe that the research sector in Romer (1990) invents new varieties of capital goods. It is in this sense that technical change is capital-augmenting in his setting. However, as Romer's aggregate production function is of the Cobb-Douglas type, the distinction between capital-augmenting and labor-augmenting becomes meaningless. Technical change can always be represented as labor-augmenting. This is a direct consequence of Uzawa's theorem (see, Uzawa (1961) and Irmen (2013b) for applications to models of endogenous technical change).

framework technical change is not costless but has a cost in terms of current output.[2]

In addition, the present paper gives a different answer to the question about whether and why technical change is eventually only labor-augmenting. In the literature of the 1960s and in Funk (2002) or Acemoglu (2003b) the answer hinges on the elasticity of substitution between capital and labor being greater or smaller than unity. This elasticity determines the (local) stability of the balanced growth path because innovation incentives hinge on the factor shares in final-good production. To the contrary, our steady state is locally stable irrespective of the elasticity of substitution.[3] As argued above, local stability is due to innovation incentives that hinge on relative factor prices which, in turn, respond to changes in the relative scarcity of factors.[4]

Finally, it is worth mentioning that we contribute the first welfare analysis to the literature on endogenous capital- and labor-augmenting technical change. Due to two intertemporal externalities the competitive equilibrium allocation is not Pareto-efficient. In spite of the intricate equilibrium interdependency between capital- and labor-augmenting technical change, we establish the subsidy rates that implement the Pareto-efficient steady state.

The third strand of the literature to which we contribute depicts endogenous technical change in competitive economies (see, e. g., Zeira (1998), Hellwig and Irmen (2001), Boldrin and Levine (2002), or Boldrin and Levine (2008)). The present paper shows that capital- and labor-augmenting technical change can arise endogenously in the competitive neoclassical growth model. As we detail in Section 2.2.3 competitive firms maximize infra-marginal rents which in equilibrium cover their factor costs and investment outlays. This provides another counter-example to the commonly held view that perfect competition is incompatible with endogenous innovation investments (see, e. g., Romer (2015)).

The remainder of this paper is organized as follows. Section 2 presents the model. In particular, we detail the micro-foundation of the competitive production sector and jus-

---

[2] Arguably, with these properties the present paper overcomes the main weaknesses of the literature of the 1960s that according to Nordhaus (1973), Burmeister and Dobell (1970), Funk (2002), or Acemoglu (2003a) include (i) the arbitrary optimization problem solved by firms to determine the endogenous growth rate, (ii) the ad hoc assumption of an exogenous Kennedy-von Weizsäcker Innovation Possibilities Frontier, and (iii) the fact that technical progress is costless in terms of real resources.

[3] Klump, McAdam, and Willman (2007) find empirical evidence for persistent exponential labor-augmenting technical change and positive, but declining rates of capital-augmenting technical change using US data for the period between 1953 and 1998. This pattern is consistent with near steady-state behavior under local stability.

[4] Local stability also obtains in the models of Irmen (2011) and Irmen (2013a) where the savings rate is either exogenously given or derived for two-period lived individuals.

tify why innovation investments are possible in our perfectly competitive economy. Section 3 studies the dynamic competitive equilibrium. Its definition is given and explained in Section 3.1. Section 3.2 sets up the canonical dynamical system. The steady state analysis is presented in Section 3.3. Section 3.4 clarifies the role of capital-augmenting technical change for our findings. The focus of Section 4 is on the positive implications of fiscal policies. Policies considered include a linear taxation of capital (Section 4.1), a subsidy to capital-augmenting innovation investments (Section 4.2), and a subsidy to labor-augmenting innovation investments (Section 4.3). Section 5 discusses the normative implications of endogenous capital- and labor-augmenting technical change. Here, we study the choices of a benevolent planner (Section 5.1), solve for the optimal steady-state allocation (Section 5.2), and show that it can be implemented with an appropriate choice of subsidies to innovation investments (Section 5.3). Section 6 concludes. All proofs are contained in Section 7, the Appendix.

2 The Model

Consider a competitive closed economy in an infinite sequence of periods $t = 0, 1, 2, ..., \infty$. The economy consists of a household sector and a production sector. In each period there is a single *final good* that can be consumed or invested. If invested, it may either become future capital or an input in contemporaneous innovation investments that raise the productivity of capital or labor. Households supply *labor* and *capital*. Each period has a market for all three objects of exchange. The final good serves as numéraire.

2.1 Households

The economy is populated by a single representative household comprising one member.[5] In each period, the household cares about the level of consumption, C_t, and supplies inelastically the labor endowment, L.

The per-period utility function is logarithmic, i. e., $u(C_t) = \ln C_t$. Moreover, the household evaluates sequences of consumption $\{C_t\}_{t=0}^{\infty}$ according to

$$\sum_{t=0}^{\infty} \beta^t \ln C_t, \tag{2.1}$$

[5]For ease of exposition we make a few simplifying assumptions that are, however, innocuous with respect to our main qualitative results. They include a constant household size, i. e., there is no population growth, an inelastic labor supply, and logarithmic utility. We discuss in detail the role of these assumptions in the concluding Section 6.

where $0 < \beta < 1$ is a discount factor. The household owns all firms and the capital stock. Since profits, i. e., dividends, vanish in equilibrium, we do not explicitly account for the profit distribution. Capital is the only asset in the economy. Capital at t, K_t, is installed at $t-1$, and firms pay a real rental rate, R_t, per unit of capital they work with. After use the capital stock decays at rate $\delta^K \in [0,1]$. Accordingly, the household's flow budget constraint at t is given by

$$K_{t+1} = \left(R_t + 1 - \delta^K\right) K_t + w_t L - C_t, \tag{2.2}$$

where w_t is the real wage.

Given $K_0 > 0$ and $L > 0$, the representative household maximizes (2.1) subject to (2.2), $C_t \geq 0$, $K_{t+1} \geq 0$, and an appropriate no-Ponzi game condition by choosing a sequence $\{C_t\}_{t=0}^{\infty}$. By standard arguments, the solution to this problem satisfies for all t the flow budget constraint (2.2), the Euler condition,

$$\frac{C_{t+1}}{C_t} = \beta \left(R_{t+1} + 1 - \delta^K\right), \tag{2.3}$$

and the transversality condition

$$\lim_{t\to\infty} \beta^t C_t^{-1} K_{t+1} = 0. \tag{2.4}$$

2.2 The Production Sector

The production sector has a continuum of identical, competitive firms of measure one. Without loss of generality, the analysis proceeds through the lens of a single representative firm.

2.2.1 Technology

To produce output two types of tasks need to be performed. The first type needs capital, the second labor as the only input. Denote by $m \in \mathbb{R}_+$ a task performed by capital, and let $n \in \mathbb{R}_+$ be a task performed by labor. Further, let M_t and N_t denote the measure of all tasks of the respective type performed at time t so that $m \in [0, M_t]$ and $n \in [0, N_t]$. Tasks of the respective type are identical. Therefore, total output hinges only on M_t and N_t. The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ assigns the maximum output, Y_t, to each pair $(M_t, N_t) \in \mathbb{R}_+^2$, i. e.,

$$Y_t = F(M_t, N_t), \tag{2.5}$$

where F has constant returns to scale in its arguments and is $\mathcal{C}^2$ on $\mathbb{R}_{++}$ with $F_1 > 0 > F_{11}$ and $F_2 > 0 > F_{22}$.[6] Let κ_t denote the period-t task intensity,

$$\kappa_t = \frac{M_t}{N_t}. \tag{2.6}$$

Then, output in intensive form is $F(\kappa_t, 1) \equiv f(\kappa_t)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$, with $f'(\kappa_t) > 0 > f''(\kappa_t)$ for all $\kappa_t > 0$.

At t, a task m requires $k_t(m) = 1/b_t(m)$ units of capital whereas a task n needs $l_t(n) = 1/a_t(n)$ units of labor. Hence, $b_t(m)$ and $a_t(n)$ denote the productivity of capital and labor in the respective task. The levels of productivity are given by

$$\begin{aligned} b_t(m) &= B_{t-1}\left(1-\delta^B\right)\left(1+q_t^B(m)\right), \\ a_t(n) &= A_{t-1}\left(1-\delta^A\right)\left(1+q_t^A(n)\right). \end{aligned} \tag{2.7}$$

Here, $B_{t-1}\left(1-\delta^B\right)$ and $A_{t-1}\left(1-\delta^A\right)$ represent stocks of technological knowledge that the firm inherits from the previous period. More precisely, B_{t-1} and A_{t-1} denote the respective stocks of technological knowledge used in the production sector at period $t-1$, and $\delta^j \in (0,1)$, $j = A, B$, is the rate of depreciation of the respective knowledge stock. Finally, $\left(q_t^B(m), q_t^A(n)\right) \in \mathbb{R}_+^2$ are indicators of productivity growth at t associated with task m and n, respectively.

To fix ideas, one may think of $\delta^B > 0$ as capturing the loss in technical functionality that comes along with the physical depreciation of capital at rate $\delta^K > 0$. In the same vein, depreciation at rate $\delta^A > 0$ makes sense if, e. g., some specific knowledge required to master the labor-augmenting technology represented by A_{t-1} gets lost due to labor turnover.

To achieve positive productivity growth, i. e., $q^j > 0$, $j = A, B$, the firm must engage in an innovation investment. More precisely, at t it must invest $i\left(q_t^B(m)\right) > 0$ units of the final good to achieve $q_t^B(m) > 0$ and, similarly, $i\left(q_t^A(n)\right) > 0$ units of the final good to obtain $q_t^A(n) > 0$.

The function $i : \mathbb{R}_+ \to \mathbb{R}_+$ is the same for all tasks, time invariant, $\mathcal{C}^2$ on $\mathbb{R}_{++}$, strictly increasing and strictly convex in q. Moreover, it satisfies the following regularity conditions for $j = A, B$:

$$i(0) = 0, \quad \lim_{q^j \to 0} i'\left(q^j\right) = 0, \quad \lim_{q^j \to \infty} i'\left(q^j\right) = \lim_{q^j \to \infty} i\left(q^j\right) = \infty. \tag{2.8}$$

[6] Throughout, we denote derivatives of functions with several arguments using subscripts. For functions of a single argument we use either primes or subscripts. Hence, the first derivative of $f(x)$ is either $f'(x)$ or $f_x(x)$, its second derivative is either $f''(x)$ or $f_{xx}(x)$, and so on.

Any new piece of technological knowledge is proprietary knowledge of a particular firm only in the period when it occurs. Subsequently, it becomes public and embodied in aggregate economy-wide productivity indicators $(A_t, B_t), (A_{t+1}, B_{t+1}), \ldots$. The details will be specified below.[7]

2.2.2 Firm Behavior

The representative firm takes the sequence $\{R_t, w_t, A_{t-1}, B_{t-1}\}_{t=0}^{\infty}$ of real rental rates of capital, real wages, and aggregate productivity indicators as given. Its choice involves a production plan comprising a sequence

$$\left\{M_t, N_t, k_t(m), l_t(n), q_t^B(m), q_t^A(n)\right\}_{t=0}^{\infty}$$

for $m \in [0, M_t]$ and $n \in [0, N_t]$, respectively. This plan maximizes the sum of the present discounted values of profits in all periods. Because an innovation investment generates proprietary knowledge only in the period when it is made, the inter-temporal profit maximization problem of the firm boils down to the maximization of per-period profits given by

$$F(M_t, N_t) - TC_t, \tag{2.9}$$

where TC_t is the firm's total cost, comprising factor costs and investment outlays for all performed tasks, i. e.,

$$\begin{aligned} TC_t &= \int_0^{M_t} \left[R_t k_t(m) + i\left(q_t^B(m)\right)\right] dm \\ &+ \int_0^{N_t} \left[w_t l_t(n) + i\left(q_t^A(n)\right)\right] dn. \end{aligned} \tag{2.10}$$

Here,

$$\begin{aligned} k_t(m) &= \frac{1}{B_{t-1}\left(1-\delta^B\right)\left(1+q_t^B(m)\right)}, \\ l_t(n) &= \frac{1}{A_{t-1}\left(1-\delta^A\right)\left(1+q_t^A(n)\right)} \end{aligned} \tag{2.11}$$

are the respective input coefficients.

---

[7] If at t the firm makes no investment in a productivity enhancing technology, it nevertheless has access to the economy-wide technology represented by $B_{t-1}\left(1-\delta^B\right)$ and $A_{t-1}\left(1-\delta^A\right)$. Then, its task-specific productivity of capital and labor is given by $b_t(m) = B_{t-1}\left(1-\delta^B\right)$ and $a_t(n) = A_{t-1}\left(1-\delta^A\right)$. However, since $\lim_{q^j \to 0} i'\left(q^j\right) = 0$ the option not to invest will not be chosen in equilibrium.

The maximization of (2.9) can be split up into two parts. First, for each $m \in [0, M_t]$ and $n \in [0, N_t]$ the choices of $q_t^B(m)$ and $q_t^A(n)$ minimize TC_t, i. e., they minimize the cost of each task. This leads to the first-order (sufficient) conditions

$$q_t^B(m): \quad \frac{-R_t}{B_{t-1}\left(1-\delta^B\right)\left(1+q_t^B(m)\right)^2} + i'(q_t^B(m)) = 0, \tag{2.12}$$

$$q_t^A(n): \quad \frac{-w_t}{A_{t-1}\left(1-\delta^A\right)\left(1+q_t^A(n)\right)^2} + i'(q_t^A(n)) = 0. \tag{2.13}$$

Intuitively, for each task, faster productivity growth means lower factor costs. At the margin, this advantage is equal to the required additional investment outlays. In light of (2.8), and assuming $R_t > 0$ and $w_t > 0$, the conditions (2.12) and (2.13) determine unique values $q_t^B(m) = q_t^B > 0$ and $q_t^A(n) = q_t^A > 0$ to which we refer as the cost-minimizing growth rates of productivity.

Combining respectively (2.12) and (2.13) with (2.11) delivers minimized factor costs as $R_t k_t = \left(1+q_t^B\right) i'\left(q_t^B\right)$ and $w_t l_t = \left(1+q_t^A\right) i'\left(q_t^A\right)$. Let $c\left(q_t^B\right)$ and $c\left(q_t^A\right)$ denote the minimized costs per task featuring factor costs and investment outlays, i. e.,

$$c\left(q_t^B\right) = \left(1+q_t^B\right) i'\left(q_t^B\right) + i\left(q_t^B\right), \tag{2.14}$$

$$c\left(q_t^A\right) = \left(1+q_t^A\right) i'\left(q_t^A\right) + i\left(q_t^A\right). \tag{2.15}$$

Accordingly, total cost of (2.10) boils down to

$$TC_t = M_t c\left(q_t^B\right) + N_t c\left(q_t^A\right). \tag{2.16}$$

Second, the firm determines how many tasks of either type to perform. Using (2.16), it solves

$$\max_{(M_t, N_t) \in \mathbb{R}_+^2} F(M_t, N_t) - M_t c\left(q_t^B\right) - N_t c\left(q_t^A\right). \tag{2.17}$$

The respective first-order (sufficient) conditions are

$$M_t \quad : \quad f'\left(\kappa_t\right) = c\left(q_t^B\right), \tag{2.18}$$

$$N_t \quad : \quad f\left(\kappa_t\right) - \kappa_t f'\left(\kappa_t\right) = c\left(q_t^A\right). \tag{2.19}$$

Hence, for the marginal task of each type the marginal value product is equal to the minimized cost per task.

Equations (2.12), (2.13), (2.18), and (2.19) fully characterize the equilibrium behavior of the representative firm at all t. The following proposition shows that these profit-maximizing conditions allow us to express in an intuitive way the productivity growth rates $\left(q_t^B, q_t^A\right)$ in terms of κ_t and the corresponding factor prices (R_t, w_t) in terms of κ_t, B_{t-1}, and A_{t-1}.

Proposition 1 *Suppose equations (2.12), (2.13), (2.18), and (2.19) are satisfied. Then, the following holds:*

1. *There are maps $g^A : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ and $g^B : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ such that*

$$q_t^B = g^B(\kappa_t) \quad \text{with} \quad g_\kappa^B(\kappa_t) < 0, \tag{2.20}$$

$$q_t^A = g^A(\kappa_t) \quad \text{with} \quad g_\kappa^A(\kappa_t) > 0. \tag{2.21}$$

2. *There are maps $R : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ and $w : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ such that the real rental rate of capital and the real wage satisfy*

$$R_t = R\left(\kappa_t, B_{t-1}\right) \quad \text{with} \quad R_\kappa\left(\kappa_t, B_{t-1}\right) < 0 \quad \text{and} \quad R_B\left(\kappa_t, B_{t-1}\right) > 0, \tag{2.22}$$

$$w_t = w\left(\kappa_t, A_{t-1}\right) \quad \text{with} \quad w_\kappa\left(\kappa_t, A_{t-1}\right) > 0 \quad \text{and} \quad w_A\left(\kappa_t, A_{t-1}\right) > 0. \tag{2.23}$$

The intuition for Claim 1 can be gained from the first-order conditions (2.18) and (2.19). Roughly speaking, the functions $g^B(\kappa_t)$ and $g^A(\kappa_t)$ exist since $c(q_t^B)$ and $c(q_t^A)$ are strictly increasing on their respective domain. Moreover, an increase in κ_t has the following simultaneous effects. First, the marginal value product of task M_t falls and so must q_t^B. Second, the marginal value product of task N_t increases and so will q_t^A. Hence, $g_\kappa^A(\kappa_t) > 0 > g_\kappa^B(\kappa_t)$.

To understand Claim 2 consider the first-order conditions (2.12) and (2.13). Solving these conditions for the respective factor price replacing q_t^B by $g^B(\kappa_t)$ and q_t^A by $g^A(\kappa_t)$ delivers $R\left(\kappa_t, B_{t-1}\right)$ and $w\left(\kappa_t, A_{t-1}\right)$. Recall that an increase in κ_t implies a smaller q_t^B and a greater q_t^A. To make these adjustments consistent with the minimization of the cost of each task, R_t must decrease in (2.12) to reduce the marginal advantage associated with an innovation investment in capital-augmenting technology. Similarly, w_t must increase in (2.13) so that the marginal advantage associated with an innovation investment in labor-augmenting technology increases. Accordingly, $R_\kappa\left(\kappa_t, B_{t-1}\right) < 0$ and $w_\kappa\left(\kappa_t, A_{t-1}\right) > 0$. Finally, to support some given $q_t^B = g^B(\kappa_t)$ condition (2.12) requires R_t to increase in B_{t-1}

to keep the ratio R_t/B_{t-1} constant. Similarly, to support some given $q_t^A = g^A(\kappa_t)$ condition (2.13) requires w_t to increase in A_{t-1} to keep the ratio w_t/A_{t-1} constant. Hence, $R_B(\kappa_t, B_{t-1}) > 0$ and $w_A(\kappa_t, A_{t-1}) > 0$.

The following corollary shows that Proposition 1 implies an equilibrium innovation possibility frontier and an equilibrium factor price frontier.

Corollary 1 *Suppose equations (2.12), (2.13), (2.18), and (2.19) are satisfied. Then, the following holds:*

1. *There is an* equilibrium innovation possibility frontier $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ *such that*

$$q_t^A = g\left(q_t^B\right) \quad \textit{with} \quad g'\left(q_t^B\right) < 0. \tag{2.24}$$

2. *There is an* equilibrium factor price frontier $h : \mathbb{R}_{++}^3 \to \mathbb{R}_{++}$ *such that*

$$R_t = h\left(w_t, A_{t-1}, B_{t-1}\right) \quad \textit{with} \quad h_w\left(w_t, A_{t-1}, B_{t-1}\right) < 0. \tag{2.25}$$

According to Claim 1 of Corollary 1 profit-maximization implies a functional relationship between both cost-minimizing productivity growth rates. We refer to this relationship as the *equilibrium* innovation possibility frontier (EIPF) to distinguish this concept from the *exogenous* innovation possibility frontier stipulated by the induced innovations literature of the 1960s: in the present model, this frontier is endogenous as it results from the profit-maximizing choices of firms.

To develop a heuristic for the EIPF it proves useful to take a look at the dual of the profit-maximization problem (2.17). Accordingly, suppose the firm seeks to find the pair (M_t, N_t) that minimizes total costs, TC_t, of (2.16) for a quantity of output at least equal to Y_t. Then, the firm solves

$$\min_{(M_t,N_t)\in\mathbb{R}_+^2} M_t c\left(q_t^B\right) + N_t c\left(q_t^A\right) \quad \text{s.t.} \quad F\left(M_t, N_t\right) \geq Y_t. \tag{2.26}$$

The solution to this problem delivers "conditional demand functions",

$$M_t = M\left(c\left(q_t^B\right), c\left(q_t^A\right)\right) Y_t \quad \text{and} \quad N_t = N\left(c\left(q_t^B\right), c\left(q_t^A\right)\right) Y_t.$$

Linearity in Y_t follows since F has constant returns to scale. Plugging these functions into the objective function of (2.26) delivers the cost function

$$TC\left(c\left(q_t^B\right), c\left(q_t^A\right), Y_t\right) = tc\left(c\left(q_t^B\right), c\left(q_t^A\right)\right) Y_t, \tag{2.27}$$

where $tc\left(c\left(q_t^B\right), c\left(q_t^A\right)\right)$ is the minimum cost per unit of Y_t. From Euler's law firms earn zero-profits. Hence, it must hold that

$$tc\left(c\left(q_t^B\right), c\left(q_t^A\right)\right) = 1. \tag{2.28}$$

The latter condition defines the EIPF implicitly. Hence, the EIPF may be seen as an equilibrium constraint on $\left(q_t^B, q_t^A\right)$ resulting from the zero-profit condition of a cost-efficient firm operating under constant returns to scale. As $tc\left(\cdot,\cdot\right)$ as well as $c\left(q_t^B\right)$ and $c\left(q_t^A\right)$ are time-invariant, so is the EIPF. Moreover, as $tc\left(\cdot,\cdot\right)$ is strictly increasing in both arguments, $c'\left(q_t^B\right) > 0$, and $c'\left(q_t^A\right) > 0$, it follows that the slope of the EIPF is negative, i. e., $g'\left(q_t^B\right) < 0$.

The flip side of the EIPF is the equilibrium factor price frontier (EFPF) of Claim 2 of Corollary 1. It also reflects the firm's zero-profit condition (2.28), however, now in the space of factor prices.[8] To see this, consider condition (2.12) which pins down q_t^B as a function of $\left(R_t, B_{t-1}\left(1-\delta^B\right)\right)$. We call this function $z_t^B = z^B\left(R_t, B_{t-1}\left(1-\delta^B\right)\right)$. Similarly, one finds that condition (2.13) pins down q_t^A as a function of $\left(w_t, A_{t-1}\left(1-\delta^A\right)\right)$, and we call this function $z_t^A = z^A\left(w_t, A_{t-1}\left(1-\delta^A\right)\right)$. Substitution of q_t^B by z_t^B and of q_t^A by z_t^A in (2.28) delivers the EFPF. As the relevant partial derivatives are strictly positive, i. e., $z_R^B\left(R_t, B_{t-1}\left(1-\delta^B\right)\right) > 0$ and $z_w^A\left(w_t, A_{t-1}\left(1-\delta^A\right)\right) > 0$, it follows that $h_w\left(w_t, A_{t-1}, B_{t-1}\right) < 0$. Pairs of factor prices consistent with the EFPF depend on A_{t-1} and B_{t-1} and, therefore, will change over time.

The following example shows that the position and the shape of the EIPF will be determined in an intuitive way by parameters that capture geographical, technical, or institutional properties of the economy in which firms operate. Moreover, the EIPF turns out to be a convex function which contrasts with the (strictly) concave innovation possibility frontier stipulated by the induced innovation literature of the 1960s.[9]

Example 1 *Suppress time arguments and let*

$$F\left(M,N\right) = \Gamma \cdot M^{\alpha} \cdot N^{1-\alpha} \quad \textit{and} \quad i^j\left(q^j\right) = \gamma^j \cdot \frac{\left(q^j\right)^2}{2}, \quad \gamma^j > 0, \quad j = A, B.$$

The parameter, $\Gamma > 0$ may reflect cross-country differences in geography, technical and social infrastructure that affect the transformation of tasks into the final good. Here, we allow for innovation outlays to differ across task types, i. e., to achieve $q^B(m) > 0$ the firm must invest

[8]This line of reasoning is familiar from the factor price frontier introduced by Samuelson (1962).

[9]Concavity is necessary in this literature to turn the frontier into a suitable constraint for the maximization of the instantaneous rate of output growth (see, e. g., Burmeister and Dobell (1970), Chapter 3, for details).

$\gamma^B\left(q^B(m)\right)^2/2$ units of the final good whereas to achieve $q^A(n) > 0$ the firm must invest $\gamma^A\left(q^A(n)\right)^2/2$ units and $\gamma^A \neq \gamma^B$ is permissible.

Then, the cost-minimizing productivity growth rates (2.20) and (2.21) of Proposition 1 are equal to

$$q^B = g^B(\kappa) = \frac{1}{3}\left(-1+\sqrt{1+\frac{6\Gamma\alpha}{\gamma^B}\kappa^{\alpha-1}}\right),$$

$$q^A = g^A(\kappa) = \frac{1}{3}\left(-1+\sqrt{1+\frac{6\Gamma(1-\alpha)}{\gamma^A}\kappa^{\alpha}}\right).$$

As expected, $g^A_\kappa(\kappa) > 0 > g^B_\kappa(\kappa)$. Moreover, both productivity growth rates increase in Γ and decline in the cost parameters γ_B or γ_A.

The time-invariant equilibrium innovation possibility frontier is equal to

$$q^A = \frac{1}{3}\left(-1+\sqrt{1+\frac{6\Gamma(1-\alpha)}{\gamma^A}\left(\frac{2\Gamma\alpha}{\gamma^B q^B\left(3q^B+2\right)}\right)^{\frac{\alpha}{1-\alpha}}}\right).$$

Hence, given q^B, a greater Γ and lower values for the cost parameters γ_B and γ_A imply a greater q^A. Some tedious but straightforward algebra also reveals that the EIPF is indeed strictly convex.

2.2.3 Perfect Competition and the Cost of Innovation Investments

It is frequently argued that perfect competition is inconsistent with innovation investments, hence with endogenous economic growth (see, e. g., Romer (2015)). The argument is that perfectly competitive firms will not engage in such investment activities because they have no way to recover the costs of investment. The purpose of this section is to show that this opinion is untenable. Indeed, we establish that the firms of the competitive production sector outlined above compete for infra-marginal rents. Moreover, their choices are meant to maximize these rents.

Without loss of generality we focus on the representative firm. The rent that this firm earns from performing tasks with capital is[10]

$$\int_0^{M_t}\left[\frac{\partial F\left(m,N_t\right)}{\partial m} - R_t k_t(m) - i\left(q_t^B(m)\right)\right]dm. \tag{2.29}$$

[10]Mutatis mutandis, an analogous argument holds for the rent earned on tasks performed by labor.

This expression sums up the differences between the marginal value product and the cost of all M_t tasks. At the intensive margin, this rent is maximized if for all $m \in [0, M_t]$ the first-order condition (2.12) holds. Then, $q_t^B(m) = q_t^B$ and (2.29) becomes

$$\int_0^{M_t} \left[\frac{\partial F(m, N_t)}{\partial m} - c\left(q_t^B\right) \right] dm. \tag{2.30}$$

At the extensive margin, (2.30) is maximized when M_t takes on the value that is consistent with the first-order condition (2.18). The resulting configuration is illustrated in Figure 2.1. The shaded area shows the infra-marginal rent on all tasks performed by capital. Obviously, the firm earns a strictly positive rent on all but the marginal task, M_t. The shaded area corresponds to the maximum of (2.30) and is equal to

$$F(M_t, N_t) - F(0, N_t) - M_t c\left(q_t^B\right). \tag{2.31}$$

The term $F(0, N_t)$ will be strictly positive unless tasks performed by capital constitute an essential input. Conceptually, it is subtracted here since (2.31) is meant to state the rent earned on tasks performed by capital. However, $F(0, N_t) > 0$ constitutes the part of $F(M_t, N_t)$ that can be fully attributed to the N_t tasks performed by labor and is, thus, unrelated to the performance of any task by capital. Nevertheless, this correction has the flavor of a bookkeeping convention since the *gross* rent that the firm actually earns on tasks performed by capital is

$$F(M_t, N_t) - M_t c\left(q_t^B\right). \tag{2.32}$$

Observe that profits of (2.17) exhibit constant returns to scale in (M_t, N_t). Therefore, conditions (2.18) and (2.19) in conjunction with Euler's law imply that maximum firm profits are zero. Hence, profit-maximization implies that the gross rent of (2.32) is equal to the total cost incurred from tasks performed by labor, i. e.,

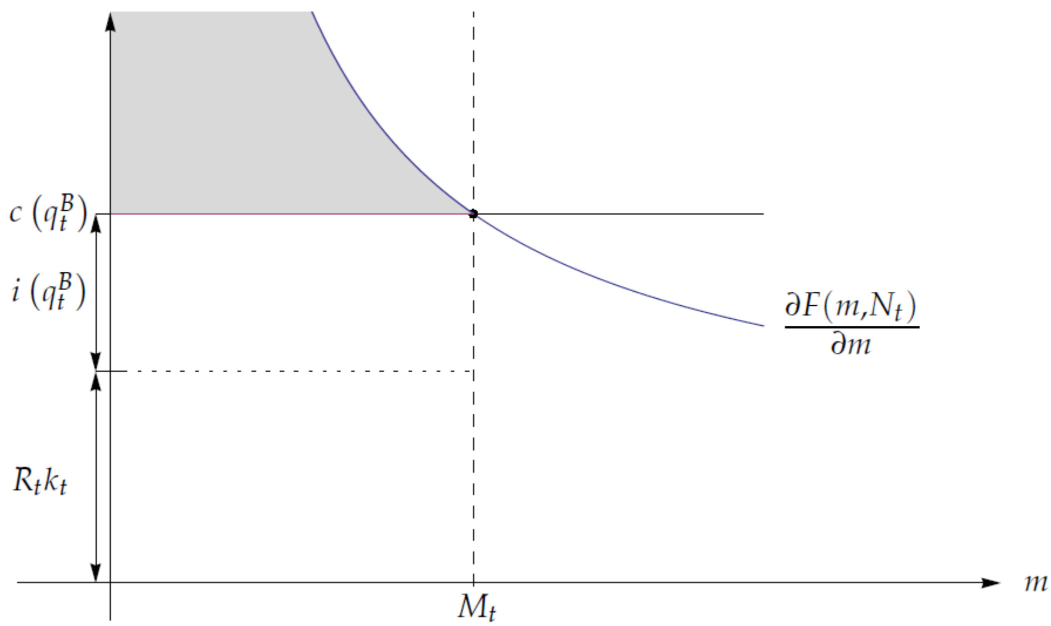
$$F(M_t, N_t) - M_t c\left(q_t^B\right) = N_t c\left(q_t^A\right). \tag{2.33}$$

In sum, there is a rent earned on each infra-marginal task performed by capital since the marginal value product of each infra-marginal task is greater than the minimized cost of performing it. Moreover, profit-maximization implies that the sum of these rents gross of $F(0, N_t)$ just covers the entire cost of performing N_t tasks by labor.

2.2.4 The Evolution of Technological Knowledge

The evolution of the technological knowledge to which firms have access is given by the evolution of the aggregate task-specific productivity indicators A_t and B_t. We stipulate

Figure 2.1: Inframarginal Rents on Tasks Performed by Capital.



that A_t and B_t correspond to the highest level of labor productivity and capital productivity attained across all tasks of the respective type at t, i. e.,

$$A_t = \max\left\{a_t(n) = A_{t-1}\left(1-\delta^A\right)\left(1+q_t^A(n)\right) \mid n \in [0, N_t]\right\},$$
$$B_t = \max\left\{b_t(m) = B_{t-1}\left(1-\delta^B\right)\left(1+q_t^B(m)\right) \mid m \in [0, M_t]\right\}. \tag{2.34}$$

Firm's optimization implies $q_t^B(m) = q_t^B$ and $q_t^A(n) = q_t^A$, as well as $a_t(n) = a_t$ and $b_t(m) = b_t$ so that

$$A_t = a_t = A_{t-1}\left(1-\delta^A\right)\left(1+q_t^A\right),$$
$$B_t = b_t = B_{t-1}\left(1-\delta^B\right)\left(1+q_t^B\right), \tag{2.35}$$

for all $t = 0, 1, 2, \cdots$ with $A_{-1} > 0$ and $B_{-1} > 0$ given.

3 Dynamic Competitive Equilibrium

3.1 Definition

The dynamic competitive equilibrium is defined as follows.

Definition 1 *Given $L_t = L$, initial values of the capital stock, $K_0 > 0$, and of technological knowledge, $A_{-1} > 0$ and $B_{-1} > 0$, a dynamic competitive equilibrium is a sequence*

$$\left\{M_t, N_t, k_t(m), l_t(n), q_t^B(m), q_t^A(n), A_t, B_t, w_t, R_t, C_t, K_{t+1}, Y_t\right\}_{t=0}^{\infty},$$

for all $m \in [0, M_t]$ and $n \in [0, N_t]$, such that

(E1) the behavior of the representative household is described by (2.3) and (2.4).

(E2) the production sector satisfies Proposition 1,

(E3) for all t, both factor markets clear, i. e.,

$$\int_0^{M_t} k_t(m)dm \leq K_t \quad and \quad \int_0^{N_t} l_t(n)dn \leq L,$$

each holding as equality if the corresponding factor price is strictly positive,

(E4) for all t, the market for the final good clears,

(E5) the productivity indicators A_t and B_t evolve according to equation (2.35).

Condition $(E1)$ requires household optimization while $(E2)$ ensures optimal behavior of firms and zero profits. Since equilibrium factor prices will be strictly positive, there will be full employment of capital and labor. Moreover, profit-maximization implies $k_t(m) = k_t = 1/b_t = 1/B_t$ and $l_t(n) = l_t = 1/a_t = 1/A_t$. Hence, condition $(E3)$ determines the total number of each task type to be equal to the amount of the respective production factor in efficiency units, i. e.,

$$M_t = B_t K_t \quad \text{and} \quad N_t = A_t L_t. \tag{3.1}$$

Let V_t denote the economy's *net output* at t defined as the difference between total output of the final good, the investment outlays for all tasks performed by capital, $M_t i\left(q_t^B\right)$, and the investment outlays for all tasks performed by labor, $N_t i\left(q_t^A\right)$. Then,

$$V_t = F(M_t, N_t) - M_t i\left(q_t^B\right) - N_t i\left(q_t^A\right). \tag{3.2}$$

Accordingly, the market clearing condition for the market of the final good, (E4), requires

$$K_{t+1} = V_t - C_t + \left(1 - \delta^K\right) K_t, \tag{3.3}$$

i. e., next period's capital stock is equal to the surviving capital plus the difference between net output, V_t, and consumption, C_t.

Finally, observe that $(E2)$, $(E3)$, and $(E5)$, imply that in equilibrium the task intensity of equation (2.6), may be expressed as

$$\kappa_t = \frac{B_t K_t}{A_t L} = \frac{B_{t-1}\left(1-\delta^B\right)\left(1+g^B(\kappa_t)\right)K_t}{A_{t-1}\left(1-\delta^A\right)\left(1+g^A(\kappa_t)\right)L}. \tag{3.4}$$

Thus, in equilibrium the task intensity is equal to the ratio of efficient capital to efficient labor, or, for short, to the *efficient capital intensity*. Since innovations are induced, the respective efficiency units depend on the task intensity. The following proposition establishes that there is a unique value $\kappa_t > 0$ that satisfies (3.4). We refer to this value as the *equilibrium task intensity*. To simplify the notation let us introduce

$$\Theta_t \equiv \frac{B_{t-1}\left(1-\delta^B\right)K_t}{A_{t-1}\left(1-\delta^A\right)L}. \tag{3.5}$$

Proposition 2 *There is a unique* equilibrium task intensity $\kappa_t > 0$ *that satisfies (3.4). Moreover, there is a function* $\kappa : \mathbb{R}_{++} \to \mathbb{R}_{++}$ *such that*

$$\kappa_t = \kappa\left(\Theta_t\right) \quad \textit{with} \quad \kappa'\left(\Theta_t\right) > 0.$$

The ratio Θ_t has an interpretation as the efficient capital intensity of period t *before any investment activity is undertaken*. It is the correct measure of the relative scarcity of factors of production at t to which firms' investment behavior responds. In line with Hicks' famous assertion (Hicks (1932), p. 124) this ratio induces the degree to which firms will engage in capital- and labor-augmenting technical change. To see this, suppose the economy enters period t with $\Theta_t > \Theta_{t-1}$. Then, (efficient) labor has become scarcer between period $t-1$ and t. Moreover, as $\kappa'\left(\Theta_t\right) > 0$ we have $\kappa_t > \kappa_{t-1}$, and, in accordance with Proposition 1, $q_t^A > q_{t-1}^A$, $q_t^B < q_{t-1}^B$, $w_t > w_{t-1}$, and $R_t < R_{t-1}$.

3.2 The Canonical Dynamical System

The canonical dynamical system comprises two state variables, the equilibrium task intensity, κ_t, and the stock of capital-augmenting technological knowledge, B_t, and one control variable, the level of consumption per unit of efficient labor, $c_t \equiv C_t / \left(A_t L\right)$. The following proposition has the complete description of this system. To simplify the notation, we denote by $v_t \equiv V_t / \left(A_t L\right)$ the net output per unit of efficient labor at t. Using the two factor market clearing conditions of (3.1) and replacing $\left(q_t^B, q_t^A\right)$ by $\left(g^B\left(\kappa_t\right), g^A\left(\kappa_t\right)\right)$ in accordance with Proposition 1 one obtains

$$v_t = v\left(\kappa_t\right) \equiv f\left(\kappa_t\right) - \kappa_t i\left(g_t^B\left(\kappa_t\right)\right) - i\left(g_t^A\left(\kappa_t\right)\right), \tag{3.6}$$

where $v : \mathbb{R}_{++} \to \mathbb{R}_{++}$.

Proposition 3 *(Canonical Dynamical System)*

Given $L > 0$ and initial conditions $(A_{-1}, B_{-1}, K_0) > 0$, the transitional dynamics of the dynamic competitive equilibrium is given by a unique sequence $\{\kappa_t, c_t, B_t\}_{t=0}^{\infty}$ that satisfies

$$\kappa_{t+1} = \frac{\left(1-\delta^B\right)\left(1+g^B(\kappa_{t+1})\right)}{\left(1-\delta^A\right)\left(1+g^A(\kappa_{t+1})\right)} \cdot \left[B_t\left(v\left(\kappa_t\right)-c_t\right)+\left(1-\delta^K\right)\kappa_t\right], \tag{3.7}$$

$$\frac{c_{t+1}}{c_t} = \beta \cdot \frac{B_t\left(1-\delta^B\right)\left(1+g^B(\kappa_{t+1})\right)\left(f'\left(\kappa_{t+1}\right)-i\left(g^B\left(\kappa_{t+1}\right)\right)\right)+\left(1-\delta^K\right)}{\left(1-\delta^A\right)\left(1+g^A(\kappa_{t+1})\right)}, \tag{3.8}$$

$$B_t = B_{t-1}\left(1-\delta^B\right)\left(1+g^B\left(\kappa_t\right)\right), \tag{3.9}$$

the transversality condition

$$\lim_{t\to\infty} \beta^t \frac{\kappa_{t+1}\left(1+g^A\left(\kappa_{t+1}\right)\right)}{c_t B_{t+1}} = 0, \tag{3.10}$$

and for $t = 0$,

$$\kappa_0 = \kappa\left(\Theta_0\right). \tag{3.11}$$

Intuitively, equation (3.7) reflects the economy's resource constraint (3.3). It is obtained using (3.1), $K_t = (A_t L/B_t)\,\kappa_t$, $C_t = A_t L c_t$, (2.35), and Claim 1 of Proposition 1. Equation (3.8) restates the Euler equation (2.3) using (2.12), (2.18), $C_t = A_t L c_t$, (2.35) and Claim 1 of Proposition 1. Finally, equation (3.9) states the evolution of capital-augmenting technological knowledge. It obtains from (2.35) and Claim 1 of Proposition 1. In conjunction with the transversality condition and the set of initial conditions, these equations form a three-dimensional system of first-order, non-linear difference equations and characterize a unique sequence $\{\kappa_t, c_t, B_t\}_{t=0}^{\infty}$.

To develop an intuitive understanding of the mechanics of the dynamical system start with the initial conditions. They deliver κ_0 from equation (3.11). Using this and $B_{-1} > 0$ in (3.9) gives a unique $B_0 > 0$. The resource constraint describes a relation between c_0 and κ_1 for any given pair $(\kappa_0, B_0) \in \mathbb{R}^2_{++}$. For any such pair the transversality condition pins down the initial choice of consumption c_0. Then, the resource constraint delivers a unique $\kappa_1 > 0$, whereas the Euler equation determines a unique c_1. Mutatis mutandis, the same reasoning applies to all periods $t > 0$. Hence, (κ_t, B_t) are indeed the state variables of the canonical dynamical system.

3.3 Steady State Analysis

Definition 2 *A steady state is a path along which all variables grow at constant, but possibly different rates.*

We use an asterisk to denote steady-state variables, e. g., g^* is the steady-state growth rate of the economy.

3.3.1 Existence

To guarantee the existence of a steady state with strictly positive and finite state variables we make the following two assumptions:

Assumption 1 *It holds that*

$$\lim_{\kappa \to 0} f'(\kappa) > c\left(\frac{\delta^B}{1-\delta^B}\right) > \lim_{\kappa \to \infty} f'(\kappa). \tag{3.12}$$

Assumption 2 *It holds that*

$$1-\delta^A > \beta\left(1-\delta^K\right). \tag{3.13}$$

The discussion of the following proposition will reveal the significance of these assumptions.

Proposition 4 *(Steady State)*

1. *The dynamical system of Proposition 3 has a unique steady state involving $(\kappa^*, B^*, c^*) \in \mathbb{R}^3_{++}$ if Assumptions 1 and 2 hold. The steady state is a solution to*

$$c^* = v(\kappa^*) - \frac{\kappa^*}{B^*}\left(g^* + \delta^K\right), \tag{3.14}$$

$$B^* = \frac{(1+g^*) - \beta\left(1-\delta^K\right)}{\beta\left(f'(\kappa^*) - i\left(g^B(\kappa^*)\right)\right)}, \tag{3.15}$$

$$g^B(\kappa^*) = \frac{\delta^B}{1-\delta^B}. \tag{3.16}$$

2. *The steady-state growth rate of the economy is*

$$g^* \equiv \frac{A_{t+1}}{A_t} - 1 = \left(1 - \delta^A\right)\left(1 + g^A\left(\kappa^*\right)\right) - 1.$$

Moreover, along the steady-state path, it holds that

a) $\frac{Y_{t+1}}{Y_t} = \frac{V_{t+1}}{V_t} = \frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{M_{t+1}}{M_t} = \frac{N_{t+1}}{N_t} = \frac{w_{t+1}}{w_t} = 1 + g^*,$

b) $R^* = B^*\left(f'\left(\kappa^*\right) - i\left(\frac{\delta^B}{1-\delta^B}\right)\right), \quad k^* = \frac{1}{B^*}, \quad \frac{l_{t+1}}{l_t} = \frac{1}{1+g^*}.$

According to Statement 1 of Proposition 4 there is a unique steady state if Assumptions 1 and 2 hold. To develop the explanation for this finding, start with the evolution of capital-augmenting technological knowledge of (3.9). Accordingly, any trajectory with $B_t/B_{t-1} - 1 = const.$ requires $\kappa_t = \kappa_{t+1} = \kappa^*$. Intuitively, the level of κ^* must be such that profit-maximizing firms undertake innovation investments that generate new capital-augmenting technological knowledge just enough to offset its depreciation. Invoking Proposition 1 this is the case if $\left(q^B\right)^* = g^B\left(\kappa^*\right) = \delta^B/(1-\delta^B)$ as stated in (3.16).

Assumption 1 assures that (3.16) has a solution $\kappa^* \in \mathbb{R}_{++}$. To see this, observe that a choice of $\left(q^B\right)^*$ means that (minimized) costs per task performed by capital are equal to $c\left(\left(q^B\right)^*\right) = c\left(\delta^B/(1-\delta^B)\right)$. Hence, for tasks $m < M_t$ the marginal value product, $f'(\kappa)$, must exceed, for $m > M_t$ it must fall short of these costs. This is what condition (3.12) guarantees.[11]

In steady state, the Euler equation makes sure that the household's desired consumption growth rate is equal to the growth rate of the economy. As discussed below, the latter satisfies $(1+g^*) = \left(1-\delta^A\right)\left(1+g^A\left(\kappa^*\right)\right)$. Then, using (2.22) and the fact that κ^* is determined by (3.16), the Euler equation pins down B^* as a solution to $(1+g^*) = \beta\left(R\left(\kappa^*, B\right) + 1 - \delta^K\right)$. Assumption 2 provides a sufficient condition for a solution $B^* > 0$ to exist. Intuitively, it assures that the numerator in (3.15) is strictly positive even if κ^* and $g^A\left(\kappa^*\right)$ are very small which is the case if δ^B is very large.

Finally, given (κ^*, B^*), the resource constraint (3.14) determines a finite $c^* > 0$ as the difference between net output per unit of efficient labor and the required capital investment per unit of efficient labor needed to keep κ^* constant.

Statement 2 of Proposition 4 establishes that the steady-state growth rate of the economy is given by the growth rate of labor-augmenting technological knowledge. Absent

[11]Obviously, Assumption 1 is also necessary for $\kappa^* \in \mathbb{R}_{++}$ to exist. Notice that a value $\kappa^* \in (0, \infty)$ would always exist if we had imposed the usual Inada conditions on F.

population growth, this is the growth rate of final-good output, net output, consumption, capital, the respective total number of tasks, and of the real wage. The steady-state rental rate of capital is constant. Moreover, the input coefficients of capital are constant, whereas the one of labor hinges on g^*.

Why is it that in steady state only the stock of labor-augmenting technological knowledge evolves whereas the one of capital-augmenting technological knowledge stagnates? The conceptual answer to this question provides the "Generalized Steady-State Growth Theorem" developed in Irmen (2013b). This theorem generalizes Uzawa's Theorem (Uzawa (1961)) to settings where technical change has a cost in terms of current final-good output. Roughly speaking, it says that an economy where capital accumulates and the equilibrium net output function has constant returns to scale in capital and labor, technical change must be labor-augmenting in steady state whereas capital-augmenting technical change must vanish. With full employment of capital and labor as stated in (3.1) net output of (3.2) becomes $V_t = F(B_t K_t, A_t L_t) - B_t K_t i\left(q_t^B\right) - A_t L_t i\left(q_t^A\right)$ and satisfies this property. Hence, in steady state, $B_t = B^*$ and A_t evolves at rate g^*.

Observe that the steady-state growth rate of the economy may be negative, i. e., $g^* \leq 0$. Intuitively, this is the case if $g^A(\kappa^*) \leq \delta^A / \left(1 - \delta^A\right)$ which may be satisfied if κ^* is small due to a large δ^B. However, as $B^* > 0$ is required, the Euler equation (3.15) implies a lower bound on steady-state growth rate, i. e., $g^* > \beta(1 - \delta^K) - 1$. Finally, observe that Assumption 2 is sufficient (but not necessary) for $B^* > 0$.

3.3.2 Comparative Statics

Proposition 5 *(Comparative Statics of the Steady State)*

1. *Consider two economies that differ only with respect to their discount factor such that $\beta' > \beta$. Then, their steady states satisfy*

$$\begin{aligned} (\kappa^*)' &= \kappa^*, \quad (g^*)' = g^*, \quad (B^*)' < B^*, \\ (R^*)' &< R^*, \quad (c^*)' < c^*. \end{aligned} \tag{3.17}$$

2. *Consider two economies that differ only with respect to their depreciation rate of the stock of capital-augmenting technological knowledge such that $\left(\delta^B\right)' > \delta^B$. Then, their steady states satisfy*

$$\begin{aligned} (\kappa^*)' &< \kappa^*, \quad (g^*)' < g^*, \quad (B^*)' < B^*, \\ (R^*)' &< R^*, \quad (c^*)' \lesseqgtr c^*. \end{aligned} \tag{3.18}$$

3. *Consider two economies that differ only with respect to their depreciation rate of the stock of labor-augmenting technological knowledge such that* $\left(\delta^A\right)' > \delta^A$. *Then, their steady states satisfy*

$$\begin{aligned} (\kappa^*)' &= \kappa^*, \quad (g^*)' < g^*, \quad (B^*)' < B^*, \\ (R^*)' &< R^*, \quad (c^*)' > c^*. \end{aligned} \tag{3.19}$$

Arguably, Statement 1 has the most important result of Proposition 4. The steady-state growth rate of the economy, g^*, does not hinge on characteristics of the household sector. Intuitively, this is so since the discount factor neither interferes with the incentives to engage in innovation investments as summarized by the function g^B nor does it directly affect the evolution of the stocks of technological knowledge. However, differences in the discount factor have level effects. The more patient economy is predicted to have the lower steady-state level of capital-augmenting technological knowledge, or, equivalently, a lower rental rate of capital. This follows from the steady-state Euler equation (3.15) that requires both economies to have consumption grow at the same rate $(g^*)' = g^*$. Then, a greater discount factor must be entirely offset by a decline in the rental rate of capital. This is accomplished through a decline in capital-augmenting technological knowledge. Finally, with $(B^*)' < B^*$ the resource constraint (3.14) requires the more patient economy to reduce its consumption per unit of efficient labor. Intuitively, since κ^* is the same in both economies it must be that $\left((K_t/A_tL)^*\right)' > (K_t/A_tL)^*$, i. e., capital per efficient labor is greater in the more patient economy. Therefore, more current output is needed to keep this ratio constant. Accordingly, $(c^*)' < c^*$.

Statements 2 and 3 highlight that differences in the depreciation rate of the stock of factor-augmenting technological knowledge generate growth effects. First, consider the steady-state effects of changing δ^B. A higher depreciation rate of capital-augmenting technological knowledge requires stronger private incentives to engage in innovation investments that raise the productivity of capital. Hence, in line with Proposition 1 and (3.16) the efficient capital intensity must fall, i. e., $(\kappa^*)' < \kappa^*$. This weakens the private incentives to engage in innovation investments that raise the productivity of labor so that $(g^*)' < g^*$. As consumption must grow at the latter rate the Euler condition requires $(R^*)' < R^*$. This adjustment implies $(B^*)' < B^*$. Finally, the effect on the steady-state consumption per unit of efficient labor remains indeterminate in general.

Second, consider changes in δ^A of Statement 3. As these changes leave (3.16) unaffected, we have $(\kappa^*)' = \kappa^*$, and $(g^*)' < g^*$ is due to faster depreciation of the stock of labor-augmenting technological knowledge. As above, slower growth of consumption requires adjustments in the Euler equation and the resource constraint. They lead to $(B^*)' < B^*$, $(R^*)' < R^*$, and $(c^*)' > c^*$.

Additional and intuitive comparative static results are obtained if we impose more structure.

Example 2 *Reconsider the setup of Example 1, and suppose that Assumption 2 holds. Then, a unique steady state exists and involves*

$$\kappa^* = \left(\frac{2\Gamma\alpha\tilde{\delta}^B}{\gamma^B} \right)^{\frac{1}{1-\alpha}}, \tag{3.20}$$

where $\tilde{\delta}^B = \left(1-\delta^B\right)^2 / \left(\delta^B \left(2+\delta^B\right)\right)$*.*

To interpret this equation recall that κ^* *is the task intensity necessary to sustain innovation investments such that* $\left(q^B\right)^* = \delta^B / \left(1-\delta^B\right)$*. Moreover, from Proposition 1 a higher* κ *reduces* q^B*. Then, it is straightforward to see why* κ^* *increases in the productivity parameter* Γ *and decreases in the cost parameter* γ^B *or in the depreciation rate* δ^B*. Since* $g^* = \left(1-\delta^A\right)\left(1+g^A\left(\kappa^*\right)\right) - 1$ *we arrive at*

$$g^* = \frac{\left(1-\delta^A\right)}{3} \left(2 + \sqrt{1 + \frac{6\Gamma(1-\alpha)}{\gamma^A} \left(\frac{2\Gamma\alpha\tilde{\delta}^B}{\gamma^B} \right)^{\frac{\alpha}{1-\alpha}}} \right) - 1. \tag{3.21}$$

Here, Γ *exerts two positive effects on* g^**. First, there is a direct effect since innovation incentives are higher the more productive the aggregate production function is. Second, there is a general equilibrium effect since* κ^* *also increases. A greater* γ^A *or* δ^A *has a direct negative effect on* g^**, greater values for* γ^B *or* δ^B *reduce* g^* *through negative general equilibrium effects on* κ^**.*

Finally, let us note that the steady state of Proposition 4 is consistent with Kaldor's facts if $g^* > 0$ (see, Kaldor (1961)). Indeed, one readily verifies that the productivity of labor, measured either by a_t, V_t/L, or Y_t/L and capital per worker, K_t/L, grow at rate $g^* > 0$. Moreover, the capital coefficient in aggregate output, K_t/Y_t, or in aggregate net output, K_t/V_t, and the return on capital are stable. Since in steady state $R_t = R^*$, the factor shares are also stable. Hence, the steady state is consistent with Kaldor's facts.

3.3.3 Local Stability

Proposition 6 *(Local Stability of the Steady State)*

The steady-state equilibrium of Proposition 4 is asymptotically locally stable in the state space.

To establish Proposition 6 the nonlinear dynamical system of Proposition 3 is linearly approximated about the steady state (κ^*, c^*, B^*). Given $L > 0$ and the initial values $(A_{-1}, B_{-1}, K_0) > 0$ the state of the economy in period t is fully described by the two state variables κ_t and B_t, so that c_t is the only control variable. Hence, given the two state variables we need a stable eigenspace of dimension two for the dynamical system to exhibit a unique convergent path toward the steady state. The proof of Proposition 6 shows that the linearized dynamical system has two stable eigenvalues and one unstable eigenvalue. The convergence toward the steady state may be monotonic or oscillatory.

To gain intuition for Proposition 6 consider the case of a monotonic convergence. Initially, the economy is in its steady-state equilibrium (κ^*, c^*, B^*). Suppose that at the beginning of some period t an exogenous event destroys a part of its capital-augmenting technological knowledge so that $B'_{t-1} < B^*$. Then, compared to the steady state Θ of (3.5) falls. In other words, at the beginning of period t efficient capital is relatively scarcer than along the steady-state path. Accordingly, from Proposition 2 we have $\kappa'_t < \kappa^*$, and with Proposition 1 it follows that $\left(q_t^B\right)' > \left(q^B\right)^*$ and $\left(q_t^A\right)' < \left(q^A\right)^*$. This leads to $B'_{t-1} < B'_t < B^*$ and $A'_t < A^*$. Hence, the immediate effect of induced technical change is to partly offset the initial loss in capital-augmenting technological knowledge. For the periods that follow, $\kappa'_t < \kappa^*$ triggers a process of capital accumulation so that the sequences $\left\{\kappa'_{t+i}\right\}_{i=1}^{\infty}$ and $\left\{B'_{t+i}\right\}_{i=1}^{\infty}$ monotonically converge to κ^* and B^*, respectively.

In contrast to the existing literature on endogenous capital- and labor-augmenting technical change, the local stability of the steady-state equilibrium does not require the elasticity of substitution of the production function to be less than unity.[12] Indeed, Proposition 6 holds irrespective of the elasticity of substitution. To highlight this point consider the following numerical example. Notice in passing that here the elasticity of substitution in the aggregate production function plays a role for the type of convergence to the steady-state equilibrium and also for the steady-state growth rate.

Example 3 *Reconsider the economy described in Example 1. We now choose the following parameter values:* $\Gamma = \gamma^A = \gamma^B = 1$, $\alpha = 1/3$, $\delta^B = 1/4$, $\delta^A = 1/4$, $\beta = 0.99$, *and* $\delta^K = 0.06$. *Furthermore, we allow for the production function to be of the CES-type, i. e.,*

$$F(M,N) = \begin{cases} \Gamma\left(\alpha M^{\frac{\sigma-1}{\sigma}} + (1-\alpha) N^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} & \text{if} \quad \sigma \neq 1, \\ \\ \Gamma \cdot M^{\alpha} \cdot N^{1-\alpha} & \text{if} \quad \sigma = 1, \end{cases}$$

[12] The contributions of the induced innovations literature that involve capital accumulation (e. g., von Weizsäcker (1962), Drandakis and Phelps (1966), Samuelson (1966)), and the more recent analysis of Acemoglu (2003b) all require an elasticity of substitution smaller than unity for the stability of the steady state. This result is driven by the fact that the direction of technical change hinges on the factor shares.

where $\sigma > 0$ is the elasticity of substitution between the two types of tasks. We emphasize that the aim of our numerical example is not to calibrate the model, but rather to show the qualitative results.

Table 1 shows the eigenvalues of the dynamical system of Proposition 3 linearized around the steady state of Proposition 4 for various values of the elasticity of substitution, σ. For different values of σ the convergence to the steady-state equilibrium may be monotonic or oscillatory. It is worth emphasizing that $\sigma = 1$ does not represent a critical value below which convergence is necessarily monotonic and above which it is oscillatory. Notice also that, naturally, the variables do not need to converge at the same speed to their steady-state levels. Finally, observe that the steady-state growth rate increases in the elasticity of substitution thus confirming the analytical finding established in Irmen (2011).

Table 1: Elasticity of Substitution of the Production Function, Local Stability of the Steady State, and Steady-State Growth.

σ	eigenvalues			$g^*(\%)$
0.25	1.07941,	0.949732,	0.561591	0.51
0.5	1.09029,	0.939364,	0.71533	0.94
0.75	1.10048,	0.926904,	0.790202	1.31
1	1.11007,	0.908942,	0.841108	1.64
1.25	1.11917,	0.884550 ± 0.0280296i		1.92
1.5	1.12791,	0.890272 ± 0.0448385i		2.18
1.75	1.13637,	0.893694 ± 0.0535632i		2.41
2	1.14464,	0.895618 ± 0.0593213i		2.61

3.4 The Role of Capital-Augmenting Technical Change

What is the role of capital-augmenting technical change for the results derived so far? To address this question we contrast the model of the previous sections with a version that altogether dispenses with capital-augmenting technical change. To accomplish this, reconsider the model of Section 2 for $B_t = b_t = k_t = 1$, $q_t^B = \delta^B = i\left(q_t^B\right) = 0$, and $M_t = K_t$. Then, the efficient capital intensity of (3.4) boils down to $\kappa_t = K_t / \left(A_t L\right)$. Mutatis mutandis, the latter ratio also satisfies Proposition 2 where now $\Theta_t \equiv K_t / \left(A_{t-1}\left(1-\delta^A\right) L\right)$.

With these changes, the dynamical system of Proposition 3 reduces to a two-dimensional system of non-linear first-order difference equations involving one state variable, κ_t, and one control variable, c_t. These difference equations include the resource constraint and the Euler condition.

Proposition 7 *(Canonical Dynamical System Without Capital-Augmenting Technical Change)*

Given $L > 0$ and initial conditions $(A_{-1}, K_0) > 0$, the transitional dynamics of the dynamic competitive equilibrium is given by a unique sequence $\{\kappa_t, c_t\}_{t=0}^{\infty}$ that satisfies

$$\kappa_{t+1} = \frac{v(\kappa_t) - c_t + \left(1 - \delta^K\right)\kappa_t}{(1 - \delta^A)\left(1 + g^A(\kappa_{t+1})\right)}, \tag{3.22}$$

$$\frac{c_{t+1}}{c_t} = \beta \cdot \frac{f'(\kappa_{t+1}) + 1 - \delta^K}{(1 - \delta^A)\left(1 + g^A(\kappa_{t+1})\right)}, \tag{3.23}$$

the transversality condition

$$\lim_{t \to \infty} \beta^t \frac{\kappa_{t+1}\left(1 + g^A\left(\kappa_{t+1}\right)\right)}{c_t} = 0, \tag{3.24}$$

and for $t = 0$,

$$\kappa_0 = \kappa\left(\Theta_0\right). \tag{3.25}$$

From Definition 2 a steady state involves $A_{t+1}/A_t - 1 = const$. Hence, with (2.35) and Proposition 1 we have $\kappa_t = \kappa_{t+1} = \kappa^*$ and $\left(q^A\right)^* = g^A\left(\kappa^*\right)$. Moreover, from (3.22) we obtain $c_t = c_{t+1} = c^*$. To guarantee the existence of a steady state with $\kappa^* \in \mathbb{R}_{++}$, we replace Assumptions 1 and 2 by

Assumption 3 *It holds that*

$$\lim_{\kappa \to 0} f'(\kappa) > \frac{1 - \delta^A}{\beta} - \left(1 - \delta^K\right) > \lim_{\kappa \to \infty} f'(\kappa). \tag{3.26}$$

The significance of Assumption 3 will become clear below.

Proposition 8 *(Steady State Without Capital-Augmenting Technical Change)*

1. *The dynamical system of Proposition 7 has a unique steady state involving $(\kappa^*, c^*) \in \mathbb{R}^2_{++}$ if Assumption 3 holds. The steady state is a solution to*

$$c^* = v(\kappa^*) - \kappa^*\left(g^* + \delta^K\right), \tag{3.27}$$

$$1 + g^* = \beta\left(f'\left(\kappa^*\right) + 1 - \delta^K\right). \tag{3.28}$$

2. *The steady-state growth rate of the economy is*

$$g^* \equiv \frac{A_{t+1}}{A_t} - 1 = \left(1 - \delta^A\right)\left(1 + g^A\left(\kappa^*\right)\right) - 1.$$

Moreover, along the steady-state path, it holds that

$$a) \qquad \frac{Y_{t+1}}{Y_t} = \frac{V_{t+1}}{V_t} = \frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{N_{t+1}}{N_t} = \frac{w_{t+1}}{w_t} = 1 + g^*,$$

$$b) \qquad R^* = f'\left(\kappa^*\right), \quad \frac{l_{t+1}}{l_t} = \frac{1}{1+g^*}.$$

The comparison of the steady state with and without capital-augmenting technical change (Proposition 4 versus Proposition 8) reveals two important structural differences. First, in a world without capital-augmenting technical change the steady-state capital-labor ratio in efficiency units is pinned down by the Euler condition (3.28). Hence, κ^* is such that the desired growth rate of consumption (and of the economy as a whole), g^*, is supported by the steady-state rental rate of capital, $R^* = f'(\kappa^*)$. If Assumption 3 holds then this alignment has a solution $\kappa^* > 0$ which will naturally depend on β.

On the contrary, in the world with capital-augmenting technical change, the role of κ^* is to induce innovation investments so that the stock of capital-augmenting technological knowledge remains constant over time. The necessary adjustments to meet this requirement reflect only the characteristics of the production sector including the way how technological knowledge accumulates. As a consequence condition (3.16) is independent of β.

Second, the comparative statics with respect to β and δ^A of the steady state of Proposition 8 involve adjustments in κ^*. In contrast, for the steady state with capital-augmenting technical change of Proposition 4 changes in the same parameters induce adjustments in B^*. The following proposition documents the resulting differences.

Proposition 9 *(Steady-State Comparative Statics Without Capital-Augmenting Technical Change)*

1. *Consider two economies that differ only with respect to their discount factor such that $\beta' > \beta$. Then, their steady states satisfy*

$$(\kappa^*)' > \kappa^*, \quad (g^*)' > g^*, \quad (R^*)' < R^*, \quad \textit{and} \quad (c^*)' \gtreqless c^*. \tag{3.29}$$

2. *Consider two economies that differ only with respect to their depreciation rate of the stock of capital-augmenting technological knowledge such that $\left(\delta^A\right)' > \delta^A$. Then, their steady states satisfy*

$$(\kappa^*)' > \kappa^*, \quad (g^*)' < g^*, \quad (R^*)' < R^*, \quad \textit{and} \quad (c^*)' \gtreqless c^*. \tag{3.30}$$

The comparison between Proposition 9 and Claim 1 and 3 of Proposition 5 highlights two important differences. First, without capital-augmenting technical change the more patient economy grows faster in steady state. Intuitively, this economy saves more and, as a result ends up with a greater efficient capital intensity. According to Proposition 1 this induces faster labor-augmenting technical change, hence, faster steady-state growth. Second, the negative effect on the steady-state growth rate of the economy associated with a greater depreciation rate of labor-augmenting technological knowledge is partly offset by an increase in κ^*.

4 Positive Implications: Fiscal Policy

This section establishes the effect of three different fiscal policy measures on the steady-state growth rate. First, we study the effects of a linear tax on the return to capital. Second, we analyze the role of subsidies for innovation investments that increase the productivity of capital. Finally, we turn to subsidies for innovation investments that increase the productivity of labor. We follow the literature and assume that the government redistributes its tax revenues in a lump-sum fashion to balance its budget. Similarly, it finances its subsidies through a lump-sum tax. Accordingly, none of the considered policies affects the household's flow budget constraint (2.2).

4.1 A Linear Tax on Capital

Suppose the government levies a tax on capital such that the net after-tax rate of return per unit of capital at t is $(1-\tau)\left(R_t-\delta^K\right)$, $\tau \in [0,1)$. Then, the Euler condition (2.3) becomes $C_{t+1}/C_t = \beta\left((1-\tau)\left(R_{t+1}-\delta^K\right)+1\right)$, i. e., from the household's point of view, the relative price of consumption tomorrow increases in τ.

However, such a tax does not affect the steady-state growth rate of the economy. According to Proposition 4, the latter is equal to the growth rate of labor-augmenting technological knowledge and hinges on $\left(q^A\right)^* = g^A(\kappa^*)$. As κ^* is determined by (3.16) it reflects only the production side of the economy. Accordingly, the tax on capital leaves the economy's steady-state growth rate unchanged.

4.2 A Subsidy to Capital-Augmenting Innovation Investments

Suppose the government pays a subsidy $\sigma^B i\left(q_t^B(m)\right)$ for all innovation investments that raise the productivity of capital at t. Here, $\sigma^B \in [0,1)$ is the subsidy rate. The subsidy reduces the (minimized) cost per task performed by capital and, therefore, renders innovation investments more attractive. This shows up in the conditions for profit-maximization that now involve

$$f'(\kappa_t) = \left(1 - \sigma^B\right) c\left(q_t^B\right). \tag{4.1}$$

The latter generalizes (2.18) to the case where $\sigma^B > 0$ and implicitly defines

$$q_t^B = g^B(\kappa_t, \sigma^B), \quad \text{with} \quad g_\kappa^B(\kappa_t, \sigma^B) < 0 \quad \text{and} \quad g_{\sigma^B}^B(\kappa_t, \sigma^B) > 0. \tag{4.2}$$

Hence, a greater subsidy means a higher productivity growth rate q_t^B. The subsidy also increases the steady-state growth rate of the economy. To see this, recall that the latter is equal to the growth rate of labor-augmenting technological knowledge and hinges on $\left(q^A\right)^* = g^A(\kappa^*)$. However, here κ^* depends on and increases in σ^B. Indeed, in light of (4.2) condition (3.16) must be replaced by

$$g^B(\kappa_{\sigma^B}^*, \sigma^B) = \frac{\delta^B}{1 - \delta^B}, \tag{4.3}$$

where $\kappa_{\sigma^B}^*$ is the steady-state equilibrium task intensity consistent with a subsidy rate $\sigma^B > 0$. Implicit differentiation of (4.3) reveals that $\kappa_{\sigma^B}^*$ increases in σ^B as

$$\frac{d\kappa_{\sigma^B}^*}{d\sigma^B} = -\frac{g_\sigma^B\left(\kappa_{\sigma^B}^*, \sigma^B\right)}{g_\kappa^B\left(\kappa_{\sigma^B}^*, \sigma^B\right)} > 0. \tag{4.4}$$

The intuition is the following. In steady state q^B must be equal to $\delta^B / \left(1 - \delta^B\right)$ to keep B constant. A subsidy rate $\sigma^B > 0$ reduces the (effective) cost per task, $\left(1 - \sigma^B\right) c\left(q_t^B\right)$, on the right-hand side of (4.1). Given κ_t, this tends to increase q_t^B. To offset this tendency κ_t must increase so that the left-hand side of (4.1) can fall until q^B is again equal to $\delta^B / \left(1 - \delta^B\right)$.

These adjustments increase the steady-state growth rate which is now give by

$$g_{\sigma^B}^* = \left(1 - \delta^A\right)\left(1 + g^A\left(\kappa_{\sigma^B}^*\right)\right) - 1. \tag{4.5}$$

Then, with (4.4) we have

$$\frac{dg_{\sigma^B}^*}{d\sigma^B} = \left(1 - \delta^A\right) g_\kappa^A\left(\kappa_{\sigma^B}^*\right) \frac{d\kappa_{\sigma^B}^*}{d\sigma^B} > 0. \tag{4.6}$$

The following proposition summarizes the result of this reasoning.

Proposition 10 *(Subsidy to Capital-Augmenting Innovation Investments and Steady-State Growth)*

Consider two economies that differ only with respect to the subsidy rate to capital-augmenting innovation investments such that $1 > \left(\sigma^B\right)' > \sigma^B \geq 0$. *Then, it holds that*

$$\left(g_{\sigma^B}^*\right)' \quad > \quad g_{\sigma^B}^*. \tag{4.7}$$

4.3 A Subsidy to Labor-Augmenting Innovation Investments

Suppose the government pays a subsidy $\sigma^A i\left(q_t^A(n)\right)$ for all innovation investments that raise the productivity of labor at t. Here, $\sigma^A \in [0,1)$ denotes the subsidy rate. This subsidy reduces the cost per task performed by labor and, therefore, renders innovation investments more attractive. This shows up in the conditions for profit-maximization that now involve

$$f(\kappa_t) - \kappa_t f'(\kappa_t) = \left(1 - \sigma^A\right) c\left(q_t^A\right). \tag{4.8}$$

The latter generalizes (2.19) to the case where $\sigma^A > 0$ and leads to changes in Proposition 1. In particular, we now have

$$q_t^A = g^A(\kappa_t, \sigma^A), \quad \text{with} \quad g_\kappa^A(\kappa_t, \sigma^A) > 0 \quad \text{and} \quad g_{\sigma^A}^A(\kappa_t, \sigma^A) > 0, \tag{4.9}$$

The subsidy for labor-augmenting innovation investments increases the the steady-state growth rate of the economy. However, the channel is quite different from the one identified for a subsidy of capital-augmenting innovation investments. To see this, observe that the subsidy leaves κ^* unchanged which is still determined by (3.16). However, in light of (4.9) and Proposition 4, the steady-state growth rate of the economy may now be written as

$$g_{\sigma^A}^* = \left(1 - \delta^A\right)\left(1 + g^A\left(\kappa^*, \sigma^A\right)\right) - 1. \tag{4.10}$$

Then, using (4.9) we obtain indeed that

$$\frac{dg_{\sigma^A}^*}{d\sigma^A} = \left(1 - \delta^A\right) g_\sigma^A\left(\kappa^*, \sigma^A\right) > 0. \tag{4.11}$$

This leads to the following proposition.

Proposition 11 *(Subsidy to Labor-Augmenting Innovation Investments and Steady-State Growth)*

Consider two economies that differ only with respect to the subsidy rate for labor-augmenting innovation investments such that $1 > \left(\sigma^A\right)' > \sigma^A \geq 0$. Then, it holds that

$$\left(g_{\sigma^A}^*\right)' \quad > \quad g_{\sigma^A}^*. \tag{4.12}$$

5 Optimal Growth

This section studies the welfare properties of the dynamic competitive equilibrium and establishes two main results. First, we show that the equilibrium is not Pareto optimal since both forms of technical change give rise to an inter-temporal knowledge externality. Second, we prove that the Pareto-efficient steady state may be implemented with an appropriate policy of investment subsidies.

5.1 The Planner's Problem

To derive the Pareto-efficient allocation, we assess allocations with regard to their effects on the overall utility of the representative household (2.1). Moreover, we focus on allocations with the same structural properties as in the decentralized equilibrium. In particular, we restrict attention to symmetric configurations that involve $q_t^B(m) = q_t^B$ and $q_t^A(n) = q_t^A$.[13] To save space, we directly take capital and labor as fully employed.

Then, given $L > 0$, initial values of the capital stock, $K_0 > 0$, and of technological knowledge, $A_{-1} > 0$ and $B_{-1} > 0$, the planner solves

$$\max_{\left\{q_t^B, q_t^A, C_t\right\}_{t=0}^{\infty} \in \mathbb{R}_+^3} \quad \sum_{t=0}^{\infty} \beta^t \ln C_t, \tag{5.1}$$

subject to the resource constraint

$$C_t + K_{t+1} = F\left(B_t K_t, A_t L\right) - B_t K_t i(q_t^B) - A_t L i(q_t^A) + \left(1 - \delta^K\right) K_t, \tag{5.2}$$

the evolution of the two stocks of technological knowledge of (2.35) and a set of appropriate non-negativity constraints. Besides these constraints and three transversality constraints,[14] the planner's problem satisfies the following first-order (sufficient) conditions for K_{t+1}, q_t^B, and q_t^A, respectively:

---

[13]This excludes, e. g., asymmetric patterns where the planner chooses, say, $q_t^B(m) > 0$ for a small subset of tasks $m \in [0, \bar{m}_t]$, $\bar{m}_t < M_t$, and does not undertake innovation investments in all other tasks performed by capital. This strategy reduces current outlays for innovation investments and, at the same time, allows to start period $t+1$ with a high level of $B_t\left(1 - \delta^B\right)$ as knowledge accumulates according to (2.34). Of course, such a strategy also has a downside since the productivity of capital at t in all tasks $m \in [\bar{m}_t, M_t]$ is $B_{t-1}\left(1 - \delta^B\right)$. At all events, in a decentralized economy such patterns cannot arise in equilibrium and would even be very difficult to implement by a planning authority.

[14]The transversality constraints are $\lim_{t\to\infty} \beta^t \mu_t^K K_{t+1} = 0$, $\lim_{t\to\infty} \beta^t \mu_t^A A_t = 0$, and $\lim_{t\to\infty} \beta_t^t \mu_t^B B_t = 0$, respectively, where μ_t^K, μ_t^A, μ_t^B are the Lagranage multipliers associated with the resource constraint and the appropriate technological constraint, respectively.

$$\frac{C_{t+1}}{C_t} = \beta \left[B_t(1-\delta^B)(1+q^B_{t+1}) \left(f'\left(\kappa_{t+1}\right) - i\left(q^B_{t+1}\right)\right)\right] + \beta\left(1-\delta^K\right), \tag{5.3}$$

$$0 = \frac{K_t}{C_t}\left[f'\left(\kappa_t\right) - i\left(q^B_t\right) - \left(1+q^B_t\right) i'\left(q^B_t\right)\right] + \beta \cdot \frac{K_{t+1}}{C_{t+1}}\left(1-\delta^B\right)\left(1+q^B_{t+1}\right)^2 i'\left(q^B_{t+1}\right), \tag{5.4}$$

$$0 = \frac{K_t}{C_t}\left[f\left(\kappa_t\right) - \kappa_t f'\left(\kappa_t\right) - i\left(q^A_t\right) - \left(1+q^A_t\right) i'\left(q^A_t\right)\right] + \beta \cdot \frac{K_{t+1}}{C_{t+1}}\left(1-\delta^A\right)\left(1+q^A_{t+1}\right)^2 i'\left(q^A_{t+1}\right). \tag{5.5}$$

Condition (5.3) is the Euler condition of the planner's problem. The comparison with the Euler condition (3.8) of the competitive equilibrium reveals that the inter-temporal equilibrium allocation of capital is the efficient one. This, however, is not the case for the equilibrium choices of q^B_t and q^A_t. To see this compare (5.4) and (5.5) to their respective equilibrium counterparts (2.18) and (2.19). The decentralized equilibrium has productivity growth rates such that the minimum costs per task are equal to the respective value product of the marginal task. If these conditions hold for t then the first lines of (5.4) and (5.5) vanish whereas the second remain positive. In other words, evaluated at the equilibrium allocation the (marginal) value of q^B_t and q^A_t is strictly positive for the planner. The presence of β suggests that the additional advantage is of an inter-temporal nature. To see this more clearly, consider the following variational argument applied to the first-order condition (5.4) that describes the social planner's choice of q^B_t. Mutatis mutandis, an analogous argument applies to (5.5).

Suppose the economy evolves along an optimal path given by $\left\{A_t, B_t, K_{t+1}, C_t, q^B_t, q^A_t\right\}_{t=0}^{\infty}$. Now, the planner considers an increase in q^B_t at some period $t \geq 0$ in conjunction with a decrease in q^B_{t+1} such that the sequence $\left\{A_{\tau-1}, B_\tau, K_\tau, q^B_{\tau+1}, q^A_{\tau-1}\right\}_{\tau=t+1}^{\infty}$ remains unchanged. To study the effects of this variation, consider the planner's net output at time t,

$$V_t = F\left(B_t K_t, A_t L\right) - A_t L i\left(q^A_t\right) - B_t K_t i\left(q^B_t\right), \tag{5.6}$$

in conjunction with the evolution of the two stocks of technological knowledge of (2.35). Then, a small increase in q_t^B changes V_t by

$$dV_t = B_{t-1}\left(1-\delta^B\right)K_t\left[f'\left(\kappa_t\right) - i\left(q_t^B\right) - \left(1+q_t^B\right)i'\left(q_t^B\right)\right]dq_t^B. \tag{5.7}$$

Here, the first term in the bracketed expression represents the increase in net output due to a greater productivity of capital. The second term captures the additional investment outlays that arise since a greater productivity of capital increases the number of tasks performed by capital under full employment of the capital stock. The third term represents the additional investment outlays that arise since the investment outlays of each performed task increase.

Since B_{t+1} is unaffected by the variation in q_t^B and q_{t+1}^B, the changes in these variables must satisfy $dq_{t+1}^B = -\left(1+q_{t+1}^B\right)/\left(1+q_t^B\right)dq_t^B$. Then, the effect of dq_t^B on net output in $t+1$ is

$$\begin{aligned} dV_{t+1} &= -B_{t+1}K_{t+1}i'\left(q_{t+1}^B\right)dq_{t+1}^B, \\ &= B_{t+1}K_{t+1}i'\left(q_{t+1}^B\right)\left(\frac{1+q_{t+1}^B}{1+q_t^B}\right)dq_t^B, \\ &= B_{t-1}\left(1-\delta^B\right)^2 K_{t+1}\left(1+q_{t+1}^B\right)^2 i'\left(q_{t+1}^B\right)dq_t^B. \end{aligned} \tag{5.8}$$

Hence, net output in $t+1$ increases since $dq_t^B > 0$ and the concomitant decline in q_{t+1}^B reduces the investment outlays for all $M_{t+1} = B_{t+1}K_{t+1}$ tasks performed by capital. To link these findings to the first-order condition (5.4) observe that the latter may be written as

$$0 = \frac{dV_t}{C_t} + \beta\frac{dV_{t+1}}{C_{t+1}}. \tag{5.9}$$

Hence, along the optimal path the sum of the contemporaneous and the inter-temporal effect of a variation in dq_t^B vanishes when compared in "utils" of period t. In other words, the respective second terms in conditions (5.4) and (5.5) represent the inter-temporal advantage of greater values for q_t^B and q_t^A that are not taken into account in the decentralized equilibrium.

5.2 Steady-State Analysis

The economy of the planner involves the net output function (5.6) with constant returns to scale in capital and labor and the resource constraint (5.2). Therefore, we may invoke the generalized steady state growth theorem of Irmen (2013b). Hence, a steady state has $B_t = B^{**}$ and the growth rate of the economy is given by the growth rate of labor-augmenting technological knowledge, $g^{**} = g_A^{**}$. Moreover, $g_V^{**} = g_Y^{**} = g_C^{**} = g_K^{**} = g^{**}$. The evolution of technological knowledge (2.35) is consistent with this pattern. The task of this section is to establish the existence of such a steady state and to compare it to the steady state of the dynamic competitive economy.

To support $B_t = B^{**}$, the evolution of the stock of capital-augmenting technological knowledge of (2.35) requires

$$\left(q^B\right)^{**} = \frac{\delta^B}{1-\delta^B} = \left(q^B\right)^{*}. \tag{5.10}$$

Hence, the steady state of the planner's problem involves the same q^B as the competitive equilibrium. This is so, even though the planner internalizes the inter-temporal knowledge spill-over associated with innovation investments that increase the productivity of capital.

Constant consumption growth in (5.3) and (5.10) imply $\kappa_t = \kappa^{**}$. Then, in steady state conditions (5.4) and (5.5) boil down to

$$c\left(\frac{\delta^B}{1-\delta^B}\right) = f'\left(\kappa^{**}\right) + \beta \cdot \frac{i'\left(\frac{\delta^B}{1-\delta^B}\right)}{1-\delta^B}, \tag{5.11}$$

$$\begin{aligned} c\left(\left(q^A\right)^{**}\right) &= f\left(\kappa^{**}\right) - \kappa^{**} f'\left(\kappa^{**}\right) \\ &+ \beta\left(1-\delta^A\right)\left(1+\left(q^A\right)^{**}\right)^2 i'\left(\left(q^A\right)^{**}\right). \end{aligned} \tag{5.12}$$

Hence, the planner's steady-state choice of q^B and q^A is such that the minimized costs per task are equal to the sum of the contemporaneous marginal product of the respective task and the inter-temporal advantage arising from the knowledge spill-over. The comparison with (2.18) and (2.19) of the competitive equilibrium shows that the inter-temporal effect is neglected by the competitive production sector. The reason for this is straightforward. While innovation investments give rise to new technological knowledge that increases the productivity of the factors of production, this advantage is confined to the period in which the innovation investment is made. For all subsequent periods the newly created technological knowledge becomes publicly available and can be used by any firm free of charge. It is in this sense that the investment behavior of firms does not account for the future.

One readily verifies that condition (5.11) determines $\kappa^{**} > 0$. The comparison with (2.18) evaluated at the steady state reveals immediately that $\kappa^{**} > \kappa^{*}$. This reflects the presence of the inter-temporal effect that forces the marginal product of tasks performed by capital to fall. Condition (5.12) determines $\left(q^A\right)^{**} > \left(q^A\right)^{*}$. There are two reinforcing reasons for this inequality. First, $\kappa^{**} > \kappa^{*}$ means that the marginal product of tasks performed by labor increases. Therefore, the cost-minimizing level of q^A will be higher. Second, the inter-temporal advantage is itself increasing in q^A. In steady state, the Euler condition, (5.3), pins down B^{**} such that consumption grows at the same rate as the economy. The following proposition summarizes these results.

Proposition 12 *(Planner's Steady-State Allocation)*

1. *Suppose Assumptions (1) and (2) hold. Then, the planner's problem has a unique steady state involving* $\left(\kappa^{**}, \left(q^B\right)^{**}, \left(q^A\right)^{**}\right) \in \mathbb{R}^3_{++}$, *and*

$$\begin{aligned} c^{**} &= f\left(\kappa^{**}\right) - \kappa^{**} i \left(\frac{\delta^B}{1-\delta^B}\right) - i\left(\left(q^A\right)^{**}\right) \\ &- \frac{\kappa^{**}}{B^{**}}\left(g^{**} + \delta^K\right), \end{aligned} \tag{5.13}$$

$$B^{**} = \frac{\left(1-\delta^A\right)\left(1+\left(q^A\right)^{**}\right) - \beta\left(1-\delta^K\right)}{\beta\left(f'\left(\kappa^{**}\right) - i\left(\frac{\delta^B}{1-\delta^B}\right)\right)} > B^{*}, \tag{5.14}$$

2. *The welfare-maximizing steady-state growth rate of the economy is*

$$g^{**} \equiv \frac{A_{t+1}}{A_t} - 1 = \left(1-\delta^A\right)\left(1+\left(q^A\right)^{**}\right) - 1 > g^{*}.$$

Moreover, along the planner's steady-state path, it holds that

a) $$\frac{Y_{t+1}}{Y_t} = \frac{V_{t+1}}{V_t} = \frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{M_{t+1}}{M_t} = \frac{N_{t+1}}{N_t} = 1 + g^{**},$$

b) $$k^{*} = \frac{1}{B^{**}}, \quad \frac{l_{t+1}}{l_t} = \frac{1}{1+g^{**}}$$

5.3 Pareto-Improving Fiscal Policy

The discrepancy between the steady-state growth rates of the dynamic competitive equilibrium and the planner's solution suggests the possibility of Pareto-improving policy measures. The following proposition establishes that appropriately chosen investment subsidies accompanied by a lump-sum tax to balance the government's budget may close the gap between the allocation chosen by the planner and the one obtained under laissez-faire.

Proposition 13 *Suppose the government subsidizes innovation investments at rates*

$$\left(\sigma^{B}\right)^{**}=\frac{\beta i'}{i'+\left(1-\delta^{B}\right)i}\in(0,1), \tag{5.15}$$

where i is evaluated at $q^{B}=\delta^{B}/\left(1-\delta^{B}\right)$, and

$$\left(\sigma^{A}\right)^{**}=\frac{\beta\left(1-\delta^{A}\right)\left(1+\left(q^{A}\right)^{**}\right)i'\left(\left(q^{A}\right)^{**}\right)}{c\left(\left(q^{A}\right)^{**}\right)}\in(0,1), \tag{5.16}$$

and balances its budget with a lump-sum tax. Then, $\left(\left(\sigma^{B}\right)^{**},\left(\sigma^{A}\right)^{**}\right)$ implements the planner's steady-state allocation.

Notice that the two subsidies differ. The reason is that they fulfill different purposes. The subsidy to innovation investments that increase the productivity of capital induces a higher steady-state value of the equilibrium task intensity. To see this consider (4.1) in steady state and (5.15). Since $q^{B}=\delta^{B}/(1-\delta^{B})$, the optimal subsidy rate is such that $\left(\sigma^{B}\right)^{**}c(\delta^{B}/(1-\delta^{B}))$ is just equal to the inter-temporal spill-over on the right-hand side of (5.11). Next, consider (4.8) in steady state and (5.12). Since $\kappa=\kappa^{**}$, the optimal subsidy rate is such that $\left(\sigma^{A}\right)^{**}c((q^{A})^{**})$ is just equal to the inter-temporal spill-over on the right-hand side of (5.16). Accordingly, firms internalize both inter-temporal effects associated with their innovation investments so that the planner's steady-state allocation is implemented and the economy grows at rate g^{**}.

6 Concluding Remarks

This paper shows that endogenous capital- and labor-augmenting technical change can be incorporated into the neoclassical growth model with both infinitely lived dynasties and firms behaving competitively. We establish that due to the presence of endogenous capital-augmenting technical change positive and normative implications substantially change. We conclude that the neglect of capital-augmenting technical change is not benign.

To a large extent, this is due to the "straightjacket" imposed by the necessity for capital-augmenting technical progress to vanish in the steady state (Irmen (2013b)). This requirement pins down the steady-state value of one of the economy's state variables, namely, the efficient capital intensity. In steady state this variable must induce innovation vestments in capital-augmenting technological knowledge just enough to offset its depreciation. At the same time, the efficient capital intensity determines the growth rate of labor-augmenting technological knowledge, thus, the economy's steady-state growth rate.

The reader may recall that our results are obtained for a logarithmic per-period utility function of the representative household. This begs the question of whether our qualitative findings are different under a more general utility function allowing for a constant inter-temporal elasticity of substitution (CIES) different from unity. In fact, little will change. To see this, remember that neither the steady-state efficient capital intensity nor the steady-state growth rate hinge on household preferences. Both are entirely determined by the production sector of the economy. Therefore, the CIES will only affect steady-state levels as well as the transition while leaving the local stability property of the steady state intact.

For a CIES exceeding unity an additional restriction on permissible parameter constellations is called for. This condition assures that the steady-state growth rate is not too large so that the household's problem remains well-defined. The presence of a CIES also introduces an additional parameter for comparative static exercises. However, since the household becomes less willing to accept deviations from a uniform consumption profile the smaller the CIES, the comparative-static effect of a decline in the CIES delivers the same sign as a decline in the discount factor, β.

In the main text we also abstract from population growth. Let us now consider the main consequences of a constant population growth rate, g_L. Again, since the steady-state growth rate is determined within the production sector it will neither depend on g_L nor on the size of the population. Moreover, from (3.6) evaluated at the steady state it becomes apparent that in steady state the equilibrium net output per unit of efficient labor is also independent of g_L and the size of the population. Hence, in the terminology of Jones (2005) there are neither "strong" nor "weak" scale effects. Clearly, there will be level effects along the transition associated with g_L that, however, do not affect the local stability property of the steady state.

It is worth noting that the respective roles of the CIES and of g_L are quite different in the model variant of Section 3.4 where only labor-augmenting technical change is feasible. As both parameters appear in the Euler condition they will affect the steady-state growth rate.[15] This observation corroborates our conclusion that the neglect of capital-augmenting technical change is not benign.

Our paper leaves some important issues unresolved. They include the plausibility of a time-invariant equilibrium innovation possibility frontier (EIPF). Should this frontier

[15]To be precise, g_L will appear in the Euler condition if the representative household evaluates sequences of per-capita consumption $\{\tilde{c}_t\}_{t=0}^{\infty}$ according to $\sum_{t=0}^{\infty} \beta^t \ln \tilde{c}_t$. If the household cares for the per-period utility of all household members at t then the objective functional is $\sum_{t=0}^{\infty} \beta^t \left(1+g_L\right)^t \ln \tilde{c}_t$ and the Euler condition is independent of g_L.

move over time? Empirical studies such as Caselli and Coleman (2006) suggest the existence of country-specific frontiers which may actually change over time. One way to think about this is in terms of investment-specific technical change that lowers the relative price of the resources used as innovation investments. This route may also open the door to a new mechanism that links the empirically observed decline of labor shares to the relative price of investment goods (see, e. g., Karabarbounis and Neiman (2014)).

Another desirable feature would be a more flexible role for tasks. So far, we restrict attention to time-invariant factor-specific tasks. However, in practice the boundary between tasks performed by labor and those performed by capital shifts over time. Technical change may tend to transfer tasks from one factor of production to another. Moreover, history shows that technical change may make certain tasks redundant altogether and eliminate them from the production process. We leave these challenging questions for future research.

7 Appendix: Proofs

The numbering within each proof refers to the respective claims. Without loss of generality, we may suppress the time argument when appropriate.

7.1 Proof of Proposition 1

1. Consider $c(q^j)$, $j = A, B$, of (2.14) and (2.15). The properties of $i(q^j)$ ensure that $c(q^j)$ is strictly increasing with $\lim_{q^j \to 0} c(q^j) = 0$ and $\lim_{q^j \to \infty} c(q^j) = \infty$. Next observe that $f'(\kappa)$ and $f(\kappa) - \kappa f'(\kappa)$ are strictly monotonic. Therefore, (2.18) defines some function $q_t^B = g^B(\kappa_t)$ where $g^B : \mathbb{R}_{++} \to \mathbb{R}_{++}$ whereas (2.19) defines some function $q_t^A = g^A(\kappa_t)$ where $g^A : \mathbb{R}_{++} \to \mathbb{R}_{++}$. Implicit differentiation of these equations reveals that $g_\kappa^A(\kappa_t) > 0 > g_\kappa^B(\kappa_t)$ for all $\kappa > 0$ as claimed in (2.20) and (2.21).
2. Solving (2.12) and (2.13) for the respective factor price R and w and using Claim 1 delivers

$$
\begin{aligned}
R &= B_{-1}\left(1-\delta^B\right)\left(1+g^B(\kappa)\right)^2 i'\left(g^B(\kappa)\right) \equiv R(\kappa, B_{-1}), \\
w &= A_{-1}\left(1-\delta^A\right)\left(1+g^A(\kappa)\right)^2 i'\left(g^A(\kappa)\right) \equiv w(\kappa, A_{-1}),
\end{aligned}
$$

where $R : \mathbb{R}_{++}^2 \to \mathbb{R}_{++}$ and $w : \mathbb{R}_{++}^2 \to \mathbb{R}_{++}$. The partial derivatives indicated in (2.22) and (2.23) follow immediately from Claim 1 and the properties of the function i. ■

7.2 Proof of Corollary 1

1. Consider equations (2.20) and (2.21). Since g^B is strictly decreasing on its domain it is invertible. Let $G^B : \mathbb{R}_{++} \to \mathbb{R}_{++}$ denote the inverse of g^B. Then, from (2.20), $\kappa = G^B\left(q^B\right)$. Hence, with (2.21), we may write

$$q^A = g^A\left(G^B\left(q^B\right)\right) \equiv g\left(q^B\right).$$

The slope of the function $g(q^B)$ is given by

$$g'\left(q^B\right) \equiv \frac{dq^A}{dq^B} = \frac{dg^A(\kappa)}{d\kappa}\frac{dG^B\left(q^B\right)}{dq^B} = \frac{g_\kappa^A(\kappa)}{g_\kappa^B(\kappa)} < 0. \tag{7.1}$$

2. Consider equations (2.22) and (2.23). From equation (2.23), the function $w(\kappa, A_{-1})$ is strictly increasing in κ on its domain. Hence, given A_{-1}, this function is invertible in κ. Let $W : \mathbb{R}_{++}^2 \to \mathbb{R}_{++}$ denote this inverse. Then, $\kappa = W(w : A_{-1})$. Hence, with (2.22), we may write

$$R = R(W(w : A_{-1}), B_{-1}) \equiv h(w, A_{-1}, B_{-1}),$$

where $h : \mathbb{R}_{++}^3 \to \mathbb{R}_{++}$. The partial derivative of $h(w, A_{-1}, B_{-1})$ with respect to w is given by

$$
\begin{aligned}
h_w(w, A_{-1}, B_{-1}) \equiv \frac{dR}{dw} &= \frac{dR(\kappa, B_{-1})}{d\kappa}\frac{dW(w : A_{-1})}{dw} \\
&= \frac{R_\kappa(\kappa, B_{-1})}{w_\kappa(\kappa, A_{-1})} < 0.
\end{aligned} \tag{7.2}
$$

■

7.3 Proof of Proposition 2

Consider equation (3.4). Given $\Theta > 0$, its right-hand side defines a continuous function, $RHS(\kappa, \Theta) > 0$ for all $\kappa > 0$. Moreover, following Proposition 1, the properties of $g^A(\kappa)$ and $g^B(\kappa)$ imply that $RHS(\kappa, \Theta)$ is continuous and strictly decreasing in $\kappa > 0$. Hence, $\lim_{\kappa \to 0} RHS(\kappa, \Theta) > 0$. By continuity of $RHS(\kappa, \Theta)$, there is a unique $\kappa > 0$ that satisfies $\kappa = RHS(\kappa, \Theta)$. Implicit differentiation delivers that this value increases whenever Θ increases. ■

7.4 Proof of Proposition 3

We show in the main text how to derive the three-dimensional system (3.7) - (3.9). There, we also argue that κ_t and B_t describe the state of the economy at all t. Here, it remains to be shown that for any set of initial values (A_{-1}, B_{-1}, K_0, L) the transitional dynamics is given by a unique sequence $\{\kappa_t, c_t, B_t\}_{t=0}^{\infty}$.

To accomplish this we introduce

$$\Omega(\kappa_{t+1}) \equiv \frac{\left(1-\delta^A\right)\left(1+g^A\left(\kappa_{t+1}\right)\right)}{\left(1-\delta^B\right)\left(1+g^B\left(\kappa_{t+1}\right)\right)}\kappa_{t+1}. \tag{7.3}$$

Then, equation (3.7) may be rewritten as

$$\Omega(\kappa_{t+1}) = B_t\Big(v(\kappa_t) - c_t\Big) + \Big(1-\delta^K\Big)\,\kappa_t. \tag{7.4}$$

For any triple $(\kappa_t, B_t, c_t) \in \mathbb{R}^3_{++}$ such that the right-hand side of (7.4) is strictly positive, there will be a unique value of $\kappa_{t+1} > 0$ satisfying equation (7.4) if $\Omega(\kappa_{t+1})$ is strictly positive, continuous and monotonically increasing in $\kappa_{t+1} > 0$ and may take any value in $\mathbb{R}_{++}$.

Observe first that $\Omega(\kappa_{t+1}) > 0$ indeed holds for all $\kappa_{t+1} > 0$. Moreover, $\Omega(\kappa_{t+1})$ is continuous. These properties follow from the properties of the functions g^A and g^B, as established in Proposition 1. It remains to be shown that $\lim_{\kappa\to 0}\Omega(\kappa_{t+1}) = 0$ and $\lim_{\kappa\to\infty}\Omega(\kappa_{t+1}) = \infty$. To establish this, consider the right-hand side of (7.3). Recall from Proposition 1 that $g^B(\kappa)$ is decreasing on $\mathbb{R}_{++}$ and bounded below by zero. Hence, $\lim_{\kappa\to\infty} g^B(\kappa)$ is finite, while $\lim_{\kappa\to 0} g^B(\kappa)$ is either finite or infinite. Further, Proposition 1 implies that $\lim_{\kappa\to 0} g^A(\kappa)$ is finite and bounded below by zero while $\lim_{\kappa\to\infty} g^A(\kappa)$ is finite or infinite since g^A is increasing on $\mathbb{R}_{++}$. Consequently, as κ tends to zero we have $\lim_{\kappa\to 0}\Omega(\kappa_{t+1}) = 0$ and as κ tends to infinity we have $\lim_{\kappa\to\infty}\Omega(\kappa_{t+1}) = \infty$.

It follows that $\Omega(\kappa_{t+1})$ is increasing in $\kappa_{t+1} > 0$, approaches zero as $\kappa \to 0$ and approaches infinity as $\kappa \to \infty$. Therefore, there is a unique $\kappa_{t+1} > 0$ that satisfies equation (3.7) given $(\kappa_t, B_t, c_t) \in \mathbb{R}^3_{++}$.

Given a unique $\kappa_{t+1} > 0$ equation (3.8) delivers a unique $c_{t+1} > 0$ and equation (3.9) a unique $B_{t+1} > 0$. ■

7.5 Proof of Proposition 4

1. Equations (3.14) - (3.16) follow immediately from the corresponding equations (3.7) - (3.9) of the dynamical system for the reasons discussed in the main text. Obviously, in steady state the transversality condition (3.10) is also satisfied. As explained in the main text, Assumption 1 and Assumption 2 guarantee a strictly positive solution to (3.16) and (3.15), respectively. It remains to be shown that $c^* > 0$ or $B^* v(\kappa^*) > \kappa^*\left(g^* + \delta^K\right)$. Using (3.15) the latter inequality may be written as

$$\frac{\left[(1+g^*) - \beta\left(1-\delta^K\right)\right] v(\kappa^*)}{\beta\left(f'(\kappa^*) - i\left(g^B(\kappa^*)\right)\right)} > \kappa^*\Big(g^* + \delta^K\Big) \tag{7.5}$$

A sufficient condition for this to hold is obtained for $\beta = 1$. This gives

$$v(\kappa^*) > \left(f'(\kappa^*) - i\left(g^B(\kappa^*)\right)\right)\kappa^*. \tag{7.6}$$

Since equilibrium profits are zero, we have $f(\kappa) = c\left(q^A\right) + \kappa c\left(q^B\right)$. Using the latter in (3.6) delivers in steady state

$$v^*(\kappa^*) = \left(1 + g^A(\kappa)^*\right) i'\left(g^A(\kappa)^*\right) + \kappa^*\left(1 + g^B(\kappa)^*\right) i'\left(g^B(\kappa)^*\right).$$

Then, with the understanding that both g^A and g^B are evaluated at κ^*, inequality (7.6) becomes

$$\begin{aligned}\left(1+g^A\right) i'\left(g^A\right) + \kappa^*\left(1+g^B\right) i'\left(g^B\right) &> \left(f'(\kappa^*) - i\left(g^B\right)\right)\kappa^*, \\ \left(1+g^A\right) i'\left(g^A\right) &> 0,\end{aligned}$$

as $f'(\kappa^*) - c\left(g^B\right) = 0$ from (2.18).

2. The expression for the steady-state growth rate, g^*, follows from (2.35) and Proposition 1. The derivation of the remaining findings is explained in the main text. ■

7.6 Proof of Proposition 5

1. From (3.16), κ^* is independent of β. Therefore, g^* does not depend on β either. From (3.15) it is immediate that a higher β requires a lower B^*. According to the expression for R^* in Claim 2 of Proposition 4 the rental rate of capital must also fall. From (3.14) the same is true for c^*.
2. Implicit differentiation of (3.16) delivers

$$\frac{d\kappa^*}{d\delta^B} = \frac{1}{(1-\delta^B)^2 g_\kappa^B(\kappa^*)} < 0 \tag{7.7}$$

as $g_\kappa^B < 0$. Hence, $(\kappa^*)' < \kappa^*$. The concomitant effect on the steady-state growth rate, g^*, is

$$\frac{dg^*}{d\delta^B} = \frac{dg^*}{d\kappa}\frac{d\kappa^*}{d\delta^B} < 0.$$

The sign follows since $dg^*/d\kappa = \left(\partial g^*/\partial q^A\right)\left(dg^A(\kappa^*)/d\kappa\right) > 0$. Hence, $(g^*)' < g^*$.

To obtain the effect of δ^B on B^* consider (3.15). Then,

$$\begin{aligned}\frac{dB^*}{d\delta^B} &= \frac{dB^*}{d\kappa}\frac{d\kappa^*}{d\delta^B} \\ &= \frac{1}{\beta\left(f' - i\left(g^B\right)\right)}\left[\frac{dg^*}{d\kappa} - \frac{\left(g^* + \delta^K\right)\left(f'' - i'\left(g^B\right)g_\kappa^B\right)}{f' - i\left(g^B\right)}\right]\frac{d\kappa^*}{d\delta^B} < 0,\end{aligned} \tag{7.8}$$

where f and g^B are evaluated at κ^*. To verify the sign of this expression note from (2.14) that

$$g_\kappa^B(\kappa) \equiv \frac{dq^B}{d\kappa} = \frac{f''(\kappa)}{2i'\left(q^B\right) + \left(1 + q^B\right)i''\left(q^B\right)}. \tag{7.9}$$

Hence, $f''(\kappa^*) - i'\left(g^B(\kappa^*)\right)g_\kappa^B(\kappa^*) < 0$ and $(B^*)' < B^*$.

Using (3.15) the steady-state rental rate of capital may be written as $R^* = (1 + g^*)/\beta - \left(1 - \delta^K\right)$. Following $dg^*/d\delta^B < 0$ it is immediate that $(R^*)' < R^*$.

Finally, consider the effect of δ^B on c^* of (3.14). It is given by

$$\begin{aligned}\frac{dc^*}{d\delta^B} &= \frac{dc^*}{d\kappa}\frac{d\kappa*}{d\delta^B} \\ &= \left[v'(\kappa^*) - \frac{\left(g^* + \delta^K + \kappa^*\frac{dg^*}{d\kappa}\right)B^* - \frac{dB^*}{d\kappa}\left(g^* + \delta^K\right)\kappa^*}{\left(B^*\right)^2}\right]\frac{d\kappa*}{d\delta^B}\end{aligned} \tag{7.10}$$

and is indeterminate in general.

3. From (3.16), κ^* is independent of δ^A. Therefore, it holds that $(\kappa')^* = \kappa^*$. Since $g^* = \left(1-\delta^A\right)\left(1+g^A\left(\kappa^*\right)\right) - 1$ it is immediate that $\left(\delta^A\right)' > \delta^A \Rightarrow (g^*)' < g^*$.
   From (3.15) we have $\text{sign}[dB^*/d\delta^A] = \text{sign}[dg^*/d\delta^A] < 0$. Hence, $(B^*)' < B^*$. Invoking $R^* = (1+g^*)/\beta - \left(1-\delta^K\right)$ and (3.14) one finds that $\text{sign}[dR^*/d\delta^A] = -\text{sign}[dc^*/d\delta^A] = \text{sign}[dg^*/d\delta^A] < 0$. Hence, $(R^*)' < R^*$ and $(c^*)' > c^*$. ■

7.7 Proof of Proposition 6

We characterize the local stability of the dynamical system of Proposition 3 in the proximity of its steady-state equilibrium, (κ^*, c^*, B^*).[16]

Consider first the system of autonomous, nonlinear, first-order difference equations given by (3.7), (3.8) and (3.9), and notice that equations (3.7) and (3.8) define continuously differentiable functions, $\Phi^i : \mathbb{R}^3_{++} \to \mathbb{R}_{++}$, where $i = \kappa, c$, such that

$$\kappa_{t+1} = \Phi^\kappa(\kappa_t, c_t, B_t), \quad c_{t+1} = \Phi^c(\kappa_t, c_t, B_t).$$

Next, forward equation (3.9) one period and use $\kappa_{t+1} \equiv \Phi^\kappa(\kappa_t, c_t, B_t)$ to obtain

$$B_{t+1} = B_t(1-\delta)(1+g^B(\Phi^\kappa(\kappa_t, c_t, B_t))) \equiv \Phi^B(\kappa_t, c_t, B_t),$$

where Φ^B is also a continuously differentiable function.

With this notation the dynamical system may be rewritten as

$$\begin{aligned} \kappa_{t+1} &= \Phi^\kappa(\kappa_t, c_t, B_t), \\ c_{t+1} &= \Phi^c(\kappa_t, c_t, B_t), \\ B_{t+1} &= \Phi^B(\kappa_t, c_t, B_t). \end{aligned}$$

This nonlinear dynamical system is approximated locally about its steady-state equilibrium, (κ^*, c^*, B^*), by the following linear system:

$$\begin{bmatrix} \kappa_{t+1} \\ c_{t+1} \\ B_{t+1} \end{bmatrix} = J \begin{bmatrix} \kappa_t \\ c_t \\ B_t \end{bmatrix} + X,$$

where J is the Jacobian matrix of the dynamical system evaluated at (κ^*, c^*, B^*), and X is a constant column vector.

To obtain the elements of J take the total differential of

$$\Omega(\kappa_{t+1}) = B_t\left(v(\kappa_t) - c_t\right) + (1-\delta^K)\kappa_t,$$

$$c_{t+1} = \beta \frac{\left[B_{t+1}\left(f'(\kappa_{t+1}) - i\left(g^B(\kappa_{t+1})\right)\right) + (1-\delta^K)\right] c_t}{\left(1-\delta^A\right)\left(1+g^A(\kappa_{t+1})\right)},$$

$$B_{t+1} = B_t(1-\delta^B)\Big(1+g^B(\kappa_{t+1})\Big),$$

[16]See Tabaković (2015) for a general discussion of the local stability analysis for three-dimensional discrete dynamical systems.

where $\Omega(\kappa_{t+1})$ is given by (7.3). This gives

$$\Phi_\kappa^\kappa = \frac{B^* v'(\kappa^*) + 1 - \delta^K}{(1+g^*)\left(1+\epsilon_\kappa^A + \epsilon_\kappa^B\right)} > 0,$$

$$\Phi_c^\kappa = -\frac{B^*}{(1+g^*)\left(1+\epsilon_\kappa^A + \epsilon_\kappa^B\right)} < 0,$$

$$\Phi_B^\kappa = \frac{v(\kappa^*) - c^*}{(1+g^*)\left(1+\epsilon_\kappa^A + \epsilon_\kappa^B\right)} > 0,$$

$$\Phi_\kappa^c = \frac{\beta B^* \left[\left(f''(\kappa^*) - i' g_\kappa^B\right) + \left(f'(\kappa^*) - i(g^B(\kappa^*))\right)(1-\delta^B) g_\kappa^B\right]}{1+g^*} c^* \Phi_\kappa^\kappa$$
$$- \frac{\beta B^* (1-\delta^A) g_\kappa^A}{1+g^*} c^* \Phi_\kappa^\kappa < 0,$$

$$\Phi_c^c = 1 + \frac{\Phi_\kappa^c \Phi_c^\kappa}{\Phi_\kappa^\kappa} > 1,$$

$$\Phi_B^c = \frac{\beta\left(f'(\kappa^*) - i(g^B(\kappa^*))\right) c^*}{(1+g^*)} + \frac{\Phi_\kappa^c \Phi_B^\kappa}{\Phi_\kappa^\kappa} \quad \text{(sign indeterminate)},$$

$$\Phi_\kappa^B = B^*(1-\delta^B) g_\kappa^B \Phi_\kappa^\kappa < 0,$$

$$\Phi_c^B = B^*(1-\delta^B) g_\kappa^B \Phi_c^\kappa > 0,$$

$$\Phi_B^B = 1 + B^*(1-\delta^B) g_\kappa^B \Phi_B^\kappa = 1 + \frac{\Phi_\kappa^B \Phi_B^\kappa}{\Phi_\kappa^\kappa} \in (0,1).$$

The local stability properties of our three-dimensional system are fully determined by the eigenvalues λ_1, λ_2 and λ_3 of the Jacobian matrix. The eigenvalues of the Jacobian result as the solution to the following characteristic polynomial:

$$c(\lambda) \equiv \lambda^3 - tr(J)\lambda^2 + \sum M_2(J)\lambda - \det(J), \tag{7.11}$$

where $tr(J)$ denotes the trace, $\sum M_2(J)$ the sum of principal minors of order two and $\det(J)$ the determinant of the Jacobian matrix. One can show that

$$tr(J) = \lambda_1 + \lambda_2 + \lambda_3 = \Phi_\kappa^\kappa + \Phi_c^c + \Phi_B^B > 0, \tag{7.12}$$
$$\sum M_2(J) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 2\Phi_\kappa^\kappa + \Phi_c^c \Phi_B^B - \Phi_B^c \Phi_c^B > 0, \tag{7.13}$$
$$det(J) = \lambda_1\lambda_2\lambda_3 = \Phi_\kappa^\kappa > 0. \tag{7.14}$$

By Descartes' rule of signs we know that if the terms of a polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Moreover, the number of negative roots is at most equal to the number of continuations in the signs of the coefficients. Inspection of equation (7.11) reveals that it has

(α) either three real positive roots,

(β) or one real positive root and one pair of complex conjugate roots.

Next, evaluate the characteristic polynomial at $\lambda = 1$ to obtain $c(1) = -\Phi_c^B c^*/B^* < 0$, implying that one eigenvalue is of magnitude greater than 1, say $\lambda_1 > 1$. If the remaining two eigenvalues are real then

($\alpha 1$) either both have magnitude greater than one,

($\alpha 2$) or both have magnitude smaller than one.

Otherwise, the remaining two eigenvalues are complex and

($\beta 1$) either have modulus greater than one, i. e., $\mid x \pm \omega i \mid > 1$,

($\beta 2$) or have modulus smaller than one, i. e., $\mid x \pm \omega i \mid < 1$.

To determine the magnitude of the remaining two eigenvalues use equations (7.12) - (7.14) to obtain

$$\mathcal{C}(\lambda_p \lambda_q) = (\lambda_p \lambda_q)^3 - \sum M_2(J)(\lambda_p \lambda_q)^2 + tr(J)det(J)(\lambda_p \lambda_q) - det(J)^2 = 0. \tag{7.15}$$

Here, $(\lambda_p \lambda_q)$ represents any of the three product pairs of the eigenvalues of $c(\lambda)$. Moreover, we have that, by construction, the three roots of $\mathcal{C}(\lambda_p \lambda_q)$ are the three product pairs of the eigenvalues of $c(\lambda)$ and therefore all roots of $\mathcal{C}(\lambda_p \lambda_q)$ are greater than zero.

(α) Consider the case in which all eigenvalues are real. Without loss of generality, let $\lambda_1 > 1$. We can determine the magnitude of the remaining eigenvalues by evaluating $\mathcal{C}(\lambda_p \lambda_q)$ at $\lambda_p \lambda_q = 1$. Some algebra delivers

$$\mathcal{C}(1) = \underbrace{B^* \left(1 - \delta^B\right) g_\kappa^B}_{<0} \underbrace{\left[\frac{\beta\,(f' - i)}{1 + g^*} c^* \Phi_c^\kappa\right]}_{<0} + \underbrace{\Phi_B^\kappa}_{>0} \underbrace{\left[\Phi_\kappa^B - B^* \left(1 - \delta^B\right) g_\kappa^B\right]}_{>0} > 0.$$

This implies that either all three roots of $\mathcal{C}$ are smaller than one, or that one is smaller than one and two are greater than one. Since the three roots of $\mathcal{C}(\lambda_p \lambda_q)$ are given by the three product pairs of the eigenvalues of $c(\lambda)$ it follows that only alternative ($\alpha 2$) is compatible with $c(1) < 0$ *and* $\mathcal{C}(1) > 0$. Therefore, we may conclude that if all eigenvalues are real and positive, the system is asymptotically locally stable in the state space.

(β) Consider now the case of one real eigenvalue and a pair of complex eigenvalues. Without loss of generality let $\lambda_1 > 1$ be the real eigenvalue and let λ_2, λ_3 be the complex conjugate pair. First notice that in this case only one root of $\mathcal{C}(\lambda_p \lambda_q)$ is real, namely, the product of the two complex conjugate eigenvalues of $c(\lambda)$. Then the fact that $\mathcal{C}(1) > 0$ implies that the only real root of $\mathcal{C}$ must be smaller than one which is only possible if $\mid \lambda_{2,3} \mid = \mid x \pm \omega i \mid < 1$. Therefore, we may conclude that if the system features one real eigenvalue and a pair of complex conjugate eigenvalues, it will be asymptotically locally stable in the state space. ■

7.8 Proof of Proposition 7

Proposition 7 claims that for given $L > 0$ and initial values $(A_{-1}, K_0) > 0$ the transitional dynamics of the dynamic competitive equilibrium is given by a unique sequence $\{\kappa_t, c_t\}_{t=0}^{\infty}$ that satisfies equations (3.22) - (3.25). Mutatis mutandis, the proof is essentially the same as the one of Proposition 3. Without capital-augmenting technical change $\Omega(\kappa_{t+1})$ of (7.3) boils down to

$$\Omega(\kappa_{t+1}) = \left(1 - \delta^A\right)\left(1 + g^A\left(\kappa_{t+1}\right)\right)\kappa_{t+1}. \tag{7.16}$$

Then, equation (3.22) may be rewritten as

$$\Omega(\kappa_{t+1}) = v(\kappa_t) - c_t + \left(1 - \delta^K\right)\kappa_t. \tag{7.17}$$

However, $\Omega(\kappa_{t+1})$ of (7.16) inherits all relevant properties of its counterpart of (7.3). Hence, there is a unique $\kappa_{t+1} > 0$ that satisfies equation (3.22) given $(\kappa_t, c_t) \in \mathbb{R}^2_{++}$. With this value of κ_{t+1} and $c_t > 0$, equation (3.23) delivers a unique $c_{t+1} > 0$. ■

7.9 Proof of Proposition 8

1. Equations (3.27) and (3.28) follow immediately from the corresponding equations (3.22) - (3.23) of the dynamical system for reasons discussed in the main text. Obviously, in the steady state the transversality condition (3.24) is also satisfied. Assumption 3 ensures that (3.28) is satisfied for a unique value $\infty > \kappa^* > 0$. It remains to be shown that $c^* > 0$, i. e., $v(\kappa^*) > \kappa^*(g^* + \delta^K)$. A sufficient condition for this to hold may be obtained for $\beta = 1$. Using $\beta = 1$ in (3.28) the latter inequality may be expressed as

$$v(\kappa^*) > \kappa^* f'(\kappa^*). \tag{7.18}$$

Since equilibrium profits are zero we have $f(\kappa) = \kappa f'(\kappa) + c(q^A)$. Evaluating the latter at the steady state yields $v(\kappa^*) = \kappa^* f'(\kappa^*) + \left(1 + (g^A)\right) i'(g^A)$, where g^A is evaluated at κ^*. Then, inequality (7.18) becomes

$$\begin{aligned}\kappa^* f'(\kappa^*) + \left(1 + \left(g^A\right)\right) i'\left(\left(g^A\right)\right) &> \kappa^* f'(\kappa^*),\\ \left(1 + \left(g^A\right)\right) i'\left(\left(g^A\right)\right) &> 0.\end{aligned}$$

2. The expression for the steady-state growth rate follows from (2.35) and Proposition 1. We give the explanation of the remaining findings in the main text. ■

7.10 Proof of Proposition 9

1. Implicit differentiation of (3.28) reveals that $d\kappa^*/d\beta > 0$, hence $(\kappa^*)' > \kappa^*$. Further, $dg^*/d\beta = \left(dg^*/dq^A\right)\left(dg^A\left(\kappa^*\right)/d\kappa\right)\left(d\kappa^*/d\beta\right) > 0$ as all three derivatives are strictly positive. Hence, $(g^*)' > g^*$. Diminishing returns to capital and $d\kappa^*/d\beta > 0$ deliver $(R^*)' < R^*$. Since $v'\left(\kappa^*\right)$ cannot be signed in general, the effect of the discount factor on c^* is indeterminate in general.
2. Implicit differentiation (3.28) yields $d\kappa^*/d\delta^A > 0$, hence $(\kappa^*)' > \kappa^*$. In conjunction with diminishing returns to capital, we have $(R^*)' < R^*$. The effect of δ^A on g^* is immediate from $\partial g^*/\partial\delta^A = -\left(1 + g^A\left(\kappa^*\right)\right) < 0$. Again, the effect on c^* through equation (3.27) remains indeterminate in general. ■

7.11 Proof of Proposition 10

To be found in the main text. ■

7.12 Proof of Proposition 11

To be found in the main text. ■

7.13 Proof of Proposition 12

1. Equations (5.13) and (5.14) follow immediately from equations (5.2) and (5.3) for reasons given in the main text. Moreover, the tranversality conditions are satisfied. Assumption 1 and Assumption 2 guarantee a strictly positive solution to (5.11) and (5.14). Showing that $c^{**} > 0$ is analogous to showing that $c^{*} > 0$ in the competitive equilibrium.

2. The expression for the steady-state growth rate, g^{**}, follows from (2.35) and Proposition 1. The remaining results follow from the arguments given in the main text. ■

7.14 Proof of Proposition 13

With $\sigma^B \in (0, 1)$ the relevant first-order condition is (4.1). In steady state, the latter gives rise to a function $\kappa\left(\sigma^B\right)$ with $\kappa'\left(\sigma^B\right) > 0$ that satisfies

$$f'\left(\kappa\left(\sigma^B\right)\right) = \left(1 - \sigma^B\right) c\left(\frac{\delta^B}{1 - \delta^B}\right). \tag{7.19}$$

Hence, the desired value for σ^B is such that $\kappa\left(\sigma^B\right) = \kappa^{**}$. Using (5.4) gives $\left(\sigma^B\right)^{**}$ of (5.15).

With $\sigma^A \in (0, 1)$ the relevant first-order condition is now (4.8). Using (5.5) at $\kappa_t = \kappa^{**}$ and $q_t^A = \left(q^A\right)^{**}$, the latter determines $\left(\sigma^A\right)^{**}$ as stated in (5.16). ■

References

ACEMOGLU, D. (2003a): "Factor Prices and Technical Change: From Induced Innovations to Recent Debates," in *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*, ed. by P. Aghion, R. Frydman, J. Stiglitz, , and M. Woodford, pp. 464–491. Princeton University Press, Princeton, New Jersey.

——— (2003b): "Labor- and Capital-Augmenting Technical Change," *Journal of the European Economic Association*, 1(1), 1–37.

——— (2009): *Introduction to Modern Economic Growth*. Princeton University Press, Princeton, New Jersey.

AGHION, P., AND P. HOWITT (1992): "A Model of Growth through Creative Destruction," *Econometrica*, 60(2), 323–351.

BOLDRIN, M., AND D. K. LEVINE (2002): "Factor Saving Innovation," *Journal of Economic Theory*, 105(1), 18–41.

——— (2008): "Perfectly Competitive Innovation," *Journal of Monetary Economics*, 55(3), 435–453.

BURMEISTER, E., AND R. A. DOBELL (1970): *Mathematical Theories of Economic Growth*. Collier-Macmillian, London.

CASELLI, F., AND W. J. COLEMAN (2006): "The World Technology Frontier," *American Economic Review*, 96(3), 499–522.

CASS, D. (1965): "Optimum Growth in an Aggregative Model of Capital Accumulation," *Review of Economic Studies*, 32, 233–240.

DRANDAKIS, E. M., AND E. S. PHELPS (1966): "A Model of Induced Invention, Growth, and Distribution," *The Economic Journal*, 76, 823–840.

FUNK, P. (2002): "Induced Innovation Revisited," *Economica*, 69, 155–171.

GROSSMAN, G. M., AND E. HELPMAN (1991): *Innovation and Growth in the Global Economy*. MIT Press, Cambridge, MA.

HELLWIG, M., AND A. IRMEN (2001): "Endogenous Technical Change in a Competitive Economy," *Journal of Economic Theory*, 101, 1–39.

HICKS, J. R. (1932): *The Theory of Wages*. Macmillan, London, 1st edn.

IRMEN, A. (2011): "Steady-State Growth and the Elasticity of Substitution," *Journal of Economic Dynamics and Control*, 35(8), 1215–1228.

——— (2013a): "Capital- and Labor-Saving Technical Change in an Aging Economy," *International Economic Review,* forthcoming.

——— (2013b): "A Generalized Steady-State Growth Theorem," CREA Discussion Paper Series 2013-26, Center for Research in Economic Analysis, University of Luxembourg.

JONES, C. I. (2005): "Growth and Ideas," in *Handbook of Economic Growth*, ed. by P. Aghion, and S. Durlauf, vol. 1 of *Handbook of Economic Growth*, chap. 16, pp. 1063–1111. Elsevier.

KALDOR, N. (1961): "Capital Accumulation and Economic Growth," in *The Theory of Capital*, ed. by F. A. Lutz, and D. C. Hague, pp. 177–222. Macmillan & Co. LTD., New York: St. Martin's Press.

KARABARBOUNIS, L., AND B. NEIMAN (2014): "The Global Decline of the Labor Share," *The Quarterly Journal of Economics*, 129(1), 61–103.

KENNEDY, C. (1964): "Induced Bias in Innovation and the Theory of Distribution," *The Economic Journal*, 74, 541–547.

KLUMP, R., P. MCADAM, AND A. WILLMAN (2007): "Factor Substitution and Factor Augmenting Technical Progress in the US: A Normalized Supply-Side System Approach," *Review of Economics and Statistics*, 89(1), 183–192.

KOOPMANS, T. (1965): "On the Concept of Optimal Economic Growth," in *(Study Week on) The Economic Approach to Development Planning*, ed. by P. A. S. S. Varia, pp. 225–287. North Holland, Amsterdam.

NORDHAUS, W. D. (1973): "Some Sceptical Notes Thoughts on the Theory of Induced Innovation," *Quarterly Journal of Economics*, 87(2), 208–219.

RAMSEY, F. P. (1928): "A Mathematical Theory of Savings," *The Economic Journal*, 38, 543–559.

ROMER, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98(5), S71–S102.

——— (2015): "Mathiness in the Theory of Economic Growth," *American Economic Association*, 105(5), 89–93.

SAMUELSON, P. (1965): "A Theory of Induced Innovation along Kennedy – Weisäcker Lines," *Review of Economics and Statistics*, 47(3), 343–356.

——— (1966): "Rejoinder: Agreements, Disagreements, Doubts, and the Case of Induced Harrod-Neutral technical Change," *Review of Economics and Statistics*, 48(4), 444–448.

SAMUELSON, P. A. (1962): "Parable and Realism in Capital Theory: The Surrogate Production Function," *Review of Economic Studies*, 29, 193 – 206.

TABAKOVIĆ, A. (2015): "On the Characterization of Steady States in Three-Dimensional Discrete Dynamical Systems," CREA Discussion Paper Series 2015-16, Center for Research in Economic Analysis, University of Luxembourg.

UZAWA, H. (1961): "Neutral Inventions and the Stability of Growth Equilibrium," *Review of Economic Studies*, 28(2), 117–124.

VON WEIZSÄCKER, C. C. (1962): "A New Technical Progress Function," *mimeo, MIT; published in: German Economic Review (2010)*, 11, 248–265.

ZEIRA, J. (1998): "Workers, Machines, And Economic Growth," *Quarterly Journal of Economics*, 113(4), 1091–1117.