CESifo Working Papers

The Optimal Defense of Network Connectivity

Dan Kovenock
Brian Roberson

CESIFO WORKING PAPER NO. 5653
CATEGORY 12: EMPIRICAL AND THEORETICAL METHODS
DECEMBER 2015

ISSN 2364-1428

CESifo *Working Paper No. 5653*

The Optimal Defense of Network Connectivity

Abstract

Maintaining the security of critical infrastructure networks is vital for a modern economy. This paper examines a game-theoretic model of attack and defense of a network in which the defender's objective is to maintain network connectivity and the attacker's objective is to destroy a set of nodes that disconnects the network. The conflict at each node is modeled as a contest in which the player that allocates the higher level of force wins the node. Although there are multiple mixed-strategy equilibria, we characterize correlation structures in the players' multivariate joint distributions of force across nodes that arise in all equilibria. For example, in all equilibria the attacker utilizes a stochastic 'guerrilla warfare' strategy in which a single random [minimal] set of nodes that disconnects the network is attacked.

JEL-codes: C720, D740.

Keywords: allocation game, asymmetric conflict, attack and defense, Colonel Blotto Game, network connectivity, weakest-link, best-shot.

Dan Kovenock
Economic Science Institute
Argyros School of Business and
Economic /, Chapman University
One University Drive
USA – Orange, CA 92866
kovenock@chapman.edu

*Brian Roberson**
Purdue University
Department of Economics
Krannert School of Management
403 W. State Street
USA – West Lafayette, IN 47907
brobers@purdue.edu

*corresponding author

November 12, 2015
Earlier versions of this paper circulated under the titles "Terrorism and the Optimal Defense of Networks of Targets" and "The Optimal Defense of Networks of Targets." We have benefited from the helpful comments of Kai Konrad, Daniel G. Arce, and participants in the 15th SAET Conference in Cambridge. The authors, of course, remain solely responsible for any errors or omissions.

1 Introduction

In the literature on game-theoretic models of attack and defense there has been a growing interest in the attack and defense of networks of targets. One focus of the work on the strategic role of network structure in this context is the role that strategic complementarities among targets play in creating structural asymmetries between the attack and defense of a network. For example in complex infrastructure networks — such as communication systems, electrical power grids, water and sewage systems, oil pipeline systems, transportation systems, and cyber security systems — there often exist particular targets or combinations of targets which if destroyed would be sufficient to disconnect the network and create a terrorist "spectacular."

The focus of this article is on strategic behavior in the attack and defense of network connectivity. A network (or graph) is connected if there exists a path between every pair of nodes. Similarly, a network is k-connected if there exist at least k internally disjoint paths (i.e. paths that do not share internal nodes) between every pair of nodes. Note that connectivity provides a measure of the robustness of a network with respect to node failure/destruction in that a k-connected network remains connected when any $k-1$ nodes are destroyed. Connectivity is a classic graph theory problem, dating back to Menger's (1927) work on cuts and internally-disjoint paths.[1] In our game of attack and defense of network connectivity, the defender's objective is to maintain network connectivity and the attacker's objective is to successfully attack a set of nodes that disconnects the network. Note that each minimal set of nodes that disconnects the network — known as a minimal cut set or minimal separator (and by minimal it is meant that no proper subset of the minimal cut set disconnects the network) — may be thought of as possessing *best-shot* redundancy in that the minimal cut set is successfully defended if and only if the defender successfully defends at least one node within the set.[2] Conversely, the network may be thought of as having *weakest-link* exposure in that the network is successfully defended if and only if the defender successfully defends all minimal cut sets within the network.

We examine properties arising in the set of Nash equilibria of a simultaneous-move game of attack and defense of a network. Each node in the network is vulnerable to attack and at each node the conflict is modeled as a deterministic contest in which the player who allocates the higher level of force wins the node with probability one. In this game the attacker's

[1]For further details, see Ford and Fulkerson (1962).

[2]See Hirshleifer (1983), who coins the terms best-shot and weakest-link in the context of the voluntary provision of public goods.

objective is to maximize the product of the probability of disconnecting the network by winning at least one cut set and his payoff for the successful attack of at least one cut set, v_A, net of the expenditure on forces, which are allocated at a constant unit cost. Conversely, the defender's objective is to maximize the product of the probability of maintaining connectivity and his payoff for maintaining connectivity, v_D, net of his expenditure on forces, also allocated at constant unit cost. A distinctive feature of this environment is that a mixed strategy is a joint distribution function in which the randomization in the force allocation to each node is represented as a separate dimension. A pair of equilibrium joint distribution functions specifies not only each player's randomization in force expenditure to each node, but also the correlation structure of the force expenditures across the node set. We construct a Nash equilibrium pair of distribution functions and, in the case in which the network has disjoint minimal cut sets, completely characterize the unique set of Nash equilibrium univariate marginal distributions and the unique equilibrium payoff of each player. Furthermore, we show that in any equilibrium the attacker launches an attack on at most one minimal cut set. Similarly, the defender randomly chooses one node from each minimal cut set to defend. Noting that a network is connected if and only if the network contains a spanning tree (a subgraph of the network that is a tree and contains all of the nodes of the tree), the defender's equilibrium strategy may be interpreted as choosing a random spanning tree to defend, where the spanning tree is implicitly determined by the choice of the one node in each minimal cut set that is defended.

Closely related is the literature on sequential-move models of the attack and defense of a network of targets such as Dziubiński and Goyal (2013a, b).[3] In the two-stage game examined in Dziubiński and Goyal (2013a) the defender moves first and chooses which nodes to defend, where defense is perfect in the sense that a defended node survives an attack with probability one. Then, the attacker observes which nodes have been defended and chooses which nodes to attack. Any undefended node that is attacked is destroyed with probability one. Note that the defender is an exogenously imposed leader — implying that the attacker's

[3]See also Goyal and Vigier (2014) who examine a zero-sum model of the attack and defense of a network of targets with contagion in which the conflict at each node is determined by the Tullock contest success function. Also related are sequential-move reliability-theoretic models such as Bier, Oliveros, and Samuelson (2007), and Powell (2007a, b). In these models, defensive resources increase the stochastic reliability of a target in the event of an attack, the defender's payoff is additive with respect to the values of the surviving targets, and it is exogenously specified that the attacker uses a guerrilla warfare strategy consisting of an attack on a single target. In the case in which the defender has private information concerning the vulnerability of the individual targets, as in Powell (2007b), this sequential-move structure gives rise to an interesting signaling problem in that the defender would like to protect the most vulnerable targets but does not want to signal to the attacker which targets are the most vulnerable.

force allocations can be made contingent on the defender's allocation. In equilibrium, the defender chooses either to protect the minimum set of nodes that ensures the survival of a spanning tree, or to leave the network undefended. In Dziubiński and Goyal (2013b) the defender first designs the network and allocates defensive resources across nodes, where defense is again assumed to be perfect. Then, the attacker chooses an exogenously specified number of nodes to attack at zero cost. Although these papers simplify the conflict at each node by focusing on the case of sequential moves and perfect defense, these models feature a rich network structure that, in addition to providing insights on the connectivity problem, allows for a general value function for the residual network and provides insight into the defender's choice of network structure.

In contrast, our simultaneous-move model is motivated by applications such as information or transportation network defense or border defense, where attackers must either take actions before being certain of the allocation of defensive resources or where strategies like random monitoring or deployment may be employed by defenders and, thus, defensive resources can either be concealed or randomly allocated with sufficient speed that it is difficult to argue that attacker allocations can be made contingent on defensive allocations. Although we take the network as given and restrict our focus to only the connectivity problem, we allow for a contest structure with nontrivial conflict at each node that results in rich equilibrium behavior in which mixed strategies involve multivariate joint distributions specifying the force expenditure to each node and the correlation structure of the force expenditures across the node set. Furthermore, by endogenizing the attacker's entry and force expenditure decisions, our approach sheds light not only on the conditions under which the assumption of one attack is likely to hold, but also related issues such as how the defender's actions can decrease the number of attacks.

Our results on endogenous force correlation structures in games of attack and defense are closely related to the literature on the classic Colonel Blotto game.[4] Originating with Borel (1921), the Colonel Blotto game is a two-player game in which each player allocates his fixed level of forces across a finite number of battlefields, within each battlefield the higher allocation wins, and each player maximizes the number of battlefield wins. As in our game of attack and defense, a mixed strategy is a joint distribution function. However, in the Colonel

[4]Recent work on Blotto-type games includes extensions such as: asymmetric players (Roberson 2006, Hart 2008, Weinstein 2012, Dziubiński 2013, Macdonell and Mastronardi 2015), non-constant-sum variations (Kvasov 2007, Hortala-Vallve and Llorente-Saguer 2010, 2012, Roberson and Kvasov 2012), alternative definitions of success (Golman and Page 2009, Tang, Shoham, and Lin 2010, Rinott, Scarsini, and Yu 2012), and political economy applications (Laslier 2002, Laslier and Picard 2002, Roberson 2008, Thomas 2012).

Blotto game it is the budget constraint that creates a linkage between the force allocations to the individual battlefields. Allocating force to a specific battlefield reduces the level of forces that can be allocated to other battlefields. Conversely, the linkages in our game of attack and defense arise because of the definition of success for each of the players.[5] There are a number of related games that display similar objective-based linkages. For example, Szentes and Rosenthal (2003a) examine the so-called "chopstick auction" in which three identical objects are separately, but simultaneously, auctioned and each of two players wins a fixed prize of known and common value if and only if he wins at least two of the three objects. The player placing the highest bid on a given object wins the object. Szentes and Rosenthal examine both winner-pay and all-pay versions of this auction. In the winner-pay version, a bid that does not win an object is refunded. In the all-pay version, all bids are forfeited. Szentes and Rosenthal (2003b) extend this analysis to a related n-player game in which each player's objective is to secure a super-majority of auction wins. The model we examine here differs in that the objective-based linkages are asymmetric across players.[6]

Our model features an environment in which random noise plays little role in determining the outcomes at the nodes — at each node the player with the larger resource expenditure for the node wins the node with certainty.[7] Closely related is the literature on simultaneous-move multidimensional resource allocation games in which the conflict at each node features a softer form of competition that emphasizes the role of random noise in determining the outcomes at the nodes.[8] For example, under the Tullock contest success function (henceforth, CSF) the probability that a player wins a node is equal to the ratio of the player's resource

[5]In a related attack and defense game, Bernhardt and Polborn (2010) examine a cost-based asymmetry between attack and defense. In that case, the "committed" attacker experiences no opportunity costs from allocating forces and continues attacking targets until either he runs out of targets or is defeated. Also related is Kovenock, Sarangi, and Wiser (2015) which examines a modified form of the board game Hex in which a contest arises at each cell on the board with the contest winner determined by the Tullock contest success function.

[6]See Kovenock and Roberson (2012) for a survey of cost- and objective-based linkages in multidimensional resource allocation games. See also Arce, Kovenock and Roberson (2012) which examines a related game with multiple attack technologies.

[7]This corresponds to the limiting case of the general ratio-form contest success function $a^m/(a^m + d^m)$ where m is set to ∞ and a and d are the two players' allocations of force. The parameter $m \in \mathbb{R}_+$ is inversely related to the level of noise in the conflict: low values imply a large amount of noise and high values correspond to low or no noise. Because (for a single contest with linear costs) pure-strategy equilibria fail to exist for all m greater than 2 and there exist equilibria in one-shot contests that are payoff equivalent to the $m = \infty$ case whenever $m > 2$ (Baye, Kovenock, De Vries 1994, Alcalde and Dahm 2010), the case of $m = \infty$ is viewed as an important theoretical benchmark that is relevant for all $m > 2$.

[8]See for example Snyder (1989) and Klumpp and Polborn (2006) which examine related games featuring the symmetric majoritarian objective in the context of politicians engaged in a campaign resource allocation game.

expenditure at the node to the sum of all of the players' expenditures at the node. The case of the attack and defense of a network in which each minimal cut set consists of a single node with the outcome at each node determined by the Tullock CSF is examined by Clark and Konrad (2007), who find that under this softer form of competition the attacker optimally chooses a complete coverage strategy in which each and every node that is a minimal cut set is attacked with certainty. In contrast, we find that when the factors influencing node outcomes are explicitly captured in the model, with unmodeled factors or "noise" playing little or no role, attackers utilize a stochastic guerilla warfare strategy in all equilibria. For the special case in which the network consists of only singleton minimal cut sets, this involves a single random node that is a cut set being attacked, but with a positive probability that each cut set is chosen as the one to be attacked.[9]

Section 2 presents the model of attack and defense with network connectivity. Section 3 characterizes a Nash equilibrium in mixed strategies and explores properties shared by all Nash equilibrium mixed strategies. Section 4 concludes.

2 The Model

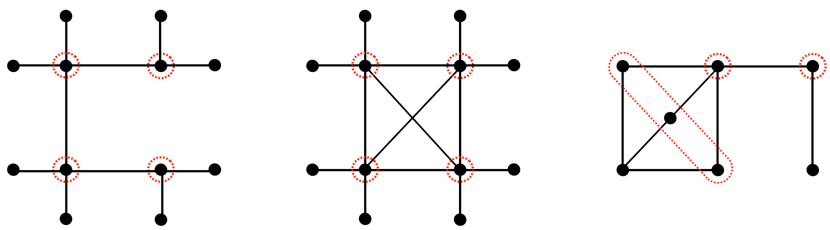
Players

The model is formally described as follows. Two players, an attacker, A, and a defender, D, simultaneously allocate their forces across the $\widehat{n} \geq 2$ nodes in the network $G = (\widehat{N}, E)$ with node-set $\widehat{N}$ and edge-set E. G is a *connected* network, and thus, there exists at least one path between every pair of nodes in $\widehat{N}$. A network is said to be disconnected if it is not connected.

For a connected network G, a (node) cut set is a set of nodes $C \subset \widehat{N}$ whose removal from G results in a disconnected network. A *minimal cut set* is a cut set satisfying the property that no proper subset forms a cut set. Let $\mathcal{B}$ denote the index set consisting of indices for all minimal cut sets of network G. Let N_j denote the index set consisting of indices for all nodes in minimal cut set $j \in \mathcal{B}$, and let $n_j \equiv |N_j|$ denote the number of nodes in minimal

[9]Kovenock, Roberson, and Sheremeta (2010) experimentally examine behavior in a specification of the game of attack and defense of a network with only singleton minimal cut sets where the conflict at each target is modeled either by the Tullock CSF or the specification given in this paper, the auction CSF. Consistent with the theoretical prediction under the auction CSF, attackers utilize a stochastic guerilla warfare strategy — in which a single random node is attacked — more than 80% of the time. Under the lottery CSF, attackers utilize the stochastic guerilla warfare strategy almost 45% of the time, in contrast to the theoretical prediction that the attacker covers all of the nodes, which is observed less than 30% of the time.

cut set j. Figure 1 provides three examples — a tree network, a core-periphery network,[10] and an arbitrary network — with their respective minimal cut sets. Note that in each of these examples the corresponding minimal cut sets are disjoint. In the results section we focus first on networks with disjoint minimal cut sets, and then show how these results can be extended to networks with overlapping minimal cut sets.



(a) Tree Network (b) Core-Periphery Network (c) Arbitrary Network

Figure 1: Example Networks with Disjoint Minimal Cut Sets

A minimal cut set is successfully defended if the defender allocates at least as high a level of force as the attacker to at least one node within the set. Conversely, an attack on a minimal cut set is successful if the attacker allocates a strictly higher level of force than the defender to each node in the set. Let x_A^i (x_D^i) denote the level of force allocated by the attacker (defender) to node i. For each $j \in \mathcal{B}$ define

$$\iota^j = \begin{cases} 1 & \text{if } \forall\ i \in N_j \mid x_A^i > x_D^i \\ 0 & \text{otherwise} \end{cases}.$$

Observe that for each node, the player that allocates the strictly higher level of force wins that node (with ties going to the defender), but in order to win the minimal cut set the

[10]Here we use the definition of a core-periphery network from Bramoullé (2007): in a core-periphery network the node set E can be partitioned into two subsets, core nodes and periphery nodes, such that each core node is directly connected to all other core nodes, while each periphery node is connected to one core node and is not connected to any periphery node.

attacker must win all of the nodes.

The players are risk neutral and have asymmetric objectives. The attacker's objective is to successfully attack at least one minimal cut set, and the attacker's payoff for the successful attack of at least one minimal cut set is $v_A > 0$. The attacker's payoff function is given by

$$\pi_A(\mathbf{x}_A, \mathbf{x}_D) = v_A \max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) - \sum_{i\in\widehat{N}} x_A^i.$$

The defender's objective is to preserve connectivity, and the defender's payoff for successfully defending connectivity is $v_D > 0$. The defender's payoff function is given by

$$\pi_D(\mathbf{x}_A, \mathbf{x}_D) = v_D\left(1 - \max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right)\right) - \sum_{i\in\widehat{N}} x_D^i.$$

For each player, the force allocated to each node must be nonnegative.

Because nodes that are not contained in any minimal cut set ($\widehat{N} \setminus \cup_{j\in\mathcal{B}} N_j$) do not factor into the players' payoff functions except insofar as resources may be wasted on them, it is clear that in any equilibrium $x_A^i = x_D^i = 0$ for all $i \in \widehat{N} \setminus \cup_{j\in\mathcal{B}} N_j$. Henceforth, we restrict our focus to the set of nodes that are contained in the union of the minimal cut sets, denoted by $N \equiv \cup_{j\in\mathcal{B}} N_j$, with $n \equiv |N|$ denoting the number of such nodes.

It is important to note that our formulation utilizes an auction contest success function.[11] It is well known that, because behavior is invariant with respect to positive affine transformations of utility, all-pay auctions in which players have different constant unit costs of resources may be transformed into behaviorally equivalent all-pay auctions with identical unit costs of resources, but suitably modified valuations. This result extends directly to the environment examined here, and thus, our focus on asymmetric valuations also covers the case in which the players have different constant unit costs of resources.

Also observe that in the formulation described above the network features a weakest-link exposure problem for the defender. That is, if the defender loses a single minimal cut set then connectivity is lost. Conversely, each minimal cut set features best-shot redundancy in that the minimal cut set is successfully defended if the defender wins at least one node in the set.

[11] See Baye, Kovenock, and de Vries (1996).

Strategies

It is clear that there is no pure-strategy equilibrium for this class of games. For player $k \in \{A, D\}$, a mixed strategy is an n-variate distribution function $P_k : \mathbb{R}^n_+ \to [0, 1]$ with one-dimensional marginal distribution functions $\{P^i_k\}_{i \in N}$, one univariate marginal distribution function for each node $i \in N$. The n-tuple of player k's allocation of force across the n nodes is a random n-tuple drawn from the n-variate distribution function P_k.

Model of Attack and Defense of Network Connectivity

The model of attack and defense of network connectivity, which we label

$$ADN\{G, v_A, v_D\},$$

is the one-shot game in which players compete by simultaneously announcing mixed strategies, each node is won by the player that provides the higher allocation of force for that node, ties are resolved as described above, and players' payoffs, π_A and π_D, are specified above.

3 Results

It is useful to introduce a simple summary statistic that captures both the asymmetry in the players' valuations and the structural asymmetries arising from the network structure.

Definition 1. *Let $\alpha = v_D/(v_A[\sum_{j \in \mathcal{B}} \frac{1}{n_j}])$ denote the normalized relative strength of the defender.*

Several properties of this summary statistic should be noted. First, the normalized relative strength of the defender is increasing in the relative valuation of the defender to the attacker (v_D/v_A) and, for each minimal cut set $j \in \mathcal{B}$, is increasing in the number of nodes (or best-shot redundancy) n_j in cut set j. In particular, the normalized relative strength of the defender is increasing in the total best-shot redundancy arising in G, as measured by $1/\sum_{j \in \mathcal{B}} \frac{1}{n_j}$.

For all G with disjoint minimal cut sets, Theorem 1 establishes the uniqueness of: (i) the players' equilibrium expected payoffs and (ii) the players' sets of univariate marginal distributions. Theorem 1 also provides a pair of equilibrium mixed strategies. Case (1) of Theorem 1 examines the parameter configurations for which the defender has a normalized relative strength advantage, i.e. $\alpha \geq 1$. Case (2) of Theorem 1 addresses the parameter configurations for which the defender has a normalized relative strength disadvantage, i.e. $\alpha < 1$. It is important to note that the stated equilibrium mixed strategies (n-variate distributions) are not unique. However, in Propositions 1-3 we characterize properties of optimal attack and defense that hold in all equilibria.

Theorem 1. *For any parameter configuration of the game $ADN\{G, v_A, v_D\}$ in which G has disjoint minimal cut sets there exists a Nash equilibrium. Moreover, there exists a unique set of Nash equilibrium univariate marginal distributions and a unique equilibrium payoff for each player.*

(1) *If $\alpha \geq 1$, then for each $j \in \mathcal{B}$ and $i \in N_j$, player A's univariate marginal is, for $x^i \in [0, \frac{v_A}{n_j}]$,*

$$P_A^i\left(x^i\right) = 1 - \frac{v_A}{n_j v_D} + \frac{x^i}{v_D}.$$

Similarly, player D's univariate marginal is

$$P_D^i\left(x^i\right) = 1 - \frac{1}{n_j} + \frac{x^i}{v_A}.$$

The expected payoff for player A is 0, and the expected payoff for player D is $v_D(1 - \frac{1}{\alpha})$.

(2) *If $\alpha < 1$, then for each $j \in \mathcal{B}$ and $i \in N_j$, player A's univariate marginal is, for $x^i \in [0, \frac{\alpha v_A}{n_j}]$,*

$$P_A^i\left(x^i\right) = 1 - \frac{\alpha v_A}{n_j v_D} + \frac{x^i}{v_D}.$$

Similarly, player D's univariate marginal is

$$P_D^i\left(x^i\right) = 1 - \frac{\alpha}{n_j} + \frac{x^i}{v_A}.$$

The expected payoff for player D is 0, and the expected payoff for player A is $v_A(1 - \alpha)$.

One Nash equilibrium of $ADN\{G, v_A, v_D\}$ is for each player to allocate his forces according to the following n-variate distribution functions:

(1) *If $\alpha \geq 1$, then for player A and $\boldsymbol{x} \in \prod_{j \in \mathcal{B}} [0, \frac{v_A}{n_j}]^{n_j}$*

$$P_A\left(\boldsymbol{x}\right) = 1 - \frac{1}{\alpha} + \frac{\sum_{j \in \mathcal{B}} \min_{i \in N_j}\{x^i\}}{v_D}.$$

Similarly for player D and $\boldsymbol{x} \in \prod_{j \in \mathcal{B}} [0, \frac{v_A}{n_j}]^{n_j}$

$$P_D(\boldsymbol{x}) = \min \left(\left\{ \frac{\sum_{i \in N_j} x^i}{v_A} \right\}_{j \in \mathcal{B}} \right).$$

(2) *If $\alpha < 1$, then for player A and* $\boldsymbol{x} \in \prod_{j \in \mathcal{B}} [0, \frac{\alpha v_A}{n_j}]^{n_j}$

$$P_A(\boldsymbol{x}) = \frac{\sum_{j \in \mathcal{B}} \min_{i \in N_j} \{x^i\}}{v_D}.$$

Similarly for player D and $\boldsymbol{x} \in \prod_{j \in \mathcal{B}} [0, \frac{\alpha v_A}{n_j}]^{n_j}$

$$P_D(\boldsymbol{x}) = 1 - \alpha + \min \left(\left\{ \frac{\sum_{i \in N_j} x^i}{v_A} \right\}_{j \in \mathcal{B}} \right).$$

Proof. The proof of the uniqueness of the players' equilibrium expected payoffs and sets of univariate marginal distributions is given in the Appendix. We now establish that the pair of n-variate distribution functions given in case (1) constitute an equilibrium for $\alpha \geq 1$. The proof of case (2) is analogous. The Appendix (see the proof of Proposition 1) establishes that in any n-tuple drawn from any equilibrium n-variate distribution P_A player A allocates a strictly positive level of force to at most one minimal cut set. Although not a necessary condition for equilibrium, the P_A described in Theorem 1 displays the property that for the minimal cut set which receives the strictly positive level of force the force allocated to each node in that set is an almost surely increasing function of the force allocated to any other node in that cut set. The Appendix (see the proof of Proposition 1) also establishes that in any n-tuple drawn from any equilibrium n-variate distribution P_D player D allocates a strictly positive level of force to at most one node in each minimal cut set.

We will now show that, for each player, each point in the support of their equilibrium n-variate distribution function, P_A or P_D, given in case (1) of Theorem 1 results in the same expected payoff, and then show that there are no profitable deviations from this support.[12] When $x_A^i > 0$ for every $i \in N_j$, the probability that player A wins every node in minimal cut set $j \in \mathcal{B}$ is given by the n_j-variate marginal distribution $P_D(\{x_A^i\}_{i \in N_j}, \{\{\frac{v_A}{n_{j'}}\}_{i \in N_{j'}}\}_{j' \in \mathcal{B} | j' \neq j})$, which we denote as $P_D^{N_j}(\{x_A^i\}_{i \in N_j})$. Given that player D is using the equilibrium strategy P_D

[12] Except for possibly at points of discontinuity of his expected payoff function, each player k must make his equilibrium expected payoff at each point in the support of his equilibrium strategy, P_k.

described above, the payoff to player A for any allocation of force $\mathbf{x}_A \in \mathbb{R}^n_+$ which allocates a strictly positive level of force only to the nodes in minimal cut set $j \in \mathcal{B}$, and allocates zero force to every other node is

$$\pi_A(\mathbf{x}_A, P_D) = v_A P_D^{N_j}\left(\{x_A^i\}_{i\in N_j}\right) - \sum_{i\in N_j} x_A^i.$$

Simplifying,

$$\pi_A(\mathbf{x}_A, P_D) = v_A\left(\frac{\sum_{i\in N_j} x_A^i}{v_A}\right) - \sum_{i\in N_j} x_A^i = 0.$$

Thus, the expected payoff to player A from allocating a strictly positive level of force to only one minimal cut set is 0 regardless of which minimal cut set is attacked.

For player A, the only possible payoff increasing deviation from his mixed strategy is to allocate a strictly positive level of force to two or more minimal cut sets. Beginning with the case in which player A attacks two minimal cut sets $j, j' \in \mathcal{B}$ with $x_A^i > 0$ for every $i \in N_j \cup N_{j'}$, the probability that player A wins all of the nodes in both cut sets $j,\, j' \in \mathcal{B}$ is given by the $(n_j + n_{j'})$-variate marginal distribution $P_D(\{x_A^i\}_{i\in N_j\cup N_{j'}}, \{\{\frac{v_A}{n_{j''}}\}_{i\in N_{j''}}\}_{j''\in\mathcal{B}|j''\neq j,j'})$, which we denote as $P_D^{N_j,N_{j'}}(\{x_A^i\}_{i\in N_j\cup N_{j'}})$. The payoff to player A for any allocation of force $\mathbf{x}_A \in \mathbb{R}^n_+$ which allocates a strictly positive level of force to every node in exactly two minimal cut sets $j,\, j' \in \mathcal{B}$ is

$$\begin{aligned}&\pi_A(\mathbf{x}_A, P_D) = \\ &\quad v_A P_D^{N_j}\left(\{x_A^i\}_{i\in N_j}\right) + v_A P_D^{N_{j'}}\left(\{x_A^i\}_{i\in N_{j'}}\right) - v_A P_D^{N_j,N_{j'}}\left(\{x_A^i\}_{i\in N_j\cup N_{j'}}\right) - \sum_{i\in N_j\cup N_{j'}} x_A^i.\end{aligned}$$

Simplifying,

$$\pi_A(\mathbf{x}_A, P_D) = -v_A \min\left\{\frac{\sum_{i\in N_j} x_A^i}{v_A}, \frac{\sum_{i\in N_{j'}} x_A^i}{v_A}\right\} < 0.$$

The case of player A allocating a strictly positive level of force to more than two minimal cut sets follows directly. Clearly, in any optimal strategy player A never allocates a strictly positive level of force to more than one minimal cut set.

The case for player D follows along similar lines. If player D allocates a strictly positive level of force to a subset $\phi \subseteq N$ of nodes with one node, denoted ϕ_j, from each minimal cut set $j \in \mathcal{B}$, then the probability that player D preserves connectivity is given by $P_A(\{x_D^i\}_{i\in\phi}, \{\{\frac{v_A}{n_j}\}_{i\in N_j\setminus\phi_j}\}_{j\in\mathcal{B}})$, which we denote as $P_A^{\phi}(\{x_D^i\}_{i\in\phi})$. Given that player A is using the equilibrium strategy P_A described above the payoff to player D for any allocation of force $\mathbf{x}_D \in \mathbb{R}_+^n$ which allocates a strictly positive level of force to only the nodes in the set ϕ is

$$\pi_D\left(\mathbf{x}_D, P_A\right) = v_D P_D^{\phi}(\{x_D^i\}_{i\in\phi}) - \sum_{i\in\phi} x_D^i.$$

Simplifying,

$$\pi_D\left(\mathbf{x}_D, P_A\right) = v_D\left(1-\frac{1}{\alpha}\right) + v_D\left(\frac{\sum_{i\in\phi} x_D^i}{v_D}\right) - \sum_{i\in\phi} x_D^i$$

and thus

$$\pi_D\left(\mathbf{x}_D, P_A\right) = v_D\left(1-\frac{1}{\alpha}\right)$$

for all $\mathbf{x}_D$ in which a strictly positive level of force is allocated to each node in some set $\phi \subseteq N$ of nodes with one node from each minimal cut set $j \in \mathcal{B}$. Following lines similar to those given above in the demonstration that player A attacks at most one minimal cut set (see the proof of Proposition 1 in the Appendix for further details), player D cannot increase his expected payoff by deviating to an allocation with a strictly positive level of force at two or more nodes within one or more minimal cut sets. $\square$

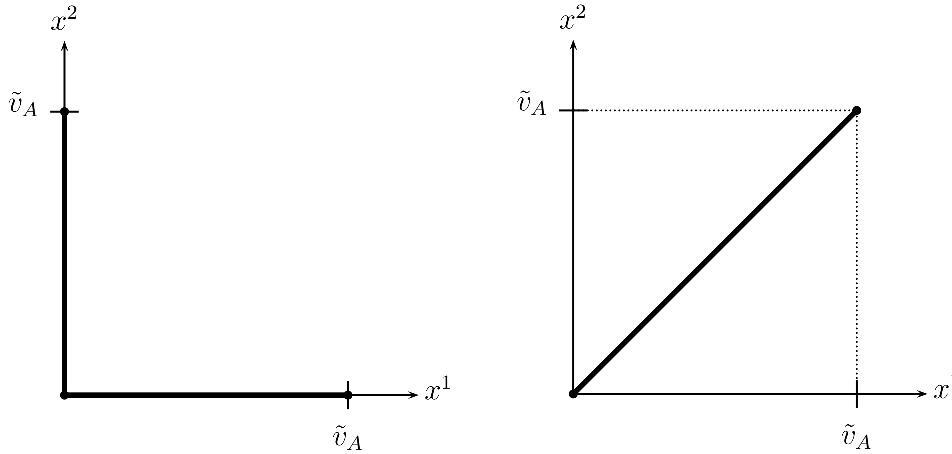
Although the equilibrium mixed strategies stated in Theorem 1 are not unique,[13] it is useful to provide some intuition regarding the existence of this particular equilibrium before moving on to the characterization of properties of optimal attack and defense that hold in all equilibria (Propositions 1-3). The supports of the equilibrium mixed strategies stated in Theorem 1 are given in Figure 2 in an example with a specific parameter configuration. Panels (a) and (b) of Figure 2 provide the supports for the attacker and defender, respectively, in the case in which there are two minimal cut sets each with one node ($i = 1, 2$). Panels (c) and (d) of Figure 2 provide the supports for the attacker and defender, respectively, in the case in which there is one minimal cut set with two nodes ($i = 1, 2$) and one minimal cut set with one node ($i = 3$).

Across all of the Panels (a)-(d), if $\alpha = 1$ then each player randomizes continuously over their respective shaded line segments. In the event that the defender has a normalized relative strength advantage ($\alpha > 1$), the defender's strategy stays the same, but the attacker

[13]For example, in the case (1) parameter range of Theorem 1 another equilibrium strategy for player D is to use the mixed strategy

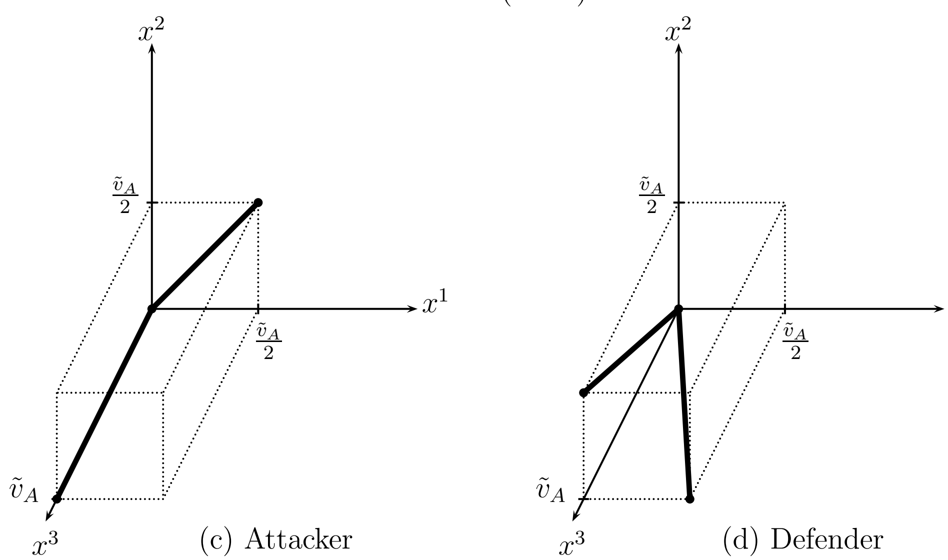
$$P_D\left(\mathbf{x}\right) = \prod_{j\in\mathcal{B}}\left\{\frac{\sum_{i\in N_j} x^i}{v_A}\right\}.$$

Network with two minimal cut sets, each with one node ($i = 1, 2$)



(a) Attacker (b) Defender

Network with two minimal cut sets, one with two nodes ($i = 1, 2$) and one with one node ($i = 3$)



(c) Attacker (d) Defender

Figure 2: Supports of the Theorem 1 pair of equilibrium mixed strategies ($\tilde{v}_A = \min\{\alpha v_A, v_A\}$).

now places a mass point of size $1-(1/\alpha)$ at the origin and randomizes continuously over the respective line segments with the remaining probability. Conversely, if the defender has a normalized relative strength disadvantage ($\alpha < 1$), then it is the defender who places a mass point (of size $1-\alpha$) at the origin.

Beginning with Panels (a) and (b), recall that if the attacker successfully attacks a single minimal cut set the network is disconnected. As shown in Panel (a) the support of the attacker's equilibrium mixed strategy, P_A, lies on the axes, and, thus, the attacker launches an attack on at most one minimal cut set. To successfully defend a set of singleton minimal cut sets, the defender must win every node within the set. As shown in Panel (b) the support of the defender's (Theorem 1) equilibrium mixed strategy, P_D, lies on the 45° line, and, thus, the defender's allocation of force to node i is an almost surely strictly increasing function of the force allocated to node $-i$. Note that if the attacker launches an attack on at most one minimal cut set, then the probability that any single attack is successful depends only on the univariate marginal distributions of the defender's mixed strategy (an n-variate joint distribution). In addition, the defender's expected force expenditure depends only on his set of univariate marginal distributions, and, for a given set of univariate marginal distributions, is invariant to the correlation structure.[14] Finally, note that given the defender's choice of correlation structure [Panel (b)], the attacker's probability of at least one successful attack depends only on the maximum of his force allocations across the two nodes. That is, given the defender's equilibrium mixed strategy, if the set of points such that $x_A^i \geq x_A^{-i} > 0$ for some $i \in \{1,2\}$ has positive measure, then the attacker can strictly increase his expected payoff by reducing x_A^{-i} to $x_A^{-i} = 0$ for all such points. In such a deviation, the probability of at least one successful attack is unaffected, but the attacker's expected force expenditure decreases. Thus, at each point in the support of the equilibrium mixed strategy the attacker launches at most one attack.

Panels (c) and (d) examine a simple network with one minimal cut set with two nodes and one minimal cut set with one node. In Panel (c), note that the attacker launches an attack on at most one minimal cut set. In the event in which the minimal cut set with two nodes is attacked, the attacker's allocation of force to node i in the cut set is an almost surely strictly increasing function of the force allocated to node $-i$ in the cut set. In Panel (d), note that for this parameter configuration there are two spanning trees of the network and the defender randomly chooses a spanning tree to protect by allocating a strictly positive

[14]More formally, for a given set of univariate marginal distribution functions, the expected force expenditure is invariant to the mapping into a joint distribution function, i.e. the n-copula. For further details see Nelsen (1999).

level of force to at most one of the nodes $i \in \{1, 2\}$ in the minimal cut set with two nodes and choosing to set the level of force allocated to the single node in the remaining minimal cut set as an almost surely increasing function of the level of force allocated to the minimal cut set with two nodes. That is, the force allocated to each node in the randomly chosen spanning tree is an almost surely strictly increasing function of the level of force allocated to the other node in the spanning tree. Given these correlation structures, the intuition for why the attacker launches an attack on at most one minimal cut set in the network follows along the lines given above for the network with singleton minimal cut sets.

We have already pinned down equilibrium payoffs and univariate marginal distributions arising in all equilibrium mixed strategies. We now characterize several other qualitative features arising in all equilibria. Proposition 1 examines the number of minimal cut sets that are simultaneously attacked as well as the number of nodes within each minimal cut set that are simultaneously attacked and defended. Propositions 2 and 3 examine the likelihood that the defender leaves the network undefended, the likelihood that the attacker launches an attack on the network, and conditional on an attack, the likelihood that a given minimal cut set $j \in \mathcal{B}$ is attacked. Formal proofs for Propositions 1-3 are given in the Appendix.

Proposition 1. *In any equilibrium $\{P_A, P_D\}$ of the game $ADN\{G, v_A, v_D\}$ with disjoint minimal cut sets:*

(1) *If $\mathbf{x}_A$ is an n-tuple contained in the support of P_A, then $\mathbf{x}_A$ allocates a strictly positive level of force to at most one minimal cut set.*

(2) *If $\mathbf{x}_D$ is an n-tuple contained in the support of P_D, then $\mathbf{x}_D$ allocates a strictly positive level of force to at most one node within each minimal cut set.*

For intuition on part (1) of Proposition 1, recall that success for the attacker requires winning at least one minimal cut set. Because attacking multiple cut sets is costly, it must be the case that doing so increases the probability of success. Then note that the defender has the ability to utilize correlation structures in his mixed strategy for which the attacker's probability of success is non-increasing as the number of minimal cut sets that are attacked increases. For example in the equilibrium strategies given in part (1) of Theorem 1 the defender randomly chooses one node from each minimal cut set to defend and the mixed strategy specifies that across this set of nodes the allocation of force has perfect positive correlation. Given that in equilibrium the defender employs such a strategy, the attacker allocates a strictly positive level of force to at most one minimal cut set. The intuition for part (2) of Proposition 1 follows along similar lines. In particular, the attacker now has the ability to utilize a mixed strategy for which the defender's probability of successfully defending the cut set is non-increasing as the number of defended nodes in the cut set increases.

Proposition 2. *If $\alpha \geq 1$, then in any equilibrium $\{P_A, P_D\}$ of the game $ADN\{G, v_A, v_D\}$ with disjoint minimal cut sets:*

(1) *With probability $1 - \frac{1}{\alpha}$, the network is not attacked.*

(2) *Conditional on an attack on the network, the probability that player A attacks minimal cut set $j \in \mathcal{B}$ is $1/(n_j[\sum_{j' \in \mathcal{B}} \frac{1}{n_{j'}}])$, which is decreasing in the number of nodes in cut set j.*

(3) *Player D allocates a strictly positive level of force to each minimal cut set $j \in \mathcal{B}$ with certainty.*

For $\alpha > 1$, the normalized relative strength of the defender is high enough that all equilibria involve the attacker refraining from attack with positive probability. Conversely, the defender defends the network with certainty. Because each minimal cut set j provides the defender with a form of best-shot redundancy in proportion to the number of nodes, n_j, in cut set j, larger minimal cut sets are more difficult to successfully attack. Conditional on launching an attack on the network, the probability that minimal cut set j is attacked is decreasing in the number of nodes n_j.

Proposition 3 addresses the case of $\alpha < 1$. In this case, the attacker optimally launches an attack with certainty and the defender leaves the network undefended with positive probability.

Proposition 3. *If $\alpha < 1$, then in any equilibrium $\{P_A, P_D\}$ of the game $ADN\{G, v_A, v_D\}$ with disjoint minimal cut sets:*

(1) *The network is attacked with certainty.*

(2) *The probability that player A attacks minimal cut set $j \in \mathcal{B}$ is $1/(n_j[\sum_{j' \in \mathcal{B}} \frac{1}{n_{j'}}])$, which is decreasing in the number of nodes in cut set j.*

(3) *Player D leaves the network undefended with probability $1 - \alpha$ and, with probability α, allocates a strictly positive level of force to every cut set $j \in \mathcal{B}$.*

If $\alpha < 1$, then the normalized relative strength of the defender is sufficiently low that all equilibria involve the defender leaving the network undefended with positive probability, and the likelihood that the defender leaves the network undefended is decreasing in the normalized relative strength of the defender.

To summarize, the following conditions hold in all equilibria. If $\alpha > 1$, then in any equilibrium the attacker chooses, with positive probability, not to launch an attack. Regardless of the value of α, the attacker launches an attack on at most one minimal cut set. Conditional on an attack, the likelihood that any individual minimal cut set is attacked depends on the number of nodes within the minimal cut set. For each minimal cut set the likelihood of attack is decreasing in the number of nodes. If $\alpha < 1$, then in any equilibrium the defender leaves the network undefended with positive probability. Lastly, regardless of the value of α, conditional on the network being defended, the defender randomly chooses one node within each minimal cut set to defend, i.e. the defender chooses a random spanning tree to defend. Figure 3 below illustrates in the case of network (c) from Figure 1 how randomizing over which node in each minimal cut set to defend is equivalent to randomizing over defense of a spanning tree. In this example there exist two singleton minimal cut sets and a minimal cut set with three nodes. For each of the nodes in the non-singleton minimal cut set Figure 3 provides the corresponding spanning tree that is implicitly preserved if each of the defended nodes are preserved, where the defended nodes are surrounded by a dashed circle and the undefended nodes are denoted by hollow nodes. Note that if one or both of the undefended nodes are destroyed the network is still connected by the spanning tree that is created by the nodes that are defended.

Networks with Overlapping Minimal Cut Sets

Figure 4 provides two examples of networks with overlapping minimal cut sets. In cases such as in panel (a) of Figure 4 where the overlapping minimal cut sets are symmetric with respect to the number of nodes in each cut set, it is straightforward to extend the specific pair of equilibrium mixed strategies in Theorem 1. If, as in panel (b) of Figure 4, the overlapping minimal cut sets differ with respect to the number of nodes, then it may be possible to judiciously extend the specific pair of equilibrium mixed strategies in Theorem 1 to cover this case, but this extension will depend critically on the network structure.

Beginning with the case of cycle networks as in panel (a) of Figure 4, consider an arbitrary cycle network with $n \geq 4$ nodes. Let the nodes be sequentially indexed (clockwise) from 1 to n. The following corollary provides an extension of the mixed strategies in Theorem 1

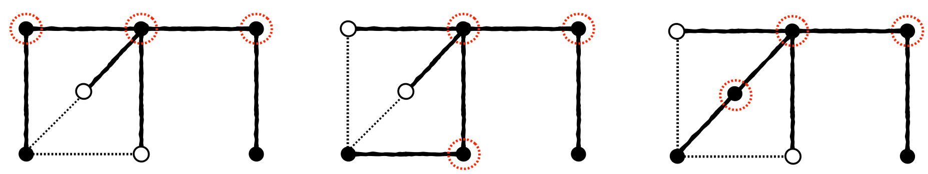
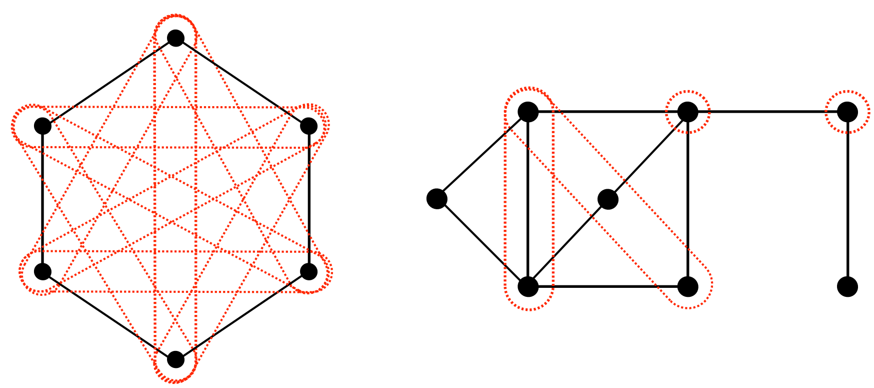


Figure 3: Randomly Defended Spanning Trees



(a) Cycle Network

(b) Arbitrary Network

Figure 4: Example Networks with Overlapping Minimal Cut Sets

that applies to cycle networks. In constructing the equilibrium, define $\alpha^* = \frac{v_D}{v_A(n/2)}$ and let the index $i+2$ refer to $(i+2)(\text{mod } n)$.

Corollary 1. *For any parameter configuration of the game $ADN\{G, v_A, v_D\}$ with G a cycle network, there exists an equilibrium in which each player allocates his forces according to the following n-variate distribution functions:*

(1) *If $\alpha^* \geq 1$, then for player A and $\boldsymbol{x} \in [0, \frac{v_A}{2}]^n$*

$$P_A(\boldsymbol{x}) = 1 - \frac{1}{\alpha^*} + \frac{\sum_{i\in N} \min\{x^i, x^{i+2}\}}{v_D}$$

Similarly for player D and $\boldsymbol{x} \in [0, \frac{v_A}{2}]^n$

$$P_D(\boldsymbol{x}) = \left(\frac{2\min_{i\in N} x^i}{v_A}\right)$$

In this equilibrium, the expected payoff for player A is 0, and the expected payoff for player D is $v_D(1 - \frac{1}{\alpha^*})$.

(2) *If $\alpha^* < 1$, then for player A and $\boldsymbol{x} \in [0, \frac{\alpha^* v_A}{2}]^n$*

$$P_A(\boldsymbol{x}) = \frac{\sum_{i\in N} \min\{x^i, x^{i+2}\}}{v_D}$$

Similarly for player D and $\boldsymbol{x} \in [0, \frac{\alpha^* v_A}{2}]^n$

$$P_D(\boldsymbol{x}) = 1 - \alpha^* + \left(\frac{2\min_{i\in N} x^i}{v_A}\right)$$

In this equilibrium, the expected payoff for player D is 0, and the expected payoff for player A is $v_A(1 - \alpha^*)$.

For a cycle network, any pair of nonadjacent nodes forms a minimal cut set, the destruction of which disconnects the network. In the following comments, we provide a sketch of the proof that the mixed strategies in Corollary 1 form an equilibrium but omit the arguments ruling out the attack of multiple minimal cut sets which follow along the same line as in the proof of Theorem 1. If $\alpha^* \geq 1$, then the expected payoff to the attacker from attacking any pair of nonadjacent nodes i and $i' = (i+2)(\text{mod } n)$ with $(x_A^i, x_A^{i'}) \in [0, \frac{v_A}{2}]^2$ is

$$v_A \left(\frac{2 \min\{x_A^i, x_A^{i'}\}}{v_A} \right) - x_A^i - x_A^{i'} \leq 0 \tag{1}$$

which holds with equality if $x_A^i = x_A^{i'}$. In the mixed strategy equilibrium identified in Corollary 1, the attacker chooses a random node i and its first nonadjacent node i' in a clockwise direction to attack, and sets $x_A^i = x_A^{i'}$ at each point in the support. Thus, (1) holds with equality at each point in the support of the attacker's equilibrium mixed strategy.

When $\alpha^* \geq 1$ the defender's Corollary 1 equilibrium mixed strategy specifies that the allocation to each node i in the network is strictly positive and is an almost surely strictly increasing function of the allocation to each other node. Specifically, at each point in the support of the defender's Corollary 1 equilibrium mixed strategy $x_D^i = x_D^{i'}$ for all $i, i' \in N$. If the defender defends each node i in the network with $x_D^i \in (0, \frac{v_A}{2}]$, then the defender's expected payoff is

$$v_D \left(1 - \frac{1}{\alpha^*} + \frac{\sum_{i \in N} \min\{x_D^i, x_D^{i+2}\}}{v_D} \right) - \sum_{i \in N} x_D^i \leq v_D \left(1 - \frac{1}{\alpha^*} \right)$$

which holds with equality if $x_D^i = x_D^{i+2}$ for all i as in the support of the defender's Corollary 1 mixed strategy. And, given the attacker's Corollary 1 equilibrium mixed strategy, there exist no profitable deviations for the defender. The case of $\alpha^* < 1$ follows along similar lines.

As noted, the panel (b) network in Figure 4 illustrates that when minimal cut sets overlap and differ with respect to the number of nodes that they contain, the qualitative nature of Nash equilibrium may depend critically on the details of the network structure. Nonetheless, it may be possible to derive a Nash equilibrium profile.

In the case of the network in panel (b), let the nodes be sequentially indexed (left to right with any nodes equidistant from the leftmost node being indexed from top to bottom) from 1 to n. Note that in panel (b), the 4 minimal cut sets are $(2,3)$, $(2,4,6)$, (5), and (7), and thus, nodes 1 and 8 are never attacked or defended. Let $\mathbf{x} = (x^2, x^3, x^4, x^5, x^6, x^7) \in \mathbb{R}_+^6$ be a 6-tuple that corresponds to the allocation of force to nodes 2-7, and for this network define $\alpha^* = \frac{3v_D}{8v_A}$.

Corollary 2. *For any parameter configuration of the game $ADN\{G, v_A, v_D\}$ with G given by panel (b) of Figure 4, there exists an equilibrium in which each player allocates his forces according to the following n-variate distribution functions:*

(1) *If $\alpha^* \geq 1$, then for player A and* $\mathbf{x} \in [0, \frac{2v_A}{3}] \times [0, \frac{v_A}{3}] \times [0, \frac{v_A}{6}] \times [0, v_A] \times [0, \frac{v_A}{6}] \times [0, v_A]$

$$P_A(\boldsymbol{x}) = 1 - \frac{1}{\alpha^*} + \frac{\min\left\{\frac{x^2}{2}, x^3\right\} + \min\left\{\frac{x^2}{2}, x^4 + x^6\right\} + x^5 + x^7}{v_D}.$$

Similarly for player D and $\boldsymbol{x} \in [0, \frac{2v_A}{3}] \times [0, \frac{v_A}{3}] \times [0, \frac{v_A}{6}] \times [0, v_A] \times [0, \frac{v_A}{6}] \times [0, v_A]$

$$P_D(\boldsymbol{x}) = \left(\frac{\min\{x^2 + \min\{x^3, x^4 + x^6\}, x^5, x^7\}}{v_A}\right).$$

In this equilibrium, the expected payoff for player A is 0, and the expected payoff for player D is $v_D(1 - \frac{1}{\alpha^*})$.

(2) *If $\alpha^* < 1$, then for player A and* $\boldsymbol{x} \in [0, \frac{2\alpha^* v_A}{3}] \times [0, \frac{\alpha^* v_A}{3}] \times [0, \frac{\alpha^* v_A}{6}] \times [0, \alpha^* v_A] \times [0, \frac{\alpha^* v_A}{6}] \times [0, \alpha^* v_A]$

$$P_A(\boldsymbol{x}) = \frac{\min\left\{\frac{x^2}{2}, x^3\right\} + \min\left\{\frac{x^2}{2}, x^4 + x^6\right\} + x^5 + x^7}{v_D}.$$

Similarly for player D and $\boldsymbol{x} \in [0, \frac{2\alpha^* v_A}{3}] \times [0, \frac{\alpha^* v_A}{3}] \times [0, \frac{\alpha^* v_A}{6}] \times [0, \alpha^* v_A] \times [0, \frac{\alpha^* v_A}{6}] \times [0, \alpha^* v_A]$

$$P_D(\boldsymbol{x}) = 1 - \alpha^* + \left(\frac{\min\{x^2 + \min\{x^3, x^4 + x^6\}, x^5, x^7\}}{v_A}\right).$$

In this equilibrium, the expected payoff for player D is 0, and the expected payoff for player A is $v_A(1 - \alpha^*)$.

To preserve a spanning tree in panel (b) of Figure 4, the defender must win nodes 5 and 7, combined with either node 2 or a combination of node 3 with node 4 and/or node 6, which is exactly what happens in the support of the equilibrium mixed strategy for the defender in Corollary 2. If $\alpha^* \geq 1$ and the defender only defends nodes 2, 5, and 7 with any $x_D^2 \in (0, \frac{2v_A}{3}]$, and $x_D^5, x_D^7 \in (0, v_A]$ respectively, then the defender's expected payoff is

$$v_D\left(1 - \frac{1}{\alpha^*} + \frac{x_D^2 + x_D^5 + x_D^7}{v_D}\right) - x_D^2 - x_D^5 - x_D^7 = v_D\left(1 - \frac{1}{\alpha^*}\right).$$

Similarly, if D only defends nodes $3-7$ with any $x_D^3 \in (0, \frac{v_A}{3}]$, $x_D^4, x_D^6 \in (0, \frac{v_A}{6}]$, and $x_D^5, x_D^7 \in (0, v_A]$ respectively, then the defender's expected payoff is

$$v_D\left(1 - \frac{1}{\alpha^*} + \frac{x_D^3 + x_D^4 + x_D^6 + x_D^5 + x_D^7}{v_D}\right) - x_D^3 - x_D^4 - x_D^6 - x_D^5 - x_D^7 = v_D\left(1 - \frac{1}{\alpha^*}\right).$$

Recall that the 4 minimal cut sets are $(2,3)$, $(2,4,6)$, (5), and (7). Consistent with part (1) of Proposition 1, at each point in the support of A's Corollary 2 equilibrium mixed strategy A attacks at most one minimal cut set. We focus on the case in which the $(2,3)$ minimal cut set is attacked, and note that the three remaining cases follow along similar lines. If $\alpha^* \geq 1$, then the expected payoff to the attacker from attacking nodes 2 and 3 with $(x_A^2, x_A^3) \in [0, \frac{2v_A}{3}] \times [0, \frac{v_A}{3}]$ is

$$v_A \left(\frac{x_A^2 + x_A^3}{v_A} \right) - x_A^2 - x_A^3 = 0.$$

As in the case of disjoint minimal cut sets, in both of the examples in Figure 4 the corresponding Corollary 1 and 2 equilibrium mixed strategies involve the defender choosing a random spanning tree to defend and the attacker choosing a random minimal cut set to attack.

4 Conclusion

This paper examines a game-theoretic model of attack and defense of network connectivity. The model features asymmetric objectives: the defender wishes to successfully defend network connectivity and the attacker's objective is to successfully attack at least one minimal cut set. Although the model allows for general correlation structures for force expenditures within and across the nodes, for the case in which the minimal cuts sets are disjoint we derive the unique equilibrium expected payoffs of the attacker and defender and demonstrate that there exists a unique equilibrium univariate marginal distribution of forces to each node. An equilibrium pair of mixed strategies for the attacker and defender, each of which is a joint distribution governing the allocation of forces to all nodes, is also constructed, although these are generally non-unique.

Our approach leads to a wealth of interesting extensions and applications. Because the game examined here is a set of complete information all-pay auctions linked by payoff complementarities, almost any extension of the standard one-dimensional strategic allocation problem represented by the standard all-pay auction with complete information has a corresponding extension in this game. Examples include incomplete information about values or unit costs of forces, affine handicapping of players within contests at nodes, and nonlinear costs of forces.[15] In addition, as in other models of strategic multidimensional resource

[15]Examples of these extensions for the one-dimensional strategic allocation problem include Amann and

allocation, such as Colonel Blotto games, interesting extensions arise by introducing more heterogeneity across nodes, such as allowing for differential node values for the attacker and defender within the network structure, or other factors that might link allocations across nodes, such as budget constraints or "infrastructure technologies" that allow lumpy force expenditure across sets of nodes. Although this general research agenda is in its early stage, we believe these issues deserve further development.

References

[1] Alcalde, J., and M. Dahm (2010), "Rent seeking and rent dissipation: a neutrality result," *Journal of Public Economics* 94:1-7.

[2] Amann, E., and W. Leininger (1996), "Asymmetric all-pay auctions with incomplete information: the two-player case," *Games and Economic Behavior* 14:1-18.

[3] Arce, D.G., Kovenock, D., and B. Roberson (2012), "Weakest-link attacker-defender games with multiple attack technologies," *Naval Research Logistics* 59:457-469.

[4] Baye, M.R., Kovenock, D., and C.G. de Vries (1994), "The solution to the Tullock rent-seeking game when $R > 2$: mixed-strategy equilibria and mean dissipation rates," *Public Choice* 81:363-380.

[5] Baye, M.R., Kovenock, D., and C.G. de Vries (1996), "The all-pay auction with complete information," *Economic Theory* 8:291-305.

[6] Bernhardt, D. and M.K. Polborn (2010), "Non-convexities and the gains from concealing defense from committed terrorists," *Economic Letters* 107:52-54.

[7] Bier, V.M., Oliveros, S., and L. Samuelson (2007), "Choosing what to protect: strategic defensive allocation against an unknown attacker," *Journal of Public Economic Theory* 9:563-587.

[8] Borel, E. (1921), "La théorie du jeu les équations intégrales à noyau symétrique," *Comptes Rendus de l'Académie* 173:1304-1308; English translation by L. Savage (1953), "The theory of play and integral equations with skew symmetric kernels," *Econometrica* 21:97-100.

Leininger (1996), Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006), Gale and Stegeman (1994), Konrad (2002), Kaplan and Wettstein (2006), Che and Gale (2006), and Siegel (2009).

[9] Bramoullé, Y. (2007) "Anti-Coordination and Social Interactions," *Games and Economic Behavior* 58:30-49.

[10] Che, Y.K., and I.L. Gale (2006), "Caps on political lobbying: reply," *American Economic Review* 96:1355-1360.

[11] Clark, D.J., and K.A. Konrad (2007), "Asymmetric conflict: weakest link against best shot," *Journal of Conflict Resolution* 51:457-469.

[12] Dziubiński, M. (2013), "Non-symmetric discrete General Lotto games," *International Journal of Game Theory* 42: 801-833.

[13] Dziubiński, M., and S. Goyal (2013a), "How to defend a network," University of Cambridge working paper.

[14] Dziubiński, M., and S. Goyal (2013b), "Network design and defence," *Games and Economic Behavior* 79: 30-43.

[15] Ford, L.R., and D.R. Fulkerson (1962) *Flows in Networks*, Princeton University Press, Princeton NJ.

[16] Gale, I.L., and M. Stegeman (1994), "Exclusion in all-pay auctions," working paper 9401, Federal Reserve Bank of Cleveland.

[17] Golman, R., and S.E. Page (2009), "General Blotto: games of strategic allocative mismatch," *Public Choice* 138:279-299.

[18] Goyal, S., and A. Vigier (2014), "Attack, Defence, and Contagion in Networks," *Review of Economic Studies* 81:1518-1542.

[19] Hart, S. (2008), "Discrete Colonel Blotto and General Lotto games," *International Journal of Game Theory* 36:441-460.

[20] Hirshleifer, J. (1983), "From weakest-link to best-shot: the voluntary provision of public goods," *Public Choice* 41:371-386.

[21] Hortala-Vallve, R., and A. Llorente-Saguer (2010), "A simple mechanism for resolving conflict," *Games and Economic Behavior* 70:375-391.

[22] Hortala-Vallve, R., and A. Llorente-Saguer (2012), "Pure-strategy Nash equilibria in non-zero sum Colonel Blotto games," *International Journal of Game Theory* 40:331-343.

[23] Kaplan, T.R., and D. Wettstein (2006), "Caps on political lobbying: comment," *American Economic Review* 96:1351-1354.

[24] Klumpp, T., and M.K. Polborn (2006), "Primaries and the New Hampshire effect," *Journal of Public Economics* 90:1073-1114.

[25] Konrad, K.A. (2002), "Investment in the absence of property rights: the role of incumbency advantages," *European Economic Review* 46:1521-1537.

[26] Kovenock, D. and B. Roberson (2012) "Conflicts with multiple battlefields," in Garfinkel, M.R., & Skaperdas, S., eds. *Oxford Handbook of the Economics of Peace and Conflict*, Oxford: Oxford University Press.

[27] Kovenock, D., Roberson, B., and R. Sheremeta (2010), "The attack and defense of weakest link networks," CESifo working paper No. 3211.

[28] Kovenock, D., Sarangi, S., and M. Wiser (2015), "All-pay 2×2 Hex: a multibattle contest with complementarities," *International Journal of Game Theory* 44:571-597.

[29] Krishna, V., and J. Morgan (1997), "An analysis of the war of attrition and the all-pay auction," *Journal of Economic Theory* 72:343-362.

[30] Kvasov, D. (2007), "Contests with limited resources," *Journal of Economic Theory* 136:738-748.

[31] Laslier, J.F. (2002), "How two-party competition treats minorities," *Review of Economic Design* 7:297-307.

[32] Laslier, J.F., and N. Picard (2002), "Distributive politics and electoral competition," *Journal of Economic Theory* 103:106-130.

[33] Macdonell, S., and N. Mastronardi (2015), "Waging simple wars: a complete characterization of two-battlefield Blotto equilibria," *Economic Theory* 58:183-216.

[34] Menger, K. (1927), "Zur allgemeinen Kurventheorie," *Fundamenta Mathematicae* 10:96-115.

[35] Moldovanu, B., and A. Sela (2001), "The optimal allocation of prizes in contests," *American Economic Review* 91:542-558.

[36] Moldovanu, B., and A. Sela (2006), "Contest architecture," *Journal of Economic Theory* 126:70-96.

[37] Nelsen, R. B. (1999) *An Introduction to Copulas*. Springer, New York.

[38] Powell, R. (2007a), "Defending against terrorist attacks with limited resources," *American Political Science Review* 101:527-541.

[39] Powell, R. (2007b), "Allocating defensive resources with private information about vulnerability," *American Political Science Review* 101:799-809.

[40] Rinott, Y., Scarsini, M., and Y. Yu (2012), "A Colonel Blotto gladiator game," *Mathematics of Operations Research* 37:574-590.

[41] Roberson, B. (2006), "The Colonel Blotto game," *Economic Theory* 29:1-24.

[42] Roberson, B. (2008), "Pork-barrel politics, targetable policies, and fiscal federalism," *Journal of the European Economic Association* 6:819-844.

[43] Roberson, B., and D. Kvasov (2011) "The non-constant-sum Colonel Blotto game," *Economic Theory* 51:397-433.

[44] Schweizer, B., and A. Sklar (1983), *Probabilistic Metric Spaces*. Elsevier Science Publishing Co., New York.

[45] Siegel, R. (2009), "All-pay contests," *Econometrica* 77:71-92.

[46] Snyder, J.M. (1989), "Election goals and the allocation of campaign resources," *Econometrica* 57:637-660.

[47] Szentes, B., and R.W. Rosenthal (2003a), "Three-object two-bidder simultaneous auctions: chopsticks and tetrahedra," *Games and Economic Behavior* 44:114-133.

[48] Szentes, B., and R.W. Rosenthal (2003b), "Beyond chopsticks: symmetric equilibria in majority auction games," *Games and Economic Behavior* 45:278-295.

[49] Tang, P., Shoham, Y., and F. Lin (2010), "Designing competitions between teams of individuals," *Artificial Intelligence* 174:749-766.

[50] Thomas, C. (2012), "N-dimensional Colonel Blotto games with asymmetric valuations," University of Texas working paper.

[51] Weinstein, J. (2012), "Two notes on the Blotto Game," *B.E. Journal of Theoretical Economics* 12: DOI: 10.1515/1935-1704.1893.

Appendix

This appendix characterizes, for the case of disjoint minimal cut sets as in Theorem 1, the unique equilibrium payoffs, the unique sets of equilibrium univariate marginal distributions, and properties of the supports of all equilibrium joint distributions. The proofs of Theorem 1 and Propositions 2 and 3 follow from the 9 lemmas formulated in this characterization. In particular, Theorem 1 follows from Lemmas 8 and 9 and the formal proof is stated after Lemma 9. Similarly, Propositions 2 and 3 follow from Lemmas 5, 6 and 8, and the formal proof is stated after Lemma 8. Note also that the proof of Proposition 1, which is stated after Lemmas 1-4, factors prominently in the proofs of Lemmas 5-8.

Before proceeding, observe the following notational conventions which will be used throughout the appendix. For points in $\mathbb{R}^n$, we will use the vector notation $\mathbf{x} = (x^1, x^2, \dots, x^n)$. For $a^i \leq b^i$ for all $i = 1, 2, \dots, n$, let $[\mathbf{a}, \mathbf{b}]$ denote the n-box $B = [a^1, b^1] \times [a^2, b^2] \times \dots \times [a^n, b^n]$, the Cartesian product of n closed intervals. The vertices of the n-box B are the points $(c^1, c^2, \dots, c^n)$ where c^i is equal to a^i or b^i. Lastly, let $\bar{s}^i_k$ and $\underline{s}^i_k$ denote the upper and lower bounds, respectively, for player k's marginal distribution for node i.

Given that the defender is using the mixed strategy (joint distribution function) P_D, let

$$Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1\middle|P_D, \mathbf{x}_A\right) \tag{2}$$

denote the probability that with a force allocation of $\mathbf{x}_A$ the attacker wins at least one minimal cut set. Thus, the attacker's expected payoff from any pure strategy $\mathbf{x}_A$ is

$$v_A Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1\middle|P_D, \mathbf{x}_A\right) - \sum_{i\in N} x^i_A. \tag{3}$$

It will also be useful to note that the attacker's expected payoff from any mixed strategy P_A is

$$v_A E_{P_A}\left[Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1\middle|P_D, \mathbf{x}_A\right)\right] - \sum_{i\in N} E_{P^i_A}\left[x^i_A\right] \tag{4}$$

where E_{P_A} denotes the expectation with respect to the mixed strategy P_A and $E_{P^i_A}$ denotes the expectation with respect to the univariate marginal distribution for node i, P^i_A, of the mixed strategy P_A.

Similarly, given that the attacker is using the mixed strategy P_A, let

$$Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 0\middle|P_A, \mathbf{x}_D\right) \tag{5}$$

denote the probability that with a force allocation of $\mathbf{x}_D$ the defender wins all of the minimal cut sets in the network. Thus, the defender's expected payoff from any pure strategy $\mathbf{x}_D$ is

$$v_D Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right)=0\middle|P_A,\mathbf{x}_D\right)-\sum_{i\in N}x_D^i. \tag{6}$$

Lastly, the defender's expected payoff from any mixed strategy P_D is

$$v_D E_{P_D}\left[Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right)=0\middle|P_A,\mathbf{x}_D\right)\right]-\sum_{i\in N}E_{P_D^i}\left[x_D^i\right] \tag{7}$$

where E_{P_D} and $E_{P_D^i}$ denote the expectation with respect to the mixed strategy P_D and the expectation with respect to the univariate marginal distribution for node i, P_D^i, respectively.

We begin by showing that for each node i within minimal cut set j, both players' mixed strategies have the same upper bound, denoted $\bar{s}^j$, and a lower bound of 0.

Lemma 1. *In any equilibrium, for each $j\in\mathcal{B}$, $\bar{s}_A^i=\bar{s}_D^i=\bar{s}^j>0$ and $\underline{s}_A^i=\underline{s}_D^i=0$ for all $i\in N_j$.*

Proof. We begin with the proof that $\underline{s}_A^i=\underline{s}_D^i=0$ for all i. By way of contradiction, suppose $\underline{s}_A^i\neq\underline{s}_D^i$. Let $\hat{\underline{s}}^i\equiv\max\{\underline{s}_A^i,\underline{s}_D^i\}$, and let k be the identity of the player attaining $\hat{\underline{s}}^i$ (that is $\hat{\underline{s}}^i=\underline{s}_k^i$ and $\hat{\underline{s}}^i>\underline{s}_{-k}^i$).

If $\underline{s}_{-k}^i>0$, when player $-k$ allocates $\underline{s}_{-k}^i$ to node i player $-k$ loses node i with certainty and can strictly increase his payoff by setting $\underline{s}_{-k}^i=0$. It follows directly, that player $-k$ does not randomize over the open interval $(0,\hat{\underline{s}}^i)$, and thus player $-k$ must have a mass point at 0.

In the case in which $\underline{s}_{-k}^i=0$ (where player $-k$ does not randomize over the open interval $(0,\hat{\underline{s}}^i)$ and has a mass point at 0), we know that (i) both players cannot have a mass point at $\underline{s}_k^i$, (ii) player $-k$ cannot place mass at $\underline{s}_k^i$, and (iii) player k can strictly increase his payoff by lowering $\underline{s}_k^i$ to a neighborhood above 0. Thus, we conclude that $\underline{s}_A^i=\underline{s}_D^i=0$ for all i.

Lastly, for the proof that for each $j\in\mathcal{B}$, $\bar{s}_A^i=\bar{s}_D^i=\bar{s}^j>0$ for all $i\in N_j$, it is clearly the case that in any equilibrium $\bar{s}_A^i=\bar{s}_D^i$ for all $i\in N$. For the proof that for every $j\in\mathcal{B}$ and $i\in N_j$ $\bar{s}_A^i=\bar{s}_D^i>0$, by way of contradiction, suppose that there exists a minimal cut set $j\in\mathcal{B}$ and node $i\in N_j$ such that $\bar{s}_A^i=\bar{s}_D^i=0$. Player A loses minimal cut set j with certainty and, thus, it must be the case that $\bar{s}_A^{i'}=\bar{s}_D^{i'}=0$ for all $i'\in N_j$. However, player A can strictly increase his payoff by reducing to zero any allocation of force outside of the minimal cut set $j\in\mathcal{B}$ and allocating an arbitrarily small, but strictly positive, level of force to each node $i'\in N_j$. Thus, we have a contradiction. Similarly, for any pair $i'',i'''\in N_j$ it follows that if $\bar{s}_A^{i''} = \bar{s}_D^{i''} < \bar{s}_A^{i'''} = \bar{s}_D^{i'''}$ then player D would do better by moving mass from $\bar{s}_D^{i'''}$ to $\bar{s}_D^{i''}$. Thus, $\bar{s}_A^i = \bar{s}_D^i = \bar{s}^j > 0$ for all $i \in N_j$ and all $j \in \mathcal{B}$, which completes the proof. $\square$

Lemma 2. *In any equilibrium $\{P_A, P_D\}$ with the set of univariate marginal distributions $\{P_A^i, P_D^i\}_{i=1}^n$, for every $j \in \mathcal{B}$ and every node $i \in N_j$ neither player's univariate marginal distribution places positive mass on any point except possibly at zero.*

Proof. If for node i, $x_k^i > 0$ is such a point for player k, then player $-k$ must best-respond by placing no mass in $(x_k^i - \epsilon, x_k^i]$ for some $\epsilon > 0$ sufficiently small. If not, player $-k$ would benefit from moving mass from this interval to either zero or to a δ-neighborhood above x_k^i. Therefore player k can increase his payoff by moving mass from x_k^i to the ϵ-neighborhood below x_k^i. $\square$

Lemma 3. *In any equilibrium, each player's expected payoff (equations (3) and (6) for the attacker and defender respectively) is constant over the support of his mixed strategy (joint distribution) except possibly at points of discontinuity of his expected payoff function.*

Proof. Except for possibly at points of discontinuity of his expected payoff function, each player k must make his equilibrium expected payoff at each point in the support of his equilibrium strategy, P_k. Otherwise, player k would benefit by moving mass to the n-tuple(s) in his support with the highest expected payoff. $\square$

Lemma 4. *In any equilibrium $\{P_A, P_D\}$ with the set of univariate marginal distributions $\{P_A^i, P_D^i\}_{i=1}^n$, for every $j \in \mathcal{B}$ and every node $i \in N_j$, each player k's univariate marginal distribution P_k^i randomizes continuously over the interval $(0, \bar{s}^j]$.*

Proof. For every node $i \in N_j$, Lemma 1 implies that $\bar{s}_k^i = \bar{s}^j > 0$, for $k = A, D$. Lemma 2 rules out mass points of P_k^i in the interval $(0, \bar{s}^j]$. To rule out gaps, by way of contradiction, suppose that there exists an equilibrium in which for some such node $i \in N_j$, player k's univariate marginal distribution for node i, P_k^i, is constant over the interval $[\underline{\beta}, \overline{\beta}) \subset (0, \bar{s}^j]$. For this to be an equilibrium, it must be the case that P_{-k}^i is also constant over the interval $[\underline{\beta}, \overline{\beta})$. Otherwise, player $-k$ could increase his payoff by moving mass in the interval to the lower bound.

However, if $P_{-k}^i(\underline{\beta}) = P_{-k}^i(\overline{\beta})$, then for sufficiently small $\epsilon > 0$ spending $\overline{\beta} + \epsilon$ in node i cannot be optimal for player k. Indeed, by discretely reducing his expenditure from $\overline{\beta} + \epsilon$ to $\underline{\beta} + \epsilon$ player k's payoff would strictly increase. Consequently, if P_k^i is constant over $[\underline{\beta}, \overline{\beta})$ it must also be constant over $[\underline{\beta}, \bar{s}^j]$, a contradiction to the fact that $\bar{s}_k^i = \bar{s}^j$ and the definition of $\bar{s}_k^i$. $\square$

Proposition 1. *In any equilibrium* $\{P_A, P_D\}$ *of the game* $ADN\{G, v_A, v_D\}$ *with disjoint minimal cut sets:*

(1) *If* $\mathbf{x}_A$ *is an* n*-tuple contained in the support of* P_A*, then* $\mathbf{x}_A$ *allocates a strictly positive level of force to at most one minimal cut set.*

(2) *If* $\mathbf{x}_D$ *is an* n*-tuple contained in the support of* P_D*, then* $\mathbf{x}_D$ *allocates a strictly positive level of force to at most one node within each minimal cut set.*

Proof. We begin with the proof of part (1). By way of contradiction suppose that there exists an equilibrium $\{P_A, P_D\}$ such that for a positive measure of points in the support of P_A at least two minimal cut sets simultaneously receive strictly positive levels of force (henceforth, simultaneously attacked).

We first introduce some notation. Let $\mathbf{x}_A^j$ denote the restriction of the vector $\mathbf{x}_A$ to the set of nodes contained in minimal cut set j (i.e., $\{x_A^i\}_{i \in N_j}$). Denote the set of points in the support of P_A that simultaneously attack at least two minimal cut sets as

$$\Omega_A \equiv \{\mathbf{x}_A \in \text{Supp}(P_A) \big| \mathbf{x}_A^j \neq \mathbf{0} \text{ for at least two } j \in \mathcal{B}\}.$$

For each point $\mathbf{x}_A \in \Omega_A$ let $\mathcal{P}_{\mathbf{x}_A}(j \in \mathcal{B} | \mathbf{x}_A^j \neq \mathbf{0})$ denote the power set of the indices of minimal cut sets that player A simultaneously attacks at the point $\mathbf{x}_A$. Let ψ denote an arbitrary element of this power set, let $|\psi|$ denote the cardinality of the set ψ, and let $\mathbf{x}_A^{\psi}$ denote the restriction of the vector $\mathbf{x}_A$ to the set of nodes contained in the minimal cut sets in ψ (i.e., $\{x_A^i\}_{i \in \cup_{j \in \psi} N_j}$). For each point $\mathbf{x}_A \in \Omega_A$ define $\mathcal{J}(\mathbf{x}_A) = \{j \in \mathcal{B} | \mathbf{x}_A^j \neq \mathbf{0}\}$ as the set of the indices of minimal cut sets that player A simultaneously attacks at the point $\mathbf{x}_A$.

If at the point $\mathbf{x}_A \in \Omega_A$ player A simultaneously attacks two minimal cut sets j' and j'', then the probability that at $\mathbf{x}_A \in \Omega_A$ player A wins at least one minimal cut set is given by Claim 1.

Claim 1. *If at* $\mathbf{x}_A \in \Omega_A$ *player* A *simultaneously attacks two minimal cut sets* j' *and* j''*, then the probability that player* A *wins at least one minimal cut set is*

$$\begin{aligned} &Pr\left(\max\left(\left\{\iota^j\right\}_{j \in \mathcal{B}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) = \\ &\qquad Pr\left(\iota^{j'} = 1 \middle| P_D, \mathbf{x}_A^{j'}\right) + Pr\left(\iota^{j''} = 1 \middle| P_D, \mathbf{x}_A^{j''}\right) - Pr\left(\iota^{j'}, \iota^{j''} = 1 \middle| P_D, \mathbf{x}_A^{j',j''}\right) \end{aligned} \quad (8)$$

Note that the third term in the second line of (8) corrects for the first two terms' multiple counting of player A winning at least one minimal cut set. Next, we consider the probability that player A wins at least one minimal cut set in the special case in which player A simultaneously attacks three minimal cut sets.

Claim 2. *If at $\mathbf{x}_A \in \Omega_A$ player A simultaneously attacks three minimal cut sets j', j'', and j''', then*

$$\begin{aligned} &Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) = \\ &\qquad Pr\left(\iota^{j'} = 1 \middle| P_D, \mathbf{x}_A^{j'}\right) + Pr\left(\iota^{j''} = 1 \middle| P_D, \mathbf{x}_A^{j''}\right) + Pr\left(\iota^{j'''} = 1 \middle| P_D, \mathbf{x}_A^{j'''}\right) \\ &- Pr\left(\iota^{j'}, \iota^{j''} = 1 \middle| P_D, \mathbf{x}_A^{j',j''}\right) - Pr\left(\iota^{j'}, \iota^{j'''} = 1 \middle| P_D, \mathbf{x}_A^{j',j'''}\right) - Pr\left(\iota^{j''}, \iota^{j'''} = 1 \middle| P_D, \mathbf{x}_A^{j'',j'''}\right) \\ &\qquad + Pr\left(\iota^{j'}, \iota^{j''}, \iota^{j'''} = 1 \middle| P_D, \mathbf{x}_A^{j',j'',j'''}\right). \end{aligned} \tag{9}$$

Again, note that the third and fourth lines of (9) correct for the second line's multiple counting of player A winning at least one minimal cut set. Given Claims 1 and 2, a straightforward proof by induction can be used to establish that for any arbitrary point $\mathbf{x}_A \in \Omega_A$ the probability that player A wins at least one minimal cut set is given as follows.

Claim 3. *At an arbitrary point $\mathbf{x}_A \in \Omega_A$ the probability that player A wins at least one minimal cut set is given by*

$$\begin{aligned} &Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) = \\ &\qquad \sum_{\psi \in \mathcal{P}_{\mathbf{x}_A}(j\in\mathcal{B}|\mathbf{x}_A^j \neq \mathbf{0})\setminus\emptyset} (-1)^{(|\psi|-1)}\, Pr\left(\iota^j = 1 \;\forall\; j \in \psi \middle| P_D, \mathbf{x}_A^{\psi}\right). \end{aligned} \tag{10}$$

Continuing with the proof, by way of contradiction suppose that there exists an equilibrium $\{P_A, P_D\}$ in which two or more minimal cut sets are simultaneously attacked. Recall that in order to win a minimal cut set player A has to allocate a strictly higher level of force to every node in the minimal cut set. Thus, it is strictly suboptimal in a minimal cut set j for player A to have $\mathbf{x}_A^j \neq \mathbf{0}$ with $x_A^i = 0$ for some $i \in N_j$, and, in the discussion that follows, we focus on the case in which $\mathbf{x}_A^j \gg \mathbf{0}$. For any $\mathbf{x}_A \in \mathrm{Supp}(P_A)$ such that $\mathbf{x}_A^j \gg \mathbf{0}$ for some minimal cut set j, the probability that player A wins every node in cut set j, and hence wins cut set j is

$$Pr(\iota^j = 1 | P_D^{N_j}, \mathbf{x}_A^j) = P_D^{N_j}(\mathbf{x}_A^j), \tag{11}$$

where $P_D^{N_j}$ is the n_j-variate marginal distribution for minimal cut set j. For each $\mathbf{x}_A \in \Omega_A$, the probability that at the point $\mathbf{x}_A \in \Omega_A$ player A wins every node in each minimal cut set

$j \in \psi$ is

$$Pr\left(\iota^j = 1\ \forall\ j \in \psi \middle| P_D, \mathbf{x}_A\right) = P_D^{\psi}(\mathbf{x}_A^{\psi}) \tag{12}$$

where P_D^{ψ} is the $(\sum_{j\in\psi} n_j)$-variate marginal distribution over all of the minimal cut sets $j \in \psi$.

In the construction of the proof we will make use of a joint distribution function, $\hat{P}_D$, that involves the application of the Fréchet-Hoeffding upper-bound n-copula[16] to player D's set of (multivariate) marginal distributions for each of the minimal cut sets, $\{P_D^{N_j}\}_{j\in\mathcal{B}}$, under the equilibrium strategy P_D. That is, $\hat{P}_D(\mathbf{x}_A) = \min_{j\in\mathcal{B}}\{P_D^{N_j}(\mathbf{x}_A^j)\}$. Clearly this is a valid joint distribution function; for each $j \in \mathcal{B}$ the n_j-variate marginal distribution $P_D^{N_j}(\mathbf{x}_A^j)$ is preserved; for each $i \in \cup_{j\in\mathcal{B}} N_j$ the univariate marginal distribution $P_D^i(x_A^i)$ is preserved; and for each $\mathbf{x}_A \in \Omega_A$

$$\hat{P}_D^{N_{\mathcal{J}(\mathbf{x}_A)}}(\mathbf{x}_A^{\mathcal{J}(\mathbf{x}_A)}) = \min_{j\in\mathcal{J}(\mathbf{x}_A)}\{P_D^{N_j}(\mathbf{x}_A^j)\} \tag{13}$$

where $P_D^{N_{\mathcal{J}(\mathbf{x}_A)}}$ is the $(\sum_{j\in\mathcal{J}(\mathbf{x}_A)} n_j)$-variate marginal distribution over all of the minimal cut sets $j \in \mathcal{J}(\mathbf{x}_A)$. Because the expected cost of the equilibrium strategy P_D — given in the second term in (7) — depends on only the set of univariate marginal distributions $\{P_D^i\}_{i\in\cup_{j\in\mathcal{B}}N_j}$, the strategy $\hat{P}_D(\mathbf{x}_A)$ has the same expected cost as $P_D(\mathbf{x}_A)$.

Inserting (11), (12), and (13) into (10) yields the following result.

Claim 4. *If player D uses the strategy $\hat{P}_D(\mathbf{x}_A)$, then for each $\mathbf{x}_A \in \Omega_A$ the probability that player A successfully attacks at least one of the minimal cut sets is*

$$Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1 \middle| \hat{P}_D, \mathbf{x}_A\right) = \max_{j\in\mathcal{B}|\mathbf{x}_A^j \gg \mathbf{0}}\{P_D^{N_j}(\mathbf{x}_A^j)\}. \tag{14}$$

We now have the following result regarding $\hat{P}_D$.

Claim 5. *For each $\mathbf{x}_A \notin \Omega_A$,*

$$Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) = Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1 \middle| \hat{P}_D, \mathbf{x}_A\right).$$

For each $\mathbf{x}_A \in \Omega_A$,

$$Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) \geq Pr\left(\max\left(\left\{\iota^j\right\}_{j\in\mathcal{B}}\right) = 1 \middle| \hat{P}_D, \mathbf{x}_A\right)$$

[16] See Nelsen (1999) and Schweizer and Sklar (1983) for more details.

where if $P_D^{N_{\mathcal{J}(\mathbf{x}_A)}}(\mathbf{x}_A^{\mathcal{J}(\mathbf{x}_A)}) \neq \hat{P}_D^{N_{\mathcal{J}(\mathbf{x}_A)}}(\mathbf{x}_A^{\mathcal{J}(\mathbf{x}_A)})$ *then this inequality is strict.*

If $\mathbf{x}_A \notin \Omega_A$, then player A attacks at most one minimal cut set and the first part of the claim follows directly. For the second part of the claim, we begin with the case in which player A attacks two minimal cut sets j' and j''. Suppose, without loss of generality, that $P_D^{N_{j'}}(\mathbf{x}_A^{j'}) \geq P_D^{N_{j''}}(\mathbf{x}_A^{j''})$. Inserting (11) into Claim 1, the second part of Claim 5 follows from Claim 4.[17] For the case in which player A attacks more than two minimal cut sets, there exists a $j' \in \mathcal{J}(\mathbf{x}_A)$ such that $P_D^{N_{j'}}(\mathbf{x}_A^{j'}) \geq P_D^{N_j}(\mathbf{x}_A^j)$ for all $j \in \mathcal{J}(\mathbf{x}_A)$. Because player A cannot strictly increase his probability of winning at least one minimal cut set by modifying $\mathbf{x}_A$ so that $\mathbf{x}_A^j = \mathbf{0}$ for all $j \neq j', j''$ (where $j', j'' \in \mathcal{J}(\mathbf{x}_A)$ are such that $j'' \neq j'$ and $P_D^{N_{j'}}(\mathbf{x}_A^{j'}) \geq P_D^{N_j}(\mathbf{x}_A^j)$ for all $j \in \mathcal{J}(\mathbf{x}_A)$) and we know that the second part of Claim 5 applies in the case in which player A only attacks two minimal cut sets j' and j'', the second part of Claim 5 extends directly to the case in which player A attacks more than two minimal cut sets.[18]

Returning to the proof of part (1) of Proposition 1, if for almost every $\mathbf{x}_A \in \Omega_A$, $P_D^{N_{\mathcal{J}(\mathbf{x}_A)}}(\mathbf{x}_A^{\mathcal{J}(\mathbf{x}_A)}) = \hat{P}_D^{N_{\mathcal{J}(\mathbf{x}_A)}}(\mathbf{x}_A^{\mathcal{J}(\mathbf{x}_A)})$, then from (3) and Claim 4 player A can strictly increase his payoff by modifying each subset of Ω_A that has positive measure by choosing, at each $\mathbf{x}_A$ in such subsets, a minimal cut set j such that $P_D^{N_j}(\mathbf{x}_A^j) \geq P_D^{N_{j'}}(\mathbf{x}_A^{j'})$ for all $j' \in \mathcal{J}(\mathbf{x}_A)$ and setting $\mathbf{x}_A^{j'} = \mathbf{0}$ for all $j' \neq j$. This contradicts the assumption that $\{P_A, P_D\}$ is an equilibrium. Conversely, Claim 5 implies that if there exists a subset of $\mathbf{x}_A \in \Omega_A$ with positive measure such that $P_D^{N_{\mathcal{J}(\mathbf{x}_A)}}(\mathbf{x}_A^{\mathcal{J}(\mathbf{x}_A)}) \neq \hat{P}_D^{N_{\mathcal{J}(\mathbf{x}_A)}}(\mathbf{x}_A^{\mathcal{J}(\mathbf{x}_A)})$, then there exists a strictly payoff increasing deviation for player D, which contradicts the assumption that $\{P_A, P_D\}$ is an equilibrium. This completes the proof of part (1) of Proposition 1.

For the proof of part (2) of Proposition 1, we can modify the argument given above for part (1) as follows. At node i in minimal cut set j let $\iota^{j,i} = 0$ if $x_D^i \geq x_A^i$ and $\iota^{j,i} = 1$ otherwise. Note that $\iota^j \equiv \min_{i \in N_j}\{\iota^{j,i}\}$, and the probability that player D wins minimal cut set j is given by $Pr\left(\iota^j = 0 | P_A^{N_j}, \mathbf{x}_D^j\right)$. For each $j \in \mathcal{B}$, let

$$\Omega_D^j \equiv \{\mathbf{x}_D^j \in \mathrm{Supp}(P_D^{N_j}) \big| x_D^i \neq 0 \text{ for at least two } i \in N_j\}.$$

[17] For the strict inequality in Claim 5, note that by assumption P_D is an equilibrium strategy and $\mathbf{x}_A \in \Omega_A$ is a point in $\mathrm{Supp}(P_A)$, also an equilibrium strategy. If at $\mathbf{x}_A$ player A attacks two minimal cut sets j' and j'', then because the cost-decreasing deviation of attacking only minimal cut set j' with $\mathbf{x}_A^{j'}$ (and setting $\mathbf{x}_A^{j''} = \mathbf{0}$) cannot be payoff increasing and the right-hand side of the inequality in Claim 5 is invariant to such a deviation, the inequality in Claim 5 must be strict.

[18] The argument for the strict inequality in Claim 5 when player A attacks more than two minimal cut sets follows along the same lines as for the case in which player A attacks two minimal cut sets.

For each $j \in \mathcal{B}$ and each point $\mathbf{x}_D^j \in \Omega_D^j$ define $\mathcal{I}^j(\mathbf{x}_D^j) = \{i \in N_j | x_D^i \neq 0\}$ as the set of the indices of nodes in minimal cut set j that player D simultaneously defends at the point $\mathbf{x}_D^j$. Letting $\hat{P}_A^{N_j}(\mathbf{x}_D^j) = \min_{i\in N_j}\{P_A^i(x_D^i)\}$ and $\hat{P}_A(\mathbf{x}_D)$ denote any joint distribution with the set of n_j-variate marginal distributions $\{\hat{P}_A^{N_j}(\mathbf{x}_D^j)\}_{j\in\mathcal{B}}$, we can then use the arguments in claims 1-5 to construct the following modified forms of Claims 4 and 5, which we denote as Claim 4* and Claim 5*, respectively.

Claim 4*. *If player A uses the strategy $\hat{P}_A(\mathbf{x}_D)$ with n_j-variate marginal distribution $\hat{P}_A^{N_j}(\mathbf{x}_D^j)$ for each minimal cut set $j \in \mathcal{B}$, then for each $\mathbf{x}_D^j \in \Omega_D^j$ the probability that player D successfully defends minimal cut set j is*

$$Pr\left(\iota^j = 0 \middle| \hat{P}_A^{N_j}, \mathbf{x}_D^j\right) = \max_{i\in N_j | x_D^i > 0}\{P_A^i(x_D^i)\}. \tag{15}$$

Claim 5*. *For each $\mathbf{x}_D^j \notin \Omega_D^j$,*

$$Pr\left(\iota^j = 0 \middle| P_A^{N_j}, \mathbf{x}_D^j\right) = Pr\left(\iota^j = 0 \middle| \hat{P}_A^{N_j}, \mathbf{x}_D^j\right).$$

For each $\mathbf{x}_D^j \in \Omega_D^j$,

$$Pr\left(\iota^j = 0 \middle| P_A^{N_j}, \mathbf{x}_D^j\right) \geq Pr\left(\iota^j = 0 \middle| \hat{P}_A^{N_j}, \mathbf{x}_D^j\right).$$

where if $P_A^{N_j}(\mathbf{x}_D^j) \neq \hat{P}_A^{N_j}(\mathbf{x}_D^j)$ then this inequality is strict.

Returning to the proof of part (2) of Proposition 1, if for almost every $\mathbf{x}_D^j \in \Omega_D^j$, $P_A^{N_j}(\mathbf{x}_D^j) \neq \hat{P}_A^{N_j}(\mathbf{x}_D^j)$, then from Claim 5* it follows that there exists a strictly payoff increasing deviation for player A, which contradicts the assumption that $\{P_A, P_D\}$ is an equilibrium. Conversely, if for almost every $\mathbf{x}_D^j \in \Omega_D^j$, $P_A^{N_j}(\mathbf{x}_D^j) = \min_{i\in N_j}\{P_A^i(x_D^i)\}$, then from (6) and Claim 4*, it follows that player D can strictly increase his payoff by modifying each subset of Ω_D^j that has positive measure by choosing, at each $\mathbf{x}_D^j$ in such subsets, a node i such that $P_A^i(x_D^i) \geq P_A^{i'}(x_D^{i'})$ for all $i' \in N_j$ and setting $x_D^{i'} = 0$ for all $i' \in N_j | i' \neq i$. This contradicts the assumption that $\{P_A, P_D\}$ is an equilibrium. This completes the proof of part (2) of Proposition 1. □

Lemma 5. *In any equilibrium $\{P_A, P_D\}$, for any minimal cut set j and every n_j-tuple $\mathbf{x}_A^j \in [0, \bar{s}^j]^{n_j}$ there exists a $\kappa_A^j \geq 0$ such that, $P_D^{N_j}(\mathbf{x}_A^j) = \frac{\kappa_A^j}{v_A} + \frac{\sum_{i\in N_j} x_A^i}{v_A}$. Moreover, κ_A^j satisfies $\kappa_A^j = v_A - n_j\bar{s}^j \geq 0$.*

Proof. From Proposition 1 part (2) in the support of any equilibrium strategy player D allocates a strictly positive level of force to at most one node in minimal cut set j, and thus the support of player D's n_j-variate marginal distribution for minimal cut set j, $P_D^{N_j}$, is located on the axes in $\mathbb{R}_+^{n_j}$. Combining this with Lemma 4 — each of player D's univariate marginal distributions randomizes continuously over the interval $(0, \bar{s}^j]$ — and Lemma 2 — each of player D's univariate marginals has no mass points except for possibly at zero — it follows that there are no mass points in the support of player D's n_j-variate marginal distribution for minimal cut set j, $P_D^{N_j}$, except for possibly at the origin in $\mathbb{R}_+^{n_j}$.

Combining Proposition 1 part (1) — in the support of any equilibrium strategy player A attacks at most one minimal cut set — with Lemma 3 — each player has a constant expected payoff over the support of his mixed strategy (except for possibly at points of discontinuity of the expected payoff function) — it follows that for each minimal cut set j there exists a $\kappa_A^j \geq 0$ such that for each $\mathbf{x}_A$ in the support of P_A in which $\mathbf{x}_A^j \in (0, \bar{s}^j]^{n_j}$

$$Pr\left(\iota^j = 1 \middle| P_D^{N_j}, \mathbf{x}_A\right) = P_D^{N_j}(\mathbf{x}_A^j) = \frac{\kappa_A^j}{v_A} + \frac{\sum_{i \in N_j} x_A^i}{v_A}. \tag{16}$$

Moreover, from the definition of ι^j it is clear that for each $\mathbf{x}_A$ in the support of any equilibrium strategy P_A such that $\mathbf{x}_A^j \neq \mathbf{0}$, it must be that $\mathbf{x}_A^j \in (0, \bar{s}^j]^{n_j}$. Otherwise, player A could increase his payoff by setting $\mathbf{x}_A^j = \mathbf{0}$.

The proof that follows shows that the second equality in equation (16) holds not only for each $\mathbf{x}_A$ in the support of P_A such that $\mathbf{x}_A^j \neq \mathbf{0}$, but for all n_j-tuples $\mathbf{x}^\mathbf{j} \in [0, \bar{s}^j]^{n_j}$.

Consider an arbitrary point $\mathbf{x}_A \in \text{Supp}(P_A)$ in which $x_A^{i'} \in (0, \bar{s}^j)$ for $i' \in N_j$. Because $\mathbf{x}_A \in \text{Supp}(P_A)$ and $\mathbf{x}_A^j \neq \mathbf{0}$, we know that $\mathbf{x}_A^j \in (0, \bar{s}^j]^{n_j}$, and thus, equation (16) applies. From Lemma 4, there exists an $\epsilon^{i'} > 0$ such that $(x_A^{i'} + \epsilon^{i'}) \in (0, \bar{s}^j]$. Furthermore, there exists a point $\tilde{\mathbf{x}}_A \in \text{Supp}(P_A)$ such that $\tilde{x}_A^{i'} = (x_A^{i'} + \epsilon^{i'})$. Similarly, for each $i \in N_j$ such that $i \neq i'$ define ϵ^i as $\epsilon^i = \tilde{x}_A^i - x_A^i$.

Because from Proposition 1 part (1) player A attacks at most one minimal cut set and in both $\mathbf{x}_A$ and $\tilde{\mathbf{x}}_A$ player A attacks minimal cut set j, we know that for each $i \notin N_j$, $\tilde{x}_A^i = x_A^i = 0$, and we can restrict our focus to player D's n_j-variate marginal distribution for minimal cut set j, $P_D^{N_j}$. Recall that for any $\mathbf{x}^j \in \mathbb{R}_+^{n_j}$, $P_D^{N_j}(\mathbf{x}^j)$ is equal to the $P_D^{N_j}$-volume of the n_j-box $[\mathbf{0}, \mathbf{x}^j]$.

Let $\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}^j)$ denote the first-order differences of the function $P_D^{N_j}$ as follows:

$$\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}^j) = P_D^{N_j}(x^1, \ldots, x^{i-1}, \tilde{x}_A^i, x^{i+1}, \ldots, x^{n_j}) - P_D^{N_j}(x^1, \ldots, x^{i-1}, x_A^i, x^{i+1}, \ldots, x^{n_j}). \tag{17}$$

Because the support of $P_D^{N_j}$ is located on the axes in $\mathbb{R}_+^{n_j}$, the expression $\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}_A^j)$ is the measure of the support of $P_D^{N_j}$ over the interval $(x_A^i, \tilde{x}_A^i)$ on the ith axis.[19] Note that the difference in (17) involves one point in the support of P_A, $(x^1, \ldots, x^{i-1}, x_A^i, x^{i+1}, \ldots, x^{n_j})$, and one point, $(x^1, \ldots, x^{i-1}, \tilde{x}_A^i, x^{i+1}, \ldots, x^{n_j}) \in (0, \bar{s}^j]^{n_j}$, that may or may not be in the support of P_A. Because the expected payoff from the n_j-tuple $(x^1, \ldots, x^{i-1}, \tilde{x}_A^i, x^{i+1}, \ldots, x^{n_j})$ must be less than or equal to the equilibrium expected payoff and from Lemma 4 the first equality in equation (16) holds at this point we know that

$$\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}_A^j) \leq \frac{\epsilon^i}{v_A}. \tag{18}$$

Because the support of $P_D^{N_j}$ is located on the axes in $\mathbb{R}_+^{n_j}$, we also know that

$$P_D^{N_j}(\tilde{\mathbf{x}}_A^j) = P_D^{N_j}(\mathbf{x}_A^j) + \sum_{i \in N_j} \Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}_A^j). \tag{19}$$

That is, the $P_D^{N_j}$-volume of the n_j-box $[\mathbf{0}, \tilde{\mathbf{x}}_A^j]$ is equal to the $P_D^{N_j}$-volume of the n_j-box $[\mathbf{0}, \mathbf{x}_A^j]$ plus the measure of the support of $P_D^{N_j}$ over the interval $(x_A^i, \tilde{x}_A^i)$ on each of the $i \in N_j$ axes, where the caveat in footnote 19 applies.

Because both $\mathbf{x}_A$ and $\tilde{\mathbf{x}}_A$ are contained in the support of P_A and $\mathbf{x}_A, \tilde{\mathbf{x}}_A \in (0, \bar{s}^j]^{n_j}$ it follows from equation (16), Lemma 1, and Lemma 2 that

$$P_D^{N_j}(\tilde{\mathbf{x}}_A^j) - P_D^{N_j}(\mathbf{x}_A^j) = \sum_{i \in N_j} \frac{\epsilon^i}{v_A} \tag{20}$$

Combining equations (19) and (20) it follows that for each $i \in N_j$ equation (18) holds with equality. That is the measure of the support of $P_D^{N_j}$ over the interval $(x_A^i, \tilde{x}_A^i)$ on the ith axis is equal to ϵ^i / v_A.

Given that the points $\mathbf{x}_A$ and $\tilde{\mathbf{x}}_A$ were arbitrarily chosen from the support of P_A and that

[19] This interval is for the case that $x_A^i \leq \tilde{x}_A^i$, or equivalently $\epsilon^i \geq 0$, for all $i \in N_j$. If $x_A^i > \tilde{x}_A^i$ for one or more $i \in N_j$, then $\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}_A^j)$ should be replaced with $\Delta_{\min\{x_A^i, \tilde{x}_A^i\}}^{\max\{x_A^i, \tilde{x}_A^i\}} P_D^{N_j}(\mathbf{x}_A^j)$ and the relevant interval is $(\min\{x_A^i, \tilde{x}_A^i\}, \max\{x_A^i, \tilde{x}_A^i\})$.

there are no mass points in the support of player D's n_j-variate marginal distribution for minimal cut set j, $P_D^{N_j}$, except for possibly at the origin, it follows directly that the measure of the support of $P_D^{N_j}$ over any interval $[a,b] \subset (0, \bar{s}^j]$ on the ith axis is equal to $(b-a)/v_A$. Furthermore, player D must place a mass point of size κ_A^j/v_A at the point $\mathbf{x}^j = \mathbf{0}$, and from (16), Lemma 1, and Lemma 2, $\kappa_A^j = v_A - n_j\bar{s}^j \geq 0$. This concludes the proof of Lemma 5. $\square$

Lemma 6. *In any equilibrium* $\{P_A, P_D\}$*, there exists a* $\kappa_D \geq 0$ *such that for any set* ϕ *of nodes with one node from each minimal cut set* $j \in \mathcal{B}$ *and every* $|\mathcal{B}|$*-tuple* $\mathbf{x}_D^\phi \in \prod_{j\in\mathcal{B}}[0, \bar{s}^j]$, $P_A^\phi(\mathbf{x}_D^\phi) = \frac{\kappa_D}{v_D} + \frac{\sum_{i\in\phi} x_D^i}{v_D}$. *Moreover,* κ_D *satisfies* $\kappa_D = v_D - \sum_{j\in\mathcal{B}} \bar{s}^j$.

The proof of Lemma 6 is analogous to the proof of Lemma 5, and we thus provide only a brief sketch of the proof. First, from part (2) of Proposition 1 we know that at each point in the support of any equilibrium strategy P_D player D allocates a strictly positive level of force to at most one node in each minimal cut set. For an arbitrary point $\mathbf{x}_D \in \mathrm{Supp}(P_D)$ let ϕ denote a set of $|\mathcal{B}|$ nodes with one node from each minimal cut set $j \in \mathcal{B}$ that includes each node i for which $x_D^i > 0$. From Proposition 1 part (1) in the support of any equilibrium strategy player A allocates a strictly positive level of force to at most one minimal cut set j, and thus the support of player A's $|\mathcal{B}|$-variate marginal distribution for node set ϕ, P_A^ϕ, is located on the axes in $\mathbb{R}_+^{|\mathcal{B}|}$. Combining this with Lemma 4 — each of player A's univariate marginals randomizes continuously over the interval $(0, \bar{s}^j]$ — and Lemma 2 — each of player A's univariate marginals has no mass points except for possibly at zero — it follows that there are no mass points in the support of player A's $|\mathcal{B}|$-variate marginal distribution for node set ϕ, P_A^ϕ, except for possibly at the origin in $\mathbb{R}_+^{|\mathcal{B}|}$.

Combining Proposition 1 part (2) — in the support of any equilibrium strategy P_D player D allocates a strictly positive level of force to at most one node in each minimal cut set — with Lemma 3 — each player has a constant expected payoff over the support of his mixed strategy (except for possibly at points of discontinuity of the expected payoff function) — it follows that there exists a $\kappa_D \geq 0$ such that for each $\mathbf{x}_D$ in the support of P_D with corresponding node set ϕ

$$Pr\left(\max\left(\{\iota^j\}_{j\in\mathcal{B}}\right) = 0 \middle| P_A, \mathbf{x}_D\right) = P_A^\phi(\mathbf{x}_D^\phi) = \frac{\kappa_D}{v_D} + \frac{\sum_{i\in\phi} x_D^i}{v_D}. \tag{21}$$

The remainder of the proof follows along lines similar to the proof of Lemma 5.

Lemma 7. *In any equilibrium,* $n_j\bar{s}^j = n_{j'}\bar{s}^{j'}$, $\forall\ j, j' \in \mathcal{B}$.

Proof. Recall that, in any equilibrium, from Lemma 5, for any $j \in \mathcal{B}$ and for any $\mathbf{x}_A^j \in [0, \bar{s}^j]^{n_j}$, $P_D^{N_j}(\mathbf{x}_A^j) = \frac{v_A - n_j \bar{s}^j}{v_A} + \frac{\sum_{i \in N_j} x_A^i}{v_A}$. From the combination of Proposition 1 part (1), at most one minimal cut set is attacked, and Lemma 5 it follows that player A's expected payoff from any attack on a minimal cut set j is $v_A - n_j \bar{s}^j$.

From Lemma 3, player A's expected payoff is constant across all points in the support of P_A, except possibly at points of discontinuity of the expected payoff function. Thus, $\forall$ $j, j' \in \mathcal{B}$, $v_A - n_j \bar{s}^j = v_A - n_{j'} \bar{s}^{j'}$ or equivalently $n_j \bar{s}^j = n_{j'} \bar{s}^{j'}$. □

Lemma 8. *In any equilibrium* $\{P_A, P_D\}$*,*

(1) *If* $\alpha \geq 1$*, player A's expected payoff is 0 and player D's expected payoff is* $v_D(1 - \frac{1}{\alpha})$*.*

(2) *If* $\alpha < 1$*, player D's expected payoff is 0 and player A's expected payoff is* $v_A(1 - \alpha)$*.*

(3) $n_j \bar{s}^j = \min\{v_A, v_D/[\sum_{j' \in \mathcal{B}}(1/n_{j'})]\}$ $\forall$ $j \in \mathcal{B}$*.*

Proof. Recall part (2) of Proposition 1 — in any equilibrium $\{P_A, P_D\}$ if $\mathbf{x}_D \in \text{Supp}(P_D)$ then $\mathbf{x}_D$ allocates a strictly positive level of force to at most one node in each minimal cut set j. Letting ϕ denote a set of $|\mathcal{B}|$ nodes with one node from each minimal cut set $j \in \mathcal{B}$ that includes each node i for which $x_D^i > 0$, it follows from Lemma 6 and part (2) of Proposition 1 that player D's expected payoff at each point $\mathbf{x}_D \in \text{Supp}(P_D)$ is $v_D - \sum_{j \in \mathcal{B}}(\bar{s}^j)$. Similarly for player A, recall part (1) of Proposition 1 — in any equilibrium $\{P_A, P_D\}$ if $\mathbf{x}_A \in \text{Supp}(P_A)$ then $\mathbf{x}_A$ allocates a strictly positive level of force to at most one minimal cut set j. From Lemma 5 and part (1) of Proposition 1, it follows that player A's expected payoff at each point $\mathbf{x}_A \in \text{Supp}(P_A)$ is $v_A - n_j \bar{s}^j$.

There are two cases to consider. In case (i) $v_A - n_j \bar{s}^j > 0$, then from Lemma 5 player D places an atom of size $(v_A - n_j \bar{s}^j)/v_A$ on the n_j-tuple $\mathbf{x}_D^j = \mathbf{0}$. From Lemma 7, it follows that if player D places an atom on the n_j-tuple $\mathbf{x}_D^j = \mathbf{0}$ for some minimal cut set j, then for all $j' \in \mathcal{B}$ player D places an atom on the $n_{j'}$-tuple $\mathbf{x}_D^{j'} = \mathbf{0}$. Because, player D wins the minimal cut set in the event of a tie at one or more nodes player A does not place an atom at the origin. Thus, player D's expected payoff is necessarily 0, $v_D - \sum_{j \in \mathcal{B}}(\bar{s}^j) = 0$.

In case (ii) $v_A - n_j \bar{s}^j = 0$, then from Lemma 5 player D does not place an atom on the n_j-tuple $\mathbf{x}_D^j = \mathbf{0}$, and player D's expected payoff is weakly positive $v_D - \sum_{j \in \mathcal{B}}(\bar{s}^j) \geq 0$. Combining Lemma 7 with cases (i) and (ii), it follows that, $n_j \bar{s}^j = \min\{v_A, v_D/[\sum_{j' \in \mathcal{B}}(1/n_{j'})]\}$. This completes the proof of part (3) and parts (1) and (2) follow directly, using the fact that $\alpha \geq 1$ implies $n_j \bar{s}^j = v_A$ and $\alpha < 1$ implies $n_j \bar{s}^j = v_D/[\sum_{j' \in \mathcal{B}}(1/n_{j'})]$. □

Proofs of Propositions 2 and 3:

Proof. Suppose $\{P_A, P_D\}$ is an equilibrium. For Propositions 2 and 3 parts (1) and (2), it follows from Lemma 6 that the probability that the network is not attacked is $P_A^{\phi}(\mathbf{0}) = \frac{\kappa_D}{v_D}$, where $\kappa_D = v_D - \sum_{j\in\mathcal{B}} \bar{s}^j$ and ϕ is any arbitrary set of nodes with one node from each minimal cut set $j \in \mathcal{B}$. From part (3) of Lemma 8, if $\alpha \geq 1$ then $\frac{\kappa_D}{v_D} = 1 - \frac{1}{\alpha}$ and if $\alpha < 1$ then $\frac{\kappa_D}{v_D} = 0$. Part (1) of Proposition 2 and Part (1) of Proposition 3 follow directly. Part (2) of Proposition 2 and Part (2) of Proposition 3 follow from the probability that, conditional on an attack, minimal cut set $j \in \mathcal{B}$ is attacked, which from Lemma 6 is equal to

$$\frac{1 - P_A^{\phi}(0, \{\bar{s}^{j'}\}_{j'\neq j})}{1 - P_A^{\phi}(\mathbf{0})} = \frac{\frac{\bar{s}^j}{v_D}}{1 - \frac{\kappa_D}{v_D}} = \frac{\bar{s}^j}{\sum_{j'\in\mathcal{B}} \bar{s}^{j'}} = \frac{1}{n_j[\sum_{j'\in\mathcal{B}} \frac{1}{n_{j'}}]} \tag{22}$$

where the last equality in (22) follows from Lemma 7.

Next, part (3) of Proposition 2 follows from Lemma 5 and part (3) of Lemma 8. In particular, Lemma 5 specifies that, for all $j \in \mathcal{B}$, $P_D^{N_j}(\mathbf{x}_A^j) = \frac{\kappa_A^j}{v_A} + \frac{\sum_{i\in N_j} x_A^i}{v_A}$, with $\kappa_A^j = v_A - n_j\bar{s}^j \geq 0$. Then, part (3) of Lemma 8 specifies that if $\alpha \geq 1$, then $n_j\bar{s}^j = v_A$ for all $j \in \mathcal{B}$. Thus, for $\alpha \geq 1$ it follows that $P_D^{N_j}(\mathbf{0}) = 0$ for each $j \in \mathcal{B}$. That is, the defender allocates a strictly positive level of force to each minimal cut set $j \in \mathcal{B}$, and this completes the proof of part (3) of Proposition 2.

Lastly, for part (3) of Proposition 3, it follows from Lemma 5 and part (3) of Lemma 8 that when $\alpha < 1$, $P_D^{N_j}(\mathbf{0}) = 1 - \alpha$ for all $j \in \mathcal{B}$. That is, the defender leaves each minimal cut set $j \in \mathcal{B}$ undefended with probability $1 - \alpha > 0$. Note that if $P_D^{N_j}(\mathbf{0}) = 1 - \alpha$ for all $j \in \mathcal{B}$, then $P_D(\mathbf{0}) \leq 1 - \alpha$. If $P_D(\mathbf{0}) = 1 - \alpha$, then the defender leaves the entire network undefended with probability $1 - \alpha > 0$. We now introduce three claims that will be used to show that $P_D(\mathbf{0}) < 1 - \alpha$ does not arise in equilibrium.

Claim 6. *For each minimal cut set j, any node $i \in N_j$, and any $\underline{\beta} \in (0, \bar{s}^j)$ there exists a set of points of positive measure in the support of any equilibrium strategy P_A with $\mathbf{x}_A^j \neq \mathbf{0}$ and $x_A^i \in (0, \underline{\beta}]$.*

Note that Claim 6 is a direct consequence of Lemma 6. The measure of the support of any equilibrium strategy P_A over n-tuples in which $x_A^i \in (0, \underline{\beta}]$ is calculated as follows. For any ϕ such that $i \in N_j \cap \phi$, this measure is $P_A^{\phi}(x^i = \underline{\beta}, \{\bar{s}^{j'}\}_{j'\neq j}) - P_A^{\phi}(x^i = 0, \{\bar{s}^{j'}\}_{j'\neq j})$. Recall that from Lemma 6, for any set ϕ of nodes with one node from each minimal cut set $j \in \mathcal{B}$ and every $|\mathcal{B}|$-tuple $\mathbf{x}_D^{\phi} \in \prod_{j\in\mathcal{B}}[0, \bar{s}^j]$, $P_A^{\phi}(\mathbf{x}_D^{\phi}) = \frac{\kappa_D}{v_D} + \frac{\sum_{i\in\phi} x_D^i}{v_D}$. Thus, $P_A^{\phi}(x^i = \underline{\beta}, \{\bar{s}^{j'}\}_{j'\neq j}) - P_A^{\phi}(x^i = 0, \{\bar{s}^{j'}\}_{j'\neq j}) > 0$ for $\underline{\beta} > 0$.

For the next claim note that if $P_D(\mathbf{0}) < 1-\alpha$, then, with strictly positive probability, the defender leaves some but not all minimal cut sets undefended. Without loss of generality, let j' and j'' denote a pair of minimal cut sets for which in the mixed strategy P_D it is the case that with strictly positive probability $\mathbf{x}_D^{j'} \neq \mathbf{0}$ and $\mathbf{x}_D^{j''} = \mathbf{0}$.

Claim 7. *If $P_D(\mathbf{0}) < 1 - \alpha$, then there exists at least one node $i' \in N_{j'}$ and a measurable interval $[\underline{\beta}, \overline{\beta}] \subset (0, \overline{s}^{j'}]$ such that $x_D^{i'} \in [\underline{\beta}, \overline{\beta}]$ and $\mathbf{x}_D^{j''} = \mathbf{0}$ with strictly positive probability.*

Claim 7 follows from part (2) of Proposition 1. If with strictly positive probability $\mathbf{x}_D^{j'} \neq \mathbf{0}$ and $\mathbf{x}_D^{j''} = \mathbf{0}$, then it follows that there exists an n-box $[\mathbf{a}, \mathbf{b}]$ with $\mathbf{b}^{j'} \gg \mathbf{a}^{j'} \neq 0$ and $\mathbf{b}^{j''} = \mathbf{a}^{j''} = \mathbf{0}$ such that $P_D(\mathbf{b}) - P_D(\mathbf{a}) > 0$. Then, recall that from part (2) of Proposition 1 that if $\mathbf{x}_D^{j'} \neq \mathbf{0}$, then $\mathbf{x}_D^{j'}$ allocates a strictly positive level of force to at most one node i' within minimal cut set j'. Thus, there exists at least one node $i' \in N_{j'}$ and a measurable interval $[\underline{\beta}, \overline{\beta}] \subseteq [a^{i'}, b^{i'}] \subset (0, \overline{s}^{j'}]$ such that $x_D^{i'} \in [\underline{\beta}, \overline{\beta}]$ and $\mathbf{x}_D^{j''} = \mathbf{0}$ with strictly positive probability.

The last claim regards player A's probability of winning at least one minimal cut set using a deviation from equilibrium that involves any $\mathbf{x}_A$ with $\mathbf{x}_A^{j'} \neq \mathbf{0}$ where $x_A^{i'} \in (0, \underline{\beta})$ for node $i' \in N_{j'}$, $\mathbf{x}_A^{j''} = \epsilon$ for $\epsilon > 0$, and $\mathbf{x}_A^{j} = \mathbf{0}$ for all $j \neq j', j''$.

Claim 8. *If $P_D(\mathbf{0}) < 1-\alpha$, then for any $\mathbf{x}_A$ with $\mathbf{x}_A^{j'} \neq \mathbf{0}$ where $x_A^{i'} \in (0, \underline{\beta})$ for node $i' \in N_{j'}$, $\mathbf{x}_A^{j''} = \epsilon$ for $\epsilon > 0$, and $\mathbf{x}_A^{j} = \mathbf{0}$ for all $j \neq j', j''$ there exists a $\delta > 0$ such that*

$$Pr\left(\max\left\{\iota^{j'}, \iota^{j''}\right\} = 1 \middle| P_D, \mathbf{x}_A\right) \geq Pr\left(\iota^{j'} = 1 \middle| P_D, \mathbf{x}_A\right) + \delta$$

Note that Claim 8 is a direct consequence of Claims 1 and 7. First, recall that from Claim 1 the probability that player A wins at least one minimal cut set is

$$Pr\left(\max\left\{\iota^{j'}, \iota^{j''}\right\} = 1 \middle| P_D, \mathbf{x}_A\right) = \\ Pr\left(\iota^{j'} = 1 \middle| P_D, \mathbf{x}_A^{j'}\right) + Pr\left(\iota^{j''} = 1 \middle| P_D, \mathbf{x}_A^{j''}\right) - Pr\left(\iota^{j'}, \iota^{j''} = 1 \middle| P_D, \mathbf{x}_A^{j', j''}\right)$$

Next, define δ as

$$\delta \equiv Pr\left(\iota^{j''} = 1 \middle| P_D, \mathbf{x}_A^{j''}\right) - Pr\left(\iota^{j'}, \iota^{j''} = 1 \middle| P_D, \mathbf{x}_A^{j', j''}\right).$$

Because, from Claim 7, for $P_D(\mathbf{0}) < 1 - \alpha$ there exists at least one node $i' \in N_{j'}$ such that $x_D^{i'} \in [\underline{\beta}, \overline{\beta}]$ and $\mathbf{x}_D^{j''} = \mathbf{0}$ with strictly positive probability, it follows that for an $\mathbf{x}_A$ with $\mathbf{x}_A^{j'} \neq \mathbf{0}$ where $x_A^{i'} \in (0, \underline{\beta})$ for node $i' \in N_{j'}$ and $\mathbf{x}_A^{j''} = \epsilon$ player A will, with strictly positive probability, lose minimal cut set j' but win minimal cut set j'', i.e. $Pr(\iota^{j''} = 1|P_D, \mathbf{x}_A^{j''}) > Pr(\iota^{j'}, \iota^{j''} = 1|P_D, \mathbf{x}_A^{j',j''})$ and, thus, $\delta > 0$.

We now use Claims 6 and 8 to prove part (3) of Proposition 3. By way of contradiction suppose that there exists an equilibrium with $P_D(\mathbf{0}) < 1-\alpha$. From Claim 6, we know that there exists a set of points of positive measure in the support of any equilibrium strategy P_A with $\mathbf{x}_A^{j'} \neq \mathbf{0}$, $x_A^{i'} \in (0, \underline{\beta})$ for node $i' \in N_{j'}$, and $\mathbf{x}_A^j = \mathbf{0}$ for all $j \neq j'$. Then from Claim 8, we know that player A can strictly increase his probability of winning at least one minimal cut set by deviating at each such point in the support of P_A where $\mathbf{x}_A^{j'} \neq \mathbf{0}$, $x_A^{i'} \in (0, \underline{\beta})$, and $\mathbf{x}_A^j = \mathbf{0}$ for all $j \neq j'$ by setting $\mathbf{x}_A^{j''} = \epsilon$ for $\epsilon > 0$. Furthermore, as ϵ approaches 0, such a strategy marginally increases player A's expected costs but, because $\mathbf{x}_D^{j'} \neq \mathbf{0}$ and $\mathbf{x}_D^{j''} = \mathbf{0}$ with strictly positive probability, discretely increases his probability of winning at least one minimal cut set. Thus, player A can strictly increase his expected payoff by deviating, and we have a contradiction to the assumption that there exists an equilibrium with $P_D(\mathbf{0}) < 1-\alpha$. We have, thus, shown that if $\alpha < 1$, then, in any equilibrium, $P_D(\mathbf{0}) = 1-\alpha$ and the defender leaves the entire network undefended with probability $1-\alpha > 0$. This completes the proof of part (3) of Proposition 3. $\square$

Lemma 9. *There exists a unique set of equilibrium univariate marginal distributions* $\{P_A^i, P_D^i\}_{i=1}^n$.

Proof. This proof is for the uniqueness of player D's set of univariate marginal distributions. The proof for player A is analogous. For each minimal cut set $j \in \mathcal{B}$, Lemma 5 shows that for any $\mathbf{x}_A^j \in [0, \bar{s}^j]^{n_j}$, $P_D^{N_j}(\mathbf{x}_A^j) = \frac{v_A - n_j \bar{s}^j}{v_A} + \frac{\sum_{i \in N_j} x_A^i}{v_A}$, where from part (3) of Lemma 8 $n_j \bar{s}^j = \min\{v_A, v_D/[\sum_{j' \in \mathcal{B}}(1/n_{j'})]\}$. Thus, in each minimal cut set j player D's unique univariate marginal distributions follow from player D's unique n_j-variate marginal distribution for minimal cut set j. In particular, for each $j \in \mathcal{B}$ and $i \in N_j$ player D's univariate marginal distribution for node i is calculated as $P_D^i(x^i) = P_D^{N_j}(x^i, \{\bar{s}^j\}_{i' \in N_j | i' \neq i})$:

(1) If $\alpha \geq 1$, then for each $j \in \mathcal{B}$ and $i \in N_j$, player D's univariate marginal is, for $x^i \in [0, \frac{v_A}{n_j}]$,

$$P_D^i\left(x^i\right) = 1 - \frac{1}{n_j} + \frac{x^i}{v_A}$$

(2) If $\alpha < 1$, then for each $j \in \mathcal{B}$ and $i \in N_j$, player D's univariate marginal is, for $x^i \in [0, \frac{\alpha v_A}{n_j}]$,

$$P_D^i\left(x^i\right) = 1 - \frac{\alpha}{n_j} + \frac{x^i}{v_A}$$

This completes the proof of Lemma 9. □

Proof of Theorem 1.

Proof. Immediately following the statement of Theorem 1 we showed that the joint distributions given in the Theorem constitute an equilibrium. To complete the proof, observe that parts (1) and (2) of Lemma 8 provides the unique equilibrium payoffs and Lemma 9 provides the unique sets of equilibrium univariate marginal distributions. □