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Abstract

In accordance with Basel Capital Accords, the Capital Requirements (*CR*) for market risk exposure of banks is a nonlinear function of Value-at-Risk (*VaR*). Importantly, the *CR* is calculated based on a bank's actual portfolio, i.e. the portfolio represented by its current holdings. To tackle mean-*VaR* portfolio optimization within the actual portfolio framework (*APF*), we propose a novel mean-*VaR* optimization method where *VaR* is estimated using a univariate Generalized AutoRegressive Conditional Heteroscedasticity (*GARCH*) volatility model. The optimization was performed by employing a Nondominated Sorting Genetic Algorithm (*NSGA-II*). On a sample of 40 large US stocks, our procedure provided superior mean-*VaR* trade-offs compared to those obtained from applying more customary mean-multivariate *GARCH* and historical *VaR* models. The results hold true in both low and high volatility samples.

JEL-codes: C610.

Keywords: portfolio optimization, actual portfolios, value at risk, GARCH, NSGA-II.

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1. Nomenclature

1.1. Acronyms

Actual Portfolio Framework, *APF*; Capital Requirements, *CR*; Conditional Value-at-Risk, *CVaR*; Dynamic Conditional Correlation Model, *DCC*; Evolutionary algorithm, *EA*; Expected Shortfall, *ES*; Fixed Weights Approach, *FWA*; Generalized AutoRegressive Conditional Heteroscedasticity, *GARCH*; Linear Programming, *LP*; Multi-objective Evolutionary Algorithm, *MOEA*; Maximum Likelihood Estimation, *MLE*; Nondominated Sorting Genetic Algorithm, *NSGA-II (NSGA-III)*; Pareto Envelope-based Selection Algorithm, *PESA*; Process Time, *CPU Time*; Pareto Archived Evolutionary Strategy, *PAES*; Quadratic Programming, *QP*; Standard and Poor's 100 stock market index, *S&P 100*; Strength Pareto Evolutionary Algorithm, *SPEA*; Value-at-Risk, *VaR*.

1.2. Notation

Raw return at time t , r_t ; conditional expected return, μ_t ; conditional variance, σ_t^2 ; residual (innovation) term, z_t ; portfolio, p ; probability, ρ ; dollar portfolio value at time t , $V_p(t)$; 1-day ahead conditional volatility estimate at time t , σ_{t+1} ; cumulative distribution function of returns, $F(\alpha)$; significance level, α ; time horizon, h ; degrees of freedom, d ; 1-day ahead conditional variance-covariance matrix of returns, \mathbf{H}_{t+1} ; vector of portfolio weights at time t , \mathbf{w}_t ; return on portfolio, $r_p(w)$; *GARCH VaR* of a portfolio, $VaR(r_p(w))$; expected return on portfolio, $E(r_p(w))$; price of shares in company i at time t , $P_i(t)$; portfolio holdings, n_i ; actual portfolio return at time t , $r_p(t)$.

2. Introduction

Value-at-Risk (*VaR*) is defined as the loss associated with the low (typically first or fifth) percentile of the return distribution.¹ The Basel II Capital Accord codifies *VaR* as the de-facto industry standard for the banking and insurance industries alike (see, BIS [3, 4, 5]). In particular, for the market risk exposure of banks, a bank's internal *VaR* estimates corresponding to the actual portfolio, i.e. the portfolio represented by its current holdings, translate directly into the regulatory capital charge (see Hendricks and Hirtle [6]). Motivated by this regulatory feature, we utilized the actual portfolio framework (*APF*) to determine a set of portfolios characterized by the optimal trade-off between the expected return and *VaR* (i.e. Pareto-optimal frontier). We further proposed a mean-univariate Generalized AutoRegressive Conditional Heteroscedasticity (*GARCH*) *VaR* portfolio optimization model. We assumed that portfolio returns, standardized by time varying volatility, have a conditional Student's *t* distribution, while conditional variance follows a *GARCH* (1, 1) process.² The Student's *t* distribution efficiently captures the fat tails of standardized asset returns (see Christoffersen [9], Huisman et al. [10]) whilst the *GARCH* model addresses issues related to volatility clustering observed in the data.³ To the best of our knowledge, this is the first paper that studies mean-*VaR* portfolio optimization using the actual portfolio approach and also the first paper that uses the univariate *GARCH VaR* model in this context.

Previous studies on mean-*VaR* optimization implicitly assumed fixed weights (i.e. fractions of assets) over the observed time period.⁴ Since prices change over time, maintaining the fixed portfolio weights (*FWA*) requires frequent trading and leads to changes in the number of shares of each asset in a portfolio. Regulatory capital charges, however, are determined by the *VaR* of an actual portfolio where the number of shares of an asset (rather than its weighting) is fixed over the observed period. It is, therefore, *APF*, rather than *FWA*, that is more relevant for asset managers facing regulatory *VaR* limits. To illustrate the effectiveness of our *APF* approach and univariate *GARCH* model, we compared our results with two benchmarks. Our first benchmark is the mean-historical *VaR* approach developed in Rockafellar and Uryasev [20, 21] and Krokmal et al. [22]. These authors mapped conditional *VaR* (*CVaR*) optimization into a linear programming problem and argued that the mean-*CVaR* efficient

¹ For classification and comparison of risk measures see [1] and [2], among others.

² *GARCH* was introduced in Engle [7] and *GARCH* (1, 1) in Bollerslev [8].

³ For more on superiority of *GARCH VaR* compared to historical *VaR* see [11], [12], [13], [14], [15], among others.

⁴ For example, [16], [17], [18], [19], [20], [21].

frontier provided near-optimal solutions in the context of the mean-historical *VaR* trade-off.⁵ We referred to this benchmark as the Linear Programming (*LP*) model. The second point of comparison for our approach is the mean-multivariate *GARCH VaR* optimization that can be mapped into the Quadratic Programming (*QP*) problem (see Santos et al. [26]).

Use of the univariate *GARCH* approach for *VaR* modeling, however, makes the mean-*VaR* optimization problem rather complex. Previous literature documented that Multi-Objective Evolutionary Algorithms (*MOEA*) could reliably and efficiently be applied in complex portfolio optimization problems.⁶ For example, Anagnostopoulos and Mamanis [31] studied the effectiveness of different *MOEA* (e.g. Nondominated Sorting Genetic Algorithm (*NSGA-II*), Strength Pareto Evolutionary Algorithm (*SPEA2*), Pareto Envelope-based Selection Algorithm (*PESA*), etc.) in solving various complex mean-variance optimization problems. They reported the best *NSGA-II* and *SPEA2* average performance in terms of hypervolume indicators, while *PESA* performed best in terms of the proximity to the Pareto-optimal frontier. The same authors (see [32]) also examined the mean-variance, mean-*ES*, and mean-*VaR* optimization problem with quantity, cardinality and class constraints. They showed that *NSGA-II*, *SPEA2* and *PESA* performed efficiently and their performance was independent of the risk measure used.⁷ Deb et al. [34] reported *NSGA-II*'s advantages over the Pareto Archived Evolutionary Strategy (*PAES*) and *SPEA*. Deb et al. [35] developed a hybrid *NSGA-II* procedure for handling a mean-variance portfolio optimization problem with the cardinality constraint and lower and upper bounds as investment criteria. The authors provided evidence of *NSGA-II*'s superiority over classical quadratic programming approaches. Branke et al. [36] considered the mean-variance problem with the maximum exposure constraint. The authors generated mean-variance Pareto-optimal frontiers by using a hybrid algorithm that combined *NSGA-II* with the critical line algorithm.

In this study we used the *NSGA-II* algorithm. Our choice was motivated by the above studies, whose results highlighted the advantages of *NSGA-II* in tackling complex portfolio

⁵ *CVaR* (or Expected Shortfall, *ES*) is the expected loss, conditional that loss is higher than *VaR*. Thus, *CVaR* and *VaR* are closely related risk measures. The *LP* model is widely used in *CVaR* and *VaR* optimization literature (see [23], [24], [25], among others).

⁶ For comprehensive surveys of *MOEA* applications in portfolio optimization, see [27], [28], [29], [30].

⁷ Anagnostopoulos and Mamanis also showed that *NSGA-II*, *SPEA2*, and *PESA* provide a good approximation of risk-return trade-offs in a 3 objectives optimization problem (see [33]).

optimization problems.⁸ *NSGA-II* was first introduced by Deb et al. [34] and subsequently developed in Deb et al. [37]. Recently, a new version of *NSGA* (*NSGA-III*) was developed by Deb and Jain [38] and also Jain and Deb [39]. *NSGA-III* is better suited to many-objective optimization problems (three and more objectives) and offers the ability to define the desired part of a solution space defined by reference points.⁹ Here, we prefer *NSGA-II* to *NSGA-III*, since we consider only two objectives and are interested in the entire Pareto-optimal frontier. Knowledge of the entire frontier of the risk-return trade-offs is particularly useful for asset managers in banks and other financial institutions in order to comply with different in-house and regulatory requirements.

For our numerical tests we selected 40 of the largest US stocks in the Standard and Poor's stock market index (*S&P 100*) for which sufficient data was available. First, we determined the mean-historical *VaR* Pareto-optimal frontier, in the *APF* framework, using the *NSGA-II* algorithm and compared it to the *LP* Pareto optimal frontier. The comparison showed that *NSGA-II* produced better mean-historical *VaR* trade-offs compared to the *LP* optimization approach, in both Low and High volatility samples. Second, we compared our mean-univariate *GARCH VaR* Pareto-optimal frontier with the frontiers obtained by the two benchmarks (mean-historical *VaR* and mean-multivariate *GARCH VaR*). In comparison with the two benchmarks, the proposed univariate *GARCH VaR* procedure provided actual portfolios with better mean-univariate *GARCH VaR* trade-offs, in both Low and High volatility samples.

Previous portfolio optimization studies typically neglect the differences between using portfolios with fixed weights and portfolios with fixed holdings of assets. We contribute to portfolio optimization literature by addressing recent real world regulatory changes which impose *VaR* based on actual portfolio holdings. The rare *MOEA* portfolio optimization studies that measured risk by using *VaR* tend to use historical rather than analytical *VaR* (see Anagnostopoulos and Mamanis [33], Hochreiter et al. [41], Soler et al. [42], Rankovic et al.

⁸ *NSGA-II* algorithm is also the most widely used *MOEA* in portfolio optimization literature. According to the number of studies reviewed in [30], *NSGA-II* was used in twice as many studies compared to second most popular method (*SPEA2*).

⁹ The number of the Pareto optimal solutions could also be reduced by combination of *NSGA-II* with Data Envelopment Analysis (see [40]).

[43]).¹⁰ We therefore also contribute to *MOEA* literature by examining the mean-*VaR* optimization problem with analytical univariate *GARCH VaR* instead of historical *VAR*.

The remainder of this paper is organized as follows. In Section 3 we introduce historical and *GARCH VaR* models. The optimization problem is introduced in Section 4. *MOAE* and implementation details are discussed in Section 5. Section 6 contains sample descriptive statistics. Results of mean-historical *VaR* optimization are presented in Section 7. Results of our univariate and multivariate mean-*GARCH VaR* optimization are presented and discussed in Section 8. We conclude in Section 9.

3. *VaR* Models

3.1. Traditional (historical) *VaR*

For a given portfolio, significance level α and time horizon h , portfolio *VaR* is a loss that will be exceeded, on average, only $\alpha \times 100$ percent of the time. If expressed in terms of portfolio value, *VaR* is the α -quantile of profit and loss distribution (cash *VaR*), while if expressed in terms of portfolio return r , it is the α -quantile of the return distribution (relative *VaR* or, simply, *VaR*). We focused on the α -quantile of the return distribution. Consider time horizon of $h=1$ day and return r_α such that probability $\rho(r < r_\alpha) = \alpha$. In that case, 1-day *VaR* with significance level α is:

$$VaR_\alpha = -r_\alpha \quad (1)$$

The minus sign is needed since *VaR* is defined as a loss. Given the cumulative distribution function of returns $F(\alpha)$, the α -quantile is calculated as $r_\alpha = F^{-1}(\alpha)$. The main reason for the popularity of this method is ease of its implementation and the fact that it makes no assumptions about the parametric form of return distributions. On the other hand, the historical *VaR* often slowly reacts to abrupt changes in market conditions (see Alexander [11]).

¹⁰ Only 4.27% of *MOEA* studies use *VaR* as one of the objectives in the portfolio optimization problem (see [27]).

3.2. Analytical *VaR*

A popular alternative to historical simulation is provided by analytical (or parametric) *VaR* models. These models take a stand on the shape of the return distribution and capture the following stylized facts about asset returns (see Christoffersen [9]). First, asset returns are difficult to predict based on their past realizations. Second, volatility of daily returns dominates their mean. Third, unlike returns, the variance of daily returns tends to exhibit clustering over time. Namely, days with highly volatile returns are likely followed by days with highly volatile returns and vice versa. Fourth, after periods of high or low asset volatility, volatility tends to return toward a stable long run level. Fifth, even though asset returns are often modeled using a normal distribution, an unconditional return distribution usually has much heavier tails than predicted by a normal distribution.¹¹ Using the first two conditional moments of distribution, the portfolio return can be presented in the form:

$$r_t = \mu_t + \sigma_t z_t \quad (2)$$

where r_t is the raw return at time t , μ_t is the conditional expected return, σ_t is conditional volatility of the return, and z_t is the residual (innovation term) of the process. It is assumed that z_t are independently and identically distributed and follow a known theoretical distribution D with zero mean and unit variance $D(0, 1)$.

To specify the analytical *VaR* model we specified the volatility updating model, as well as the shape of the conditional return distribution $D(0, 1)$. A commonly applied class of conditional volatility models that captures all of the above stylized facts of returns is the *GARCH* model.¹² In estimating *GARCH* parameters on daily data, we have taken into account that the mean value of daily returns is dominated by the standard deviation of returns (see Christoffersen [9], Pritsker [15], Alexander [44]).¹³ This implies:

$$r_t \approx \sigma_t z_t \quad (3)$$

We took into account non-normality of standardized asset returns by assuming they follow a standardized Student's t distribution (with zero mean and unit variance) with d degrees of

¹¹ Although this is partially rectified when returns are standardized by time-varying volatility, some residual non-normality still remains.

¹² *GARCH* process was first applied to *VaR* modeling in [14].

¹³ Alternatively, one can apply the same model to mean-adjusted returns.

freedom t (d). The degree of freedom parameter is an additional parameter estimated jointly with the other model parameters using Maximum Likelihood Estimation (*MLE*) method. The 1-day ahead analytical *VaR* estimate with significance level α , calculated at time t , is obtained as follows:

$$VaR_{t+1}^{\alpha} = -\sigma_{t+1} t_{\alpha}^{-1}(d) \quad (4)$$

where σ_{t+1} is the 1-day ahead conditional volatility estimate at time t obtained by applying the proposed model, d are degrees of freedom of the estimated Student's t distribution of standardized portfolio returns, and $t_{\alpha}^{-1}(d)$ is α -quantile of the standardized Student's t distribution with d degrees of freedom.

Conditional portfolio volatility can be modeled directly using the time series of portfolio returns (we referred to this method as the Univariate *GARCH* approach). Alternatively, one can estimate conditional volatility using the conditional variance-covariance matrix estimated via the multivariate *GARCH* model. We referred to this method as the multivariate *GARCH* approach.

3.2.1. Univariate *GARCH VaR*

Consider the (univariate) time series of portfolio returns and denote by r_t the realization of portfolio return at time t . The simplest and by far the most popular *GARCH* model of conditional volatility, commonly referred to as *GARCH (1, 1)*, has the following form (see Bollerslev [8]):

$$\sigma_{t+1}^2 = \omega + \alpha r_t^2 + \beta \sigma_t^2 \quad (5)$$

Here, $\alpha, \beta > 0$ and $\alpha + \beta < 1$.¹⁴ Thus, the univariate *GARCH VaR* is:

$$VaR_{t+1}^{\alpha} = -\left(\omega + \alpha r_t^2 + \beta \sigma_t^2\right)^{1/2} t_{\alpha}^{-1}(d) \quad (6)$$

¹⁴ The second condition assures stationarity of the conditional volatility process.

3.2.2. Multivariate GARCH VaR

In a multivariate GARCH modeling framework, the 1-day ahead portfolio conditional volatility can be estimated as a function of the constituents' returns as follows:¹⁵

$$\sigma_{t+1}^2 = \mathbf{w}'_t \mathbf{H}_{t+1} \mathbf{w}_t \quad (7)$$

where \mathbf{H}_{t+1} is $N \times N$ conditional variance-covariance matrix of returns of individual assets, and \mathbf{w}_t is the vector of portfolio weights at time t .¹⁶ Importantly, the multivariate GARCH implicitly assumes a fixed portfolio weights approach (FWA).¹⁷ The portfolio weights are thus fixed at $w_i(t) = w_i(T)$ for all, $t \leq T$, where T denotes optimization date. Thus, the multivariate GARCH VaR is:

$$VaR_{t+1}^\alpha = -(\mathbf{w}'_t \mathbf{H}_{t+1} \mathbf{w}_t)^{1/2} t_\alpha^{-1}(d) \quad (8)$$

4. Optimization problem

4.1. Mean-univariate GARCH VaR optimization (APF approach)

The problem we are trying to solve has the following general form:

$$\min VaR(r_p(\mathbf{w})) = -(\omega + \alpha r_T^2 + \beta \sigma_T^2)^{1/2} t_\alpha^{-1}(d) \quad (9)$$

$$\max E(r_p(\mathbf{w})) \quad (10)$$

$$\text{subject to} \quad \sum_{i=1}^N w_i = 1 \quad (11)$$

$$0 \leq w_i \leq 1, \quad i = 1, \dots, N \quad (12)$$

Here, \mathbf{w} denotes the vector of portfolio weights at optimization date $t=T$, its components are w_i , $r_p(\mathbf{w})$ is the return on the portfolio, $VaR(r_p(\mathbf{w}))$ denotes the portfolio risk measure that we try to minimize, and $E(r_p(\mathbf{w}))$ is the expected return on the portfolio. $t_\alpha^{-1}(d)$ here refers to a

¹⁵ See, for example, [26] and [45].

¹⁶ Boldface denotes matrices and vectors. Matrices are in upper case, whereas vectors are in lower case.

¹⁷ As discussed in [46], the fixed weights assumption further implies continuous portfolio rebalancing.

quantile of a univariate Student's t distribution with d degrees of freedom. Equation (11) describes the standard budget constraint which requires that weights sum up to 1. Equation (12) states that no short sales are allowed.¹⁸

To calculate actual portfolio VaR and the corresponding expected return, a time series of portfolio returns with fixed asset holdings is needed. We determined asset holdings at the optimization date by adopting an arbitrary dollar portfolio value (set, without loss of generality, to 1). Thus, we assumed that, at optimization date $t=T$, dollar portfolio value $V_p(t=T)=1$. Portfolio holdings n_i at time T were determined based on weights at $t=T$:

$$n_i = \frac{w_i V_p(t=T)}{P_i(t=T)} = \frac{w_i}{P_i(t=T)} \quad (13)$$

Here $P_i(t)$ denotes the price of shares in company i at time t . In the actual portfolio approach, holdings are held fixed over time. The actual portfolio return at time $t \leq T$ is, therefore, determined as follows:

$$r_p(t) = \frac{V_p(t)}{V_p(t-1)} - 1 = \frac{\sum_{i=1}^N n_i P_i(t)}{\sum_{i=1}^N n_i P_i(t-1)} - 1 \quad (14)$$

$V_p(t)$ denotes the dollar portfolio value at time t . The actual portfolio mean- VaR optimization problem simply means that we used equation (14) when determining input returns for the above optimization problem.

In the actual portfolio approach, portfolio weights at time $t \leq T$, are given by the expression:

$$w_i(t) = \frac{n_i P_i(t)}{\sum_{i=1}^N n_i P_i(t)} \quad (15)$$

Equations (14) and (15) imply that:

¹⁸ Short sale prohibition is a common constraint imposed on large institutional investors such as mutual or pension funds.

$$r_p(t) = \sum_{i=1}^N w_i(t-1)r_i(t) \quad (16)$$

where $r_i(t)$ is simple return on asset i at time t .

4.2. Mean - multivariate *GARCH VaR* optimization (*FWA* approach)

The portfolio optimization problem (Eq. (9) – (12)) can be specified as follows:

$$\min \text{VaR}(r_p(\mathbf{w})) = -(\mathbf{w}'\mathbf{H}_{T+1}\mathbf{w})^{1/2} t_\alpha^{-1}(d) \quad (17)$$

$$\text{subject to } \mathbf{w}'\boldsymbol{\mu} = \bar{r}_p \quad (18)$$

$$\text{subject to } \sum_{i=1}^N w_i = 1 \quad (19)$$

$$0 \leq w_i \leq 1, \quad i = 1, \dots, N \quad (20)$$

where \mathbf{H}_{T+1} denotes the conditional variance-covariance matrix of individual returns estimated at the time of optimization $t=T$, $\boldsymbol{\mu}$ vector of expected returns of individual assets and \bar{r}_p denotes the expected portfolio return.

A portfolio return for an arbitrary, $t \leq T$, is given by the expression:

$$r_p(t) = \sum_{i=1}^N w_i(T)r_i(t) \quad (21)$$

Quantile $t_\alpha^{-1}(d)$ refers to a quantile of a multivariate Student's t distribution with d degrees of freedom. Here, d does not depend on a portfolio composition. Namely, d and, thus, quantile $t_\alpha^{-1}(d)$ depend solely on the choice of constituent return series and not on the way in which they are combined into a particular portfolio. \mathbf{H}_{T+1} and d are obtained using the Dynamic Conditional Correlation (*DCC*) model of Engle [47].¹⁹

DCC decomposes conditional variance-covariance matrix \mathbf{H}_{t+1} into conditional standard deviations and correlations, that is:

¹⁹ We used 'rmgarch' package [48] within software R [49]. For fitting *GARCH* model we used 'dccfit' method. For the 1-day ahead estimation of conditional covariance matrix we used 'dccoforecast' method.

$$\mathbf{H}_{t+1} = \mathbf{D}_{t+1} \mathbf{R}_{t+1} \mathbf{D}_{t+1} \quad (22)$$

where $\mathbf{D}_{t+1} = \text{diag}(\sigma_{1,t+1}, \dots, \sigma_{N,t+1})$. Here $\text{diag}(\ast)$ is the operator transforming a $N \times 1$ vector into a $N \times N$ diagonal matrix. We assume that conditional variances $\sigma_{i,t+1}^2$, $i = 1, \dots, N$, follow a standard univariate *GARCH* (1,1) model.

Matrix \mathbf{R}_{t+1} is a symmetric positive definite conditional correlation matrix defined as:

$$\mathbf{R}_{t+1} = \text{diag}(\mathbf{Q}_{t+1})^{-1/2} \mathbf{Q}_{t+1} \text{diag}(\mathbf{Q}_{t+1})^{-1/2} \quad (23)$$

where \mathbf{Q}_{t+1} is a proxy process which is assumed to follow *GARCH*-type dynamics:

$$\mathbf{Q}_{t+1} = (1 - \alpha - \beta) \bar{\mathbf{Q}} + \alpha \mathbf{z}_t \mathbf{z}_t^T + \beta \mathbf{Q}_t \quad (24)$$

Vector $\mathbf{z}_t = (z_{1,t}, \dots, z_{N,t})^T$ has elements $z_{i,t} = r_{i,t} / \sigma_{i,t}$ (standardized unexpected returns or innovations), $\bar{\mathbf{Q}}$ is the $N \times N$ covariance matrix of \mathbf{z}_t and α and β are non-negative scalar parameters satisfying $\alpha + \beta < 1$.

Since $t_\alpha^{-1}(d)$ is constant, the optimization problem (Eq. 17) is equivalent to a problem represented by quadratic programming formulation (referred to as *QP* model):²⁰

$$\min \mathbf{w}' \mathbf{H}_{T+1} \mathbf{w} \quad (25)$$

It should be emphasized that quantile $t_\alpha^{-1}(d)$ in this approach is constant and therefore not portfolio specific. Thus, the multivariate *GARCH* approach is more restrictive compared to the univariate *GARCH* approach where the degrees of freedom of the estimated standardized Student's t distribution (and therefore quantile $t_\alpha^{-1}(d)$) were portfolio specific.

4.3. Summary of assumptions

4.3.1. General

- i) Short sales are not allowed (Eqs. (12) and (21));
- ii) The standard budget constraint which requires that weights must sum up to 1 (Eqs. (11) and (20)).

²⁰ This problem is of the standard Markowitz type (see [50]) with added short sales constraints.

4.3.2. Mean-univariate *GARCH VaR* optimization

- i) Portfolio returns, standardized by time varying volatility, follow a conditional standardized Student's t distribution (with zero mean and unit variance);
- ii) Conditional variance follows *GARCH (1, 1)* process, $\sigma_{t+1}^2 = \omega + \alpha r_t^2 + \beta \sigma_t^2$, (Eq. 5);
- iii) In estimating GARCH parameters on daily data, the mean value of daily returns is dominated by the standard deviation of returns and that $r_t \approx \sigma_t z_t$, (Eq. 3);
- iv) At optimization date $t=T$, without loss of generality, dollar portfolio value $V_p(t=T)=1$.

4.3.3. Mean-multivariate *GARCH VaR* optimization

- i) Portfolio weights are fixed over the observed period;
- ii) Returns of assets from the opportunity set jointly follow a multivariate Student's t distribution;
- iii) \mathbf{Q}_{t+1} (Eq. 23) follows GARCH-type dynamics: $\mathbf{Q}_{t+1} = (1 - \alpha - \beta)\bar{\mathbf{Q}} + \alpha \mathbf{z}_t \mathbf{z}_t^T + \beta \mathbf{Q}_t$
- iv) In matrix \mathbf{D}_{t+1} conditional variance $\sigma_{i,t+1}^2$ follows a univariate *GARCH (1, 1)* process

5. Portfolio optimization methodology

5.1. Evolutionary algorithms (*EA*)

EA are efficient stochastic search techniques for solving complex optimization and search problems (e.g., optimization problems with non-differentiable objective functions, large and non-convex solution spaces, complex constraints etc.) through an emulation of natural selection i.e. survival of the fittest (see Sastry et al. [51]). *EA* begin with a set of randomly generated candidate solutions, referred to as a population. In each of the iterations (i.e. generations) the following processes are performed: i) Good solutions from the current population are selected and transferred into a set of potential parents (mating pool); ii) Randomly selected parent solutions are combined (crossover) producing new solutions (offspring solutions); iii) Some randomly selected offspring solutions are slightly modified (mutation); iv) Generated offspring solutions constitute the population of the next generation. By performing these processes in each generation (until the termination condition is satisfied) the solution "evolves" and becomes even better in fulfilling the stated objectives.

Over the past two decades, significant advances have been made on *EA* methods to solve multi-objective optimization problems.²¹ This has led to the creation of multi-objective evolutionary algorithms (*MOEA*). In general, *MOEA* address two important issues: i) How to generate a Pareto-optimal frontier; and ii) How to provide diversity in alternative solutions (i.e. how to avoid convergence to a single point on the Pareto-optimal frontier). To achieve the first goal, most *MOEA* implementations use ranking based on the concept of dominance, while different diversity-preserving techniques are employed to achieve the second goal.

5.2. *MOEA* implementation

We directly borrowed the *NSGA-II* algorithm from Deb et al. [37].²² Implementation of *NSGA-II* involves adopting settings for the solution representation, the population size, the crossover and mutation probabilities and the termination condition. Solution representation depends upon the specific optimization problem. For the model at hand, the solution was defined as a non-negative real-valued vector of portfolio weights at time $t=T$, that is, the vector of fractions of the total budget invested in individual securities (see Section 4). Population size (the number of candidate portfolios in each generation), was set to 100. The Pareto-optimal frontier was, therefore, approximated with 100 points. The step by step flow chart which presents a schematic view of the proposed portfolio optimization method is shown in Figure 1.

In the preparatory phase our software generated and printed a data file (data.csv) with a time series of occurrences of assets under consideration. Next, it generated and printed an R script file (Script.R) with commands for estimation of the univariate *GARCH VaR*. It then executed the *NSGA-II* algorithm in the execution phase.²³

²¹ For a detailed introduction to *MOEA*, see [52].

²² The algorithm was coded in C# and run on a personal computer with Intel i5 processor and 4GB of RAM.

²³ For more details on the algorithm steps, see [37].

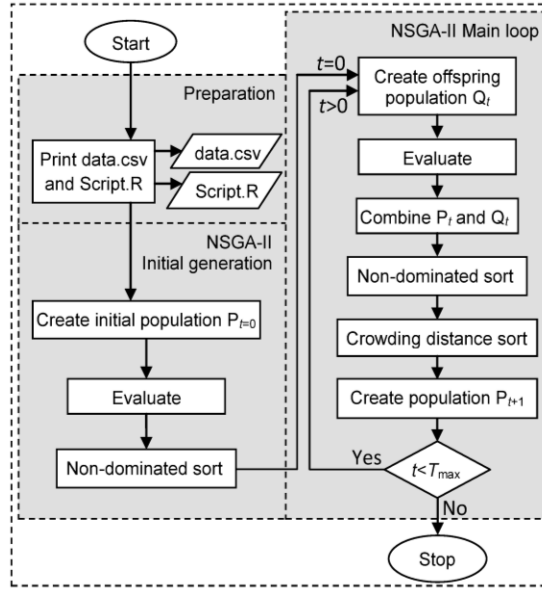


Figure 1. Flow chart of the optimization algorithm

For the breeding of an offspring population we used the uniform crossover operator. Two solutions (portfolios) from the current population were randomly selected and were recombined with a predefined crossover probability. If the two solutions undergo recombination, every chromosome (individual asset weight) is exchanged between the pair of randomly selected solutions with a certain probability, known as the swapping probability, otherwise, the two offspring are simply copies of their parents (see Sastry et al. [51]). In accordance with previous literature, we set the swapping probability to be 0.5.²⁴

We applied a uniform mutation operator (uniform replacement). When applying uniform mutation, each chromosome is selected with a predefined mutation probability and replaced with a realization of a random variable, uniformly distributed in the range defined by the lower and upper domain bounds. The selected crossover and mutation operator ensured that the constraint defined by Eq. (12) was satisfied for each offspring. However, these operators did not ensure satisfaction of the budget constraint (Eq.(11)). Hence, we had to normalize each of the offspring solutions.²⁵

In order to find the appropriate parameter values for the crossover and mutation probabilities, we performed a series of experiments. The performance was assessed using the ϵ -indicator, and the hypervolume metric (see Zitzler et al. [53]). The ϵ -indicator is a binary performance

²⁴ See [31], [32], [33], [51].

²⁵ We have done this by dividing each weight by sum of all weights.

metric which is used to measure how close the approximation set is to the reference set. As a reference set, the true or the best known Pareto-optimal frontier was used. The ϵ -indicator determines a minimum value the reference set must be multiplied by in order for every solution in the reference set to become weakly dominated by at least one solution in the approximation set. If the approximation set matches the reference set exactly then the ϵ -indicator takes the value of one. For this metric, values close to one indicate that the approximation set very closely matches the reference set. The hypervolume indicator was used to measure the diversity of the approximation set. The hypervolume measure quantifies the volume of the objective space dominated by an approximation set. For optimization problems with two objectives, it quantifies the area of the objective space dominated by the approximation set, bounded by a predefined reference point. Thus, for this metric higher values are preferable.

To determine the best performing mutation and crossover probability sets, we tested four crossover probabilities (0.7, 0.8, 0.9, 1.0) and five uniform mutation probabilities (0.001, 0.005, 0.01, 0.05, 0.1). The above tests resulted in 20 different configurations for each set of objectives (mean-historical *VaR*, mean-*GARCH VaR*) and for each data sample that we used. For each configuration, the algorithm was left to run until 100,000 solutions were generated.

The interactions between the *NSGA-II* algorithm and our statistical analysis were highlighted in Figure 2. For the evaluation of each individual solution portfolio, our software performed the following steps: i) It generated and printed a file with portfolio weights (*weights.csv*); ii) It called software R and executed the R script file (*Script.R*) created in the preparatory phase. During the execution of the script file, software R used the data file (*data.csv*) and the solution portfolio weights file (*weights.csv*) and generated a time series of actual portfolio returns (applying Eq. (13)–(14)). Then *VaR* was estimated using the univariate *GARCH* model (Eq. (4) and (5)).²⁶

²⁶ We used ‘*rugarch*’ package [54] within software R [49]. For fitting *GARCH* model we used ‘*ugarchfit*’ method; For the 1-day ahead estimation of conditional volatility we used ‘*ugarchforecast*’ method.

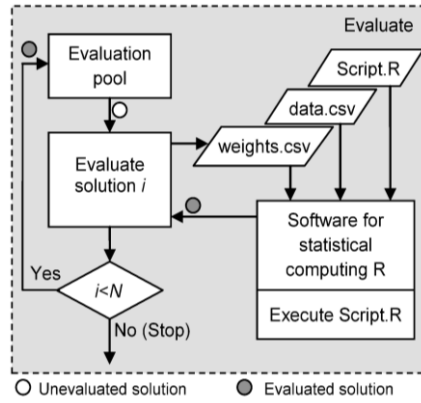


Figure 2. Interactions between the *NSGA-II* algorithm and statistical computing

In the case of the mean-historical *VaR* approach, algorithm execution requires calculating the α quantile of the empirical distribution of 100,000 candidate portfolios. However, in the case of the mean-univariate *GARCH VaR* approach, the optimization procedure implies calculating the α quantile of the parametric distribution of candidate portfolios. Additional complexity of the univariate *GARCH VaR* approach stems from the fact that the *GARCH* parameters for each of 100,000 portfolios were obtained by the *MLE* method. We ensure that all Pareto-optimal frontiers, obtained using different configurations, include solutions which provide the minimum risk and the maximum return.²⁷

6. Sample descriptive statistics

For the purposes of this study, we selected 40 constituents of the *S&P 100* with the highest market capitalization as of Sept 6th 2013.²⁸ We observed the constituents' time series of 1,421 returns in the period Jan 15th 2008-Sept 6th 2013. We used a rolling window of 1,000 returns for *VaR* estimation. For each day within the rolling estimation period of 421 days (Jan 4th 2012-Sept 6th 2013) we estimated 1-day ahead daily volatility of the *S&P 100* time series. The volatilities were estimated using the univariate *GARCH (1, 1)* model. Standardized returns were assumed to have a standardized *t*-distribution. *GARCH* volatility estimations of the *S&P 100* index were based on a rolling window of 1,000 daily returns. Maximum volatility of

²⁷ We obtained these solutions and included them into the initial population of each *MOEA* execution. Maximum return solution was obtained analytically, while minimum risk solution was obtained using single-objective genetic algorithm.

²⁸ Initially, 43 stocks were considered of which 3 were discarded (tickers: PM, V, KFT) due to incomplete data during the sample period.

0.01294 (20.5% in annual terms) was determined on June 29th 2012. Minimum volatility of 0.00583 (9.3% in annual terms) was determined on July 31st 2013.²⁹ From the chosen sample of 40 stocks we created two time series with 1,001 daily occurrences: i) First, ending on the maximum volatility date (referred to as the High volatility sample); and ii) Second, ending on the minimum volatility date (referred to as the Low volatility sample). All *VaR* values in this paper correspond to a significance level of 0.01, with a time horizon of one day. Table 1 presents a summary statistics of the time series comprising the Low and High volatility samples.

Table 1. Descriptive statistics for the Low and High volatility samples

Sample	Average expected return (annualized)	Average volatility (annualized)	Average historical <i>VaR</i>	Average <i>GARCH VaR</i>
Low volatility	16.61%	23.47%	3.96%	3.12%
High volatility	13.47%	37.94%	6.95%	4.12%

7. Mean-historical *VaR* optimization

Rockafellar and Uryasev [20, 21] and Krokmal et al. [22] argue that the mean-*CVaR* efficient frontier generated by the *LP* model also provides near-optimal solutions in the context of mean-historical *VaR* optimization. We tested this assertion in an actual portfolio framework. Namely, for each portfolio belonging to the mean-*CVaR* Pareto-optimal frontier generated by a *LP* (*FWA* approach), we created a time series of actual portfolio returns (applying Eq. (13) –(14)), calculated the corresponding 1% historical *VaR* and expected return, and represented the portfolios in the actual portfolio mean-*VaR* plane. In this way we generated an approximate Pareto-optimal frontier corresponding to the *LP* solution. Next, we determined the mean-historical *VaR* Pareto-optimal frontier (in the actual portfolio framework directly) using the *NSGA-II* optimization approach (see Figure 3). Figure 3 shows that the *NSGA-II* frontiers dominate the *LP* frontiers, especially in the High volatility sample.

²⁹ We annualize volatility using square root of time assuming 252 trading days per annum.

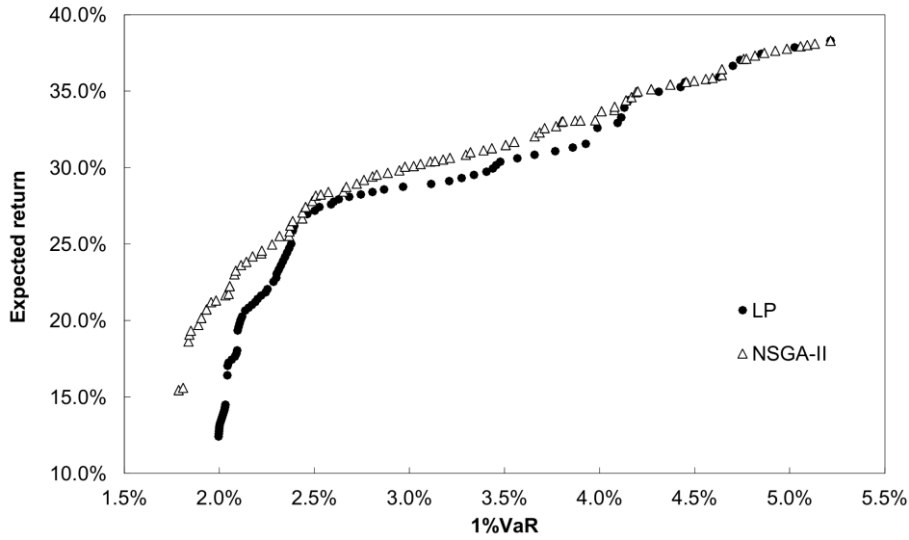


Figure 3.a. *NSGA-II* and *LP* approximate mean-historical *VaR* Pareto-optimal frontier in the Low volatility sample

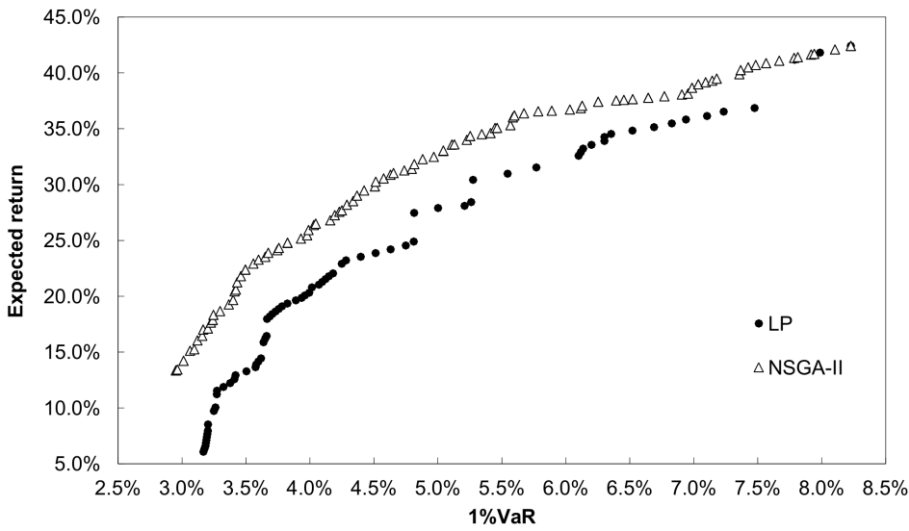


Figure 3.b. *NSGA-II* and *LP* approximate mean-historical *VaR* Pareto-optimal frontier in the High volatility sample

Best performing *NSGA-II* mean-historical *VaR* approximate of the Pareto-optimal set was selected based on the hypervolume metric. To compare the *NSGA-II* mean-historical *VaR* Pareto-optimal frontier to the *LP* Pareto-optimal set, we used the ϵ -indicator. As a reference set, we adopted the *NSGA-II* mean-historical *VaR* Pareto-optimal frontier. The comparison is presented in Table 2. The values of the hypervolume parameter and reference points, together with the ϵ -indicator, were for the best performing *NSGA-II* approximates of the mean-*VaR* Pareto-optimal frontier.

Table 2. Comparison of *NSGA-II* mean-historical *VaR* and *LP* models

Sample	Mutation probability	Crossover rate	Hypervolume	Reference point	ε -indicator
Low volatility	0.005	0.8	4.2366E-05	(0.0521,0)	1.1300
High volatility	0.05	1	6.9160E-05	(0.0823,0)	1.1526

The values for the ε -indicator confirm that the *NSGA-II* optimization approach provides better mean-historical *VaR* trade-offs of actual portfolios compared to the *LP* solutions in both the High and Low volatility samples. The relatively higher value of the ε -indicator (1.1526) suggests that *NSGA-II* performed particularly well during the High volatility period.

8. Results for mean-*GARCH VaR* optimization

8.1. Pareto-optimal frontiers in the Low and High volatility samples

In this section we compared the Pareto-optimal frontiers for: i) The mean-univariate *GARCH VaR* (or univariate *GARCH* for short); ii) The mean-multivariate *GARCH VaR* (benchmark 1); and iii) The mean-historical *VaR* (benchmark 2).³⁰ In order to obtain solutions in the actual portfolio framework for both benchmark portfolio solutions (represented by the corresponding vector of weights \mathbf{w}), we generated a times series of portfolio returns. The time series of portfolio returns was generated by employing Equations (13) and (14) within the *APF* approach. We then calculated *VaR* using the univariate *GARCH* model (Eq. (4) and (5)).³¹

The results are shown in Figure 4. The Pareto-optimal frontiers, for actual portfolios, are for the two volatility regimes when *1%VaR* was estimated using the univariate *GARCH* model. Triangle markers represent the univariate *GARCH* frontier (obtained via *NSGA-II*), while filled dots represent benchmark 1 solutions and empty dots represent benchmark 2 solutions.

³⁰ As presented in Figure 3, the *NSGA-II* method provided superior actual portfolio mean-historical *VaR* trade-offs compared to the *LP* model, in both samples. Consequently, we adopted the *NSGA-II* mean-historical *VaR* portfolios as benchmark 2.

³¹ During this process some of benchmark 1 and benchmark 2 portfolio solutions became dominated and thus have been discarded from the approximation set.

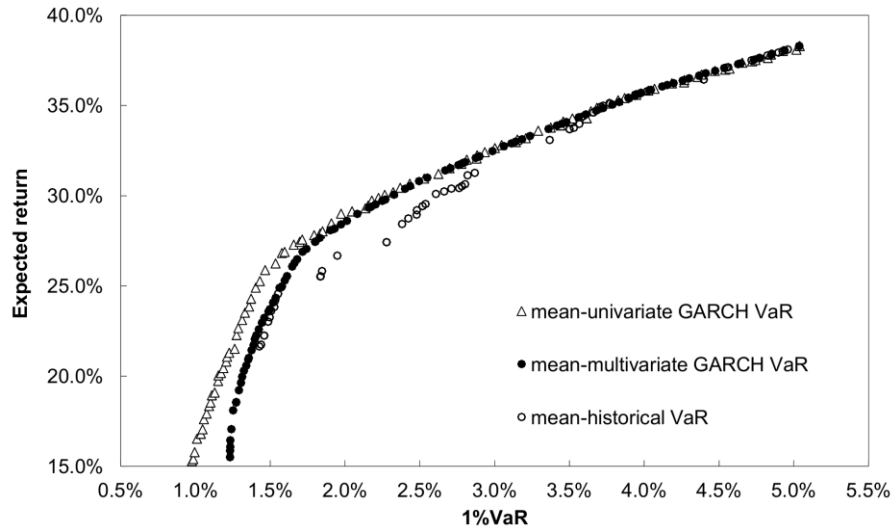


Figure 4a. Pareto-optimal frontiers in the Low volatility sample

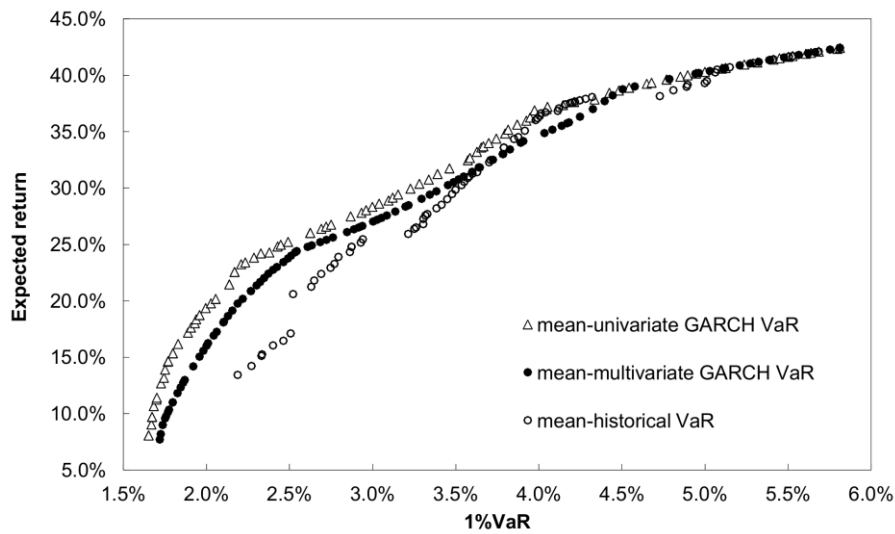


Figure 4b. Pareto-optimal frontiers in the High volatility sample

The differences are particularly prominent in the area of low returns (and low risk). In this segment of the Pareto-optimal set, thus, the opportunities for improving the mean-*VaR* trade-off through optimization are the greatest. It is worth noting that this segment of the Pareto-optimal set is associated with most diversified portfolios. In contrast, with higher expected return values, the cardinality of efficient portfolios is reduced (the highest return portfolio, by construction, corresponds to a single asset).

To further compare the univariate *GARCH* Pareto-optimal frontier to the respective benchmarks, we used the ϵ -indicator. As a reference set, we adopted the univariate *GARCH*

frontier. Table 3 presents the values of the hypervolume parameter and reference points for the best performing *NSGA-II* mean-univariate *GARCH VaR* approximate Pareto-optimal sets. The table also shows ε -indicators corresponding to the respective benchmarks in the Low and High volatility samples.

Table 3. Parameters of the best performing mean-univariate *GARCH VaR*

	Mutation probability	Crossover rate	Hypervolume	Reference Point	ε -indicator	
					Benchmark 1	Benchmark 2
Low volatility	0.05	0.7	5.0973E-05	(0.0504, 0)	1.2608	1.4618
High volatility	0.01	0.9	5.3451E-05	(0.0581, 0)	1.0866	1.3243

The above presented ε -indicators confirm superiority of the univariate *GARCH VaR* optimization approach. The advantages of the univariate *GARCH VaR* optimization approach are particularly pronounced in the Low volatility sample. Measured by the ε -indicators, the mean-historical *VaR* approach provides the worst approximations out of the three approaches. The above result is particularly important given that approximately 75% of banks tend to use the historical *VaR* models for portfolio optimization (see Pérignon and Smith [55]).

8.2. Process time (*CPU* time)

As expected, our *CPU* time is longer compared to the *CPU* time of the traditional methods (e.g. *QP*). For example, the execution of one generation of *NSGA-II*, when solving the mean-univariate *GARCH VaR* problem, lasted 41 seconds. Hence, to execute 1,000 generations of *NSGA-II* we needed 41,000 seconds.³² The total *CPU* time for mean-univariate *GARCH VaR* optimization with the *QP* solver was 424 seconds.³³ Our additional analysis, however, revealed that the *NSGA-II* *CPU* time could be shorter since the execution of 1,000 generations was not always necessary. For example, for the mean-historical *VaR* optimization, 99% of final hypervolume was achieved after 50 generations (12 seconds) in the Low volatility sample, and after 52 generations (12.5 seconds) in the High volatility sample.³⁴ The same

³² In comparison, it took only 1 second for the *LP* model to generate mean-historical *VaR* Pareto-optimal frontier consisting of 100 solutions.

³³ The mean-univariate *GARCH VaR* optimization (using multivariate *GARCH VaR* approach) consisted of three steps: i) Determination of conditional variance-covariance matrix of individual returns (H_{T+1}); ii) Execution of *QP* solver; and iii) Estimation of univariate *GARCH VaR* for each solution obtained by using *QP* solver. To determine conditional variance covariance matrix 382 seconds was needed. The execution of *QP* solver for Pareto optimal front of 100 solutions lasted 1 second. Finally the estimation of univariate *GARCH VaR*, for 100 solutions obtained by using *QP* solver, lasted 41 second.

³⁴ We define final hypervolume as a hypervolume of Pareto-optimal set generated after 1,000 generations.

analysis applied on the mean-univariate *GARCH VaR* optimization revealed that 99% of final hypervolume was achieved after 40 generations (1,640 seconds) in the Low volatility sample. The corresponding values in the High volatility sample were 20 generations and 820 seconds. Given its longer *CPU* time, the use of *NSGA-II* could be justified by its greater flexibility and ability to deal with more complex real-life portfolio optimization problems compared to the *QP* method (see Deb et al. [35]).

8.3. Out-of-sample estimates

Bank managers and regulators are interested in the out-of-sample performance of different optimization models. Thus, we compared out-of-sample performance of the different optimization models used in our study. The optimization dates for our High and Low volatility samples were dates with the highest (29th June 2012) and lowest (31st July 2013) volatilities. Based on the estimated portfolios on the optimization dates, we calculated respective mean and *VaR* values for dates which fall exactly 1 month later (30th July 2012 and 3rd September 2013 respectively).

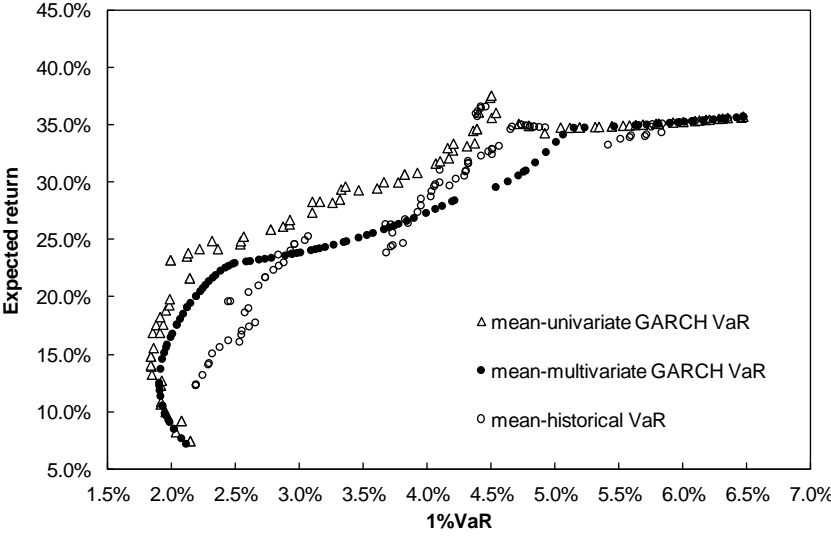


Figure 5a. Out-of-sample test in the High volatility sample

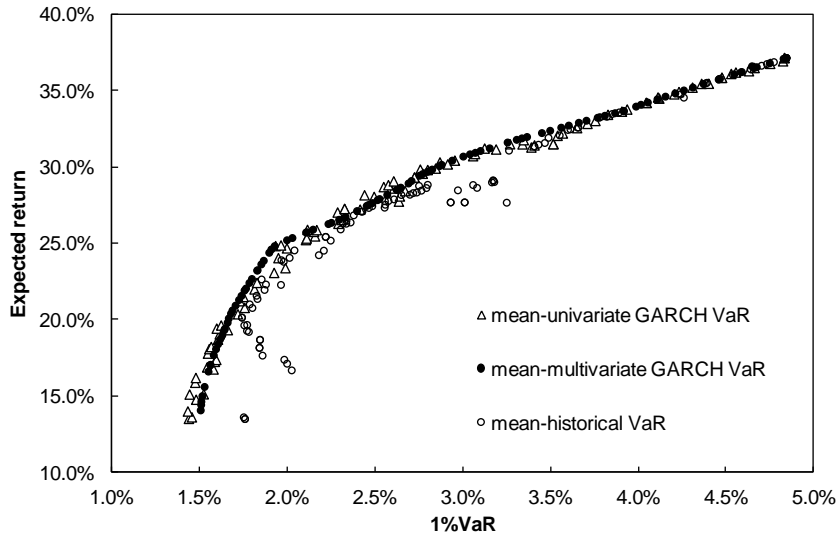


Figure 5b. Out-of-sample test in the Low volatility sample

The results of the out-of-sample analysis were consistent with the results reported in Figure 4, thus confirming the superiority of estimates based on the mean-univariate *GARCH* model in both volatility samples. As expected, the differences between the results for the mean-univariate and mean-multivariate *GARCH* models are less pronounced in the Low volatility sample.

9. Conclusions

According to Basel regulation, the riskiness of a portfolio and the implied capital charge are calculated using *VaR* estimations of an actual portfolio of a financial institution (i.e. the portfolio that corresponds to current portfolio holdings). This paper proposes a novel approach to mean-*VaR* portfolio optimization within the actual portfolio framework when *VaR* is estimated by the analytical univariate *GARCH VaR* model with the assumption of a conditional *t* distribution of standardized portfolio returns. Due to the complexity of the proposed optimization problem, we applied meta-heuristics. Specifically, we developed software which combined a *NSGA-II* multi-objective evolutionary algorithm with software for statistical computing R. In the empirical section, we examined the opportunity set consisting of 40 large US stocks belonging to the *S&P 100* index in two volatility regimes. We found that the *NSGA-II* method resulted in better actual portfolio mean-historical *VaR* trade-offs in both Low and High volatility regimes.

Next, we compared the mean-univariate *GARCH VaR* Pareto-optimal frontier to the mean-multivariate *GARCH VaR* and to the mean-historical *VaR* Pareto-optimal frontiers. In comparison to the two benchmarks, the proposed univariate *GARCH VaR* procedure, again, provided actual portfolios with a superior mean-univariate *VaR* frontier for both volatility samples. The results suggest that the multivariate *GARCH* modeling framework lacks flexibility to conform to the actual portfolio framework which is inherent in regulation. At the same time, the mean-historical *VaR* frontier provided the worst mean-univariate *GARCH VaR* trade-offs.

Overall, our results bear two important implications for financial institutions and their regulators. First, the results highlighted differences between the actual portfolio approach and the approach based on the fixed weights. Second, the results show the importance of carefully selecting amongst different *VaR* methodologies used in portfolio optimization.

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