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Aging, Social Security Design, and Capital Accumulation

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Abstract

This paper analyzes the impact of demographic aging on capital accumulation and welfare in economies with unfunded pensions. Using a two-period overlapping generation model with potentially endogenous retirement decisions, it shows that both the type of aging, i.e. declining fertility or increasing longevity, and the type of pension system, i.e. defined contributions or defined benefits, are important in understanding this impact. Results show that when aging is driven by increasing longevity, an unregulated retirement age system leads to a greater improvement in welfare. In contrast, with decreasing fertility, a mandatory retirement system with defined contributions fares better.

JEL-Codes: H200, F420, H800.

Keywords: aging, public finance sustainability, social security.

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1 Introduction

Demographic aging presents a major challenge to all industrialized economies and a large number of developing countries. According to UN Population Division projections, the total world population will increase by 40% and the median age on the planet will increase by 7.8 years within the next four decades. Compared to recent history, these changes represent a significant slowdown in population growth and a considerable acceleration in aging.

Although demographic aging is common around the world, the factors involved vary among countries. In the short and medium terms, these may be transitory events such as out-migration of the young population, wars or the aging of the baby-boom generation. However, in the long term, this phenomenon can be traced back to decreases in fertility rates and increases in longevity, albeit at different magnitudes in different economies.

The economic implications of demographic aging are complex, and they are not always well understood in public discussions. Some consequences are clearly unfavorable: aging, whether it is driven by a decrease in fertility or an increase in longevity, ceteris paribus, pushes the old-age dependency rate up, *i.e.* there are more elderly people than economically active people. This, in turn, increases the pressure on unfunded pensions. In contrast, some consequences of aging are perceived to be positive: if the decline in fertility outweighs the increase in longevity, then the total population will decrease. This outcome may be welcomed by some on the basis of environmental concerns. Finally, there are some ambiguous consequences. An example is the effect of aging on capital accumulation, a key determinant of growth, which we investigate in this paper.

Studying the effect of aging on capital accumulation is particularly difficult when a large number of discretionary policy choices can affect the outcomes. In this paper, we use a two-period overlapping generation model to show that the effect of aging on capital accumulation and welfare depends on: i) the type of aging, *i.e.* decreasing fertility or increasing longevity, ii) the type of unfunded social security system, *i.e.* defined contribution (DC) or defined benefit (DB), and finally iii) the regulation of the retirement age, *i.e.* mandatory early retirement vs. *laissez-faire*.

To fix these ideas, we set up an economic environment in which each individual lives for two periods. The first period of her life has a unitary length, while the second one has a variable longevity. In the first period, the individual works and earns a wage equal to her marginal productivity net of any social security contributions. This income is then devoted to consumption in the first period and saving for future consumption. In the second period, she works for a fraction of her remaining lifetime with her work duration being determined by the balance between the marginal income and the disutility created by the work. In the case of mandatory retirement, however, the optimal retirement choice may be overwritten and she may be forced to retire earlier than she would like. In the end, the individual's second period consumption is equal to the wage earnings from second period work, savings from the first period's with interest earnings, and, finally, pension benefits from the unfunded social security system.

This structure enables us to elaborate on our main results by introducing a number of institutional and demographic factors *vis-à-vis* the standard Diamond case, where individuals do not work in the second period of their lifetime and there is no PAYG pension system. First, we allow work in the second period of life and investigate how a mandatory early retirement rule affects the outcome. Second, we consider different types of unfunded social security systems in order to see how incentives respond to changes in demography under different pension system obligations and entitlements. Third, and finally, we also investigate these effects under different aging profiles, *i.e.* fertility driven vs. longevity-driven changes in age composition.

The main contribution of this study is, then, to show the incidence of these three factors on the effects of aging on capital accumulation and welfare. In the standard Diamond case, an increase in fertility decreases capital accumulation in the absence of a PAYG pension system, as capital is diluted by more workers. In our framework, this depressive effect is reinforced if the country has a DC pension system. In contrast, it is weakened or possibly reversed with DB pensions. Similar results are also derived with increasing longevity. A small increase in longevity has a fostering effect on capital accumulation in the standard case. Introducing PAYG pensions and the possibility of work in the second period, however, diminishes and potentially reverses the fostering effect.

The economics literature comprises a large number of studies devoted to understanding the effects of demographic aging in different settings. These could be classified on the basis of numerous criteria: explicit recognition of the distinction between different sources of aging, *e.g.* longevity and fertility changes, consideration of different social security systems, and characterization of growth in exogenous or endogenous settings. We will not provide an exhaustive review of this large field. However, a subset of these studies that investigate how institutional factors and behavioral responses may affect the impact of aging on capital accumulation is more relevant for our purposes.

An interesting discussion on the effect of longevity increase on growth is provided by Bloom et al. (2007). The authors point out that, in theory, improvements in healthy life expectancy should increase the average age of retirement with little effect on savings rates. In many countries, however, retirement incentives in social security programs prevent retirement ages from keeping pace with changes in life expectancy, increasing the need for life-cycle savings. Using a cross-country panel of macroeconomic data, the paper then finds that increased longevity raises aggregate savings rates in countries with universal pension coverage and retirement incentives. Similarly, Bloom et al. (2003) show that aging leads to more capital accumulation even if retirement is endogenous. Echevarria (2004) reaches the same conclusion. Kalemli-Ozcan et al. (2000) show that the positive effect of mortality decline on capital accumulation is larger if education decisions are endogenous.

De la Croix and Licandro (1999) and Zhang et al. (2001, 2003) argue that the effect of increasing longevity depends on its initial level. For low levels of life expectancy the effect is positive but it can turn negative for high levels. Similarly, Miyazawa (2006) also shows that the effect of an increase in longevity on economic growth has a hump-shaped pattern. This is the result of a two-effect system. First, higher longevity increases the aggregate saving rate directly by increasing precautionary saving for the prolonged retirement period and indirectly by increasing accidental bequests (the bequest-wage ratio is important because higher-income groups have a higher propensity to save). Second, it reduces the frequency of accidental bequests, which implies that the population share of the higher income group decreases. This leads to a reduction in aggregate savings. The relative shares of these factors change over the aging horizon. This is also true for income inequality (first positive, then negative). Kinugasa and Mason (2006) provide empirical evidence to show that an increase of wealth across countries is likely as mortality declines.

Let us also mention the effect of aging on human capital and hence on growth. The idea that an increase in longevity can foster investment in education became well known starting with Ben-Porath (1967). Recently, Ludwig and Vogel (2010), by using a two-period overlapping generation model similar to ours, looked at the effect of longevity on human capital accumulation but also extended their analysis to cover fertility. They showed that, whereas declining fertility stimulates education and capital accumulation, increasing longevity has an ambiguous effect on both.

Among the studies that link the impact of aging with social security systems, Ito and Tabata (2008) find that unfunded social security systems can explain the hump-shaped relationship between longevity and per capita output. Tabata (2014) looks at the effect on growth of a shifting from a DB to a DC PAYG pension on growth. He shows that this shift is growth-enhancing and alleviates the cost of aging. Heijdra and Mierau (2011) also compare the respective effects of DB and DC PAYG pensions on economic growth in an aging society. They show that the DC formula fares better than the DB one in facilitating growth. They also show that raising the retirement age as a response to an increase in longevity dampens the growth gains. The analysis in this paper compares several different social security systems, retirement age policies, and types of aging within a unified framework, thus providing a consistent survey of the welfare effects of demographic aging under various conditions.

Some of the theoretical results that we develop in this paper are known in the existing literature. Others, including the surprisingly positive effect of mandatory early retirement on welfare and the non-monotonic dynamic effects of aging on welfare, are novel. In addition, a major contribution of this paper is to organize all of these findings in a unified framework that can show us what would be the ideal social security system for a society with an aging population. We assume throughout our analysis that the pay-as-you-go (PAYG) pension system is a given and that, assuming dynamic efficiency, this system is welfare-worsening. In other words, we do not tackle the issue of shifting from a PAYG to a fully funded system of pensions.

The rest of the paper is organized as follows. In section 2, we present our basic model and main results for an economy that consists of identical individuals with a defined contribution pension system. Section 3 is devoted to steady state comparative statics where we investigate changes in capital accumulation and welfare for all pension systems, retirement schemes, and aging types. Sections 4 and 5 present static and dynamic simulations respectively. Finally, in the last section, we offer some concluding remarks.

2 Basic Model

We use a standard two-period overlapping generation model. An individual who belongs to generation t lives in two periods: t and $t+1$. The first period of her life has a unitary length, while the second one has a length $\ell \leq 1$, where ℓ reflects variable longevity.

In the first period, the individual works and earns a wage, w_t, which is devoted to first-period consumption, c_t, savings, s_t, and pension contribution, τ. In the second period, she works for an amount of time $z_{t+1} \leq \ell \leq 1$ and earns $z_{t+1}w_{t+1}$. These earnings, together with the proceeds of savings $R_{t+1}s_t$ and the PAYG pension p, finance the second period consumption d_{t+1}.

We assume that working in the second period z_{t+1} implies a disutility defined in monetary terms $v\left(z_{t+1}, \ell\right)$, where $\frac{\partial v}{\partial z} > 0$, $\frac{\partial^2 v}{\partial z^2} > 0$ are imposed for the existence of a unique solution. In addition, disutility from working in the second period of life is a decreasing function of longevity, *i.e.* $\frac{\partial v}{\partial \ell} < 0$, which reflects the idea that an increase in longevity fosters later retirement. Note that, for simplicity, earnings in the second period of life are not taxed. Any savings in a funded social security system are not modeled explicitly and are assumed to be identical to other savings. Thus, the pension contribution parameter τ measures the relative size of the unfunded pensions. In other words, $\tau = 0$ implies that the whole pension system is funded.

Denoting by $u(\cdot)$, the utility function for consumption c or d, and U, the lifetime utility, the problem of an individual of generation t is:

$$\max_{s,z} U_t = u\left(w_t - \tau - s_t\right) + \beta \ell u\left(\frac{w_{t+1} z_{t+1} + R_{t+1} s_t + p - v\left(z_{t+1}, \ell\right)}{\ell}\right), \tag{1}$$

where $p = \tau(1+n)$ is the pension benefit in period $t+1$ and β is the time discount factor. The gross rate of population growth $(1+n)$ is equivalent to the number of children per individual in this set-up. Note also that the argument of the second period utility is the net amount of resources then available divided by the length of the second period.[1]

The first order conditions for life time utility maximization are simply given by:

$$v'_{z_{t+1}}\left(z_{t+1}, \ell\right) = w_{t+1}, \tag{2}$$

$$\beta R_{t+1} u'\left(d_{t+1}\right) - u'\left(c_t\right) = 0, \tag{3}$$

where c_t and d_{t+1} denote first and second period consumption. The first condition (2) shows that the marginal disutility from the second period of work needs to be equal to the wage rate in equilibrium. The second condition is the consumption Euler equation, which shows that the individual cannot gain further utility by reallocating consumption between periods. In order to be able to show some of our results analytically, we will use simple functional forms for $u(\cdot)$ and $v(\cdot)$. Accordingly, we assume that the period utility function is logarithmic $u(x) = \ln x$ and that the monetary disutility function is quadratic in its main argument $v(x) = \frac{x^2}{2\gamma\ell}$. One clearly sees from the latter functional form that the disutility of working longer can be mitigated by an increase in longevity. We can now rewrite the problem of the individual as the following:

$$U_t = \ln\left(w_t - \tau - s_t\right) + \beta \ell \ln\left(\frac{w_{t+1} z_{t+1} + R_{t+1} s_t + p - \left(\frac{z_{t+1}^2}{2\gamma\ell}\right)}{\ell}\right). \tag{4}$$

[1] Suppose T_1 (T_2) is the number of years of the first (second) period, where $T_1 > T_2$. The life-time utility would then be: $U = T_1 u\left[\left(w - \tau - s\right)/T_1\right] + \beta T_2 u\left(\left(wz + Rs + p - v(z, T_2)\right)/T_2\right)$, where we normalize T_1 and T_2 so that $T_1 = 1$ and $T_2 = l$.

The first order condition with respect to z_{t+1} yields:

$$z_{t+1} = z^*_{t+1} = \gamma \ell w_{t+1}, \tag{5}$$

where an asterisk $(*)$ denotes an optimal solution.[2] Using this optimality condition and incorporating $p = \tau(1+n)$, we now get an explicit solution for the optimal saving rates:

$$s_t = \frac{\beta\ell}{1+\beta\ell} w_t - \frac{\gamma \ell w^2_{t+1}}{2R_{t+1}\left(1+\beta\ell\right)} - \tau\left(\frac{\beta\ell}{1+\beta\ell} + \frac{1+n}{\left(1+\beta\ell\right)R_{t+1}}\right). \tag{6}$$

In many countries, z is not the outcome of a choice without a distortion. Through an array of programs, workers are induced to retire at a different age than they would choose in the absence of these programs. We consider a case where the workers are induced to retire earlier than they wish, and denote this induced early retirement by $\bar{z}$.[3] In the case of this mandatory early retirement, we rewrite equations (5) and (6) as follows :

$$z_{t+1} = \bar{z} < z^*_{t+1},$$

$$s_t = \frac{\beta\ell}{1+\beta\ell} w_t - \frac{\bar{z}}{R_{t+1}\left(1+\beta\ell\right)}\left(w_{t+1} - \frac{\bar{z}}{2\gamma\ell}\right) - \tau\left(\frac{\beta\ell}{1+\beta\ell} + \frac{1+n}{\left(1+\beta\ell\right)R_{t+1}}\right). \tag{7}$$

This structure helps us develop one of the major contributions of this paper by contrasting the effects of aging when the choice of retirement is early and mandatory and when the choice is free. This is a widely discussed topic in the economics of retirement, where the focus is on the efficiency cost that mandatory early retirement entails. By setting the problem in a dynamic setting we show that early retirement has some virtue as it stimulates saving and gets the economy closer to the golden rule. By adopting a quasi linear utility specification in regards to work in second period, we assume away any income effect, which tends to overemphasize the efficiency incidence of mandatory retirement

[2]Note that we assume that lifetime utility is increasing in longevity: $\frac{\partial U}{\partial l} = \beta\left[u(d) - u'(d)d - u'(d)\frac{\partial z(z,l)}{\partial l}\right] > 0$, which is satisfied if $\frac{u'(d)d}{u(d)} < 1$. Intuitively speaking one more year of life is worth living. With the functional forms we use, this condition is reduced to $\frac{\partial U}{\partial l} = \beta\left[\log(d) - 1 + \frac{\gamma w^2}{d}\right] > 0$.

[3]An alternative specification could be that second period labor is subject to a proportional tax θ whose proceeds are returned to the old workers. Their problem would be to choose z such as to maximize : $wz(1-\theta) + T - v(z,\ell)$. With $T = \theta wz$ and $v = z^2/2\gamma\ell$, this yields $z = \gamma\ell w(1-\theta)$. In the case of optimal retirement with no distortions, $z = z^* = \gamma\ell w$. In the case of induced early retirement, $z = \bar{z} = \gamma\ell w(1-\theta)$, where θ is chosen such as to generate $\bar{z} < z^*$; see Gruber and Wise (1999) for this.

age. But this does not affect the qualitative nature of the message.

We now turn to the production side of the economy. The technology is characterized by a Cobb-Douglas production function:

$$Y_t = F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}, \tag{8}$$

where K is the stock of capital, A is a productivity parameter, and L is the labor force. We posit $A = 1$. We distinguish the labor force L_t from the size of generation t, N_t. The labor force comprises the young population of generation t and the labor force participation of the older generation $t-1$. Incorporating population growth $N_t = N_{t-1}(1+n)$, the labor force can then be written as $L_t = N_t + N_{t-1}z_t = N_{t-1}(1+n+z_t)$. In comparison, total population at time t is :

$$N_t + \ell N_{t-1} = N_{t-1}(1+\ell+n).$$

Denoting $K_t/L_t \equiv k_t$ and $Y_t/L_t \equiv y_t$, we obtain income per worker (not per capita):

$$y_t = f(k_t) = k_t^{\alpha}.$$

Factors of production are paid according to their marginal contributions :

$$R_t = f'(k_t) = \alpha k_t^{\alpha-1}, \tag{9}$$

$$w_t = f(k_t) - f'(k_t)k_t = (1-\alpha)k_t^{\alpha}. \tag{10}$$

Equilibrium conditions in the labor and capital markets are as follows:

$$L_t = N_{t-1}(1+n+z_t), \tag{11}$$

$$K_{t+1} = L_t s_t, \tag{12}$$

where the latter expression reflects the fact that capital is assumed to depreciate completely after each period. Although this assumption arises from convenience, it is not unrealistic considering the fact that a period denotes several decades. Using the optimality condition for savings that we defined previously, the latter expression can be rewritten as follows :

$$
\begin{aligned}
G_t \equiv (1+n+z_{t+1})\,k_{t+1} - \frac{\beta\ell}{1+\beta\ell}(1-\alpha)k_t^{\alpha} + \tau\left(\frac{\beta\ell}{1+\beta\ell} + \frac{(1+n)\,k_{t+1}^{1-\alpha}}{\alpha\,(1+\beta\ell)}\right), \\
+ \frac{z_{t+1}k_{t+1}^{1-\alpha}}{(1+\beta\ell)\,\alpha}\left((1-\alpha)k_{t+1}^{\alpha} - \frac{z_{t+1}}{2\gamma\ell}\right) = 0,
\end{aligned} \tag{13}
$$

which explicitly defines the dynamic behavior of capital stock. Note that the standard (Diamond) case with no social security and work in the second period of life can be deduced by shutting down these two sections, $z = \tau = 0$, which generates the following:

$$
G_t \equiv (1+n)\,k_{t+1} - \frac{\beta\ell}{1+\beta\ell}(1-\alpha)k_t^{\alpha}. \tag{14}
$$

Comparing (13) and (14), we observe two main differences. First, the third term on the right-hand side of (13) denotes the double burden that PAYG imposes on savings. Second, the fourth term reflects the double effect of working in the second period: a distortionary effect if z is not optimal and a savings inducement effect if $z < z^*$.

In equations (6) and (7) we assumed a pension system that relies on a defined contribution (DC) formula in which the tax $\bar{\tau}$ is given and thus the benefits p have to follow through based on demographic shifts. An alternative system can also be considered that offers constant annuity benefits $\bar{a}$ (DB) during retirement.[4] The two revenue constraints that these systems imply are as follows:

$$
\begin{aligned}
DC:&\ \ \bar{\tau}(1+n) = p, \\
DB:&\ \ \bar{a}(\ell - z) = \tau(1+n),
\end{aligned}
$$

where an upper bar denotes the defined variable. For example, $\bar{a}$ is the defined annuity and τ has to adjust to variations in z, ℓ, and n in this case. Note that for each type of pension system, the individual utility has to adjust accordingly. With DB the choice of the retirement age is not $z_{t+1}^* = \gamma\ell w_{t+1}$ as in the DC case, but $z_{t+1}^* = \gamma l\,(w_{t+1} - \bar{a})$ to take into account that working one more year implies foregoing $\bar{a}$.[5]

We have so far identified two major dimensions of a social security system: is it DC

[4]With defined benefits, there are two ways of exiting the pension system at the time of retirement: either through annuities or by receiving some capital. Most public DB systems provide annuities. Some DB private systems, on the other hand, provide the option to choose between the two types of exits.

[5]Note that we could include a defined contribution system with annuities, where $\bar{\tau}(1+n) = a(1-z)$, or a defined benefit system with lump sum benefits, where $\tau(1+n) = \bar{p}$. However, we restrict our attention to the more commonly known cases.

Table 1: Different Social Security and Retirement Regimes

	Type of Social Security System	Retirement Age Regulation Scheme
Case-1	Defined Contribution	Mandatory Early Retirement
Case-2	Defined Benefit	Mandatory Early Retirement
Case-3	Defined Contribution	Optimal Retirement
Case-4	Defined Benefit	Optimal Retirement

Notes: Optimal retirement is given by $z^* = \gamma \ell w$, and mandatory early retirement is given by $z = \bar{z} < z^*$.

or DB, and is there mandatory early retirement or not? Together, these two dimensions provide four different ways to describe equation (13). These four cases are presented in Table 1, and the corresponding equations for G_{it} are provided in Appendix 1.

Before proceeding with comparative statics, it is important to further discuss some of the assumptions made in this paper. First, we adopt simple functional forms for utility and production. This allows us to obtain analytical results to the extent possible and lets us assume the existence of a unique and stable equilibrium. This equilibrium is defined by the dynamics of capital accumulation (13), where $0 < \frac{\partial k_{t+1}}{\partial k_t} < 1$. This condition implies that $\frac{\partial G}{\partial k} > 0$ holds in a steady state.[6] Second, we assume a quasi-linear disutility from working in old age. On the one hand, this means that we neglect income effects that could clearly play some role in retirement decision. On the other hand, this assumption allows us to better contrast mandatory early retirement and freely chosen retirement. Finally, we assume away the intensive margin in the first period and just focus on the intensive margin in the second period. This simplifies the analysis and is in agreement with the literature on age taxation that focuses on the retirement decision; see, for example, Lozachmeur (2006).

3 Steady State Comparative Statics

In this section, we investigate comparative statics for the four alternative social security systems identified in the previous section. We adopt a steady state setting and are not concerned by any dynamics that may account for a move from our steady state to another one caused by demographic changes. Our main aim is to elaborate on the behavior of

[6]In this paper we refrain from discussing the issue of existence, unicity and stability of the equilibria by choosing commonly known and well behaving functional forms. For a general technical discussion on this, see de la Croix and Michel (2002), and for a discussion of a framework similar to our model, see Ludwig and Vogel (2010). Finally, Rogerson and Wallenius (2009) provide a nice discussion on preference specifications that are consistent with balanced growth.

capital accumulation when the economy experiences aging due to lower fertility or higher longevity.

3.1 Mandatory Early Retirement

We begin by showing the impact of an increase in fertility in a mandatory early retirement system:[7]

$$Case\,1\;(DC,\bar{z}):\quad \Psi\frac{\partial k}{\partial n} = -k - \frac{\bar{\tau}k^{1-\alpha}}{\alpha(1+\beta\ell)} < 0, \tag{15}$$

$$Case\,2\;(DB,\bar{z}):\quad \Psi\frac{\partial k}{\partial n} = -k + \frac{\bar{a}(\ell-\bar{z})\beta\ell}{(1+\beta\ell)(1+n)^2} \gtrless 0, \tag{16}$$

where $\Psi = \left(\frac{\partial G}{\partial k}\right)^{-1} > 0$. In a standard case (Diamond), an increase in fertility has a depressive effect on capital accumulation in the absence of a PAYG pension system. This is shown by the first term on the right-hand side of each equation above, which is often called the "capital dilution" effect. This depressive effect is reinforced in a DC pension system as shown by the negative second term in (15) but is weakened or possibly reversed in DB pension systems as shown by the positive second terms in (16). The explanation is quite intuitive. With a DC system, an increase in the fertility rate implies an increase in pensions, which discourages saving. With a DB system the pension level is kept constant and thus the contribution rate decreases, which fosters saving. We call this effect the "savings displacement" effect. Note that, in DB, the lower the $\bar{z}$, the larger the negative savings displacement effect. With a too generous pension, an increase in fertility can even lead to an increase in capital.

Next, we turn to the impact of an increase in longevity on equilibrium capital per worker in a mandatory early retirement system:

$$Case\,1\;(DC,\bar{z}):\quad \Psi\frac{\partial k}{\partial \ell} = \frac{1}{(1+\beta\ell)^2}\left[\beta w - \bar{\tau}\Pi - \bar{z}\Omega\right] \gtrless 0, \tag{17}$$

$$Case\,2\;(DB,\bar{z}):\quad \Psi\frac{\partial k}{\partial \ell} = \frac{1}{(1+\beta\ell)^2}\left[\beta w - \bar{a}\Theta - \bar{z}\Omega\right] \gtrless 0, \tag{18}$$

where $\Pi = \left(\frac{\beta}{R}\right)\left[R-(1+n)\right]$, $\Theta = \frac{\beta(\ell-\bar{z})}{(1+n)R}[R-(1+n)] + \frac{(1+\beta\ell)}{(1+n)R}[\frac{R}{\beta l} + (1+n)]$, and $\Omega = \frac{1}{2R\gamma\ell^2}\left[\beta l\bar{z} + 2(\bar{z}-z^*)\right]$.

In both (17) and (18) we have three terms in brackets. The first term, which is com-

[7] Although we are concerned with decreasing fertility, it is clearer mathematically to look at the effect of an increasing fertility.

mon to both cases, represents the capital dilution effect that is always positive. The third term is also common to the two expressions. It represents the savings displacement effect due to the mandatory retirement. When $\bar{z} = 0$, this term vanishes. The need for saving is then at its highest as there are no earnings in the second period. Thus, as $\bar{z}$ increases, the effect of longevity on capital decreases. The second term varies between the two expressions. It represents the saving displacement effect due to social security and vanishes when $\bar{\tau} = \bar{a} = 0$. Remember that we assume dynamic efficiency, $R > (1+n)$, and as shown by Aaron (1966), when the marginal productivity of capital is higher than the rate of population growth, social security depresses both capital and welfare. With DB, the savings displacement effect due to the defined annuity $\bar{a}$ comprises not only the gap between R and $(1+n)$ but also the sum of the two rates.

Next, we investigate capital accumulation in the absence of a mandatory retirement age.

3.2 Optimal Retirement

We now relax the early retirement assumption and analyze the impact of aging on capital accumulation when retirement is chosen optimally. The analysis here shows that, compared to the case with early retirement, the ability to adjust the retirement age optimally leads to less distortion in equilibrium, but it also diminishes incentives for saving.

$$Case\,3\,(DC, z*): \quad \Psi\frac{\partial k}{\partial n} = -k - \frac{\bar{\tau}k^{1-\alpha}}{\alpha(1+\beta\ell)} < 0, \tag{19}$$

$$Case\,4\,(DB, z*): \quad \Psi\frac{\partial k}{\partial n} = -k + \frac{\bar{a}\beta\ell^2(1-\gamma(w-\bar{a}))}{(1+\beta\ell)(1+n)^2} \gtrless 0. \tag{20}$$

Similar to the case with early retirement, the PAYG pension system reinforces the depressive effect of an increase in the fertility rate on capital accumulation in the DC case and weakens or possibly reverses it in the DB case. Turning to the effect of longevity when z is endogenous, we have:

$$Case\,3\,(DC, z*): \quad \Psi\frac{\partial k}{\partial \ell} = \frac{1}{(1+\beta\ell)^2}\left[\beta w - \bar{\tau}\Pi - \Phi\right] \gtrless 0, \tag{21}$$

$$Case\,4\,(DB, z*): \quad \Psi\frac{\partial k}{\partial \ell} = \frac{1}{(1+\beta\ell)^2}\left[\beta w - \bar{a}\chi - \Psi\right] \gtrless 0, \tag{22}$$

Table 2: The Effects of Aging on Equilibrium Capital Per Worker

	Standard Case	Defined Contribution	Defined Benefit
		Mandatory Early Retirement	
Decrease in Fertility	> 0	> 0	$\lesseqgtr 0$
Increase in Longevity	> 0	$\lesseqgtr 0$	$\lesseqgtr 0$
		Optimal Retirement	
Decrease in Fertility	> 0	> 0	$\lesseqgtr 0$
Increase in Longevity	> 0	$\lesseqgtr 0$	$\lesseqgtr 0$

where the expressions within brackets are defined as $\Phi = \frac{k\gamma w}{2\alpha}\left[2\alpha(1+\beta\ell)^2 + (1-\alpha)\right]$, $\chi = \left(\frac{1-\gamma(w-\bar{a})}{(1+n)R}\right)\left[(1+n)R + \beta l R(1+\beta l)\right]$, and $\Psi = \frac{k\alpha(w-\bar{a})}{2\alpha}\left[2\alpha(1+\beta l)^\alpha + \frac{\bar{a}}{k^\alpha} + (1-\alpha)\right]$.

In both expressions above, we observe a positive capital dilution effect. We also have two savings displacement effects that are both negative and different for the DC and the DB regimes. As in the mandatory retirement case, with $\bar{\tau} = \bar{a} = 0$ the savings displacement effect of social security vanishes. However, this effect increases with the two social security parameters, $\bar{\tau}$ in DC and $\bar{a}$ in DB. The terms Φ and Ψ represent the savings displacement effect due to second period activity. They vanish when $\gamma = 0$, which implies no activity in the second period. Both of these terms increase with longevity as z* increases with l. There is an additional depressing term with DB, which increases with $\bar{a}$.

We summarize these in Table 2. The only unambiguous case is the positive effect of declining fertility with DC. Increasing longevity is ambiguous but has a higher effect with $\bar{z} = 0$ than with z^*. With DB, both sources of aging have an ambiguous effect but, again, the effect is more prominent with $\bar{z} = 0$ than with z^*. These results are intuitive: with $\bar{z} = 0$ or at least $\bar{z} < z^*$, individuals anticipate a lower level of earnings in the second period and compensate for this by increasing their savings. As we consistently assume dynamic efficiency, all things being equal, utility under optimal retirement is not necessarily higher than under mandatory retirement. This is mainly because mandatory early retirement induces higher saving and capital accumulation as desired consumption in the second period of life time cannot be financed by extending work hours. As a result, mandatory early retirement presents a case that is closer to the golden rule than optimal retirement. This is a standard second-best problem where a distortion makes a second distortion desirable.

3.3 Aging and Utility

We next study the effects of aging on welfare. To do this, we need to distinguish between two main drivers of welfare: the capital stock that determines aggregate production and the PAYG system that has a differential impact under DC and DB. In order to provide some analytical results for the PAYG effect, we look at the incidence of aging in a fixed factor prices setting. We prove the following proposition in Appendix 2.

Proposition 1. *In a fixed factor price setting with PAYG social security and endogenous retirement age,*

1. *An increase in the fertility rate always improves individual welfare. In the case of DC the effect is the same whether retirement is freely chosen or mandatory. In the case of DB, the effect is smaller when the retirement age is not regulated.*

2. *Assuming that surviving one more year in the second period is desirable, the welfare incidence of an increase in longevity is consistently positive with DC. With DB the incidence is ambiguous. Yet it is higher with optimal retirement than with mandatory retirement.*

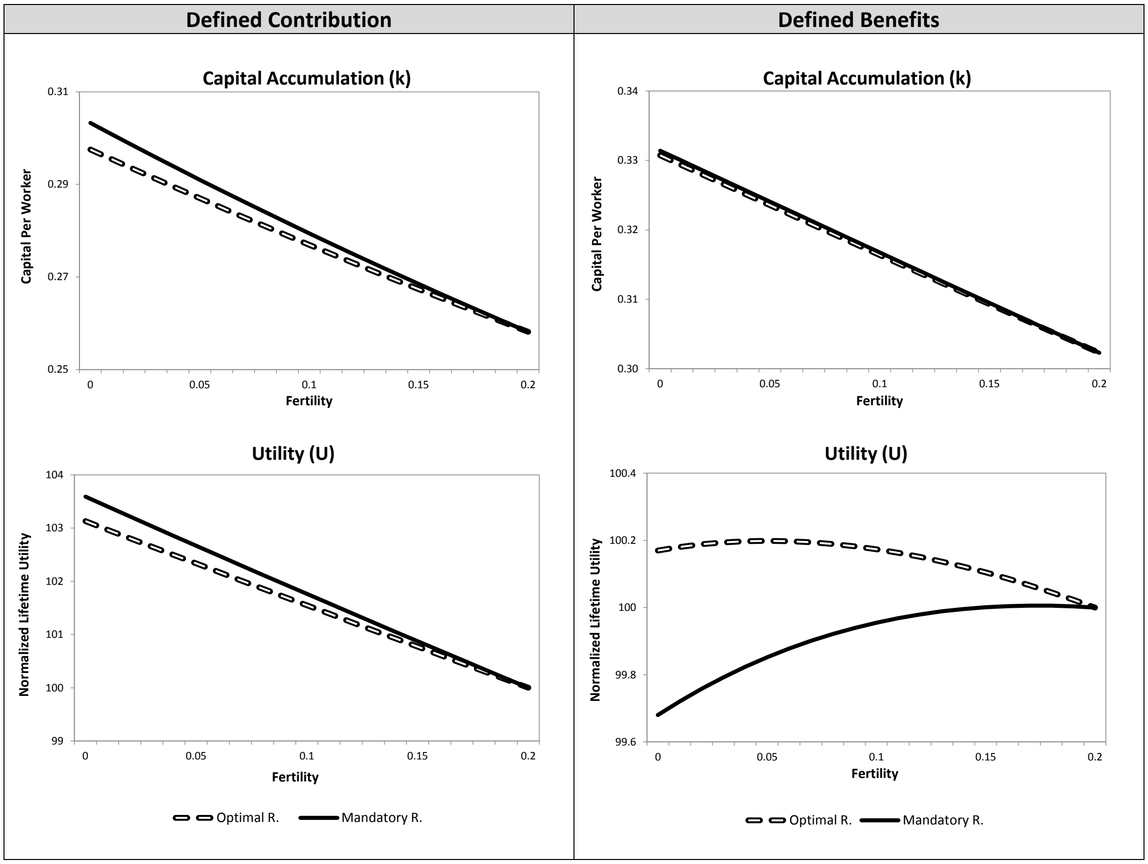
To see the effect of aging on welfare, we have to combine the capital effect that was studied in the previous subsections and the PAYG effect. We resort to numerical simulations to be able to show these combined effects. Note that in a fixed factor price setting one clearly sees the utility gain of having z^* instead of $\bar{z} < z^*$. This has to be distinguished from what the above proposition shows, namely that with DC the welfare boost of a fertility increase can be higher with $\bar{z}$ than with z^*.

4 Steady State Simulations

The analysis so far has shown that demographic change has different implications for capital accumulation under alternative PAYG systems. Analytically, these results are sufficient to show that the impact is different quantitatively. In order to better grasp these effects and to study the welfare effects, we now employ a numerical example. To this effect, we use common values of parameters from the literature to simulate the equilibrium profiles for the agent's lifetime utility U and capital per worker k with different values of fertility n and longevity ℓ.[8] Figures 1 and 2 show our results with respect to variations in

[8] Note that we do not attempt to calibrate the simulations to any specific country case. As the model is built to demonstrate our results (preferably analytically) by using the simplest possible case, it does not lend itself to replicating the complex demographic and institutional aspects of actual country examples. Nevertheless, we chose commonly used parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$, where A is a scaling parameter and plays no important role and capital's share in income in the economy

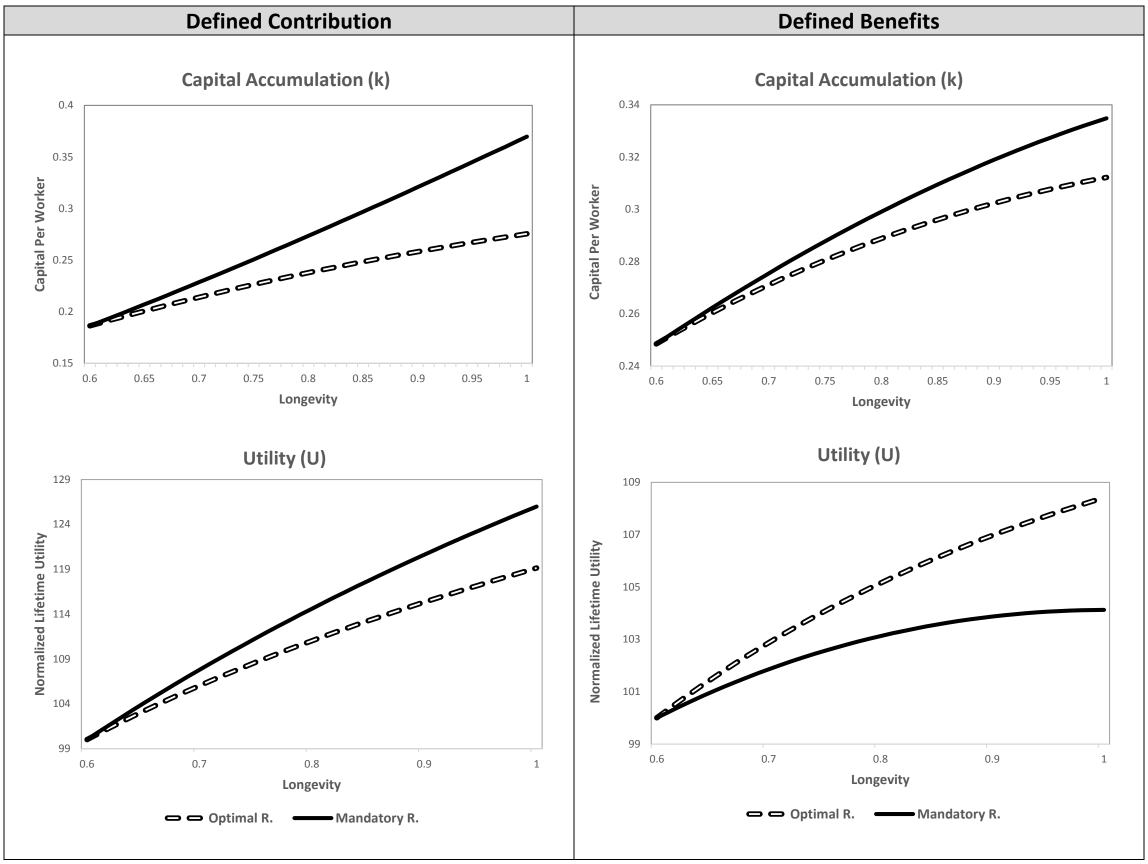
Figure 1: Fertility, Lifetime Utility and Capital per Worker



Notes: Simulations reflect the parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$. In addition, the following assumptions were made when needed: $a = 1.6$, $\bar{z} = 0.394$ in the defined benefit case with mandatory retirement and $\bar{z} = 0.579$ in the defined contribution case. These values equalize the starting points of mandatory and optimal retirement systems at $n = 0.2$.

fertility and in longevity respectively. In those figures we give the steady state values of k and U that correspond to different values of n and ℓ. Note that, to make the comparison easier, we normalize utility at 100 at the starting point of the fertility decline or the longevity increase.

Figure 2: Longevity, Lifetime Utility and Capital per Worker



Notes: Simulations reflect the parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$. In addition, the following assumptions were made when needed: $a = 1.6$, $\bar{z} = 0.394$ in the defined benefit case with mandatory retirement and $\bar{z} = 0.579$ in the defined contribution cases. These values equalize the starting points of mandatory and optimal retirement systems at $\ell = 0.6$.

4.1 Declining Fertility

We start with the interpretation of the declining fertility case. Note that we impose a condition that the mandatory age of retirement is not distortionary at the highest level of fertility level in Figure 1, *i.e.* it is set at the optimal retirement age. This explains why both the level of capital and the generational utility are identical in the mandatory and optimal retirement systems at this point. We start at $n = 0.2$ and look at the steady states that correspond to values of n declining from 0.2 to 0. At $n = 0.2$, $\bar{z} = z^*$. When n decreases, if the capital stock increases, z^* will also increase, which means that, at $n = 0$, the gap between $\bar{z}$ and z^* is at its highest with expected and contrasted implications to which we now turn.

In both DC and DB pension regimes considered, the capital stock increases regardless of the retirement age regulations when fertility decreases. This increase differs from one case to the other with clear implications for the welfare level. We first consider the DC regime. The increase in k is greater with $\bar{z}$ than with z^*. With $\bar{z}$, as k increases the labor distortion increases with the effect of pushing for additional saving. This explains why the capital stock with mandatory retirement (solid black line) dominates the capital stock under optimal retirement (dashed line) in Figure 1. The same relationship is also true for the utility profiles under the mandatory and optimal retirement systems with DC. As n is declining, the capital increases as we have seen and this is dynamically efficient. This effect dominates the static efficiency loss resulting from an increasing gap between z^* and $\bar{z}$.

Under the DB regime, the capital profile is the same for both retirement assumptions. Utility profiles, however, differ significantly between the two. A decrease in fertility leads to an initially increasing and then decreasing (hump-shaped) lifetime utility with optimal retirement. In comparison, utility always decreases with mandatory retirement. This is consistent with our theoretical findings above. The capital stock increases at the same pace with z^* and $\bar{z}$, which implies the same positive impact on utility from the capital dilution channel. The difference thus comes from the channels described in Proposition 1: the negative PAYG effect of a declining fertility is larger with $\bar{z}$ than with z^*. The capital stock evolution does not play any role, the main explanation comes from what we call the static inefficiency of mandatory retirement that increases as k increases.

(α) reflects the averages for the OECD countries, see Jones (2003). Considering the fact that each period represents approximately 35 years in our model, β is equivalent to 0.96 in annual terms. In addition, the following assumptions were made when needed: $a = 1.6$, $\bar{z} = 0.394$ in the defined benefit case with mandatory retirement and $\bar{z} = 0.579$ in the defined contribution case. These values are chosen to make the initial conditions and changes comparable with alternative social security systems. As to the demographic parameters, we took $n \in (0, 0.2)$ and $\ell \in (0.6, 1)$. The results are quite robust to an alternative range of values for either n or ℓ.

4.2 Increasing Longevity

We next turn to aging due to an increase in longevity. Note that, in all cases, capital per worker increases with longevity. However, the effect is more prominent in the mandatory retirement scheme under both DC and DB. This is also consistent with our theoretical results. With a longer life span, savings are increased to smooth consumption between two periods. The increase in savings is more dramatic when work hours cannot be adjusted in the second period of life.

Similarly, lifetime utility increases in the four cases. Whereas optimal retirement dominates mandatory retirement in the case of DB, the opposite occurs in the case of DC. The reason for this contrasting result is straightforward. We observe that with DC, the capital gap between $\bar{z}$ and z^* is higher with DC than with DB when $l = 1$. As a consequence, with DC the gain in dynamic efficiency (getting closer to the golden rule) dominates the loss in static efficiency (on the labor market). The opposite holds in the case of DB. It is important at this point to recall the key differences between DB and DC regarding aging. On the one hand, DB contributions decrease with fertility and increase with longevity. On the other hand, DC benefits increase with fertility but do not depend on longevity. These differences between DB and DC explain the results that appear on Figures 1 and 2.

Overall, the static simulations in this section show that the DC pension system seems to outperform the DB system in aging societies regardless of the retirement age regulation. This is particularly the case when the mandatory early retirement rule prevails. Note, however, that these observations compare long-term performance. Next we investigate the transition dynamics between two static equilibria brought about by a change in fertility or longevity, that have the same impact on the old age dependency ratio.

5 Dynamics

Up to now we have been concerned by the long-term (steady state) implications of aging in alternative pension systems. In this section, we investigate whether the short-term impact of a demographic shift is different from the long-term implications in alternative configurations of pension systems.

The simulations in this section use the same parameter values as in the static simulations in the previous section. However, in this case, we need to specify the magnitude of the demographic transition between two steady states as a single value. In order to make comparable the changes in n and ℓ, we characterize fertility-driven aging by decreasing the n from 0.2 to 0.137 and longevity-driven aging by increasing the ℓ from 0.9 to 0.95.

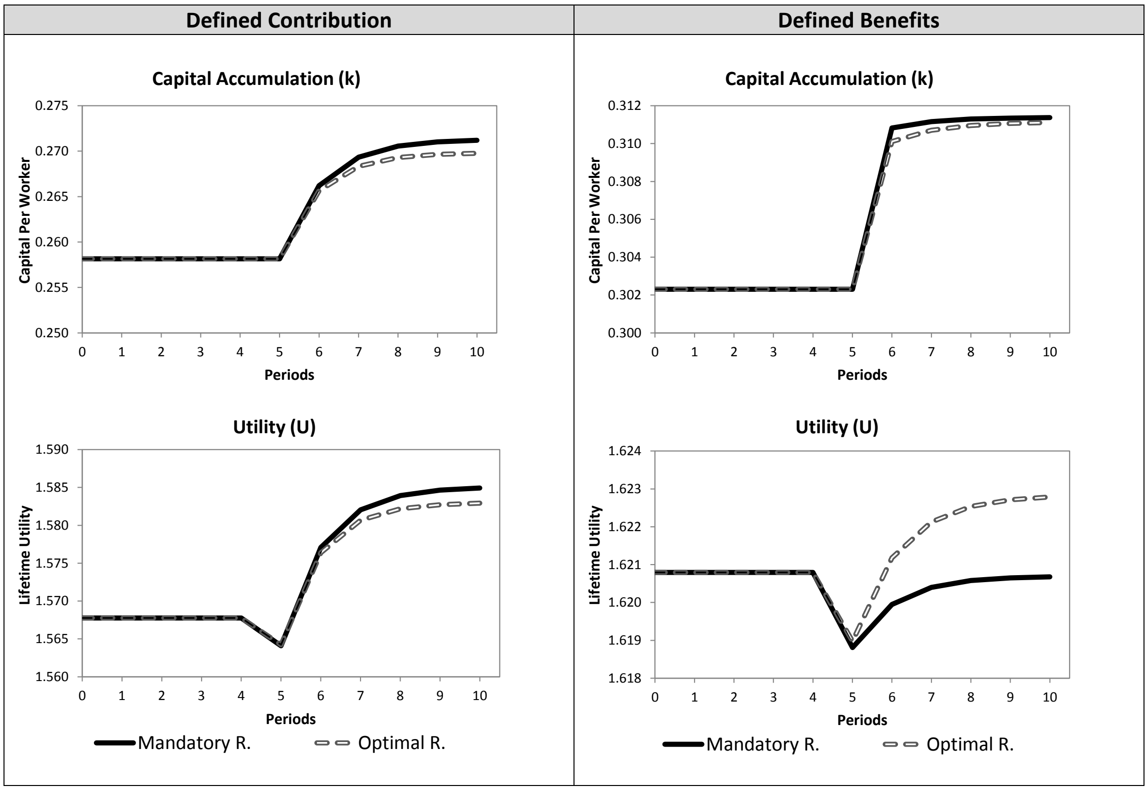
These changes, in turn, increase the rate of dependency $\frac{\ell}{1+n}$ by about 5.6 percent in comparison with the initial steady state. We assume that there is perfect foresight, *i.e.* both changes are anticipated by the agents in advance. We introduce both changes in period-6 in order to observe the anticipatory adjustments as well as the after-change propagation. These demographic shifts lead to new equilibria over time, though not necessarily in a monotonic manner as we show.

Figures 3 and 4 show the dynamic adjustment paths in different social security systems and demographic changes. In the case of capital accumulation, transitions are generally monotonic with the exception of an increase in longevity in DB, which we explain below. In all cases, capital per worker increases starting from period-6. Note that the capital per worker in period-6 is determined by a combination of two factors: savings made by the generation that is born in period-5 (which we call generation-5 from now on) and the size of the workforce in period 6 (which comprises generation-6 and the old age work hours of generation-5). In the most obvious case where the retirement age is regulated by a mandatory retirement system and aging is driven by changes in longevity, capital by worker increases as generation-5 increases its savings to smooth their consumption over a longer lifespan. With DB, generation-5 again increases its savings to finance consumption over a longer life when longevity increases in period-6. However, the members of this generation do not bear the burden of financing a larger pension bill since they contributed in period-5, before the demographic change occurs. In comparison, generation-6 and following generations pay higher social security contributions, which reduces their first period income. Thus, they cannot save as much as generation-5, which translates into a temporary overshooting in capital accumulation in period-6 as shown in Figure 4.

The other cases present more complex mechanisms. For instance, in an optimal retirement system with fertility-driven aging, capital per worker also increases due to the capital dilution effect, *i.e.* holding generation-5 savings constant, each worker gets to use more capital as there are fewer workers in generation-6. At the same time, the elderly from generation-5 increase their labor force participation, which partially offsets the decrease in n. Moreover, if the pension system is DC, then members of generation-5 may also adjust their savings in anticipation of a change in income in period-6, which comes as a result of a decrease in pensions as well as changes in interest earnings and wages as a result of the capital deepening that occurs in period-6.

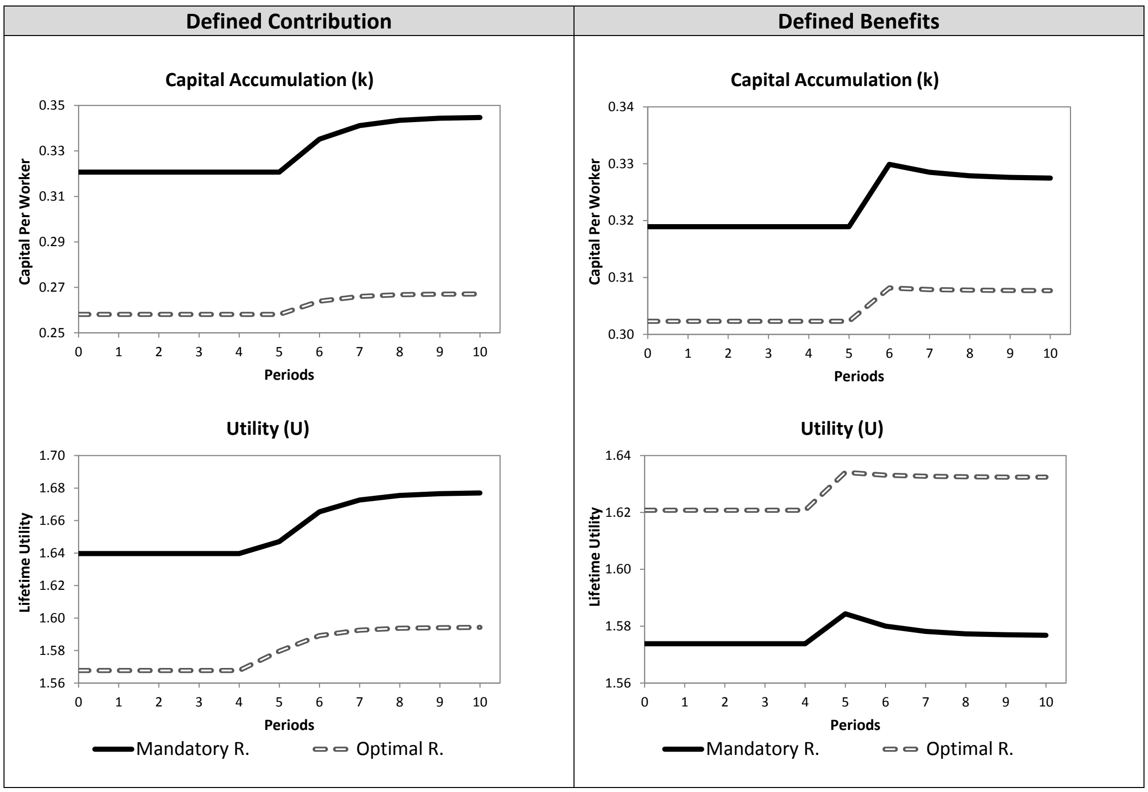
The most striking result, however, is a transitory loss in lifetime utility that comes with a fertility drop. Consider the case of fertility-driven aging in a defined contribution system for instance. Although the long-term impact of a fertility shock in period-6 on welfare is positive, generation-5 experiences a decline in its lifetime utility. In order to

Figure 3: Dynamics of Fertility-driven Aging



Notes: Simulations reflect the parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$. In addition, the following assumptions were made when needed: $\bar{a} = 1.6$, $\bar{z} = 0.394$ in the defined benefit case with mandatory retirement and $\bar{z} = 0.579$, $\tau = 0.67$ in the defined contribution case. Thus, with defined contribution, the pension contributions constitutes about a fifth of consumption. These values also equalize the mandatory and optimal retirement systems at $\ell = 0.6$. The fertility drop denotes a decrease in n from 0.2 to 0.137 in period 6. Utility level at period t shows the lifetime utility of the generation who are born in period t and live in period t and $t + 1$.

Figure 4: Dynamics of Longevity-driven Aging



Notes: Simulations reflect the parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$. In addition, the following assumptions were made when needed: $\bar{a} = 1.6$, $\bar{z} = 0.394$ in the defined benefit case with mandatory retirement and $\bar{z} = 0.579$, $\tau = 0.67$ in the defined contribution case. Thus, with defined contribution, the pension contributions constitutes about a fifth of consumption. These values also equalize the mandatory and optimal retirement systems at $\ell = 0.6$. The fertility drop denotes a decrease in n from 0.2 to 0.137 in period 6. Utility level at period t shows the lifetime utility of the generation who are born in period t and live in period t and $t + 1$.

see this, note that the fertility change reduces the pensions of generation-5 and, thus, the old-age consumption. At the same time, because this particular generation already worked and contributed to social security system in period-5, it does not benefit from efficiency gains available to future generations to the same extent. As the decrease in fertility brings the economy closer to golden rule savings, lifetime utility increases for the following generations.

The same principle is also operative in the fertility-driven aging with defined benefits case. Although pensions do not change in this case, there are other factors that affect the income of generation-5 in period-6. Note that when fertility decreases in period-6, it leads to capital deepening through adjustments in the capital dilution effect as well as the savings displacement effect. As a result, interest earnings on period-5 savings decrease and the wages increase in period-6. However, the latter is only partially beneficial to generation-5 as their work hours in period-6 are limited. Thus, the decrease in interest earnings dominates the gain in wages, leading to a loss in lifetime utility.[9] With an increase in longevity in the DB case, we observe an increase in longevity followed by a small decrease starting with generation-6. This follows from the capital accumulation dynamics that are described above.

Overall, our dynamic simulations also highlight other interesting trade-offs between alternative social security systems. The long-term welfare gain under DC is larger than under DB in all kinds of aging and retirement age regulation regimes. This can be seen by comparing the changes in equilibrium capital per worker and lifetime utility as a share of the initial equilibrium values displayed in Figures 3 and 4. For instance, when retirement age is flexible, the fertility change leads to a 1 percent increase in utility in the long term when pension system is based on defined contribution. In comparison, the increase in utility under defined benefit pensions is only about 0.1 percent. However, this ranking between the DC and DB is not necessarily true in our short-term simulations. With a fertility drop, the transition generation (generation-5) experiences a welfare reduction in the case of DC (0.2 percent). In comparison, the reduction in DB is about half of that (0.1 percent). Moreover, when aging is driven by an increase in longevity, the transition generation is better-off under DB than the future generations. This is not observed with DC, where lifetime utility gradually increases over time before it stabilizes at the new steady-state.

[9] Note that, since the underlying model is a simple two period OLG framework, changes in key variables are not expected to reflect the smooth adjustments that could happen in large calibration models where one period is approximately one calendar year.

6 Conclusions

In this paper, we evaluate the effects of aging on capital accumulation and lifetime utility. We show that these effects differ both quantitatively and qualitatively depending on the type of social security system, type of retirement regulations and the time frame of the analysis. The effects of an increase in longevity or a decrease in fertility, two phenomena that contribute to aging, change depending on a number of features in the pension system. In a standard overlapping generations model with no social security or second period work, a decrease in fertility and increase in longevity should lead to an increase in capital accumulation. However, this is not necessarily the case once we introduce those elements. The effect of a decrease in fertility on capital accumulation is still positive with a defined contribution system regardless of the retirement age regulation; however, the results are ambiguous with other unfunded social security systems and with an increase in longevity.

To show our results, we used a theoretical OLG model with simple functional forms, and when needed, we proceeded with numerical simulations by using plausible parameter values. Given this framework, it is legitimate to question the generality of our results. Would they still hold with a more complex model and more realistic calibrations? With alternative numerical examples, which we excluded here for the sake of brevity, that comprise a CES production function and wider ranges of longevity and fertility values, our results did not change qualitatively. An interesting one among those alternatives is the case where we try with the CES an elasticity that brings the economy close a fixed factor price scenario. Our results remained consistent with those presented in Proposition 1.

We believe that our findings have important implications. Demographic aging takes different forms across countries; therefore, the consequences of aging for growth and welfare are likely to be different as well. In the meantime, social security systems tend to shift progressively from a regime of defined benefits towards one of defined contributions, which is more common across countries. Our dynamic simulations show that this shift could bring long-term gains; however, the transition could impose welfare costs for current generations. The relative importance of such gains and losses, again, vary among different types of aging. In our future research, we intend to look at the joint effect of aging and changes in the social security regimes explicitly: from DC to DB and from early retirement to flexible retirement. A limitation of the current analysis is the assumption of identical individuals. With heterogeneity in wages, we might find more merits in the defined benefit formula, which is also left for future research.

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Appendix 1: Motions of Capital with Alternative Designs of Social Security

$$Case\,1: G_{1t} \equiv (1+n+\bar{z})\,k_{t+1} - \frac{\beta\ell}{1+\beta\ell}(1-\alpha)k_t^{\alpha} + \frac{\bar{\tau}}{1+\beta\ell}\left(\beta\ell + \frac{(1+n)k_{t+1}^{1-\alpha}}{\alpha}\right) + \frac{\bar{z}k_{t+1}^{1-\alpha}}{(1+\beta\ell)\,\alpha}\left((1-\alpha)k_{t+1}^{\alpha} - \frac{\bar{z}}{2\gamma\ell}\right) = 0$$

$$Case\,2: G_{3t} \equiv (1+n+\bar{z})\,k_{t+1} - \frac{\beta\ell}{1+\beta\ell}(1-\alpha)k_t^{\alpha} + \frac{\bar{a}(\ell-\bar{z})}{1+\beta\ell}\left(\frac{\beta\ell}{(1+n)} + \frac{k_{t+1}^{1-\alpha}}{\alpha}\right) + \frac{\bar{z}k_{t+1}^{1-\alpha}}{(1+\beta\ell)\,\alpha}\left((1-\alpha)k_{t+1}^{\alpha} - \frac{\bar{z}}{2\gamma\ell}\right) = 0$$

$$Case\,3: G_{3t} \equiv (1+n+(1-\alpha)k^{\alpha}\gamma\ell)\,k_{t+1} - \frac{\beta\ell}{1+\beta\ell}(1-\alpha)k_t^{\alpha} + \frac{\bar{\tau}}{1+\beta\ell}\left(\beta\ell + \frac{(1+n)k_{t+1}^{1-\alpha}}{\alpha}\right) + \frac{(1-\alpha)^2\gamma\ell k_{t+1}^{1+\alpha}}{2\,(1+\beta\ell)\,\alpha} = 0$$

$$Case\,4: G_{4t} \equiv (1+n+D_t\gamma\ell)\,k_{t+1} - \frac{\beta\ell}{1+\beta\ell}A(1-\alpha)k_t^{\alpha} + \frac{\bar{a}(\ell-D_t\gamma\ell)}{1+\beta\ell}\left(\frac{\beta\ell}{(1+n)} + \frac{k_{t+1}^{1-\alpha}}{A\alpha}\right) + \frac{D_t\gamma\ell k_{t+1}^{1-\alpha}}{(1+\beta\ell)\,A\alpha}\left((1-\alpha)k_{t+1}^{\alpha} - \frac{D_t}{2}\right) = 0$$

where $D_t = (1-\alpha)k_t^{\alpha} - \bar{a}$.

Appendix 2: Fixed Factor Price

In this section, we elaborate on the effects of aging on welfare in a fixed factor price setting. Returns to capital (R) and labor (w) are fixed. We also assume that $\bar{z} = 0$ when mandatory retirement is implemented and that $v(z,l) = \frac{z^2}{2\gamma l}$ and $R = 1$ for the sake of simplicity. These assumptions have no real bearing on the analysis, except that assuming $\bar{z} = 0$ exacerbates the utility gain of having an endogenous retirement age. Having freely chosen z implies that second period consumption is $\frac{w^2\gamma\ell}{2}$ higher. Given these assumptions, we can write down the lifetime utilities following the definitions in Table 1:

$$Case\,1:\; U_1 = u\,(w-\bar{\tau}-s) + \ell u\left(\frac{Rs+\bar{\tau}(1+n)}{\ell}\right)$$

$$Case\,2:\ U_2 = u\left(w - s - \frac{\bar{a}}{1+n}\ell\right) + \ell u\left(\frac{Rs + \bar{a}\ell}{\ell}\right)$$

$$Case\,3:\ U_3 = u\left(w - \bar{\tau} - s\right) + lu\left(\frac{Rs + \bar{\tau}(1+n) + \frac{w^2\gamma}{2}\ell}{\ell}\right)$$

$$Case\,4:\ U_4 = u\left(w - s - \frac{\bar{a}}{1+n}\ell(1-w\gamma)\right) + \ell u\left(\frac{Rs + \frac{w^2\gamma\ell}{2} + \bar{a}l(1-w\gamma)}{\ell}\right)$$

Next, we evaluate the effect of a marginal change in longevity and fertility by differentiating the definitions above. In all cases, a small increase in fertility is found to improve the welfare when wages and interest rates are unaffected.

$$Case\,1:\ \frac{\partial U_1}{\partial n} = \ell u'(d)\bar{\tau} > 0$$

$$Case\,2:\ \frac{\partial U_2}{\partial n} = u'(c)\frac{\ell\bar{a}}{(1+n)^2} > 0$$

$$Case\,3:\ \frac{\partial U_3}{\partial n} = \ell u'(d)\bar{\tau} > 0$$

$$Case\,4:\ \frac{\partial U_4}{\partial n} = u'(c)\bar{a}\frac{\ell(1-\gamma w)}{(1+n)^2} > 0$$

Similarly, the effects of longevity are given by:

$$Case\,1:\ \frac{\partial U_1}{\partial \ell} = u(d) - u'(d)d > 0$$

$$Case\,2:\ \frac{\partial U_2}{\partial \ell} = u(d) - u'(d)d + u'(d)\bar{a}\left(1 - \frac{R}{1+n}\right) \gtreqless 0$$

$$Case\,3:\ \frac{\partial U_3}{\partial \ell} = u(d) - u'(d)d + u'(d)\frac{w^2\gamma}{2} > 0$$

$$Case\,4:\ \frac{\partial U_4}{\partial \ell} = u(d) - u'(d)d + u'(d)\frac{w^2\gamma}{2} + u'(d)\bar{a}\left(1 - \frac{R}{1+n}\right)(1-\gamma w) \gtreqless 0$$

where we assume that $u(d) - u'(d)d > 0$ and that $R > 1+n$. Cases 2 and 4 are ambiguous. Longevity is welfare-improving if $\bar{a}$ is small enough and/or R is not much higher than $1+n$. Naturally if $R \leq (1+n)$, longevity is consistently welfare-improving.