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Abstract

We consider a government facing a constraint on the total consumption of a specific good with varieties, and formulate an optimal commodity tax problem under a consumption target. We obtain a uniform pricing result (similar to the familiar uniform taxation rule): setting the same consumer price for all varieties constitutes a solution if, and only if, the compensated price elasticities of the varieties with respect to an untaxed good are all equal and non-negative. If, however, this elasticity condition does not hold, the optimal policy is at variance with the well-known inverse elasticity rule and with the Corlett–Hague rule.

JEL-Codes: H210, H230.

Keywords: consumption target, Corlett-Hague Rule, inverse elasticity rule, optimal commodity taxation, uniform taxation.

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1. Introduction

The problem addressed in this paper is how a government facing a predetermined constraint on the consumption level of a specific good (with varieties) should set price-based instruments such as taxes (or subsidies). Governments frequently face quantity constraints, due to either environmental or social reasons. For example, we may consider the target level for some pollutant to protect the environment; or we may think of merit and demerit goods where the government aims at meeting some specified (minimum respectively maximum) level of consumption. Here, goods such as inoculation against a contagious disease, education, cultural goods, sporting activities and safety precaution may represent examples for the first category; and cigarettes, alcohol, drugs, prostitution, and fuel consumption, for the second.[1] Also, in view of scarce natural resources, the quantity constraint may represent the sustainable yield of renewable resources such as fish, game, timber, etc. Finally, in both trade theory and policy quantity constraints are not unusual. For example, import quotas for some classes of goods are imposed to protect domestic industries or to discriminate against a foreign country; and for similar reasons, voluntary export restrains may be applied. In all of these cases a government may consider applying familiar fiscal instruments—taxes and subsidies imposed on these goods (and their varieties)—to interfere with total consumption.[2]

[1]For example, since 2003 the WHO requests member states to increase influenza vaccination coverage of all people at high risk with the goal of attaining vaccination coverage of the elderly population of 75% by 2010 (56th WHA, 2003). In 2009 the Council of the EU recommended member states to reach this goal by the winter of 2014/2015 (European Commission, 2009). As a second example consider the governmental objective to reduce consumption of alcohol: in 2010 the Russian government, alarmed by Russians' excessive alcohol consumption, pledged to reduce alcohol consumption by 55% by 2020.

[2]For instance, Article 6 of the WHO Framework Convention on Tobacco Control (FCTC, 2003–2005) states "..., each Party should take account of its national health objectives concerning tobacco control and adopt or maintain, as appropriate, measures which may include: (a) implementing tax policies and, where appropriate, price policies, on tobacco products so as to contribute to the health objectives aimed at reducing tobacco consumption." Similarly, environmental taxes are frequently

In this paper, we assume that the government uses unit taxes imposed on the different varieties of a given basic commodity to meet a predetermined consumption level. Then, the problem of the government may be formulated as choosing consumer prices for the different varieties of this good to maximise consumer welfare subject to a restriction on the total consumption of all the varieties. Technically, this is a problem of optimal commodity taxation with a quantity constraint. We assume that the target consumption level is exogenous. It may be determined by either some political process (e. g., an international environmental agreement), or by institutional constraints (constitutional rights or laws protecting the environment), or it may represent the solution of some more general optimisation problem (welfare maximisation). The particular source of the constraint, though, is inessential here, and in this sense, our paper contributes to second-best taxation: we derive the optimal tax rates to attain a given target, which itself may or may not be socially optimal.

It may be helpful to think of the quantity constraint as the target level for the total consumption of a good with varieties generating an externality. To be specific, the target level may be seen as the maximum amount of the emissions of a specified pollutant, such as CO_2 or NO_x. Actually, many countries deploy environmental taxes to limit pollutant emissions from transport, taxing different types of fuel, such as gasoline, diesel, etc., at different rates, although the consumption of any of them results in the emission of the same pollutant, e. g., CO_2. (This observation still holds true even if we calculate tax rates on the basis of their carbon content.)[3] It is thus apparent that different varieties of fuel (or more precisely, carbon contained in different types of fuel) may be, and actually are taxed at different rates, while

employed to meet predetermined emission or consumption targets. Notable examples are taxes on different types of fuel to accomplish specified CO_2- or NO_x-emission levels.

[3]Actual tax rates on different types of fuel (and also on energy products and electricity) in the EU are documented by the European Commission (2015). Using these figures it is simple to calculate prices of carbon contained in a specific type of fuel by using the fact that, for example, each gallon of petrol contains 2421 gram of carbon; and each gallon of diesel, 2778 gram (U. S. Environmental Protection Agency, 2005).

the political interest focuses on total consumption, e. g., CO_2, rather than on its composition.

In this regard, the issue considered here is related to Pigouvian taxation.[4] Yet, we do not restrict ourselves to the case of externalities. Rather, as mentioned above, we allow for a broad interpretation of the predetermined constraint on the total consumption level of a specific good with varieties. Accordingly, we explore the fundamental fiscal problem in a general setting: given the target level of total consumption, how should taxes on different varieties be determined in order to accomplish that target at lowest cost (or highest utility)? This approach is parallel to, though still different from, the standard problem in the optimal taxation literature, where the utility maximisation is subject to a constraint on tax revenue. Since the constraint we consider here is on consumption, rather than on tax revenue (or expenditure), we thus explore how the presence of a quantity constraint affects the existing results of the optimal taxation literature obtained under a revenue constraint. To our knowledge, the role of this type of a constraint has not yet been explored in the literature since the seminal work of Ramsey (1927), whose contribution has recently been thoroughly reviewed and re-evaluated by Stiglitz (2015). Yet, the apparently unnoticed issue of a quantity constraint is too important to be neglected from a practical point of view as explained above, and also from a theoretical point of view since this modification has unconventional implications as we will demonstrate below.

For the standard problem in public finance—utility maximisation subject to a revenue constraint—the literature has established a series of fundamental and well-known results. Among them, the following are the most notable. The inverse elasticity rule (initiated by Ramsey, 1927): higher tax rates should be imposed on goods with a less elastic demand.[5] The Corlett–Hague rule (originating with Corlett and Hague, 1953): higher tax rates should be imposed on goods with a lower compensated price elasticity of demand with respect to the price of an untaxed good (e. g.,

[4]Sandmo (1975) considered the optimal taxation problem where certain negative externalities are present.

[5]The inverse elasticity rule has two versions, one for compensated demand and one for ordinary demand. See, e. g., Sandmo (1987).

leisure).[6] Uniform taxation (presented in Diamond and Mirrlees, 1971; Sandmo, 1974; and Sadka, 1977): the same tax rates should be imposed on goods if, and only if, the compensated price elasticities of the goods with respect to the price of an untaxed good are all equal.[7]

In this paper, we first establish the existence of a solution of our optimal taxation problem, and then demonstrate that the solution of this problem yields a uniform-pricing result. This result is in parallel with the well-known rule of uniform taxation: the same consumer price for all varieties is a solution if, and only if, the compensated price elasticities of the varieties with respect to the price of an untaxed good are all equal. Moreover, the equal elasticity condition implies that all varieties are weakly substitutable (i. e., non-complementary) with respect to the untaxed good.

If the elasticities under consideration are not equal, though, our problem gives rise to implications different from, and even opposite to, the inverse elasticity rule and the Corlett–Hague rule: we find that higher consumer prices should be charged (i) on varieties with a more elastic demand, and (ii) on varieties with a higher compensated price elasticity of demand with respect to the price of an untaxed good. This anti-inverse elasticity result (and similarly the anti-Corlett–Hague result) sounds counter-intuitive at first, but it is actually not: in order for the same reduction in consumption to be accomplished, elastic demand requires a smaller price increase than does inelastic demand, and therefore it is less costly (in terms of reduced consumer surplus) to reduce consumption of a variety with a more elastic demand than of an inelastic demand. For this reason, it is more efficient to meet a given target level of total consumption by taxing varieties with higher demand elasticity more heavily than those with less elastic demand. In this way, the target level can be accomplished by bringing forth rather small price distortions. This reasoning

[6]For a recent study of this rule, see Kaplow (2010).

[7]Deaton (1979), Besley and Jewitt (1995), and Barbie and Hermeling (2009) provided conditions on preferences for uniform commodity taxes to be optimal. In particular, these authors showed that uniform commodity taxes are optimal if, and only if, preferences are implicitly (or quasi) separable between leisure and consumption goods.

applies irrespective of whether the target level has to be reached from above (reduction of demand) or from below (increase of demand). (A similar intuitive reasoning applies for the anti-Corlett–Hague result.)

Hence, the policy implications of our problem depend on the values of the compensated price elasticities of the varieties with respect to the price of an untaxed good. It is thus necessary to investigate the empirical validity of the equal elasticity condition. Sandmo (1974) shows that the condition is satisfied when a consumer has homogeneous preferences which are weakly separable between the taxed goods and an untaxed good (e. g., leisure). However, weak separability is empirically rejected by many optimal tax studies (see, e. g., Jacobs and Boadway, 2014, fn. 2).

In this paper, we do not consider different goods, though, but consider different varieties of a specific good, and it seems to be an open question whether varieties of the same good are similarly related to other goods. For example, Borcherding and Silberberg (1978) and Silberberg and Suen (2001, p. 340) argue that two close substitutes such as a high and a low quality of the same good (e. g., apples) should be similarly related to other goods (a composite good) and the asymmetry in this relation seems to be "empirically insignificant." However, Minagawa (2012) questions this view, providing a qualification to this argument: suppose that there are three goods, say high and low priced (quality of) coffee beans, and milk. Then, a rise in the price of the high quality coffee leads to an exchange of one unit of this quality for more than one unit of the low quality coffee (due to the exchange rate). The total consumption of coffee beans thus increases, and correspondingly the total consumption of milk also increases provided that the consumer always takes their coffee with milk. In this case, the more expensive high-quality coffee beans and milk are substitutes while the less expensive low-quality coffee beans and milk are complements.

The rest of this paper is organised as follows. In Section 2 we set up the problem of the optimal commodity tax under a consumption target. This problem is then solved in Section 3, where we also derive its policy implications. Finally, we conclude in Section 4.

2. Model

Assume that there are n varieties of some specific good, the (generic) quantities of which we denote by $\mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$; and let $\mathbf{p} := (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n_+$ be the corresponding net prices, which we assume to be fixed throughout. Suppose that the set of these varieties is subject to unit taxes, denoted by $\mathbf{t} := (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, so that the consumer prices are given by $\mathbf{q} := (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n_{++}$ where $\mathbf{q} \equiv \mathbf{p} + \mathbf{t}$. Let $x_0 \geqq 0$ denote the (generic) quantity of a composite good, which is not subject to taxation, so that its price is simply denoted by $q_0 > 0$.

Suppose that a consumer's preference relation is represented by a continuous utility function $u : \mathbb{R}^{n+1}_+ \to \mathbb{R} : (x_0, \mathbf{x}) \mapsto u(x_0, \mathbf{x})$ which is strictly increasing and strictly quasi-concave on $\mathbb{R}^{n+1}_{++}$ with $u(x_0, \mathbf{x}) = c$ for any $(x_0, \mathbf{x}) \in \mathbb{R}^{n+1}_+ \backslash \mathbb{R}^{n+1}_{++}$, for some $c \in \mathbb{R}$. The consumer chooses consumption of the $n + 1$ goods to maximise their utility $u(x_0, \mathbf{x})$ subject to their budget constraint $\sum_{i=0}^n q_i x_i \leqq I$ where $I > 0$ represents the consumer's income. This utility maximisation problem yields ordinary (or Marshallian) demand functions, $x_0^m(q_0, \mathbf{q}, I)$ and $\mathbf{x}^m(q_0, \mathbf{q}, I)$. Correspondingly, the expenditure minimisation problem gives compensated (or Hicksian) demand functions, $x_0^h(q_0, \mathbf{q}, v)$ and $\mathbf{x}^h(q_0, \mathbf{q}, v)$, where v represents a specified utility level.

The government chooses consumer prices $\mathbf{q}$ (or unit taxes $\mathbf{t}$) to maximise the consumer's welfare, given by the indirect utility function $V(q_0, \mathbf{q}, I) := u(x_0^m(q_0, \mathbf{q}, I), \mathbf{x}^m(q_0, \mathbf{q}, I))$, and subject to the quantity constraint $\sum_{i=1}^n x_i^m(q_0, \mathbf{q}, I) = \bar{Z}$, where $\bar{Z} \in (0, \zeta]$, for some $\zeta > 0$, represents a given target amount of total consumption of the n varieties of the good.[8] Hence, the government faces the problem

$$\underset{\mathbf{q}}{\text{Maximise}} \;\; V(q_0, \mathbf{q}, I) \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i^m(q_0, \mathbf{q}, I) = \bar{Z}. \tag{1}$$

[8] Some applications require a greater-than-equal-to constraint, demanding that total consumption of n varieties of a good may not fall short of some minimum level (e. g., merit goods), while other applications require a less-than-equal-to constraint, demanding that total consumption may not exceed some maximum level (e. g., demerit goods). Since in either case a *binding* constraint becomes an equality constraint, we consider that case here.

We are interested in the properties of the solutions of this problem, and thus next derive optimal price vectors. To begin with, we first establish the existence of a solution to problem (1).

3. Results

In order to establish the existence of a solution to problem (1), we make the following assumption on the feasibility of accomplishing a consumption target.[9]

Assumption 1. For any $\bar{Z}$ there is a price vector $\mathbf{q}$ such that $\sum_{i=1}^{n} x_i^m(q_0, \mathbf{q}, I) = \bar{Z}$ with $x_i^m(q_0, \mathbf{q}, I) > 0$ for all $i = 0, 1, \ldots, n$.

Under Assumption 1, in addition to the assumptions given in Section 2, we prove the following existence result.

Proposition 1. There exists a solution to the taxation problem (1).

Proof.[10] We first show that there exists a solution to the problem

$$\underset{\mathbf{q}}{\text{Maximise}}\ V(q_0, \mathbf{q}, I)\ \text{ s.t. } \sum_{i=1}^{n} x_i^m(q_0, \mathbf{q}, I) = \bar{Z},\ q_j \geqq \varepsilon,\ j = 1, 2, \ldots, n, \qquad (2)$$

where $\varepsilon > 0$ is sufficiently small. Let $Q_{\bar{Z},\varepsilon} \equiv \{\, \mathbf{q} \in \mathbb{R}^n_{++} \mid \sum_{i=1}^{n} x_i^m(q_0, \mathbf{q}, I) = \bar{Z}, q_j \geqq \varepsilon, j = 1, 2, \ldots, n\}$. Then, according to Assumption 1, for any given value $\bar{Z}$ the set $Q_{\bar{Z},\varepsilon}$ is non-empty. That is, we can find a price vector $\bar{\mathbf{q}} \in Q_{\bar{Z},\varepsilon}$ with $x_i^m(q_0, \bar{\mathbf{q}}, I) > 0$ for all $i = 0, 1, \ldots, n$. Let $\bar{V} \equiv V(q_0, \bar{\mathbf{q}}, I)$, which is greater than $c \in \mathbb{R}$ by the assumptions for the utility function. We then define the set $Q_{\bar{V}} \equiv \{\, \mathbf{q} \in \mathbb{R}^n_{++} \,|V(q_0, \mathbf{q}, I) \geqq \bar{V}\}$. Let $Q \equiv Q_{\bar{Z},\varepsilon} \bigcap Q_{\bar{V}}$. Since $\bar{\mathbf{q}} \in Q$, the set Q is non-empty.

In the following, we will show that the set Q is compact. First, we prove that the set Q is closed. Let $Q_{\bar{Z}} \equiv \{\, \mathbf{q} \in \mathbb{R}^n_{++} \mid \sum_{i=1}^{n} x_i^m(q_0, \mathbf{q}, I) = \bar{Z}\}$. By the continuity of the ordinary demand functions, the set $Q_{\bar{Z}}$ is closed in $\mathbb{R}^n_{++}$. Similarly, by the

[9]Since the case of a single variety (i. e., $n = 1$) is trivial —in fact, it boils down to the standard textbook model with two goods: an untaxed good and a taxed good— we are interested in the case of two or more varieties (i. e., $n \geqq 2$). In this case, there are generically many price vectors leading to the same level of total consumption (see Figure 1 for a case of two varieties).

[10]The idea of this proof is borrowed from Iritani (1986, Sec. 1.6), who shows the existence of a solution for the standard optimal commodity tax problem with a revenue constraint.

continuity of the indirect utility function, the set $Q_{\bar{V}}$ is closed in $\mathbb{R}^n_{++}$. Then the set $Q_{\bar{Z}} \bigcap Q_{\bar{V}}$ is closed in $\mathbb{R}^n_{++}$. Let $Q_\varepsilon \equiv \{ \mathbf{q} \in \mathbb{R}^n_{++} | q_j \geqq \varepsilon, j = 1, 2, \ldots, n\}$. The set Q_ε is a subset of $\mathbb{R}^n_{++}$ and is closed (in $\mathbb{R}^n$). The set $Q \equiv (Q_{\bar{Z}} \bigcap Q_{\bar{V}}) \bigcap Q_\varepsilon$ is therefore closed (in $\mathbb{R}^n$).

Second, we prove that the set Q is bounded. By way of contradiction, suppose not. Then, there exists a sequence of prices $\{\mathbf{q}^\nu\}_{\nu=1}^\infty$ in Q with $||\mathbf{q}^\nu|| \to \infty$. Now, for each ν, let $r_0^\nu \equiv q_0/(q_0 + \sum_{i=1}^n q_i^\nu)$, $r_j^\nu \equiv q_j^\nu/(q_0 + \sum_{i=1}^n q_i^\nu)$, $j = 1, 2, \ldots, n$, and $I^\nu \equiv I/(q_0 + \sum_{i=1}^n q_i^\nu)$. By passing to a subsequence if necessary, we may assume that the sequence of prices and incomes $\{(r_0^\nu, \mathbf{r}^\nu, I^\nu)\}_{\nu=1}^\infty$ is such that $r_0^\nu \to 0$, $\mathbf{r}^\nu \to \mathbf{r}'$, and $I^\nu \to 0$ where each element of $\mathbf{r}'$ is in $[0, 1]$ such that r'_k is non-zero for some good k.[11]

Then, from the homogeneity of degree zero, we have for each ν, $x_i^m(q_0, \mathbf{q}^\nu, I)$ $= x_i^m(r_0^\nu, \mathbf{r}^\nu, I^\nu)$ for all $i = 0, 1, \ldots, n$. Moreover, it follows that $x_k^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) \to 0$, since, by the budget constraint, we have $0 \leqq x_k^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) \leqq I^\nu/r_k^\nu$ for each ν. But then, since for each ν, $V(q_0, \mathbf{q}^\nu, I) = V(r_0^\nu, \mathbf{r}^\nu, I^\nu) \geqq \bar{V} > c$, the assumptions on the utility function imply that there is some good l such that $x_l^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) \to \infty$ (since otherwise, $V(r_0^\nu, \mathbf{r}^\nu, I^\nu) \to c$). This implies, together with the quantity constraint $\sum_{i=1}^n x_i^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) = \bar{Z}$ for each ν, that $x_0^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) \to \infty$. On the other hand, by the budget constraint, we have for each ν, $r_0^\nu x_0^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) + \sum_{i=1}^n r_i^\nu x_i^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) = I^\nu$, which is equal to $q_0 x_0^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) + \sum_{i=1}^n q_i^\nu x_i^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) = I$. It thus follows that for each ν, $x_0^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) = [I - \sum_{i=1}^n q_i^\nu x_i^m(r_0^\nu, \mathbf{r}^\nu, I^\nu)]/q_0$. Since the right-hand side is bounded above, we obtain a contradiction. Hence, the set Q is bounded. The set Q is thus non-empty and compact, and the indirect utility function is continuous. Therefore, by Weierstrass' theorem, there exists a solution of (2).

We next establish the existence of a solution of (1). Let N be a sufficiently large integer. Consider the sequence of ε in (2): $\{\varepsilon^\nu\}_{\nu=N+1}^\infty$ where $\varepsilon^\nu \equiv 1/\nu$, and thus $\varepsilon^\nu \to 0$. Then, for each ν, there exists a solution of (2), and hence we may denote it by $\mathbf{q}_\varepsilon^\nu$. We will prove that the sequence $\{\mathbf{q}_\varepsilon^\nu\}_{\nu=N+1}^\infty$ has an accumulation point $\mathbf{q}^*$

[11] Note that r'_k does not need to be equal to 1. For example, consider $r_k^\nu = q_k^\nu/(q_0 + \sum_{i=1}^n q_i^\nu)$. If $q_1^\nu = \cdots = q_n^\nu$ for each ν, then $r_k^\nu \to 1/n$.

in $\mathbb{R}^n_{++}$. By way of contradiction, suppose not. Then, only two cases are possible: (i) $||\mathbf{q}^\nu_\varepsilon|| \to \infty$ or (ii) an accumulation point exists but is not in $\mathbb{R}^n_{++}$. In either case, by similar arguments to the above, we can derive a contradiction. Therefore, the sequence $\{\mathbf{q}^\nu_\varepsilon\}^\infty_{\nu=N+1}$ has an accumulation point $\mathbf{q}^*$ in $\mathbb{R}^n_{++}$. Clearly, $\mathbf{q}^*$ is a solution of (1). □

The situation for the case of a Cobb–Douglas utility function is illustrated in Figure 1, in which the shaded area represents the set $Q_{\bar{V}}$ and $\mathbf{q}^*$ is the utility maximising price vector, for a given level of total consumption $\bar{Z} = x_1 + x_2$.

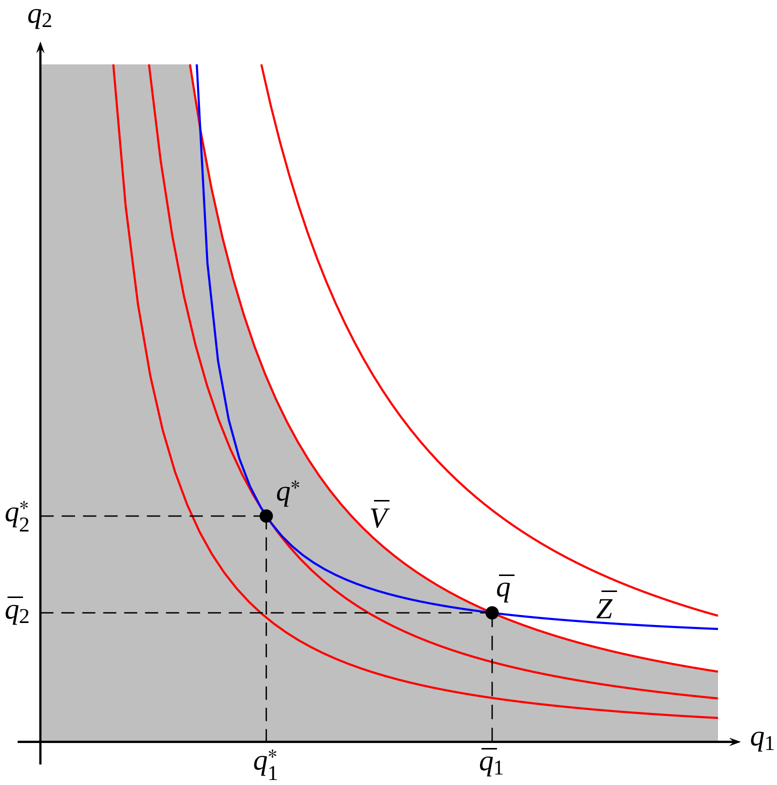

Figure 1. Curves of constant utility (red) and constant total consumption (blue)

In the following, we assume that the utility function and the demand functions are continuously differentiable, and define the compensated price elasticity of good i with respect to the consumer price of good j by $\varepsilon^h_{ij} := (q_j/x^h_i)(\partial x^h_i/\partial q_j)$. We also simply write x_i to denote the level of the demand under consideration.

Before proceeding to our next fundamental result, Proposition 2, we start with the following observation.[12]

Lemma 1. If the compensated price elasticities of all varieties with respect to the price of the composite good are equal (i. e., $\varepsilon^h_{i0} = \alpha, \ \forall i \neq 0$), then all varieties are weakly substitutable (i. e., non-complementary) with respect to the composite good (i. e., $\alpha \geqq 0$).

Proof. Given the fact that $\partial x^h_i / \partial q_j = \partial x^h_j / \partial q_i$, the assumption of this lemma implies that all ε^h_{i0} and ε^h_{0i} ($\forall i \neq 0$) have the same sign. Also, using Hicks' (1939) "third law" and the fact that $\varepsilon^h_{00} \leqq 0$, we obtain $\sum_{i=1}^n \varepsilon^h_{0i} = -\varepsilon^h_{00} \geqq 0$. Therefore, all ε^h_{0i} and ε^h_{i0} ($\forall i \neq 0$) must be non-negative. □

Remark 1. The result of uniform taxation, presented, for example, by Diamond and Mirrlees (1971), Sandmo (1974), and Sadka (1977), is formally obtained under the same elasticity condition as is Lemma 1.[13] However, the implied consequence shown in Lemma 1 (i. e., weak substitutability) is seemingly unrecognised in the optimal taxation literature. In particular, in the basic optimal commodity tax problem with *n* (or two) goods and leisure (good 0), Auerbach (1985, p. 90) and Auerbach and Hines (2002, p. 1368) interpret the case $\varepsilon^h_{i0} = \alpha$ ($\forall i \neq 0$) as meaning that all goods are equally "complementary" to leisure.[14] Hence, Lemma 1 corrects such an interpretation. That is, since the cross-price elasticities cannot be negative in this case, referring to this case as "complementary" seems to be inaccurate.

Proposition 2. Suppose that at least one of the *n* derivatives $(\partial / \partial q_j) \sum_{i=1}^n x^m_i(q_0, \mathbf{q}, I)$, $j = 1, 2, \ldots, n$ at a solution for (1) is not zero. Then the choice of equal consumer

[12] This is a straightforward generalisation of the three-good case given by Minagawa and Upmann (2013, fn. 3).

[13] Sandmo (1974) also shows that this condition is satisfied when the utility function is weakly separable, written as $u(x_0, f(\mathbf{x}))$, and the function f is homogeneous of positive degree. See also fn. 7.

[14] Similarly, Heady (1993, p. 33) states that "all goods have the same degree of complementarity or substitutability with leisure"; Sørensen (2007, p. 387), that "goods and services are equally substitutable for (complementary to) leisure"; and Boadway (2012, p. 54), that "all goods must be equally complementary with leisure."

prices for all varieties (i. e., $q_i = q,\ \forall i \neq 0$) is a solution to problem (1) if, and only if, all varieties are equally weakly substitutable with respect to the composite good (i. e., $\varepsilon^h_{i0} = \alpha \geqq 0,\ \forall i \neq 0$).

Proof. We first prove the "only if."[15] Since for some j, $(\partial/\partial q_j)\sum_{i=1}^n x_i^m(q_0, \mathbf{q}, I)$ at a solution of (1) is non-zero, the usual constraint qualification (the so-called rank condition) is satisfied. So, the Lagrangian for (1) is $L(\mathbf{q}, \lambda) := V(q_0, \mathbf{q}, I) + \lambda[\overline{Z} - \sum_{i=1}^n x_i^m(q_0, \mathbf{q}, I)]$. Applying Roy's identity to the first order conditions, we obtain

$$-\mu x_j - \lambda \sum_{i=1}^n \frac{\partial x_i^m}{\partial q_j} = 0, \quad j = 1, 2, \ldots, n, \tag{3}$$

where $\mu \equiv \partial V/\partial I$ is the marginal utility of income.[16] Using the Slutsky equation, we may express Eq. (3) as

$$-\frac{\mu}{\lambda} + \sum_{i=1}^n \frac{\partial x_i^m}{\partial I} = \frac{1}{x_j}\sum_{i=1}^n \frac{\partial x_i^h}{\partial q_j}, \quad j = 1, 2, \ldots, n. \tag{4}$$

The left-hand side of Eq. (4) is independent of j, and we denote it by θ. Then, using the fact that $\partial x_i^h/\partial q_j = \partial x_j^h/\partial q_i$ and the compensated price elasticities, we may express Eq. (4) as

$$\theta = \sum_{i=1}^n \frac{\varepsilon^h_{ji}}{q_i}, \quad j = 1, 2, \ldots, n. \tag{5}$$

For equal consumer prices, $q_i = q\ \forall i = 1, \ldots, n$, the right-hand side of Eq. (5) becomes $(1/q)\sum_{i=1}^n \varepsilon^h_{ji}$. Using Hicks' "third law," we get $(1/q)\sum_{i=1}^n \varepsilon^h_{ji} = (1/q)(-\varepsilon^h_{j0})$. Then using Lemma 1, all ε^h_{j0} $(j \neq 0)$ must be equal and non-negative.

Next, we prove the "if."[17] Suppose that all varieties are equally weakly substitutable with respect to the composite good, $\varepsilon^h_{j0} = \alpha \geqq 0,\ \forall j \neq 0$. Then, by

[15] Since problem (1) is similar to the basic optimal commodity tax problem that a government maximises a consumer's indirect utility subject to a revenue constraint, we may essentially follow the approach to the latter problem given by Diamond and Mirrlees (1971).

[16] Likewise, we may interpret the multiplier λ as the marginal utility of public consumption, since $\partial V(q_0, \mathbf{q}(\overline{Z}), I)/\partial \overline{Z} = \lambda$ where $q_j = q_j(\overline{Z})$ is a solution of (1).

[17] This is similar to the demonstration of the uniformity of tax rates in the basic optimal commodity tax problem given by Sandmo (1974). See also Sadka (1977).

definition, $\partial x_j^h/\partial q_0 = \alpha x_j/q_0$. Thus, by symmetry, $\partial x_0^h/\partial q_j = \alpha x_j/q_0$. Substituting this relation into the Slutsky equation yields

$$\frac{\partial x_0^m}{\partial q_j} = \frac{\partial x_0^h}{\partial q_j} - \frac{\partial x_0^m}{\partial I} x_j = \left(\alpha - \frac{\partial x_0^m}{\partial I} q_0\right) \frac{x_j}{q_0}, \quad j = 1, 2, \ldots, n. \tag{6}$$

The term within the parentheses in Eq. (6) is independent of j, and we denote it by β, that is, $\partial x_0^m/\partial q_j = \beta x_j/q_0$. Now, differentiating both sides of the identity $\sum_{i=0}^n q_i x_i^m(q_0, \mathbf{q}, I) \equiv I$ with respect to q_j and using the above relation, we obtain

$$x_j = -\frac{1}{1+\beta} \sum_{i=1}^n q_i \frac{\partial x_i^m}{\partial q_j}. \tag{7}$$

Substituting this into Eq. (3) yields

$$\sum_{i=1}^n \left[\frac{\mu}{(1+\beta)} q_i - \lambda\right] \frac{\partial x_i^m}{\partial q_j} = 0, \quad j = 1, 2, \ldots, n. \tag{8}$$

Then, $[\mu/(1+\beta)]q_i - \lambda = 0$, $\forall i \neq 0$ is a solution of Eq. (8).[18] That is, $q_i = (1+\beta)\lambda/\mu$, $\forall i \neq 0$. The solution is independent of i and may be denoted by q. □

Example 1. Consider the consumer's problem with two varieties. Suppose that the preference relation is represented by a Cobb–Douglas utility function $u(x_0, \mathbf{x}) := x_0 x_1 x_2$. Let $q_0 = 1$. Solving the utility maximisation problem, we have $x_i^m(q_0, \mathbf{q}, I) = I/(3q_i)$, and then $V(q_0, \mathbf{q}, I) = I^3/(27 q_1 q_2)$. Next, consider the government's problem. The first order conditions give rise to the unique solution $q_1 = q_2 = 2I/(3\overline{Z})$ and $\lambda = I\overline{Z}/6$. Moreover, the Hessian matrix of the Lagrangian is

$$\begin{pmatrix} \frac{\partial^2 L}{\partial q_1^2} & \frac{\partial^2 L}{\partial q_1 \partial q_2} & \frac{\partial^2 L}{\partial q_1 \partial \lambda} \\ \frac{\partial^2 L}{\partial q_2 \partial q_1} & \frac{\partial^2 L}{\partial q_2^2} & \frac{\partial^2 L}{\partial q_2 \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial q_1} & \frac{\partial^2 L}{\partial \lambda \partial q_2} & \frac{\partial^2 L}{\partial \lambda^2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{3\overline{Z}^4}{16I} & \frac{3\overline{Z}^2}{4I} \\ \frac{3\overline{Z}^4}{16I} & 0 & \frac{3\overline{Z}^2}{4I} \\ \frac{3\overline{Z}^2}{4I} & \frac{3\overline{Z}^2}{4I} & 0 \end{pmatrix}. \tag{9}$$

Since the determinant of this matrix is $27\overline{Z}^8/(128 I^3) > 0$, the second order condition is satisfied.[19] Therefore, the solution is indeed optimal. □

[18] The solution is unique if the matrix $(\partial x_i^m/\partial q_j)$ $(i, j = 1, 2, \ldots, n)$ is non-singular.

[19] The second order condition for the constraint maximisation problem with two variables and one equality constraint is that the determinant of the Hessian matrix of the Lagrangian is positive. See, e. g., Silberberg and Suen (2001, Sec. 6.5).

Next, we investigate the case of Lemma 1 when the equal elasticity condition does not hold. Consider the case of two varieties in the general setting. Eliminating θ and using Hicks' "third law," Eq. (5) becomes

$$\frac{q_1}{q_2} = \frac{\varepsilon^h_{21} + \varepsilon^h_{12} + \varepsilon^h_{10}}{\varepsilon^h_{12} + \varepsilon^h_{21} + \varepsilon^h_{20}}. \tag{10}$$

Hence, we have $q_1 \gtreqless q_2 \iff \varepsilon^h_{10} \gtreqless \varepsilon^h_{20}$, provided that the denominator and numerator of the right-hand side are positive. This relation suggests that higher consumer prices should be charged for the variety with the higher compensated price elasticity with respect to the composite good. Moreover, suppose that all compensated cross-price effects except with respect to good 0 are zero. Then, since $\varepsilon^h_{i0} = -\varepsilon^h_{ii},\ i = 1, 2$, the above relation becomes

$$\frac{q_1}{q_2} = \frac{\varepsilon^h_{10}}{\varepsilon^h_{20}} = \frac{\varepsilon^h_{11}}{\varepsilon^h_{22}}. \tag{11}$$

Hence, we have $q_1 \gtreqless q_2$ if, and only if, $|\varepsilon^h_{11}| \gtreqless |\varepsilon^h_{22}|$. This suggests that higher consumer prices should be charged for the variety with the more elastic demand.

On the other hand, the Corlett–Hague rule in the three good case with one untaxed good and two taxed goods (see, e. g., Auerbach and Hines, 2002, Eq. 3.21) can be written in our notation as

$$\frac{t_1/q_1}{t_2/q_2} = \frac{\varepsilon^h_{21} + \varepsilon^h_{12} + \varepsilon^h_{20}}{\varepsilon^h_{12} + \varepsilon^h_{21} + \varepsilon^h_{10}}, \tag{12}$$

and we then have $t_1/q_1 \gtreqless t_2/q_2 \iff \varepsilon^h_{20} \gtreqless \varepsilon^h_{10}$, provided that the denominator and numerator of the right-hand side are positive. That is, higher tax rates should be imposed on the good with the lower compensated price elasticity with respect to the untaxed good 0. Finally, assuming that all compensated cross-price effects except with respect to good 0 are zero, the inverse elasticity rule in the same three good case for the Corlett–Hague rule mentioned above may be expressed as

$$\frac{t_1/q_1}{t_2/q_2} = \frac{\varepsilon^h_{20}}{\varepsilon^h_{10}} = \frac{\varepsilon^h_{22}}{\varepsilon^h_{11}}, \tag{13}$$

and we have $t_1/q_1 \gtreqless t_2/q_2$ if, and only if, $|\varepsilon^h_{22}| \gtreqless |\varepsilon^h_{11}|$. That is, higher tax rates should be imposed on the good with the less elastic demand.

Comparing these results with our results, we find that if the equal elasticity condition does not hold, our problem gives rise to implications different from the Corlett–Hague rule and the inverse elasticity rule. In particular, when both net prices, p_1 and p_2, are the same, we obtain $q_1 \gtreqless q_2 \Leftrightarrow t_1 \gtreqless t_2 \Leftrightarrow t_1/q_1 \gtreqless t_2/q_2$. Hence, in this case, our implications are exactly opposite to the implications of either rule.

4. Conclusion

In this paper, we considered the problem in which a government chooses consumer prices (or unit taxes) for the different varieties of some specific good to maximise consumer welfare while achieving a specified level of total consumption of that good. This problem thus constitutes an optimal commodity tax problem under a quantity constraint.

We first showed that a solution to this problem exists; then, that the requirement that all compensated cross-price elasticities with respect to the composite good are equal and non-negative is a necessary and sufficient condition for the optimality of equal consumer prices for all varieties. This result, of equal prices, is similar to the existing result of uniform tax rates. On the other hand, if this requirement does not hold, then the results are different from, and even opposite to, the existing results of the inverse elasticity rule and the Corlett–Hague rule: we found that higher consumer prices should be charged (i) for the variety with the more elastic demand, and (ii) for the variety with the higher compensated price elasticity of demand with respect to the price of an untaxed good.

The policy implications of our problem therefore depend on the values of the cross-price elasticities, but, as mentioned in the Introduction, it seems to be an open question whether the assumption of equal elasticities is satisfied. It thus becomes an important empirical question whether varieties of the same good are similarly related to other goods.

References

A. J. Auerbach. The Theory of Excess Burden and Optimal Taxation. In: A. J. Auerbach and M. Feldstein (eds.) *Handbook of Public Economics*. Vol. 1. North-Holland, Amsterdam, 1985, pp. 61–127.

A. J. Auerbach and J. R. Hines Jr. Taxation and Economic Efficiency. In: A. J. Auerbach and M. Feldstein (eds.) *Handbook of Public Economics*. Vol. 3. Elsevier, Amsterdam, 2002, pp. 1347–1421.

M. Barbie and C. Hermeling. The Geometry of Optimal Taxation: A Primal Approach. *Economic Theory*, 39(1): 129–155, 2009.

T. Besley and I. Jewitt. Uniform Taxation and Consumer Preferences. *Journal of Public Economics*, 58(1): 73–84, 1995.

R. Boadway. *From Optimal Tax Theory to Tax Policy*. The MIT Press, Cambridge, MA, 2012.

T. E. Borcherding and E. Silberberg. Shipping the Good Apples Out: The Alchian and Allen Theorem Reconsidered. *Journal of Political Economy*, 86(1): 131–138, 1978.

W. J. Corlett and D. C. Hague. Complementarity and the Excess Burden of Taxation. *Review of Economic Studies*, 21(1): 21–30, 1953.

A. Deaton. The Distance Function in Consumer Behaviour with Applications to Index Numbers and Optimal Taxation. *Review of Economic Studies*, 46(3): 391–405, 1979.

P. A. Diamond and J. A. Mirrlees. Optimal Taxation and Public Production: I–II. *American Economic Review*, 61(1): 8–27, (3): 261–278, 1971.

European Commission. Proposal For a Council Recommendation on Seasonal Influenza Vaccination. 13 July 2009.

European Commission. Excise Duty Tables, REF 1044, 2015.

C. Heady. Optimal Taxation as a Guide to Tax Policy: A Survey. *Fiscal Studies*, 14(1): 15–41, 1993.

J. R. Hicks. *Value and Capital*. Clarendon Press, Oxford, 1939.

J. Iritani. *Kazei no Saiteki Riron (The Optimal Theory of Taxation)*. Toyo Keizai, Tokyo, 1986.

B. Jacobs and R. Boadway. Optimal Linear Commodity Taxation Under Optimal Non-linear Income Taxation. *Journal of Public Economics*, 117: 201–210, 2014.

L. Kaplow. Taxing Leisure Complements. *Economic Inquiry*, 48(4): 1065–1071, 2010.

J. Minagawa. On Giffen-like Goods. *Economics Letters*, 115(2): 282–285, 2012.

J. Minagawa and T. Upmann. A Conditional Demand Approach to the Alchian–Allen Effect. *Economics Letters*, 120(3): 546–551, 2013.

F. P. Ramsey. A Contribution to the Theory of Taxation. *Economic Journal*, 37(145): 47–61, 1927.

E. Sadka. A Theorem on Uniform Taxation. *Journal of Public Economics*, 7(3): 387–391, 1977.

A. Sandmo. A Note on the Structure of Optimal Taxation. *American Economic Review*, 64(4): 701–706, 1974.

A. Sandmo. Optimal Taxation in the Presence of Externalities. *Swedish Journal of Economics*, 77(1): 86–98, 1975.

A. Sandmo. A Reinterpretation of Elasticity Formulae in Optimum Tax Theory. *Economica*, 54(213): 89–96, 1987.

E. Silberberg and W. Suen. *The Structure of Economics: A Mathematical Analysis*. Third edition. McGraw-Hill, New York, 2001.

P. B. Sørensen. The Theory of Optimal Taxation: What is the Policy Relevance? *International Tax and Public Finance*, 14(4): 383–406, 2007.

J. E. Stiglitz. In Praise of Frank Ramsey's Contribution to the Theory of Taxation. *Economic Journal*, 125(583): 235–268, 2015.

U. S. Environmental Protection Agency. Emission Facts. EPA420-F-05-001, 2005.

World Health Assembly. Prevention and Control of Influenza Pandemics and Annual Epidemics. WHA56.19, 28 May 2003.

World Health Organisation. WHO Framework Convention on Tobacco Control. 2003–2005.