CESifo Working Papers
www.cesifo.org/wp

The Local Power of the IPS Test with Both Initial Conditions and Incidental Trends

Kajal Lahiri
Zhongwen Liang
Huaming Peng

CESifo Working Paper No. 6313
Category 12: Empirical and Theoretical Methods
January 2017

ISSN 2364-1428

CESifo
Center for Economic Studies & Ifo Institute

CESifo *Working Paper No. 6313*

The Local Power of the IPS Test with Both Initial Conditions and Incidental Trends

Abstract

This paper investigates the asymptotic local power of the the averaged *t*-test of Im, Pesaran and Shin (2003, IPS hereafter) in the presence of both initial explosive conditions and incidental trends. By utilizing the least squares detrending methods, it is found that the initial condition plays no role in determining the asymptotic local power of the IPS test, a result strikingly different from the finding in Harris et al. (2010), who examined the impact of the initial conditions on local power of IPS test without incidental trends. The paper also presents, via an application of the Fredholm method discussed in Nabeya and Tanaka (1990a, 1990b), the exact asymptotic local power of IPS test, thereby providing theoretical justifications for its lack of asymptotic local power in the neighborhood of unity with the order of $N^{-1/2}T^{-1}$ while attaining nontrivial power in the neighborhood of unity that shrinks at the rate $N^{-1/4}T^{-1}$. This latter finding is consistent with Moon et al. (2007) and extends their results to IPS test. It is also of practical significance to empirical researchers as the presence of incidental trends in panel unit root test setting is ubiquitous.

JEL-Codes: C130, C220, C230.

Keywords: panel data, unit root test, individual heterogeneity.

Kajal Lahiri
Department of Economics
University at Albany, SUNY
USA - 12222 Albany NY
klahiri@albany.edu

Zhongwen Liang
Department of Economics
University at Albany, SUNY
USA - 12222 Albany NY
zliang3@albany.edu

Huaming Peng
Department of Economics
University at Albany, SUNY
USA - 12222 Albany NY
hpeng@albany.edu

This version: Nov. 8, 2016

1 Introduction

The panel unit root tests have been extensively studied in the past two decades. Some recent comprehensive literature surveys on this topic, for example, by Phillips and Moon (2000), Choi (2006), Breitung and Pesaran (2008) and Westerlund and Breitung (2012), shed insights on various panel unit root tests, e.g., the pooled t-test proposed by Levin, Lin and Chu (2002), the IPS test and the combination tests based on Fisher-type statistics in Maddala and Wu (1999) and Choi (2001). Among these, the IPS test is the most cited and influential.

Researchers have recently realized that the initial conditions and incidental trends could have significant impact on the local power of the unit root tests in both time series and panel data models. Müller and Elliot (2003) demonstrated that the power envelopes of unit root tests in time series models display quite distinct characteristics depending on the initial values. The dependence of the power of Dickey-Fuller type tests on the initial conditions has also been found in Harvey and Leybourne (2005) and Harvey et al. (2009). In panel data setting, Moon, Perron and Phillips (2007, hereafter MPP) investigated the asymptotic power envelopes of five tests (Ploberger and Phillips (2002), Moon and Phillips (2004), Breitung (2000), Levin et al. (2002), and Moon and Perron (2008)) with or without trends under the assumption of zero initial value. Since the IPS test can not be evaluated within the MPP framework, no analytical power result for IPS is currently available, even though simulation results in MPP indicate that IPS test tends to have inferior asymptotic local power when compared to those proposed by Ploberger and Phillips (2002) and by Breitung (2000). The asymptotic local power of the IPS test under explosive initial conditions has also been examined by Harris et al. (2010), who concluded that the power declines monotonically with the increase of the magnitude of the initial value.

The main goal of this paper is to fill the void of lack of analytical asymptotic local power result for IPS test with both explosive initial conditions and incidental trends. More specifically, this paper aims to answer whether the IPS test with time trend has nontrivial local power at rates $N^{-1/4}T^{-1}$ and $N^{-1/2}T^{-1}$ respectively. Some interesting findings of our paper are as follows. First, detrending the data by least squares method will effectively take care of the initial condition by canceling out its dominant components, thereby eliminating its effect on the asymptotic local power. [1] This result will be elaborated in the next section. Second, the IPS test has no local power in the neighborhood of unity at the rate of $N^{-1/2}T^{-1}$ but gains local power at order $N^{-1/4}T^{-1}$. This result is obtained using the Fredholm method discussed in Nabeya and Tanaka (1990a, 1990b), and could not be derived within the analytical framework of Harris et al. (2010).

The rest of this paper is organized as follows. Section 2 discusses the local power of the IPS test for the model with both initial conditions and incidental trends. Monte Carlo simulation results are reported in Section 3. The detailed proofs are present in the Appendix.

[1] Moon and Perron (2004) also found that the presence of incidental trend prevents their t ratio type test statistic, which is constructed from ordinary least squares detrending, from obtaining power beyond size in a $N^{-\kappa}T^{-1}$ neighborhood of unity with $\kappa > \frac{1}{6}$.

2 Local Power of the IPS test

Consider the following standard setup for the panel autoregressive model

$$\begin{aligned} z_{it} &= d_{it} + y_{it}, \\ y_{it} &= \rho_i y_{i,t-1} + u_{it}, \\ d_{it} &= \beta_{0i} + \beta_{1i} t, \\ y_{i0} &= \xi_i, \end{aligned} \tag{2.1}$$

where for each i, d_{it} is the time trend with incidental coefficients, y_{it} is a potentially unit root process, and y_{i0} gives the random initial conditions. In panel panel unit root test setting, the hypotheses are given by

$$\begin{aligned} H_0 : \rho_i = 1, &\qquad \text{for all } i, \\ H_1 : \rho_i < 1, &\qquad \text{for } M \text{ of } i, \end{aligned}$$

where M satisfies $\lim_{N\to\infty} M/N = p$, $0 < p \leq 1$.

Following Harris et al. (2010), the following assumptions are made:

Assumption 1 The errors u_{it} are i.i.d. with $(0, \sigma^2_{u,i})$ across $t = 1, \ldots, T$ and also independently across $i = 1, \ldots, N$.

Assumption 2 Set $\xi_i = \gamma_i \sigma_{y,i}$, where γ_i is i.i.d. with $(\mu_\gamma, \sigma^2_\gamma)$ across $i = 1, \ldots, N$ with finite fourth moment, and is independent of $\{u_{it}\}$. Here $\sigma^2_{y,i}$ is the short-run variance of y_{it} for $\rho_i < 1$, hence, $\xi_i = \gamma_i \sqrt{\sigma^2_{u,i}/(1-\rho_i^2)}$.

Assumption 3 Let $\rho_i = 1 + c_i/(N^{1/4}T)$ where $c_i \leq 0$, $i = 1, \ldots, N$.

Assumption 1 and 3 are standard in the panel unit root test literature. The dependence of initial conditions on both N and T in Assumption 2 is to simplify the analysis. It is critical in Harris et al. (2010), but not pivotal in establishing the analytical results of this paper. This is because, in our analysis, as the time trend is estimated and removed, initial conditions will be effectively eliminated.

Our analysis relies on the standard local to unity asymptotics. From the model (2.1), it follows that

$$y_{it} = \rho_i^t \xi_i + \sum_{s=1}^{t} \rho_i^{t-s} u_{is}. \tag{2.2}$$

Following Müller and Elliott (2003), one can readily obtain

$$\begin{aligned} & T^{-1/2}(y_{i,[Tr]} - y_{i,0}) \\ \Rightarrow & \left\{ \begin{array}{ll} \sigma_{u,i} W_i(r) & \text{for } c_i = 0, \\ \gamma_i \sigma_{u,i} \left(e^{rc_i N^{-1/4}} - 1\right) \left(-2c_i N^{-1/4}\right)^{-1/2} + \sigma_{u,i} \int_0^r e^{c_i N^{-1/4}(r-s)} dW_i(s) & \text{else.} \end{array} \right\} \end{aligned} \tag{2.3}$$

as $T \to \infty$. It is clear from (2.2) that y_{it} consists two terms: $\rho_i^t \xi_i$ and $\sum_{s=1}^t \rho_i^{t-s} u_{is}$, where the first term characterizes individual initial condition, and the second one is a stochastic trend or near unit-root process with ρ_i close to unity. By Taylor expansion, y_{it} can be rewritten as

$$\begin{aligned} y_{it} &= \xi_i(1+(\log\rho_i)t+\frac{1}{2}(\log\rho_i)^2t^2)+\sum_{s=1}^{t}(1+(\log\rho_i)(t-s))u_{is}+O(N^{-1/2}\sqrt{t^5}/T^2) \\ &= \xi_i+(\log\rho_i)\xi_i t+\frac{1}{2}(\log\rho_i)^2\xi_i t^2+\sum_{s=1}^{t}u_{is}+(\log\rho_i)\sum_{s=1}^{t}(t-s)u_{is}+O(N^{-1/2}\sqrt{t^5}/T^2) \quad (2.4)\end{aligned}$$

Notice that when one estimates an autoregressive model and removes the time trend, both ξ_i and $(\log\rho_i)\xi_i t$ will be filtered out in the process as evident in (A.5) in Appendix. This causes our result to differ from that in Harris et al. (2010). In Harris et al. (2010), the panel autoregressive with initial conditions is considered but not with time trend, so the local power comes from $((\log\rho_i)\xi_i t)^2$,[2] the cross product term of $\sum_{s=1}^t u_{is}$ and $(\log\rho_i)\sum_{s=1}^t(t-s)u_{is}$, and the squared cross product term of $(\log\rho_i)\xi_i t$ and $\sum_{s=1}^t u_{is}$. However, in our case, $(\log\rho_i)\xi_i t$ is removed, and hence the local power sources exclusively from $((\log\rho_i)\sum_{s=1}^t(t-s)u_{is})^2$ and the cross product term of $\sum_{s=1}^t u_{is}$ and $(\log\rho_i)\sum_{s=1}^t(t-s)u_{is}$. Also, the order of $(\frac{1}{2}(\log\rho_i)^2\xi_i t^2)^2$ is $O(N^{-3/4}(t^4/T^3))$, which is smaller than $(\sum_{s=1}^t u_{is})^2=O(t)$, $((\log\rho_i)\sum_{s=1}^t(t-s)u_{is})^2=O(N^{-1/2}(t^3/T^2))$. Hence, $(\frac{1}{2}(\log\rho_i)^2\xi_i t^2)^2$ is asymptotically negligible.

The IPS test can be carried out in the following two ways. (i) One way is to run the OLS regression directly. For fixed i, one estimates a linear regression of z_{it} on 1, t and $z_{i,t-1}$ as

$$z_{it}=\alpha_i+\delta_i t+\rho_i z_{i,t-1}+u_{it},$$

then one obtains the estimator for the coefficient of $z_{i,t-1}$ as

$$\hat{\rho}_i=\frac{\left(\sum_t(t-\bar{t})^2\right)\left(\sum_t(z_{i,t-1}-\bar{z}_{i,t-1})(z_{it}-\bar{z}_{it})\right)-\left(\sum_t(t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1})\right)\left(\sum_t(t-\bar{t})(z_{it}-\bar{z}_{it})\right)}{\left(\sum_t(t-\bar{t})^2\right)\left(\sum_t(z_{i,t-1}-\bar{z}_{i,t-1})^2\right)-\left(\sum_t(t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1})\right)^2}, \quad (2.5)$$

where all the summations are taken over 2 to T, and

$$\bar{t}=\frac{1}{T-1}\sum_{s=2}^{T}s=\frac{T+2}{2},\quad \bar{z}_{i,t-1}=\frac{1}{T-1}\sum_{s=2}^{T}z_{i,s-1},\quad \bar{z}_{it}=\frac{1}{T-1}\sum_{s=2}^{T}z_{i,s}.$$

Then the t-statistic is given by

$$t_i=\frac{\hat{\rho}_i-1}{\hat{\sigma}_{u,i}\sqrt{\frac{\sum_t(t-\bar{t})^2}{\left(\sum_t(t-\bar{t})^2\right)\left(\sum_t(z_{i,t-1}-\bar{z}_{i,t-1})^2\right)-\left(\sum_t(t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1})\right)^2}}}, \quad (2.6)$$

where $\hat{\sigma}_{u,i}=\sqrt{\frac{1}{T-1}\sum_{t=2}^T(z_{it}-\hat{\alpha}_i-\hat{\delta}_i t-\hat{\rho}_i z_{i,t-1})^2}$ is a consistent estimator for σ_u.

(ii) The other way to test the unit root is to detrend first, i.e. to regress z_{it} on 1, t, then test the estimated residuals for the unit root process. Thus, one obtains

$$\begin{aligned}\hat{\beta}_{0i} &= \bar{z}_{it}-\hat{\beta}_{1i}\bar{t},\\ \hat{\beta}_{1i} &= \frac{\sum_{s=1}^T(s-\bar{s})(z_{is}-\bar{z}_{is})}{\sum_{s=1}^T(s-\bar{s})^2},\\ \hat{\varepsilon}_{it} &= z_{it}-\hat{\beta}_{0i}-\hat{\beta}_{1i}t.\end{aligned}$$

[2]It is also worthy of pointing out that that the local power is actually not from $(\log\rho_i)\xi_i t$ but from its square, which is essentially its variance.

Further, the t-statistic for the autoregressive regression of $\hat{\varepsilon}_{it}$ is constructed as

$$\begin{aligned}\tilde{t}_i &= \frac{\tilde{\rho}_i - 1}{\tilde{\sigma}_{u,i}(\sum_t \hat{\varepsilon}_{i,t-1}^2)^{-1/2}} = \frac{\sum_t \hat{\varepsilon}_{i,t-1}(\hat{\varepsilon}_{i,t} - \hat{\varepsilon}_{i,t-1})}{\tilde{\sigma}_{u,i}\sqrt{\sum_t \hat{\varepsilon}_{i,t-1}^2}} \\ &= \frac{\sum_t[(z_{i,t-1} - \bar{z}_{i,t-1}) - \hat{\beta}_{1i}(t-\bar{t})](z_{it} - \bar{z}_{it} - z_{i,t-1} + \bar{z}_{i,t-1})}{\tilde{\sigma}_{u,i}\sqrt{\sum_t[(z_{i,t-1} - \bar{z}_{i,t-1}) - \hat{\beta}_{1i}(t-\bar{t})]^2}},\end{aligned}$$

where $\tilde{\rho}_i = (\sum_t \hat{\varepsilon}_{i,t-1}^2)^{-1}\sum_t \hat{\varepsilon}_{i,t-1}\hat{\varepsilon}_{i,t}$ is the OLS estimator of the autoregression coefficient of $\hat{\varepsilon}_{it}$ and $\tilde{\sigma}_{u,i} = \sqrt{\frac{1}{T-1}\sum_{t=2}^T(\hat{\varepsilon}_{it} - \tilde{\rho}_i\hat{\varepsilon}_{i,t-1})^2}$. After some algebraic operations, one can see that, with $\hat{\beta}_{1i}$ plugged in, t_i and $\tilde{t}_i$ are essentially the same even though $\hat{\rho}_i$ is different from $\tilde{\rho}_i$. That is, t_i and $\tilde{t}_i$ are same except for $\hat{\sigma}_{u,i}$ and $\tilde{\sigma}_{u,i}$. Therefore, without loss of generality, this paper focuses on the first method.

The IPS test statistic is constructed as the standardized statistic of t-statistic, i.e.

$$Z^\tau = \frac{\sqrt{N}[N^{-1}\sum_{i=1}^N t_i - E(t_0)]}{\sqrt{V(t_0)}},$$

where $E(t_0)$ and $V(t_0)$ are the mean and the variance of the limiting distribution of the Dickey-Full statistic with both intercept and time trend, respectively.

Under H_0, one can readily see that, as $T \to \infty$,

$$t_i \Rightarrow \frac{\int_0^1 W_i^\mu(r)dW_i(r) - 12\int_0^1(r-\frac{1}{2})W_i(r)dr\int_0^1(r-\frac{1}{2})dW_i(r)}{\sqrt{\int_0^1 W_i^\mu(r)^2dr - 12\left(\int_0^1(r-\frac{1}{2})W_i(r)dr\right)^2}} \overset{def}{=} t_0 = \frac{U_3}{\sqrt{V_3}},$$

where $W_i(r)$ is a standard Brownian motion and $W_i^\mu(r) = W_i(r) - \int_0^1 W_i(s)ds$. In addition, $E(t_0) = -2.18135582$ and $\sqrt{V(t_0)} = 0.74990847$ based on the simulations in Nabeya (1999).

From the derivations in the Appendix, one can now obtain the following theorem for the asymptotic local power of the IPS test with both explosive initial conditions and incidental trends.

Theorem 2.1 *Under Assumptions 1-3, when $T \to \infty$ followed by $N \to \infty$,*

$$\begin{aligned}Z^\tau \;\Rightarrow\; & N(0,1) + c^2\Bigg[E\left(\frac{B_4 - 12B_5B_6}{\sqrt{F}}\right) + E\left(\frac{B_{12} - 12B_{10}B_9}{\sqrt{F}}\right) - E\left(\frac{(B_2 - 12B_5B_9)(B_3 - 12B_6^2)}{2\sqrt{F^3}}\right) \\ & -E\left(\frac{(B_2 - 12B_5B_9)(B_8 - 12B_5B_{10})}{\sqrt{F^3}}\right) - E\left(\frac{(B_1 - 12B_6B_9)(B_4 - 12B_5B_6)}{\sqrt{F^3}}\right) \\ & +E\left(\frac{3(B_2 - 12B_5B_9)(B_4 - 12B_5B_6)^2}{2\sqrt{F^5}}\right)\Bigg]/\sqrt{Var(t_0)},\end{aligned}$$

where $c^2 = \lim_{N\to\infty} N^{-1}\sum_{i=1}^N c_i^2$, $F = A - 12B_5^2$,

$$\begin{aligned}A &= \int_0^1 W^\mu(r)^2dr, \\ B_1 &= \int_0^1\left\{\int_0^r W(s)ds - \int_0^1\int_0^t W(s)dsdt\right\}dW(r), \\ B_2 &= \int_0^1 W^\mu(r)dW(r), \\ B_3 &= \int_0^1\left\{\int_0^r W(s)ds - \int_0^1\int_0^t W(s)dsdt\right\}^2dr,\end{aligned}$$

$$
\begin{aligned}
B_4 &= \int_0^1 W^\mu(r)\left\{\int_0^r W(s)ds - \int_0^1\int_0^t W(s)dsdt\right\}dr,\\
B_5 &= \int_0^1 (r-\frac{1}{2})W^\mu(r)dr,\\
B_6 &= \int_0^1 (r-\frac{1}{2})\left\{\int_0^r W(s)ds - \int_0^1\int_0^t W(s)dsdt\right\}dr,\\
B_7 &= \int_0^1 (r^2-\frac{1}{3})W^\mu(r)dr,\\
B_8 &= \int_0^1 W^\mu(r)\left\{\int_0^r (r-s)W(s)ds - \int_0^1\int_0^t (t-s)W(s)dsdt\right\}dr,\\
B_9 &= \int_0^1 (r-\frac{1}{2})dW(r) = \frac{1}{2}W(1) - \int_0^1 W(r)dr,\\
B_{10} &= \int_0^1 (r-\frac{1}{2})\left\{\int_0^r (r-s)W(s)ds - \int_0^1\int_0^t (t-s)W(s)dsdt\right\}dr,\\
B_{11} &= \int_0^1 (r^2-\frac{1}{3})dW(r),\\
B_{12} &= \int_0^1 \left\{\int_0^r (r-s)W(s)ds - \int_0^1\int_0^t (t-s)W(s)dsdt\right\}dW(r),
\end{aligned}
$$

with $W(r)$ a standard Brownian motion and $W^\mu(r) = W(r) - \int_0^1 W(s)ds$.

Remark 1 From (A.6) in Appendix, it is obvious that Z^τ would have local power in the neighbourhood of unity with the order of $N^{-1/2}T^{-1}$ if $E(\sqrt{F}) + E\left(\frac{B_1-12B_6B_9}{\sqrt{F}}\right) - E\left(\frac{(B_2-12B_5B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right) \neq 0$. Indeed, by utilizing the Fredholm method discussed in Nabeya and Tanaka (1990a, 1990b), Lemma A.1(i) shows that $E(\sqrt{F}) + E\left(\frac{B_1-12B_6B_9}{\sqrt{F}}\right) - E\left(\frac{(B_2-12B_5B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right) \equiv 0$, which is confirmed by simulation study in the next section.

Remark 2 The theoretical result further provides the direct calculation for the expectations in the asymptotics. It follows from the derivations in (A.8) in Appendix that

$$
\begin{aligned}
& E\left(\frac{B_4-12B_5B_6}{\sqrt{F}}\right) + E\left(\frac{B_{12}-12B_{10}B_9}{\sqrt{F}}\right) - E\left(\frac{(B_2-12B_5B_9)(B_3-12B_6^2)}{2\sqrt{F^3}}\right)\\
& -E\left(\frac{(B_2-12B_5B_9)(B_8-12B_5B_{10})}{\sqrt{F^3}}\right) - E\left(\frac{(B_1-12B_6B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right)\\
& +E\left(\frac{3(B_2-12B_5B_9)(B_4-12B_5B_6)^2}{2\sqrt{F^5}}\right)\\
= & \frac{1}{\sqrt{2\pi}}\int_0^\infty [3f_{22}(x)]^{-3/2}\Big(-\frac{\sinh(x)}{x^3} + \frac{9\cosh(x)}{x^4} - \frac{33\sinh(x)}{x^5} + \frac{48(\cosh(x)-1)}{x^6}\Big)dx, \quad (2.7)
\end{aligned}
$$

where $f_{22}(x) = 4\left(\frac{1}{x^3}\sinh(x) - \frac{2}{x^4}[\cosh(x)-1]\right)$ from (7) and page 147 in Nabeya (1999). From (2.7), one can directly compute the value of the expectations in the asymptotic local power. Specifically, using approximation scheme similar to that in Nabeya (1999), one can obtain the following.

$$
\begin{aligned}
& f_{22}(x) \approx \frac{1}{3} + \frac{1}{45}x^2 + \frac{1}{1,680}x^4 + \frac{1}{113,400}x^6 + \frac{1}{11,975,040}x^8,\\
& -\frac{\sinh(x)}{x^3} + \frac{9\cosh(x)}{x^4} - \frac{33\sinh(x)}{x^5} + \frac{48(\cosh(x)-1)}{x^6}\\
& \approx -\frac{1}{840}x^2 - \frac{192}{10!}x^4 - \frac{480}{12!}x^6.
\end{aligned}
$$

Plugging the above approximation into (2.7), then performing numerical integration with MATLAB yields

$$\frac{1}{\sqrt{2\pi}}\int_0^{\infty}[3f_{22}(x)]^{-3/2}\Big(-\frac{\sinh(x)}{x^3}+\frac{9\cosh(x)}{x^4}-\frac{33\sinh(x)}{x^5}+\frac{48(\cosh(x)-1)}{x^6}\Big)dx \approx -0.0283. \quad (2.8)$$

An immediate consequence of the calculation of the expected local power is the following corollary.

Corollary 2.2 *Under Assumptions 1-3, when $T \to \infty$ followed by $N \to \infty$,*

$$Z^{\tau} \Rightarrow N(0,1) - 0.0283c^2/\sqrt{Var(t_0)}, \quad (2.9)$$

where $\sqrt{V(t_0)} = 0.74990847$.

Remark 3 Since there is no term associated with γ_i in Theorem 2.1 and Corollary 2.2, the explosive initial condition has little, if any, impact on local asymptotic power, which differs remarkably from finding in Harris et al. (2010). As evident from the derivation, this discrepancy is due solely to the fact that the model considered in this paper assumes the the presence of both initial conditions and time trend and that removal of the trend eliminates initial conditions simultaneously. By contrast, Harris et al. (2010) consider only the explosive initial conditions without considering the existence of time trend in their model, thus the initial condition matters in Harris et al. (2010). Specifically, the asymptotic local power in this study is related to $c^2 = \lim_{N\to\infty} N^{-1}\sum_{i=1}^{N} c_i^2$ instead of $c = \lim_{N\to\infty} N^{-1}\sum_{i=1}^{N} c_i$ because all of terms related with c_i are either canceled out or asymptotically negligible, which can be seen in (A.7) in Appendix. This is also a major obstacle we faced in the proof. The local power in Harris et al. (2010) on the other hand results from the terms associated with c. Furthermore, in the model with time trend, the IPS test has no local power in the neighborhood of unity with order $N^{-1/2}T^{-1}$, but gains local power in the neighborhood of unity with order $N^{-1/4}T^{-1}$. This result agrees with Moon et al. (2007), but is unattainable in the Harris et al. (2010) framework. This hurdle was overcame by resorting to the Fredholm method shown in Lemma A.1(i) and Lemma A.1(ii) in the Appendix.

3 Monte Carlo Simulations

In this section, some simulations are conducted to verify the theoretical results. The first to be verified is a result in Lemma A.1(i):

$$E\left[\sqrt{F}+\frac{B_1-12B_6B_9}{\sqrt{F}}-\frac{(B_2-12B_5B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right]=0.$$

To do this, A, B_1, B_2, B_4, B_5, B_6, B_9 are approximated respectively by

$$\begin{aligned}
A_T &= (T-1)^{-2}\sum_{t=2}^{T}(\sum_{s=2}^{t}u_s-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=2}^{t}u_s)^2,\\
B_{1T} &= (T-1)^{2}\sum_{t=2}^{T}(\sum_{s=1}^{t-1}\sum_{k=1}^{s}u_k-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{k=1}^{s}u_k)(u_t-(T-1)^{-1}\sum_{t=2}^{T}u_t),\\
B_{2T} &= (T-1)^{-1}\sum_{t=2}^{T}(\sum_{s=1}^{t-1}u_s-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}u_s)(u_t-(T-1)^{-1}\sum_{t=2}^{T}u_t),
\end{aligned}$$

$$
\begin{aligned}
B_{4T} &= (T-1)^{-3}\sum_{t=2}^{T}(\sum_{s=1}^{t-1}u_s-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}u_s)(\sum_{s=1}^{t-1}\sum_{k=1}^{s}u_k-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{k=1}^{s}u_k),\\
B_{5T} &= (T-1)^{-5/2}\sum_{t=2}^{T}(t-(T-1)^{-1}\sum_{t=2}^{T}t)(\sum_{s=1}^{t-1}u_s-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}u_s),\\
B_{6T} &= (T-1)^{-7/2}\sum_{t=2}^{T}(t-(T-1)^{-1}\sum_{t=2}^{T}t)(\sum_{s=1}^{t-1}\sum_{k=1}^{s}u_k-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{k=1}^{s}u_k),\\
B_{9T} &= (T-1)^{-3/2}\sum_{t=2}^{T}(t-(T-1)^{-1}\sum_{t=2}^{T}t)(u_t-T^{-1}\sum_{t=2}^{T}u_t), \qquad (3.1)
\end{aligned}
$$

where u_t follows i.i.d. $N(0,1)$. Note that N times of replication are carried out and for each replication i, A_T, B_{1T}, B_{2T}, B_{4T}, B_{5T}, B_{6T}, B_{9T} are denoted by A_{iT}, B_{1iT}, B_{2iT}, B_{4iT}, B_{5iT}, B_{6iT}, B_{9iT} respectively. The sample averages

$$
\frac{1}{N}\sum_{i=1}^{N}\left[\sqrt{A_{iT}-12B_{5it}^2}+\frac{B_{1iT}-12B_{6iT}B_{9iT}}{\sqrt{A_{iT}-12B_{5iT}^2}}-\frac{(B_{2iT}-12B_{5iT}B_{9iT})(B_{4iT}-12B_{5iT}B_{6iT})}{\sqrt{(A_{iT}-12B_{5iT}^2)^3}}\right] \qquad (3.2)
$$

are reported in Table 1 for different $T = 50$, 100, 250, 5000 with the number of replications N being 50,000. Clearly, as T increases, the sample average tends to 0.

Table 1: Sample average in (3.2)

	T=50	T=100	T=250	T=5000
N=50,000	0.0213	0.0100	0.0042	0.0005

The next to be simulated is the asymptotic local power

$$
\begin{aligned}
& E\left(\frac{B_4-12B_5B_6}{\sqrt{F}}\right)+E\left(\frac{B_{12}-12B_{10}B_9}{\sqrt{F}}\right)-E\left(\frac{(B_2-12B_5B_9)(B_3-12B_6^2)}{2\sqrt{F^3}}\right)\\
& -E\left(\frac{(B_2-12B_5B_9)(B_8-12B_5B_{10})}{\sqrt{F^3}}\right)-E\left(\frac{(B_1-12B_6B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right)\\
& +E\left(\frac{3(B_2-12B_5B_9)(B_4-12B_5B_6)^2}{2\sqrt{F^5}}\right),
\end{aligned}
$$

in Theorem 2.1. In addition to (3.1), B_3, B_8, B_{10}, B_{12}, F are approximated by

$$
\begin{aligned}
B_{3T} &= (T-1)^{-4}\sum_{t=2}^{T}(\sum_{s=1}^{t-1}\sum_{k=1}^{s}u_k-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{k=1}^{s}u_k)^2,\\
B_{8T} &= (T-1)^{-4}\sum_{t=2}^{T}(\sum_{s=1}^{t-1}u_s-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}u_s)(\sum_{s=1}^{t-1}\sum_{k=1}^{s}(t-s)u_k-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{k=1}^{s}(t-s)u_k),\\
B_{10T} &= (T-1)^{-9/2}\sum_{t=2}^{T}(t-(T-1)^{-1}\sum_{t=2}^{T}t)(\sum_{s=1}^{t-1}\sum_{k=1}^{s}(t-s)u_k-(T-1)^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{k=1}^{s}(t-s)u_k),
\end{aligned}
$$

$$
\begin{aligned}
B_{12T} &= (T-1)^3 \sum_{t=2}^{T} (\sum_{s=1}^{t-1} \sum_{k=1}^{s} (t-s)u_k - (T-1)^{-1} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \sum_{k=1}^{s} (t-s)u_k)(u_t - (T-1)^{-1} \sum_{t=2}^{T} u_t), \\
F_T &= A_T - 12B_{5T}^2.
\end{aligned}
$$

Again, N times of replication in simulation are carried out, and for each replication i, B_{1T}, B_{2T}, B_{3T}, B_{4T}, B_{5T}, B_{6T}, B_{8T}, B_{9T}, B_{10T}, B_{12T}, F_T are denoted respectively by B_{1iT}, B_{2iT}, B_{3iT}, B_{4iT}, B_{5iT}, B_{6iT}, B_{8iT}, B_{9iT}, B_{10iT}, B_{12iT}, F_{iT}. The sample averages

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\Bigg[\frac{B_{4iT}-12B_{5iT}B_{6iT}}{\sqrt{F_{iT}}}+\frac{B_{12iT}-12B_{10iT}B_{9iT}}{\sqrt{F_{iT}}}-\frac{(B_{2iT}-12B_{5iT}B_{9iT})(B_{3iT}-12B_{6iT}^2)}{2\sqrt{F_{iT}^3}} \\
&-\frac{(B_{2iT}-12B_{5iT}B_{9iT})(B_{8iT}-12B_{5iT}B_{10iT})}{\sqrt{F_{iT}^3}}-\frac{(B_{1iT}-12B_{6iT}B_{9iT})(B_{4iT}-12B_{5iT}B_{6iT})}{\sqrt{F_{iT}^3}} \\
&+\frac{3(B_{2iT}-12B_{5iT}B_{9iT})(B_{4iT}-12B_{5iT}B_{6iT})^2}{2\sqrt{F_{iT}^5}}\Bigg] \qquad (3.3)
\end{aligned}
$$

are summarized in Table 2 for different $T = 50,\ 100,\ 250,\ 5000$ with the number of replications N being 50,000. As evident in Table 2, the simulated values get closer to the value calculated by the Fredholm method in (2.8) as T increases.

Table 2: Sample average in (3.3)

	T=50	T=100	T=250	T=5000
N=50,000	-0.0242	-0.0266	-0.0277	-0.0278

Finally, we want to validate the findings from our analytical derivation by simulations. More specifically, the aim is to verify the following three results: (i) The initial condition does not have any impact on the asymptotic local power when we consider the IPS test with time trend. (ii) The IPS test with time trend dose not have local power in the neighborhood of unity with the order of $N^{-1/2}T^{-1}$ but attains local power in the neighborhood of unity with the order of $N^{-1/4}T^{-1}$. (iii) The local power for the IPS test increases with c^2. In addition, the theoretical asymptotic local powers based on Corollary 2.2 are computed, and the results are presented in Table 5. To achieve these aims, the following data generating processes similar to that in Moon et al. (2007) are adopted.

$$
\begin{aligned}
z_{it} &= b_{0i} + b_{1i}t + y_{it}, \\
y_{it} &= \left(1 - \frac{c_i}{n^{\alpha}T}\right) y_{i,t-1} + \sigma_i e_{it}, \\
y_{i,0} &= \xi_i = \gamma_i \sqrt{\sigma_i^2/(1-\rho_i^2)},\ b_{0i}, b_{1i}, e_{it} \sim iid\ N(0,1), \gamma_i \sim (\mu_\gamma, \sigma_\gamma^2), \\
\sigma_i^2 &\sim U[0.5, 1.5].
\end{aligned}
$$

Various different cases are being considered where c_i follows different distributions, i.e., (1) $c_i \sim iid\ U[0,1]$; (2) $c_i \sim iid\ U[0,8]$; (3) $c_i \sim iid\ \chi^2(1)$; (4) $c_i \sim iid\ \chi^2(6)$; (7) $c_i = 1$, (8) $c_i = 8$. N and T

are selected from $\{10, 25, 100\}$ and $\{50, 100, 250\}$ respectively. The results at 5% significance level are reported in Table 3 and Table 4 with 2,000 replications. Also, we set $\alpha = 1/2$ in Table 3 and $\alpha = 1/4$ in Table 4.

Table 3: Power of IPS test in the neighborhood of $N^{-1/2}T^{-1}$

		$\gamma_i = 0$			$\gamma_i \sim U[0.5, 1.5]$		
		T=50	T=100	T=250	T=50	T=100	T=250
N=10	$c_i \sim iid\ U[0,1]$	0.0535	0.0525	0.0530	0.0535	0.0530	0.0525
	$c_i \sim iid\ U[0,8]$	0.0785	0.0745	0.0715	0.0790	0.0750	0.0730
	$c_i \sim iid\ \chi^2(1)$	0.0690	0.0505	0.0530	0.0710	0.0505	0.0545
	$c_i \sim iid\ \chi^2(6)$	0.0880	0.1030	0.0870	0.0815	0.1075	0.0820
	$c_i = 1$	0.0585	0.0595	0.0510	0.0580	0.0600	0.0505
	$c_i = 8$	0.1265	0.1140	0.1040	0.1150	0.1065	0.1035
N=25	$c_i \sim iid\ U[0,1]$	0.0545	0.0540	0.0460	0.0540	0.0535	0.0460
	$c_i \sim iid\ U[0,8]$	0.0670	0.0555	0.0530	0.0660	0.0545	0.0510
	$c_i \sim iid\ \chi^2(1)$	0.0565	0.0530	0.0505	0.0565	0.0520	0.0505
	$c_i \sim iid\ \chi^2(6)$	0.0810	0.0710	0.0705	0.0745	0.0710	0.0690
	$c_i = 1$	0.0590	0.0570	0.0460	0.0575	0.0570	0.0465
	$c_i = 8$	0.0880	0.0905	0.0850	0.0825	0.0880	0.0780
N=100	$c_i \sim iid\ U[0,1]$	0.0645	0.0580	0.0395	0.0655	0.0580	0.0395
	$c_i \sim iid\ U[0,8]$	0.0565	0.0640	0.0620	0.0565	0.0670	0.0615
	$c_i \sim iid\ \chi^2(1)$	0.0505	0.0565	0.0500	0.0490	0.0565	0.0515
	$c_i \sim iid\ \chi^2(6)$	0.0625	0.0630	0.0710	0.0640	0.0645	0.0680
	$c_i = 1$	0.0495	0.0580	0.0540	0.0490	0.0580	0.0540
	$c_i = 8$	0.0715	0.0710	0.0825	0.0715	0.0670	0.0795

Clearly, the simulation results support our theoretical findings. Firstly, by comparing the cases where $\gamma_i = 0$ and $\gamma_i \sim U[0.5, 1.5]$ within each table, there is not much difference in the local power. This agrees with our conclusion that the effect on local power from the initial condition is eliminated effectively when trends are present and removed . Secondly, Table 3 and Table 4 signal that IPS test with time trend has no local power in the neighborhood of unity with the order of $N^{-1/2}T^{-1}$ but has nontrivial local power in the neighborhood of unity with the order of $N^{-1/4}T^{-1}$, even though when c^2 is small, the difference is not substantial. Table 3 indicates that local power falls as either N or T increases for the case of $N^{-1/2}T^{-1}$. By contrast, Table 4, the local power increases with either N or T for the case of $N^{-1/4}T^{-1}$. Finally, Table 4 suggests that the local power increases substantially when c^2 is getting larger. For example, when c_i increases from 1 to 8, i.e., c^2 increases from 1 to 64, the local power jumps to 0.4380 which is almost 9 times larger than 0.0565 that is corresponding to $c^2 = 1$ when $N = 100$ and $T = 250$. When we compare $c_i \sim iid\ \chi^2(1)$ and $c_i \sim iid\ \chi^2(6)$, c^2 increases from 3 to 48, the local power increases from 0.0620 to 0.2590 when $N = 100$ and $T = 100$. Also, the values in Table 4 are consistent with the theoretical values in Table 5, even though the simulated values are slightly smaller than the theoretical values, a phenomenon also observed in Table 2 in Moon et al. (2007).

Table 4: Power of IPS test in the neighborhood of $N^{-1/4}T^{-1}$

		$\gamma_i = 0$			$\gamma_i \sim U[0.5, 1.5]$		
		T=50	T=100	T=250	T=50	T=100	T=250
N=10	$c_i \sim iid\ U[0,1]$	0.0580	0.0540	0.0455	0.0595	0.0525	0.0465
	$c_i \sim iid\ U[0,8]$	0.0975	0.1125	0.1250	0.0975	0.1150	0.1170
	$c_i \sim iid\ \chi^2(1)$	0.0690	0.0610	0.0645	0.0690	0.0610	0.0610
	$c_i \sim iid\ \chi^2(6)$	0.2350	0.1240	0.2170	0.2265	0.1190	0.2120
	$c_i = 1$	0.0700	0.0565	0.0470	0.0700	0.0555	0.0470
	$c_i = 8$	0.2800	0.2645	0.2615	0.2775	0.2670	0.2665
N=25	$c_i \sim iid\ U[0,1]$	0.0480	0.0500	0.0490	0.0480	0.0475	0.0490
	$c_i \sim iid\ U[0,8]$	0.1210	0.1220	0.1280	0.1205	0.1150	0.1155
	$c_i \sim iid\ \chi^2(1)$	0.0615	0.0560	0.0590	0.0590	0.0545	0.0585
	$c_i \sim iid\ \chi^2(6)$	0.2665	0.2060	0.1685	0.2535	0.1995	0.1520
	$c_i = 1$	0.0590	0.0645	0.0510	0.0590	0.0645	0.0505
	$c_i = 8$	0.3725	0.3710	0.3725	0.3620	0.3455	0.3465
N=100	$c_i \sim iid\ U[0,1]$	0.0500	0.0565	0.0555	0.0505	0.0565	0.0545
	$c_i \sim iid\ U[0,8]$	0.1270	0.1340	0.1505	0.1155	0.1320	0.1380
	$c_i \sim iid\ \chi^2(1)$	0.0560	0.0620	0.0580	0.0555	0.0625	0.0580
	$c_i \sim iid\ \chi^2(6)$	0.2920	0.2590	0.2390	0.2695	0.2395	0.2240
	$c_i = 1$	0.0440	0.0640	0.0565	0.0450	0.0630	0.0595
	$c_i = 8$	0.4205	0.4290	0.4380	0.3925	0.3855	0.4035

Table 5: Theoretical asymptotic local power of IPS test in the neighborhood of $N^{-1/4}T^{-1}$

$c_i \sim iid$	$U[0,1]$	$U[0,8]$	$\chi^2(1)$	$\chi^2(6)$	$c_i = 1$	$c_i = 8$
Theoretical values	0.0513	0.2005	0.0628	0.5661	0.0540	0.7794

4 Conclusion

This paper derives the analytical asymptotic local power of the IPS test when both the initial conditions and incidental trends are present in the panel data. An important empirical consequence of the present investigation is that, in the presence of incidental time trends, initial conditions no longer have any nonnegligible impact on the asymptotic local power in IPS test as least squares detrending effectively eliminates the initial conditions. Consistent with the findings in Moon et al. (2007), the IPS test has no asymptotic local power in the neighborhood of unity with the order $N^{-1/2}T^{-1}$ but gains nontrivial local power in the neighborhood of unity that shrinks at the rate $N^{-1/4}T^{-1}$ when incidental trends are fitted. Since no analytical power result is currently available for the IPS test, these results obtained by utilizing the Fredholm method proposed in Nabeya and Tanaka (1990a, 1990b), fill this void and complement those in Moon et al. (2007).

Appendix

From the model (2.1), we have that

$$z_{it} = \beta_{0i} + \beta_{1i}t + y_{it} = \beta_{0i} + \beta_{1i}t + \rho_i y_{i,t-1} + u_{it}. \tag{A.1}$$

Then,

$$z_{i,t-1} = \beta_{0i} + \beta_{1i}(t-1) + y_{i,t-1}, \tag{A.2}$$

$$y_{i,t-1} = z_{i,t-1} - \beta_{0i} - \beta_{1i}(t-1). \tag{A.3}$$

By plugging (A.3) into (A.1), we have

$$\begin{aligned} z_{i,t} &= \beta_{0i} + \beta_{1i}t + \rho_i(z_{i,t-1} - \beta_{0i} - \beta_{1i}(t-1)) + u_{it} \\ &= (1-\rho_i)\beta_{0i} + \beta_{1i}\rho_i + (1-\rho_i)\beta_{1i}t + \rho_i z_{i,t-1} + u_{it} \\ &\stackrel{def}{=} \alpha_i + \delta_i t + \rho_i z_{i,t-1} + u_{it}, \end{aligned}$$

and from (A.2), we have

$$z_{i,t-1} - \bar{z}_{i,t-1} = \beta_{1i}(t-\bar{t}) + y_{i,t-1} - \bar{y}_{i,t-1}. \tag{A.4}$$

Also, by Assumption 1 and the standard Law of Large Number, we have

$$\hat{\sigma}_{u,i} = \sqrt{\frac{1}{T-1}\sum_{t=2}^{T}(z_{it} - \hat{\alpha}_i - \hat{\delta}_i t - \hat{\rho}_i z_{i,t-1})^2} \xrightarrow{p} \sigma_u.$$

Then, by plugging (2.5) and (A.4) into (2.6) and applying (2.3), we have that as $T \to \infty$

$$\begin{aligned} t_i &= \frac{\hat{\rho}_i - 1}{\hat{\sigma}_{u,i}\sqrt{\frac{\sum_t (t-\bar{t})^2}{(\sum_t (t-\bar{t})^2)(\sum_t (z_{i,t-1}-\bar{z}_{i,t-1})^2) - (\sum_t (t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1}))^2}}} \\ &= \frac{1}{\hat{\sigma}_{u,i}}\Bigg((\rho_i - 1)\sqrt{\left(\sum_t (z_{i,t-1}-\bar{z}_{i,t-1})^2\right) - \frac{(\sum_t (t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1}))^2}{\sum_t (t-\bar{t})^2}} \\ &\quad + \frac{(\sum_t (t-\bar{t})^2)\left(\sum_t (z_{i,t-1}-\bar{z}_{i,t-1})((1-\rho_i)\beta_{1,i}(t-\bar{t}) + (u_{it}-\bar{u}_{it}))\right)}{\sqrt{\sum_t (t-\bar{t})^2}\sqrt{(\sum_t (t-\bar{t})^2)(\sum_t (z_{i,t-1}-\bar{z}_{i,t-1})^2) - (\sum_t (t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1}))^2}} \\ &\quad - \frac{(\sum_t (t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1}))\left(\sum_t (t-\bar{t})((1-\rho_i)\beta_{1,i}(t-\bar{t}) + (u_{it}-\bar{u}_{it}))\right)}{\sqrt{\sum_t (t-\bar{t})^2}\sqrt{(\sum_t (t-\bar{t})^2)(\sum_t (z_{i,t-1}-\bar{z}_{i,t-1})^2) - (\sum_t (t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1}))^2}}\Bigg) \\ &= \frac{1}{\hat{\sigma}_{u,i}}\Bigg(\frac{c_i}{TN^{1/4}}\sqrt{\left(\sum_t (z_{i,t-1}-\bar{z}_{i,t-1})^2\right) - \frac{(\sum_t (t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1}))^2}{\sum_t (t-\bar{t})^2}} \\ &\quad + \frac{(\sum_t (t-\bar{t})^2)(\sum_t (z_{i,t-1}-\bar{z}_{i,t-1})(u_{it}-\bar{u}_{it})) - (\sum_t (t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1}))(\sum_t (t-\bar{t})(u_{it}-\bar{u}_{it}))}{\sqrt{\sum_t (t-\bar{t})^2}\sqrt{(\sum_t (t-\bar{t})^2)(\sum_t (z_{i,t-1}-\bar{z}_{i,t-1})^2) - (\sum_t (t-\bar{t})(z_{i,t-1}-\bar{z}_{i,t-1}))^2}}\Bigg) \\ &= \frac{1}{\hat{\sigma}_{u,i}}\Bigg(\frac{c_i}{TN^{1/4}}\sqrt{\left(\sum_t (y_{i,t-1}-\bar{y}_{i,t-1})^2\right) - \frac{(\sum_t (t-\bar{t})(y_{i,t-1}-\bar{y}_{i,t-1}))^2}{\sum_t (t-\bar{t})^2}} \\ &\quad + \frac{(\sum_t (t-\bar{t})^2)(\sum_t (y_{i,t-1}-\bar{y}_{i,t-1})(u_{it}-\bar{u}_{it})) - (\sum_t (t-\bar{t})(y_{i,t-1}-\bar{y}_{i,t-1}))(\sum_t (t-\bar{t})(u_{it}-\bar{u}_{it}))}{\sqrt{\sum_t (t-\bar{t})^2}\sqrt{(\sum_t (t-\bar{t})^2)(\sum_t (y_{i,t-1}-\bar{y}_{i,t-1})^2) - (\sum_t (t-\bar{t})(y_{i,t-1}-\bar{y}_{i,t-1}))^2}}\Bigg) \end{aligned}$$

$$\Rightarrow \quad \frac{c_i}{N^{1/4}}\sqrt{\int_0^1 K^{\mu}_{i,c_i}(r)^2 dr - 12\left(\int_0^1 (r-\frac{1}{2})K^{\mu}_{i,c_i}(r)dr\right)^2}$$
$$+\frac{\int_0^1 K^{\mu}_{i,c_i}(r)dW_i(r) - 12\int_0^1 (r-\frac{1}{2})K^{\mu}_{i,c_i}(r)dr\int_0^1 (r-\frac{1}{2})dW_i(r)}{\sqrt{\int_0^1 K^{\mu}_{i,c_i}(r)^2 dr - 12\left(\int_0^1 (r-\frac{1}{2})K^{\mu}_{i,c_i}(r)dr\right)^2}},$$

where $K^{\mu}_{i,c_i}(r) = K_{i,c_i}(r) - \int_0^1 K_{i,c_i}(s)ds$, and

$$\begin{aligned} K_{i,c_i}(r) &= \gamma_i\left(e^{rc_iN^{-1/4}} - 1\right)\left(-2c_iN^{-1/4}\right)^{-1/2} + \int_0^r e^{c_iN^{-1/4}(r-s)}dW_i(s) \\ &= \gamma_i\left(e^{rc_iN^{-1/4}} - 1\right)\left(-2c_iN^{-1/4}\right)^{-1/2} + W_i(r) + c_iN^{-1/4}\int_0^r e^{c_iN^{-1/4}(r-s)}W_i(s)ds. \end{aligned}$$

Also, we have

$$\sum_{t=2}^{T}(t-\bar{t})^2 = \frac{1}{12}(T-1)((T-1)^2-1) = \frac{1}{12}T^3 - \frac{1}{4}T^2 + \frac{1}{6}T.$$

Similar as that in the Appendix of Harris et al. (2010), we have that for any x

$$e^{xc_iN^{-1/4}} = 1 + xc_iN^{-1/4} + \frac{1}{2}x^2c_i^2N^{-1/2} + O(N^{-3/4}),$$

and

$$\begin{aligned} K_{i,c_i}(r) &= \gamma_i rc_i(-2c_i)^{-1/2}N^{-1/8} + \frac{1}{2}\gamma_i r^2c_i^2(-2c_i)^{-1/2}N^{-3/8} + W_i(r) \\ &\quad + c_iN^{-1/4}\int_0^r W_i(s)ds + c_i^2N^{-1/2}\int_0^r (r-s)W_i(s)ds + O_p(N^{-5/8}), \\ K^{\mu}_{i,c_i}(r) &= \gamma_i(r-\frac{1}{2})c_i(-2c_i)^{-1/2}N^{-1/8} + \frac{1}{2}\gamma_i(r^2-\frac{1}{3})c_i^2(-2c_i)^{-1/2}N^{-3/8} + W_i^{\mu}(r) \\ &\quad + c_iN^{-1/4}\left\{\int_0^r W_i(s)ds - \int_0^1\int_0^t W_i(s)dsdt\right\} \\ &\quad + c_i^2N^{-1/2}\left\{\int_0^r (r-s)W_i(s)ds - \int_0^1\int_0^t (t-s)W_i(s)dsdt\right\} + O_p(N^{-5/8}). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_0^1 (r-\frac{1}{2})K^{\mu}_{i,c_i}(r)dr \\ =\ &\int_0^1 \gamma_i(r-\frac{1}{2})^2c_i(-2c_i)^{-1/2}N^{-1/8}dr + \frac{1}{2}\int_0^1 \gamma_i(r-\frac{1}{2})(r^2-\frac{1}{3})c_i^2(-2c_i)^{-1/2}N^{-3/8}dr \\ &+\int_0^1 (r-\frac{1}{2})W_i^{\mu}(r)dr + c_iN^{-1/4}\int_0^1 (r-\frac{1}{2})\left\{\int_0^r W_i(s)ds - \int_0^1\int_0^t W_i(s)dsdt\right\}dr \\ &+\int_0^1 (r-\frac{1}{2})c_i^2N^{-1/2}\left\{\int_0^r (r-s)W_i(s)ds - \int_0^1\int_0^t (t-s)W_i(s)dsdt\right\}dr + O_p(N^{-5/8}) \\ =\ &\frac{1}{12}\gamma_i c_i(-2c_i)^{-1/2}N^{-1/8} + \frac{1}{24}\gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8} \\ &+\int_0^1 (r-\frac{1}{2})W_i^{\mu}(r)dr + c_iN^{-1/4}\int_0^1 (r-\frac{1}{2})\left\{\int_0^r W_i(s)ds - \int_0^1\int_0^t W_i(s)dsdt\right\}dr \\ &+c_i^2N^{-1/2}\int_0^1 (r-\frac{1}{2})\left\{\int_0^r (r-s)W_i(s)ds - \int_0^1\int_0^t (t-s)W_i(s)dsdt\right\}dr + O_p(N^{-5/8}), \end{aligned}$$

and

$$
\begin{aligned}
& \int_0^1 K_{i,c_i}^{\mu}(r)^2 dr - 12\left(\int_0^1 (r-\frac{1}{2})K_{i,c_i}^{\mu}(r)dr\right)^2 \\
= \; & A_i + \gamma_i^2 c_i^2(-2c_i)^{-1}N^{-1/4}/12 + c_i^2 N^{-1/2}B_{3i} + \gamma_i^2 c_i^3(-2c_i)^{-1}N^{-1/2}/12 + 2\gamma_i c_i(-2c_i)^{-1/2}N^{-1/8}B_{5i} \\
& +2\gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}B_{6i} + \gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}B_{7i} + 2c_i N^{-1/4}B_{4i} + 2c_i^2 N^{-1/2}B_{8i} \\
& -12B_{5i}^2 - \gamma_i^2 c_i^2(-2c_i)^{-1}N^{-1/4}/12 - 12c_i^2 N^{-1/2}B_{6i}^2 - \gamma_i^2 c_i^3(-2c_i)^{-1}N^{-1/2}/12 - 2\gamma_i c_i(-2c_i)^{-1/2}N^{-1/8}B_{5i} \\
& -2\gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}B_{6i} - \gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}B_{5i} - 24B_{5i}c_i N^{-1/4}B_{6i} - 24B_{5i}c_i^2 N^{-1/2}B_{10i} \\
& +O_p(N^{-5/8}) \\
= \; & A_i - 12B_{5i}^2 + 2c_i N^{-1/4}(B_{4i} - 12B_{5i}B_{6i}) + \gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}(B_{7i} - B_{5i}) + c_i^2 N^{-1/2}(B_{3i} - 12B_{6i}^2) \\
& +2c_i^2 N^{-1/2}(B_{8i} - 12B_{5i}B_{10i}) + O_p(N^{-5/8}). \qquad \text{(A.5)}
\end{aligned}
$$

By plugging the previous results into t_i, we have that

$$
\begin{aligned}
& t_i \\
\Rightarrow \; & c_i N^{-1/4}\sqrt{A_i - 12B_{5i}^2 + 2c_i N^{-1/4}(B_{4i} - 12B_{5i}B_{6i}) + O_p(N^{-3/8})} \\
& +\Big(B_{2i} + (1/2)\gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}B_{11i} + c_i N^{-1/4}B_{1i} + c_i^2 N^{-1/2}B_{12i} \\
& -(1/2)\gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}B_{9i} - 12B_{5i}B_{9i} - 12c_i N^{-1/4}B_{6i}B_{9i} - 12c_i^2 N^{-1/2}B_{10i}B_{9i} + O_p(N^{-5/8})\Big)/ \\
& \Big(A_i - 12B_{5i}^2 + 2c_i N^{-1/4}(B_{4i} - 12B_{5i}B_{6i}) + \gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}(B_{7i} - B_{5i}) + c_i^2 N^{-1/2}(B_{3i} - 12B_{6i}^2) \\
& +2c_i^2 N^{-1/2}(B_{8i} - 12B_{5i}B_{10i}) + O_p(N^{-5/8})\Big)^{1/2} \\
= \; & c_i N^{-1/4}\sqrt{A_i - 12B_{5i}^2 + 2c_i N^{-1/4}(B_{4i} - 12B_{5i}B_{6i}) + O_p(N^{-3/8})} \\
& +\Big(B_{2i} - 12B_{5i}B_{9i} + c_i N^{-1/4}(B_{1i} - 12B_{6i}B_{9i}) + (1/2)\gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}(B_{11i} - B_{9i}) \\
& +c_i^2 N^{-1/2}(B_{12i} - 12B_{10i}B_{9i}) + O_p(N^{-5/8})\Big)/\Big(A_i - 12B_{5i}^2 + 2c_i N^{-1/4}(B_{4i} - 12B_{5i}B_{6i}) \\
& +\gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}(B_{7i} - B_{5i}) + c_i^2 N^{-1/2}(B_{3i} - 12B_{6i}^2) + 2c_i^2 N^{-1/2}(B_{8i} - 12B_{5i}B_{10i}) + O_p(N^{-5/8})\Big)^{1/2},
\end{aligned}
$$

where

$$
\begin{aligned}
A_i &= \int_0^1 W_i^{\mu}(r)^2 dr, \\
B_{1i} &= \int_0^1 \left\{\int_0^r W_i(s)ds - \int_0^1\int_0^t W_i(s)dsdt\right\} dW_i(r), \\
B_{2i} &= \int_0^1 W_i^{\mu}(r)dW_i(r), \\
B_{3i} &= \int_0^1 \left\{\int_0^r W_i(s)ds - \int_0^1\int_0^t W_i(s)dsdt\right\}^2 dr, \\
B_{4i} &= \int_0^1 W_i^{\mu}(r)\left\{\int_0^r W_i(s)ds - \int_0^1\int_0^t W_i(s)dsdt\right\} dr, \\
B_{5i} &= \int_0^1 (r-\frac{1}{2})W_i^{\mu}(r)dr,
\end{aligned}
$$

$$
\begin{aligned}
B_{6i} &= \int_0^1 (r-\frac{1}{2})\left\{\int_0^r W_i(s)ds - \int_0^1\int_0^t W_i(s)dsdt\right\}dr,\\
B_{7i} &= \int_0^1 (r^2-\frac{1}{3})W_i^{\mu}(r)dr,\\
B_{8i} &= \int_0^1 W_i^{\mu}(r)\left\{\int_0^r (r-s)W_i(s)ds - \int_0^1\int_0^t (t-s)W_i(s)dsdt\right\}dr,\\
B_{9i} &= \int_0^1 (r-\frac{1}{2})dW_i(r) = \frac{1}{2}W_i(1) - \int_0^1 W_i(r)dr,\\
B_{10i} &= \int_0^1 (r-\frac{1}{2})\left\{\int_0^r (r-s)W_i(s)ds - \int_0^1\int_0^t (t-s)W_i(s)dsdt\right\}dr,\\
B_{11i} &= \int_0^1 (r^2-\frac{1}{3})dW_i(r),\\
B_{12i} &= \int_0^1 \left\{\int_0^r (r-s)W_i(s)ds - \int_0^1\int_0^t (t-s)W_i(s)dsdt\right\}dW_i(r),
\end{aligned}
$$

and $W_i^{\mu}(r) = W_i(r) - \int_0^1 W_i(s)ds$. Clearly, we can see the terms involving γ_i^2 are canceled out. Recall that γ_i is from the explosive initial condition. This means the initial condition effect is eliminated by the least squares detrending. Even though, there is one term with γ_i left, we can show that this term is asymptotically negligible in Lemma A.1(ii).

Let

$$
\begin{aligned}
F_i &= A_i - 12B_{5i}^2,\\
G_i &= 2c_iN^{-1/4}(B_{4i}-12B_{5i}B_{6i}) + \gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}(B_{7i}-B_{5i})\\
&\quad + c_i^2N^{-1/2}(B_{3i}-12B_{6i}^2) + 2c_i^2N^{-1/2}(B_{8i}-12B_{5i}B_{10i}) + O_p(N^{-5/8}).
\end{aligned}
$$

We have that

$$
\begin{aligned}
&(F_i+G_i)^{-1/2}\\
=\ & \frac{1}{\sqrt{F}_i} - \frac{G_i}{2\sqrt{F_i^3}} + \frac{3G_i^2}{8\sqrt{F_i^5}} + O_p(N^{-3/4})\\
=\ & \frac{1}{\sqrt{F}_i} - \frac{2c_iN^{-1/4}(B_{4i}-12B_{5i}B_{6i}) + \gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}(B_{7i}-B_{5i})}{2\sqrt{F_i^3}}\\
& - \frac{c_i^2N^{-1/2}(B_{3i}-12B_{6i}^2) + 2c_iN^{-1/2}(B_{8i}-12B_{5i}B_{10i})}{2\sqrt{F_i^3}} + \frac{3c_i^2N^{-1/2}(B_{4i}-12B_{5i}B_{6i})^2}{2\sqrt{F_i^5}} + O_p(N^{-5/8}),
\end{aligned}
$$

and $\sqrt{F_i+G_i} = \sqrt{F_i} + c_iN^{-1/4}\frac{(B_{4i}-12B_{5i}B_{6i})}{\sqrt{F_i}} + O_p(N^{-3/8})$.

Hence, by further expansion we have that

$$
\begin{aligned}
t_i \Rightarrow\ & \frac{B_{2i}-12B_{5i}B_{9i}}{\sqrt{F_i}} + c_iN^{-1/4}\sqrt{F_i} + c_iN^{-1/4}\frac{B_{1i}-12B_{6i}B_{9i}}{\sqrt{F_i}} - c_iN^{-1/4}\frac{(B_{2i}-12B_{5i}B_{9i})(B_{4i}-12B_{5i}B_{6i})}{\sqrt{F_i^3}}\\
& +(1/2)\gamma_i c_i^2(-2c_i)^{-1/2}N^{-3/8}\left(\frac{(B_{11i}-B_{9i})}{\sqrt{F_i}} - \frac{(B_{2i}-12B_{5i}B_{9i})(B_{7i}-B_{5i})}{\sqrt{F_i^3}}\right)\\
& +c_i^2N^{-1/2}\frac{B_{4i}-12B_{5i}B_{6i}}{\sqrt{F_i}} + c_i^2N^{-1/2}\frac{B_{12i}-12B_{10i}B_{9i}}{\sqrt{F_i}}
\end{aligned}
$$

$$-c_i^2 N^{-1/2}\frac{(B_{2i}-12B_{5i}B_{9i})(B_{3i}-12B_{6i}^2)+2(B_{2i}-12B_{5i}B_{9i})(B_{8i}-12B_{5i}B_{10i})}{2\sqrt{F_i^3}}$$

$$-c_i^2 N^{-1/2}\frac{(B_{1i}-12B_{6i}B_{9i})(B_{4i}-12B_{5i}B_{6i})}{\sqrt{F_i^3}}+c_i^2 N^{-1/2}\frac{3(B_{2i}-12B_{5i}B_{9i})(B_{4i}-12B_{5i}B_{6i})^2}{2\sqrt{F_i^5}}$$

$$+O_p(N^{-5/8}). \tag{A.6}$$

From Lemma A.1, the standard CLT and LLN, we have

$$Z^\tau = \frac{\sqrt{N}(N^{-1}\sum_{i=1}^N t_i - E(t_0))}{\sqrt{Var(t_0)}}$$

$$\Rightarrow N(0,1)+c^2\Bigg[E\left(\frac{B_4-12B_5B_6}{\sqrt{F}}\right)+E\left(\frac{B_{12}-12B_{10}B_9}{\sqrt{F}}\right)-E\left(\frac{(B_2-12B_5B_9)(B_3-12B_6^2)}{2\sqrt{F^3}}\right)$$

$$-E\left(\frac{(B_2-12B_5B_9)(B_8-12B_5B_{10})}{\sqrt{F^3}}\right)-E\left(\frac{(B_1-12B_6B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right)$$

$$+E\left(\frac{3(B_2-12B_5B_9)(B_4-12B_5B_6)^2}{2\sqrt{F^5}}\right)\Bigg]/\sqrt{Var(t_0)},$$

which completes the proof.

Lemma A.1 *As $N\to\infty$, we have*

(i) $N^{-1}\sum_{i=1}^N\left[\sqrt{F_i}+\frac{B_{1i}-12B_{6i}B_{9i}}{\sqrt{F_i}}-\frac{(B_{2i}-12B_{5i}B_{9i})(B_{4i}-12B_{5i}B_{6i})}{\sqrt{F_i^3}}\right]=O_p(N^{-1/2})$,

(ii) $N^{-1}\sum_{i=1}^N\gamma_i c_i^2(-2c_i)^{-1/2}\left(\frac{(B_{11i}-B_{9i})}{\sqrt{F_i}}-\frac{(B_{2i}-12B_{5i}B_{9i})(B_{7i}-B_{5i})}{\sqrt{F_i^3}}\right)=O_p(N^{-1/2})$,

(iii)

$$N^{-1}\sum_{i=1}^N\Bigg[c_i^2\frac{B_{4i}-12B_{5i}B_{6i}}{\sqrt{F_i}}+c_i^2\frac{B_{12i}-12B_{10i}B_{9i}}{\sqrt{F_i}}-c_i^2\frac{(B_{2i}-12B_{5i}B_{9i})(B_{3i}-12B_{6i}^2)+2(B_{2i}-12B_{5i}B_{9i})(B_{8i}-12B_{5i}B_{10i})}{2\sqrt{F_i^3}}$$

$$-c_i^2\frac{(B_{1i}-12B_{6i}B_{9i})(B_{4i}-12B_{5i}B_{6i})}{\sqrt{F_i^3}}+c_i^2\frac{3(B_{2i}-12B_{5i}B_{9i})(B_{4i}-12B_{5i}B_{6i})^2}{2\sqrt{F_i^5}}\Bigg]\xrightarrow{p}c^2\Bigg[E\left(\frac{B_4-12B_5B_6}{\sqrt{F}}\right)$$

$$+E\left(\frac{B_{12}-12B_{10}B_9}{\sqrt{F}}\right)-E\left(\frac{(B_2-12B_5B_9)(B_3-12B_6^2)}{2\sqrt{F^3}}\right)-E\left(\frac{(B_2-12B_5B_9)(B_8-12B_5B_{10})}{\sqrt{F^3}}\right)$$

$$-E\left(\frac{(B_1-12B_6B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right)+E\left(\frac{3(B_2-12B_5B_9)(B_4-12B_5B_6)^2}{2\sqrt{F^5}}\right)\Bigg].$$

Proof of Lemma A.1.

(i) We need to show that

$$E\left[\sqrt{F}+\frac{B_1-12B_6B_9}{\sqrt{F}}-\frac{(B_2-12B_5B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right]=0.$$

From the simulation results, we can see that the mean of this term is close to 0. We can show this result based on the direct calculation of the characteristic function of t_i using the Fredholm method proposed in Nabeya and Tanaka (1990) or Nabeya and Sørensen (1994).

Substituting $\theta = iu/2$, $x = -v/u$ and $c = c_i N^{-1/4}$ into $\varphi_4(\theta; c, 1, x)$ of Theorem 4 in Nabeya and Tanaka (1990), we have the joint m.g.f. for (U_3, V_3) as

$$
\begin{aligned}
&\psi_3(u,v)\\
= \; & e^{-\frac{u}{2}}\Bigg[e^{-c_iN^{-\frac{1}{4}}}\Big[\frac{(c_iN^{-1/4})^5 - (c_iN^{-1/4})^4u - 4((c_iN^{-1/4})^2 + 3c_iN^{-1/4} + 27)u^2 - 8v((c_iN^{-1/4})^2 - 3c_iN^{-1/4} - 3)}{(2v - c_i^2N^{-1/2})^2}\\
&\times\frac{\sin\sqrt{2v - c_i^2N^{-1/2}}}{\sqrt{2v - c_i^2N^{-1/2}}} + \frac{24((c_iN^{-1/4})^4u + 8vu^2 - 4(c_iN^{-1/4} + 1)(v^2 - 3u^2))}{(2v - c_i^2N^{-1/2})^3}\Bigg(\frac{\sin\sqrt{2v - c_i^2N^{-1/2}}}{\sqrt{2v - c_i^2N^{-1/2}}}\\
&+\frac{\cos\sqrt{2v - c_i^2N^{-1/2}}}{2v - c_i^2N^{-1/2}} - \frac{1}{2v - c_i^2N^{-1/2}}\Bigg) + \Bigg(\frac{c_i^4N^{-1}}{(2v - c_i^2N^{-1/2})^2} - \frac{8(c_i^4N^{-1}u - c_i^3N^{-3/4}2v + 4(c_i^2N^{-1/2} + 3c_iN^{-1/4} + 6)u^2)}{(2v - c_i^2N^{-1/2})^3}\Bigg)\\
&\times\cos\sqrt{2v - c_i^2N^{-1/2}} - \frac{4(c_i^4N^{-1}u + 4(c_i^2N^{-1/2} + 3c_iN^{-1/4} - 3)u^2 - 2c_i^2N^{-1/2}v(c_iN^{-1/4} + 3))}{(2v - c_i^2N^{-1/2})^3}\Big]\Bigg]^{-1/2}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\frac{\partial}{\partial u}\psi_3(u,-v)\Big|_{u=0}\\
= \; & -\frac{1}{2}\Bigg[e^{-c_iN^{-\frac{1}{4}}}\Big[\frac{(c_iN^{-1/4})^5 + 8v((c_iN^{-1/4})^2 - 3c_iN^{-1/4} - 3)}{(-2v - c_i^2N^{-1/2})^2}\frac{\sin\sqrt{-2v - c_i^2N^{-1/2}}}{\sqrt{-2v - c_i^2N^{-1/2}}}\\
&+\frac{24(-4(c_iN^{-1/4} + 1)v^2)}{(-2v - c_i^2N^{-1/2})^3}\Bigg(\frac{\sin\sqrt{-2v - c_i^2N^{-1/2}}}{\sqrt{-2v - c_i^2N^{-1/2}}} + \frac{\cos\sqrt{-2v - c_i^2N^{-1/2}}}{-2v - c_i^2N^{-1/2}} - \frac{1}{-2v - c_i^2N^{-1/2}}\Bigg)\\
&+\Bigg(\frac{c_i^4N^{-1}}{(-2v - c_i^2N^{-1/2})^2} - \frac{8(c_i^3N^{-3/4}2v)}{(-2v - c_i^2N^{-1/2})^3}\Bigg)\cos\sqrt{-2v - c_i^2N^{-1/2}} - \frac{4(2c_i^2N^{-1/2}v(c_iN^{-1/4} + 3))}{(-2v - c_i^2N^{-1/2})^3}\Big]\Bigg]^{-1/2}\\
&-\frac{1}{2}\Bigg[e^{-c_iN^{-\frac{1}{4}}}\Big[\frac{(c_iN^{-1/4})^5 + 8v((c_iN^{-1/4})^2 - 3c_iN^{-1/4} - 3)}{(-2v - c_i^2N^{-1/2})^2}\frac{\sin\sqrt{-2v - c_i^2N^{-1/2}}}{\sqrt{-2v - c_i^2N^{-1/2}}}\\
&+\frac{24(-4(c_iN^{-1/4} + 1)v^2)}{(-2v - c_i^2N^{-1/2})^3}\Bigg(\frac{\sin\sqrt{-2v - c_i^2N^{-1/2}}}{\sqrt{-2v - c_i^2N^{-1/2}}} + \frac{\cos\sqrt{-2v - c_i^2N^{-1/2}}}{-2v - c_i^2N^{-1/2}} - \frac{1}{-2v - c_i^2N^{-1/2}}\Bigg)\\
&+\Bigg(\frac{c_i^4N^{-1}}{(-2v - c_i^2N^{-1/2})^2} - \frac{8(c_i^3N^{-3/4}2v)}{(-2v - c_i^2N^{-1/2})^3}\Bigg)\cos\sqrt{-2v - c_i^2N^{-1/2}} - \frac{4(2c_i^2N^{-1/2}v(c_iN^{-1/4} + 3))}{(-2v - c_i^2N^{-1/2})^3}\Big]\Bigg]^{-3/2}\\
&\times e^{-c_iN^{-\frac{1}{4}}}\Big[\frac{-c_i^4N^{-1}}{(-2v - c_i^2N^{-1/2})^2}\frac{\sin\sqrt{-2v - c_i^2N^{-1/2}}}{\sqrt{-2v - c_i^2N^{-1/2}}} + \frac{24c_i^4N^{-1}}{(-2v - c_i^2N^{-1/2})^3}\Bigg(\frac{\sin\sqrt{-2v - c_i^2N^{-1/2}}}{\sqrt{-2v - c_i^2N^{-1/2}}}\\
&+\frac{\cos\sqrt{-2v - c_i^2N^{-1/2}}}{-2v - c_i^2N^{-1/2}} - \frac{1}{-2v - c_i^2N^{-1/2}}\Bigg) + \Bigg(-\frac{8c_i^4N^{-1}}{(-2v - c_i^2N^{-1/2})^3}\Bigg)\cos\sqrt{-2v - c_i^2N^{-1/2}}\\
&-\frac{4c_i^4N^{-1}}{(-2v - c_i^2N^{-1/2})^3}\Big].
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
&\frac{\partial}{\partial u}\psi_3(u,-v)\Big|_{u=0} \\
= \; & -\frac{1}{2}\Bigg[(1-c_iN^{-\frac{1}{4}}+\frac{1}{2}c_i^2N^{-\frac{1}{2}}+O(N^{-1}))\Big[\frac{8v(c_i^2N^{-1/2}-3c_iN^{-1/4}-3)}{(-2v-c_i^2N^{-1/2})^2}\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}} \\
&+\frac{24(-4(c_iN^{-1/4}+1)v^2)}{(-2v-c_i^2N^{-1/2})^3}\left(\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}}+\frac{\cos\sqrt{-2v-c_i^2N^{-1/2}}}{-2v-c_i^2N^{-1/2}}-\frac{1}{-2v-c_i^2N^{-1/2}}\right) \\
&-\frac{4(6c_i^2N^{-1/2}v)}{(-2v-c_i^2N^{-1/2})^3}\Big]+O(N^{-3/4})\Bigg]^{-1/2}+O(N^{-1}) \\
= \; & -\frac{1}{2}\Bigg[\Big[\frac{8v(c_i^2N^{-1/2}-3c_iN^{-1/4}-3)}{(-2v-c_i^2N^{-1/2})^2}\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}} \\
&+\frac{24(-4(c_iN^{-1/4}+1)v^2)}{(-2v-c_i^2N^{-1/2})^3}\left(\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}}+\frac{\cos\sqrt{-2v-c_i^2N^{-1/2}}}{-2v-c_i^2N^{-1/2}}-\frac{1}{-2v-c_i^2N^{-1/2}}\right) \\
&-\frac{4(6c_i^2N^{-1/2}v)}{(-2v-c_i^2N^{-1/2})^3}\Big]-\Big[\frac{8v(-3c_i^2N^{-1/2}-3c_iN^{-1/4})}{(-2v-c_i^2N^{-1/2})^2}\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}} \\
&+\frac{24(-4(c_i^2N^{-1/2}+c_iN^{-1/4})v^2)}{(-2v-c_i^2N^{-1/2})^3}\left(\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}}+\frac{\cos\sqrt{-2v-c_i^2N^{-1/2}}}{-2v-c_i^2N^{-1/2}}-\frac{1}{-2v-c_i^2N^{-1/2}}\right)\Big] \\
&+\Big[\frac{4v(-3c_i^2N^{-1/2})}{(-2v-c_i^2N^{-1/2})^2}\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}}+\frac{24(-2(c_i^2N^{-1/2})v^2)}{(-2v-c_i^2N^{-1/2})^3}\Bigg(\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}} \\
&+\frac{\cos\sqrt{-2v-c_i^2N^{-1/2}}}{-2v-c_i^2N^{-1/2}}-\frac{1}{-2v-c_i^2N^{-1/2}}\Bigg)\Big]+O(N^{-3/4})\Bigg]^{-1/2}+O(N^{-1}) \\
= \; & -\frac{1}{2}\Bigg[\Big[\frac{4v(5c_i^2N^{-1/2}-6)}{(-2v-c_i^2N^{-1/2})^2}\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}}+\frac{24(-4v^2+2c_i^2N^{-1/2}v^2)}{(-2v-c_i^2N^{-1/2})^3}\Bigg(\frac{\sin\sqrt{-2v-c_i^2N^{-1/2}}}{\sqrt{-2v-c_i^2N^{-1/2}}} \\
&+\frac{\cos\sqrt{-2v-c_i^2N^{-1/2}}}{-2v-c_i^2N^{-1/2}}-\frac{1}{-2v-c_i^2N^{-1/2}}\Bigg)-\frac{4(6c_i^2N^{-1/2}v)}{(-2v-c_i^2N^{-1/2})^3}\Big]+O(N^{-3/4})\Bigg]^{-1/2}+O(N^{-1}). \quad \text{(A.7)}
\end{aligned}
$$

Clearly, we can see that the terms involving $N^{-1/4}$ are canceled out, and there is no $O(N^{-1/4})$ term in (A.7). Also, we know that $\sqrt{F}+\frac{B_1-12B_6B_9}{\sqrt{F}}-\frac{(B_2-12B_5B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}$ is corresponding to $N^{-1/4}$ term asymptotically. Therefore,

$$E\left[\sqrt{F}+\frac{B_1-12B_6B_9}{\sqrt{F}}-\frac{(B_2-12B_5B_9)(B_4-12B_5B_6)}{\sqrt{F^3}}\right]=0.$$

Our result here can provide the direct calculation for the expectation in Lemma A.1(iii). Recall that $t_i = \frac{U_3}{\sqrt{V}_3}$, and from Sawa (1972), we have

$$E(t_i) = E\left(\frac{U_3}{\sqrt{V}_3}\right) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{1}{\sqrt{v}}\frac{\partial}{\partial u}\psi_3(u,-v)\Big|_{u=0} dv.$$

Therefore, by the change of variable as $x = \sqrt{2v}$ and a Taylor expansion, we have

$$\begin{aligned}
& E(Z^\tau) \\
= & (Var(t_0))^{-1/2}N^{1/2}N^{-1}\sum_{i=1}^N(E(t_i) - E(t_0)) \\
= & (Var(t_0))^{-1/2}N^{1/2}\Bigg(-\frac{1}{\sqrt{2\pi}}\int_0^\infty\Bigg[3f_{22}(x) + (N^{-1}\sum_{i=1}^N c_i^2)N^{-1/2}\Big(\frac{-2\sinh(x)}{x^3} + \frac{18\cosh(x)}{x^4} - \frac{66\sinh(x)}{x^5} \\
& + \frac{96\cosh(x) - 96}{x^6}\Big) + O(N^{-3/4})\Bigg]^{-1/2} dx + O(N^{-1}) - E(t_0)\Bigg) \\
= & Var(t_0))^{-1/2}N^{1/2}\Bigg(-\frac{1}{\sqrt{2\pi}}\int_0^\infty [3f_{22}(x)]^{-1/2}dx + \frac{(N^{-1}\sum_{i=1}^N c_i^2)N^{-1/2}}{2\sqrt{2\pi}}\int_0^\infty [3f_{22}(x)]^{-3/2}\Big(-\frac{2\sinh(x)}{x^3} \\
& + \frac{18\cosh(x)}{x^4} - \frac{66\sinh(x)}{x^5} + \frac{96\cosh(x) - 96}{x^6}\Big)dx + O(N^{-3/4}) - E(t_0)\Bigg) \\
= & Var(t_0))^{-1/2}\Bigg(\frac{c^2}{\sqrt{2\pi}}\int_0^\infty [3f_{22}(x)]^{-3/2}\Big(-\frac{\sinh(x)}{x^3} + \frac{9\cosh(x)}{x^4} - \frac{33\sinh(x)}{x^5} + \frac{48(\cosh(x) - 1)}{x^6}\Big)dx \\
& + O(N^{-1/4})\Bigg),
\end{aligned} \tag{A.8}$$

where $f_{22}(x) = 4\left(\frac{1}{x^3}\sinh(x) - \frac{2}{x^4}[\cosh(x) - 1]\right)$, and $E(t_0) = -\frac{1}{\sqrt{2\pi}}\int_0^\infty [3f_{22}(x)]^{-1/2}dx$ from (7) and page 147 in Nabeya (1999).

Clearly, this shows that $Z^\tau = O_p(N^{-1/2})$ and

$$N^{-1}\sum_{i=1}^N\left[\sqrt{F_i} + \frac{B_{1i} - 12B_{6i}B_{9i}}{\sqrt{F_i}} - \frac{(B_{2i} - 12B_{5i}B_{9i})(B_{4i} - 12B_{5i}B_{6i})}{\sqrt{F_i^3}}\right] = O_p(N^{-1/2}).$$

(ii) As B_{5i}, B_{7i}, B_{9i} and B_{11i} are odd functionals and B_{2i} is the even functional of $W_i(r)$, following similar proof in Harris et al. (2010), we have the result.

(iii) From the standard Law of Large Numbers, we get the result.

References

Breitung, J. (2000) The local power of some unit root tests for panel data. In: Baltagi, B. ed. *Advances in Econometrics: Nonstationary Panels, Panel Cointegration, and Dynamic Panels* 15. Amsterdam: JAI, 161-178.

Breitung, J., M.H. Pesaran (2008) Unit roots and cointegration in panels. In: Mátyás, L., Sevestre, P. eds. *The Econometrics of Panel Data*. Springer, 279-322.

Choi, I. (2001) Unit root tests in panel data. *Journal of International Money and Finance* 20, 249-272.

Choi, I. (2006) Nonstationary panels. In: Millis, T.C., Patterson, K. eds. *Palgrave Handbook of Econometrics* Vol 1. New York: Palgrave macmillan, 511-539.

Harvey, D.I., S.J. Leybourne (2005) On testing for unit roots and the initial observation. *The Econometrics Journal* 8(1), 97-111.

Harris, D., D.I. Harvey, S.J. Leybourne, N.D. Sakkas (2010) Local asymptotic power of the Im-Pesaran-Shin panel unit root test and the impact of initial observations. *Econometric Theory* 26, 311-324.

Harvey, D.I., S.J. Leybourne, A.M.R. Taylor (2009) Unit root testing in practice: Dealing with uncertainty over the trend and initial condition. *Econometric Theory* 25, 587-636.

Im, K.S., M.H. Pesaran, Y. Shin (2003) Testing for unit roots in heterogeneous panels. *Journal of Econometrics* 115, 53-74.

Levin, A., C. Lin, C.-J. Chu (2002) Unit root tests in panel data: Asymptotic and finite-sample properties. *Journal of Econometrics* 108, 1-24.

Maddala, G.S., S. Wu (1999) A comparative study of unit root tests with panel data and a new simple test. *Oxford Bulletin of Economics and statistics* 61, 631-652.

Moon, H.R., B. Perron (2008) Asymptotic local power of pooled t-ratio tests for unit roots in panels with fixed effects. *Econometrics Journal* 11, 80-104.

Moon, H.R., P.C.B. Phillips (2004) GMM estimation of autoregressive roots near unity with panel data. *Econometrica* 72, 467-522.

Moon, H.R., B. Perron, P.C.B. Phillips (2007) Incidental trends and the power of panel unit root tests. *Journal of Econometrics* 141, 416-459.

Müller, U.K., G. Elliot (2003) Tests of unit roots and the initial condition. *Econometrica* 71, 1269-1286.

Nabeya, S. (1999) Asymptotic moments of some unit root test statistics in the null case. *Econometric Theory* 15, 139-149.

Nabeya, S., B.E. Sørensen (1994) Asymptotic distributions of the least-squares estimators and test statistics in the near unit root model with non-zero initial value and local drift and trend. *Econometric Theory* 10, 937-966.

Nabeya, S., K. Tanaka (1990a) A general approach to the limiting distribution for estimators in time series regression with nonstable autoregressive errors. *Econometrica* 58, 145-163.

Nabeya, S., K. Tanaka (1990b) Limiting power of unit-root tests in time-series regression. *Journal of Econometrics* 46, 247-271.

Phillips, P.C.B., H.R. Moon (2000) Nonstationary panel data analysis: An overview of some recent developments. *Econometric Reviews* 19, 263-286.

Ploberger, B., P.C.B. Phillips (2002) Optimal testing for unit roots in panel data. *Mimeo*, Yale

University.

Sawa, T. (1972) Finite sample properties of k-class estimators. *Econometrica* 40, 653-680.

Westerlund, J., J. Breitung (2012) Lessons from a decade of IPS and LLC. *Econometric Reviews* 32, 547-591.