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Abstract

Evaluating the sustainability of a society requires a system of shadow or accounting values derived from the sustainability objective. As a first step toward the derivation of such shadow values for a maximin objective, this paper studies an economy composed of two reproducible assets, each producing one of two consumption goods. The effect of the substitutability between goods in utility is studied by postulating, in turn, neoclassical diminishing marginal substitutability, perfect substitutability and perfect complementarity. The degree of substitutability has strong effects on the maximin solution, affecting the regularity or non-regularity of the program, and on the accounting values. This has important consequences for the computation of genuine savings and the sustainability prospects of future generations.

JEL-Codes: O440, Q560.

Keywords: sustainable development, maximin, sustainability accounting, substitutability.

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1 Introduction

The growing impacts of human activity on the environment has increased concern for the sustainability of society, of the production of goods and of the natural world. Sustainability is a complex issue, and there is not even a universally accepted definition of it (Heal, 1998; Neumayer, 2010; Martinet, 2012). To Solow (1993),[1] sustainability means the ability to continue to support a standard of living (often termed *utility*), accounting for all components of human well-being, including the consumption of manufactured goods, the flow of services from the environment and so on. The dominant view in economics is that sustainability corresponds to the requirement to sustain a measure of welfare.

A growing body of literature proposes metrics for sustainability accounting (Neumayer, 2010), among which genuine savings indicators are prominent. Genuine savings measures the evolution of the productive capacities of the economy through net investments in a comprehensive set of capital stocks, including natural assets. The accounting prices are the shadow values of an optimization problem, and a genuine savings indicator can be defined for any dynamic, forward-looking objective function satisfying the property of *independent future* (see Asheim, 2007, for details).

When the objective is defined as the standard of living of the worst-off among present and future generations, sustainability involves sustaining utility over time (Fleurbaey, 2015), and the associated accounting values are those of a maximin problem (Cairns and Martinet, 2014). The maximin value is the highest level of utility that can be sustained forever. It depends on the current stocks of manufactured capital, natural capital, human knowledge and skills, technology, etc. The shadow values of the stocks correspond to the marginal contribution of each asset to the maximin value, looking forward from current time and state. Net investment accounted for with these shadow values represents the evolution over time of the sustainable level of utility, and is interpreted as a measure of sustainability improvement or decline by Cairns and Martinet (2014). Computing net

[1]Solow (1993, pp. 167-168) has a vision that transforms sustainability from being an undisciplined, popular, contemporary expression of the conservationism that has arisen from time to time in advanced countries, into an economic prescription of durable decisions for the current generation: "If sustainability means anything more than a vague emotional commitment, it must require that something be conserved for the very long run. It is very important to understand what that thing is: I think it has to be a generalized capacity to produce economic well-being."

maximin investment is thus meaningful for sustainability accounting along any economic trajectory, efficient or not, and whether or not maximin is the pursued social objective.

In the present paper, we examine how to set a maximin accounting system in a multi-sector economy. As with other economic objectives, shadow values can be determined only through solving the corresponding optimization problem. Solving for the maximin level of utility for a modern economy, with all its various assets, consumption goods, production techniques, etc., would be a formidable task. Given the level of difficulty, taking a series of small steps seems to be the way to begin to tackle the problem. We thus consider an economy with two reproducible assets, manufactured or natural, and two arguments of the utility function, one good produced from each capital stock. The two stocks do not interact in production, but only within utility, raising the controversial question of substitutability of the components of utility. An important divide is between the notions of strong sustainability, in which it is held that it is impossible to substitute for certain environmental goods, and weak sustainability, which allows for substitutability of all components (Neumayer, 2010). We thus consider the neoclassical substitutable benchmark and the two extreme cases of perfect substitutability and non-substitutability (perfect complementarity) in utility.

Our results show that the ability to substitute has striking effects on the maximin solution. Substitutability makes it possible to use more of a less productive asset in order to let the other asset's stock build up. On the contrary, complementarity induces the redundancy of the more abundant asset, the whole sustainability pressure weighting in the limiting resource. The effects of the degree of substitutability on the shadow values are no less striking. Substitutability influences whether a program is *regular* or not. In a non-regular program, shadow values can be nil for some goods, even goods that are on the surface desirable in the economy and might be expected to have positive values. Substitutability reduces the occurrence of non-regular situations, whereas complementarity induces non-regularity associated to nil maximin shadow values.

The main contributions of the paper are (i) to begin the task of developing shadow values for sustainability accounting from scratch, a task that in principle is necessary for any particular objective and (ii) to clarify in a simple model how the degree of substitutability in utility influences regularity and the maximin shadow values.

2 Motivation

If the concerns for sustainability come from the hypothesis that society's current decisions are not sustainable, it can hardly be held that the observed, market prices can be used for sustainability accounting. Our main objective is to explore the possibility of developing a **sustainability accounting system based on maximin values**, in the spirit of genuine savings indicators. Such indicators are based on the shadow values of a dynamic optimization problem (Asheim, 2007).

Consider a stylized, general framework in which the set of decisions made in an economy (e.g., consumption, extraction of natural resource) is denoted by a vector $c = (c_1, \ldots, c_q)$ defined at any time, and the comprehensive set of producing capital stocks (e.g., manufactured capital, natural assets) by a vector $X = (X_1, \ldots, X_n)$. Economic dynamics (production and consumption leading to investment, natural resource growth and depletion, etc.) are represented by a mapping $F(X, c)$. The transition function representing the dynamics of a stock $i = 1, \ldots, n$ is denoted by F_i, with $\frac{\mathrm{d}X_i}{\mathrm{d}t} \equiv \dot{X}_i = F_i(X, c)$. Utility at any time t is derived from combinations of stocks and decisions, and defined by function $U(X, c)$. All these functions are assumed to be continuous and twice differentiable.

From a given *resource allocation mechanism*,[2] the dynamics $F(X, c)$ provide realizations of X, c and $U(X, c)$ through time that can be given a value $V(X)$. The marginal contributions of the various capital stocks $\partial V/\partial X_i$ – the *shadow values* – are used to compute comprehensive (net) investment, or genuine savings, as $\sum_i (\partial V/\partial X_i) \cdot \dot{X}_i$.

Among the possible decision criteria, maximin embodies the idea of sustainability (Cairns, 2011, 2013; Fleurbaey, 2015): it is not possible to say that a level of well being is sustainable if some generation in the future cannot enjoy it. The maximin value provides information on the highest level of utility that could be sustained in the economy, given

[2]Most of the genuine savings literature is based on the maximization of a welfare criterion (Asheim, 2007; Dasgupta, 2009). Shadow values associated to the path are then used to compute genuine savings. A notable exception is the work of Dasgupta, Mäler, and colleagues (Dasgupta and Mäler, 2000; Arrow et al., 2003) who use general, possibly non-optimal resource allocation mechanisms (*ram*) instead of maximizing welfare. The shadow values used to compute the genuine savings indicator are the marginal contribution of each capital stock to the value associated to the trajectory determined by the *ram*. Integrating the dynamic path and computing the associated value as a function of all capital stocks can be done only for simple models with strong assumptions on the *ram*.

current endowments. Maximin shadow values could be used to compute sustainable net investment, even if maximin is not taken as the objective and the economic trajectory is different from a maximin path (Cairns and Martinet, 2014). Genuine savings based on maximin provides an indication of the evolution of the sustainable utility level, and thus on the evolution of the productive capacities of the economy.

The maximin value of a state X is defined as

$$m(X) = \max_{c(\cdot)} \min_{s \geq t} U(X(s), c(s)) \ ,$$
$$\text{s.t.} \quad X(t) = X \text{ and } \dot{X}(s) = F(X(s), c(s)) \ , \forall s \geq t \ .$$

If the value function $m(X)$ can be differentiated, the shadow values are $\partial m(X)/\partial X_i$ (Cairns and Long, 2006). Net investment evaluated at the maximin shadow values is $\sum_{i=1}^{n} \partial m(X)/\partial X_i \cdot \dot{X}_i$. The links between the maximin problem and net investment have been studied since the work of Hartwick (1977) and Dixit et al. (1980) on Hartwick's rule, with recent contributions by Asheim (2007), Doyen and Martinet (2012), and Fleurbaey (2015).

There are few solved maximin problems, and the task of computing maximin shadow values for the real economy is obviously out of reach. The only way to proceed is to build up from simpler problems and to try to gain a greater understanding of the economic issues involved. When there are more than one asset, the controversial question of substitutability arises. Neumayer (2010) stresses that substitutability in production as well as in utility plays a central role in the study of sustainability.[3] The influence of substitutability in production on the maximin solution has been emphasized since the work of Solow (1974) and Dasgupta and Heal (1979), who studied interactions between sectors in the form of an extraction sector providing an input to a manufacturing sector. Some articles (e.g. Asako, 1980; Stollery, 1998; d'Autume and Schubert, 2008; d'Autume et al., 2010) study maximin problems with two arguments in the utility function, but considering the substitutability between a decision variable (consumption) and a state

[3] So-called weak sustainability postulates that there is substitutability between natural assets and other capital stocks and goods in production and well-being, while so-called strong sustainability postulates that there are natural assets that have no substitutes in the long run.

variable (either the temperature or the stock of a non-renewable resource).[4] The question of the substitutability of consumption goods in utility has received less scrutiny.

Substitutability of the flows from stocks in utility is as important a question for sustainability as substitutability of the stocks in production. We study the sustainability of an economy with two productive sectors, which interact only through utility, for the neoclassical case of smooth, diminishing marginal substitutability, as well as the extreme cases of perfect substitutability and perfect complementarity. Our aim is to characterize the maximin solution for such problems, and discuss the consequences of current decisions on a genuine savings indicator computed at maximin shadow values.

In addition to the characterization of maximin solutions in this model, we are interested in how *non-regularity* arises in a model with multiple sectors and how the substitutability in utility influences it. Non-regularity in maximin problems corresponds to situations in which the maximin path is not a constant utility path at the maximin level.[5] Non-regularity appears to be more than just an isolated anomaly in a few special cases. It emerges even in the Solow (1956) growth model with capital depreciation (as well as in the simple Fishery[6]). In this model, stocks that are beyond what is known in growth theory as the golden-rule level are redundant. The maximin value is increasing with the capital stock up to the golden-rule capital stock.[7] It is then constant above this level. Maximin shadow values are thus positive for stocks below the golden-rule level, and nil above it. This feature has implications for the definition of an accounting system.

We begin small, with a model that extends the simple fishery to an economy with two

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[4] Much of the rest of the literature has studied the axiomatic foundations of the criterion (e.g. Asheim and Zuber, 2013) or the existence of maximin solutions in specific problems (e.g. Mitra et al., 2013).

[5] Maximin can be perceived as successively raising the level of the least well-off to the extent possible, to increase equity. The end result of this sequence of redistributions can be an equalization of utility. Equity does not necessarily mean equality, however; even if a maximin solution exists, a redistribution to achieve equality is not always possible. Intergenerational equality is the outcome of the intergenerational-equity problem and not the objective. In a 'regular' maximin problem, it is optimal and efficient to distribute well being equally over time. Otherwise, the problem is non-regular (Burmeister and Hammond, 1977). Non-regularities in maximin problems have been found in the models by Solow (1974), Asako (1980), and Cairns and Tian (2010). A discussion of non-regularities is provided by Doyen and Martinet (2012). Cairns and Martinet (2014) stress its consequences for maximin shadow values, and thus for accounting.

[6] The golden-rule of consumption is replaced by Maximum Sustainable Yield. The economic implications of these two canonical models are parallel.

[7] Specifically, the maximin value is equal to the net production and the maximin path constitutes a steady state at the initial capital stock.

renewable assets such as two physically separate fisheries. Ultimately, an aim is to find conditions applicable to the maximin shadow values in an economy with several renewable assets.

3 An economy with two reproducible assets

3.1 Economic model

We consider a dynamic economy composed of infinitely many generations of identical consumers, each living for an instant in continuous time. Instantaneous utility $U(c_1, c_2)$ is derived from the consumption of two goods, denoted by c_i, $i = 1, 2$. The two goods interact only through utility.

Each consumption good i is produced by a specific sector using a reproducible asset X_i. The function $F_i(X_i)$ depends only on the stock X_i and is assumed to be continuous, (weakly) concave and differentiable. It can represent either the production function of a manufactured good or the natural growth function of a renewable natural resource.

$$\dot{X}_i(t) \equiv \frac{\mathrm{d}X_i(t)}{\mathrm{d}t} = F_i(X_i(t)) - c_i(t) \ . \tag{1}$$

Marginal productivity is assumed to be positive, at least for sufficiently low stock levels. In a model with decreasing marginal gross productivity and constant capital decay rate such as the neoclassical Solow growth model mentioned above, it may be the case that, for large stocks, the marginal gross production does not compensate for capital depreciation, so that marginal net productivity turns negative for large capital stocks. Similarly, as in the simple fishery, a natural renewable resource with a carrying capacity and a maximum sustainable yield (MSY) can have a negative marginal productivity for stocks beyond that yielding the MSY. We characterize such economic model with the following definition.

Definition 1 (Single-peaked technology)**.** *A technology $F(X)$ is single-peaked if there exists $\bar{X}$ such that $F'(X) > 0$ for $X < \bar{X}$ and $F'(X) < 0$ for $X > \bar{X}$. By differentiability, $\bar{X}$ is implicitly defined by the condition $F'\left(\bar{X}\right) = 0$.*

The capital stock $\bar{X}$ is the stock which yields the highest production level (maximum sustainable yield or golden-rule level). A single-peaked technology is a source of non-regularity in a maximin problem, as any capital stock exceeding $\bar{X}$ is potentially *redundant* (see Asako, 1980).

The maximin value of a state (X_1, X_2) is the highest level of utility that can be sustained forever from that state, as of the current date, normalized to zero:

$$
\begin{aligned}
m(X_1, X_2) = \max\ u\ , \qquad (2) \\
\text{s.t. } (X_1(0), X_2(0)) = (X_1, X_2)\ ; \\
\dot{X}_i(t) = F_i(X_i(t)) - c_i(t),\ i = 1, 2 \text{ and} \\
U(c_1(t), c_2(t)) \geq u \text{ for all } t \geq 0. \qquad (3)
\end{aligned}
$$

Differentiation of the maximin value with respect to time yields a direct link with net investment at time t (Cairns and Martinet, 2014, Lemma 1):

$$
\begin{aligned}
\dot{m}(X_1, X_2)|_t &= \left.\frac{\partial m}{\partial X_1}\right|_t \dot{X}_1(t) + \left.\frac{\partial m}{\partial X_2}\right|_t \dot{X}_2(t) \\
&= m'_{X_1}(X_1(t), X_2(t))[F_1(X_1(t)) - c_1(t)] + m'_{X_2}(X_1(t), X_2(t))[F_2(X_2(t)) - c_2(t)]\ . \qquad (4)
\end{aligned}
$$

The following lemmata are established in the Appendix.

Lemma 1 (Stationary fallback)**.** *For any state (X_1, X_2), the maximin value is at least equal to the utility derived from consumptions at the corresponding steady state:*

$$m(X_1, X_2) \geq U(F_1(X_1), F_2(X_2))\ .$$

This result means that the maximin value is at least equal to the utility which could be obtained by consuming the whole (net) production of both stocks, keeping the economy in a steady state. A dynamic maximin path may, however, yield a higher sustainable utility.

Lemma 2 (Dynamic maximin path)**.** *If the maximin value of a state (X_1, X_2) is greater than the utility from keeping the state stationary, i.e., if $m(X_1, X_2) > U(F_1(X_1), F_2(X_2))$,* *then (i) the consumption of at least one good is greater than the production of the corresponding stock and (ii) that stock decreases.*

We conjecture that if the two technologies are single-peaked, the maximin value is bounded from above by level $\bar{m} = U\left(F_1\left(\bar{X}_1\right), F_2\left(\bar{X}_2\right)\right)$. Any state $(X_1, X_2) >> (\bar{X}_1, \bar{X}_2)$ has a maximin value $\bar{m}$. This case generalizes to two dimensions the maximum sustainable yield and the associated non-regularity.[8]

To characterize the maximin value in our two sector economy, we need to make some assumptions about the interaction between the two goods in the utility function. We consider the neoclassical case of diminishing marginal substitutability as a benchmark. The results for the cases of perfect substitutability and perfect complementarity are described afterward.

3.2 The neoclassical benchmark

Consider a twice differentiable, strictly quasi-concave utility function $U(c_1, c_2)$ such that $\lim_{c_i \to 0} U'_{c_i}(c_1, c_2) = +\infty$ (both goods are essential in consumption), $U'_{c_i} > 0$ and $U''_{c_i,c_i} < 0$ (both goods contribute positively to consumption but at decreasing marginal rates) and $U''_{c_1,c_2} > 0$.[9] Assume that both initial stocks are strictly positive, i.e., $X_i(0) > 0$ for $i = 1, 2$ (otherwise, one is back to the single resource problem). Under these conditions, and given Lemma 1, one can say that consumption of both goods are positive ($c_i(t) > 0,\ i = 1, 2$) at any time along a maximin path.

We use the direct approach to maximin of Cairns and Long (2006) to characterize the regular maximin paths of problem (2). It is based on optimal control theory and requires defining the costates variables $\mu_i(t)$ for stocks $X_i(t)$, as well as a multiplier $\omega(t)$ for the maximin constraint (3). The Hamiltonian of the problem is $\mathcal{H}(X, c, \mu) =$

[8]Note that no steady state could sustain a utility larger than $\bar{m}$ due to the production peaks for the two resources. It is also easy to show that the two stocks cannot increase at the same time without reducing utility below $\bar{m}$. Asako (1980), in a two state variables model which is somewhat related to ours, states that it is obvious that there is no limit cycle.

[9]This is the set of assumptions of a standard, neoclassical economic model. If, in addition, the utility function is homothetic, substitutability can be measured by Hicksian elasticity $\chi = \frac{U'_{c_1} U'_{c_2}}{U''_{c_1,c_2}}$ (Hicks, 1932). Goods are complements in the limit where $\chi \to 0$ and perfect substitutes in the limit where $\chi \to +\infty$.

$\mu_1 (F_1(X_1) - c_1) + \mu_2 (F_2(X_2) - c_2)$ and the Langrangian is $\mathcal{L}(X, c, \mu, \omega, u) = \mathcal{H}(X, c, \mu) + \omega (U(c_1, c_2) - u)$.

Cairns and Long (2006, Proposition 1) show that, along the optimal path, the $\mu_i(t)$ are equal to the shadow values of each stock at time t, i.e., $\mu_i(t) = \left.\frac{\partial m(X_1,X_2)}{\partial X_i}\right|_t$. The Hamiltonian of the problem is interpreted as net investment at maximin shadow values, as noted in eq. (4), and it is nil as long as the necessary conditions hold:

$$\mathcal{H}(X, c, \mu) = 0 \quad \Leftrightarrow \quad \mu_1 \dot{X}_1 + \mu_2 \dot{X}_2 = 0 \,. \tag{5}$$

Equation (5) is related to Hartwick's rule and its converse, which states that if utility remains constant at the maximin level, net investments are nil over time (Dixit et al., 1980; Withagen and Asheim, 1998).

The multiplier $\omega(t)$ is interpreted as a shadow value or cost of equity, which, when positive, provides information on the difficulty of satisfying the equity constraint at time t.

The formal expression of the dynamic optimization problem as well as its optimality conditions are described in the Appendix. From these conditions, we obtain the following results.[10]

Result 1 (Shadow values of the stocks)**.** *At any time along an optimal path:*

- *The shadow value of stock X_i is equal to the marginal utility of consumption of the corresponding good weighted by the shadow value of equity.*

$$\mu_i = \omega U'_{c_i} \,, \quad i = 1, 2 \,. \tag{6}$$

- *So long as $\omega > 0$, the relative shadow value is equal to the marginal rates of substitution in consumption:*

$$\frac{\mu_1}{\mu_2} = \frac{U'_{c_1}}{U'_{c_2}} \,. \tag{7}$$

Result 2 (Evolution of the shadow values)**.** *At any time along an optimal path:*

[10] As the problem is time autonomous, we omit the time argument in the expressions.

- *Each shadow value decreases at a rate equal to the current marginal productivity of the corresponding stock:*

$$-\frac{\dot{\mu}_i}{\mu_i} = F_i'(X_i)\ , \quad i = 1, 2\ . \tag{8}$$

- *The relative shadow value $\frac{\mu_i}{\mu_j}$ decreases at a rate equal to the current difference between the stocks' levels of marginal productivity:*

$$\frac{1}{\mu_i/\mu_j}\frac{\mathrm{d}\,(\mu_i/\mu_j)}{\mathrm{d}t} = \frac{\dot{\mu}_i}{\mu_i} - \frac{\dot{\mu}_j}{\mu_j} = -\left(F_i'(X_i) - F_j'(X_j)\right)\ . \tag{9}$$

The ratio $-\frac{\dot{\mu}_i}{\mu_i}$ can be interpreted as a virtual rate of depreciation of stock i's value. Along the optimum path, this rate of depreciation is equal to cost of 'postponing' an investment on a short interval (see Dorfman, 1969, p. 821). The lower a stock, the higher its marginal productivity and therefore the more 'costly' it is to postpone investment along the optimal path.

Result 3 (Shadow value of equity)**.** *At any time along an optimal path, the shadow value of equity decreases at a rate equal to the sum of the stock marginal productivity and the rate of change of the marginal utility of consumption for the associated good:*

$$-\frac{\dot{\omega}}{\omega} = F_i'(X_i) + \frac{\dot{[U_{c_i}']}}{U_{c_i}'}\ , \quad i = 1, 2\ . \tag{10}$$

The variable $\delta(t) \equiv -\frac{\dot{\omega}(t)}{\omega(t)}$ is the rate of change of the factor $\omega(t)$. It has some of the features of a utility discount rate, related to the 'illusory' discounting factor $\omega(t)$.[11] Along the maximin path, $\omega(t)$ is endogenous, and thus so is $\delta(t)$ which is likely to be non constant, except at a steady state.

Combining this result with Result 1, we interpret the shadow value μ_i as comparable to a present-value price, evaluated at time zero, the marginal utility U_{c_i}' being the shadow current price of consumption for stock X_i.

[11] Cairns and Long (2006) interpret ω as a shadow discount factor along the maximin path.

Result 4 (Steady state)**.** *An optimal steady state* (X_1^*, X_2^*) *is characterized by an equality of the marginal productivities of all stocks:*[12]

$$F_1'(X_1^*) = F_2'(X_2^*) = \delta^* \; . \tag{11}$$

This result means that a steady state is optimal when all stocks have the same marginal productivity. At a steady state each variable is constant ($\dot{c}_1 = \dot{c}_2 = 0$ and $\dot{X}_1 = \dot{X}_2 = 0$) and $m(X_1^*, X_2^*) = U(F_1(X_1^*), F_2(X_2^*))$. From any time t such that $(X_1(t), X_2(t)) = (X_1^*, X_2^*)$ and any time $s \geq t$, one has $\omega(s) = \omega(t)e^{-(s-t)\delta^*}$ and $\mu_i(s) = U'_{c_i}(c_1^*, c_2^*)\omega(t)e^{-(s-t)\delta^*}$ for $i = 1, 2$. Note that, as δ^* is endogenous to the problem, the steady state depends on the initial state.[13]

Either stocks are at a steady state such that marginal productivities are equal or the economy follows a dynamic path with the depletion of one of the stocks (Lemma 2). Such a dynamics is characterized by the following result.

Result 5 (Transition path)**.** *Along an optimal maximin path, apart from a steady state, when* $F_i'(X_i) > F_j'(X_j)$*, the following hold.*

- *The consumption of the more productive stock* i *is lower than its production* ($c_i < F_i(X_i)$)*. This stock builds up* ($\dot{X}_i > 0$) *and its marginal productivity decreases over time* ($\mathrm{d}(F_i')/\mathrm{d}t \leq 0$)*.*

- *The consumption of the less productive stock* j *is greater than its production* ($c_j > F_j(X_j)$)*. This stock is depleted* ($\dot{X}_j < 0$) *and its marginal productivity increases over time* ($\mathrm{d}(F_j')/\mathrm{d}t \geq 0$)*.*

- *The marginal productivities converge, with* $\dot{c}_i > 0$ *and* $\dot{c}_j < 0$*.*

Utility is sustained at the maximin level with the substitution of the more productive resource by the less productive one. The stocks evolve in opposite directions until the marginal productivities are equal and the economy reaches a steady state.

From Result 1 (eq. (7)) and Result 2 (eq. (9)), we can characterize the shape of the maximin trajectory.

[12]Since conditions (6)-(10) remain valid for n stocks, this condition is general to any number of stocks.

[13]The optimal condition for a steady state in a discount utility problem with constant discount rate ρ is $F_i'(X_i^*) = F_j'(X_j^*) = \rho$, whatever the initial state.

Result 6 (Iso-maximin value curves)**.** *Along a maximin path with nil net investment at maximin shadow values and constant utility over time, at any time t, the state* $(X_1(t), X_2(t))$ *and the optimal decisions* $(c_1(t), c_2(t))$ *satisfy*

$$\frac{\dot{X}_1}{\dot{X}_2} = \frac{\dot{c}_1}{\dot{c}_2} = -\frac{\mu_2}{\mu_1} \quad \Leftrightarrow \quad \left.\frac{\mathrm{d}X_1}{\mathrm{d}X_2}\right|_{m(X_1,X_2)=\bar{u}} = \left.\frac{\mathrm{d}c_1}{\mathrm{d}c_2}\right|_{U(c_1,c_2)=\bar{u}} .$$

The slopes of the iso-utility curve in the decision map and that of the iso-maximin curve in the state map are equal. The iso-maximin curves are convex to the origin. Maximin trajectories follow these iso-value curves, toward an equilibrium at which the marginal productivities of the two stocks are equal. The relative shadow value governs the slopes $\frac{\mathrm{d}c_1}{\mathrm{d}c_2}$ and $\frac{\mathrm{d}X_1}{\mathrm{d}X_2}$ at any time along the path (including at a steady state).

Graphical representation Figure 1 is a plot of the solution, with quadratic functions used to represent production. It is a four-quadrant graph in which the East axis represents X_1, the South axis X_2, the North axis c_1 and the West axis c_2. The north-east [NE] quadrant represents production $F_1(X_1)$ and the South-West [SW] quadrant production $F_2(X_2)$. The north-west [NW] quadrant plots iso-utility curves, and the South-East [SE] quadrant is the state map (X_1, X_2) in which state trajectories can be drawn as well as level curves for the maximin value.

State $\underline{X}_1$ satisfies $F_1'(\underline{X}_1) = F_2'(0)$. The straight line[14] $\underline{X}_1 M$ in the state space [SE] corresponds to the states for which $F_1'(X_1) = F_2'(X_2)$, and thus to steady states of the maximin problem. The corresponding maximin value is given by the utility at equilibrium in the quadrant NW.

South-West of the equilibrium line $\underline{X}_1 M$, stock X_2 is less productive at the margin, i.e., $F_1'(X_1) > F_2'(X_2)$. Consumption of stock 2 exceeds its production, while consumption of stock 1 is lower than its production. The trajectory goes North-East toward the equilibrium line. Along that trajectory, utility is constant at the maximin value. The two consumption levels converge toward an equilibrium value as the states converge to an equilibrium with equal marginal productivities. The same pattern occurs (*mutatis*

[14] This is a straight line because we assumed quadratic growth functions. The marginal productivities $F_i'(X_i)$ are then proportional to the stocks X_i, with $i = 1, 2$, and this gives a linear relationship between X_1 and X_2.

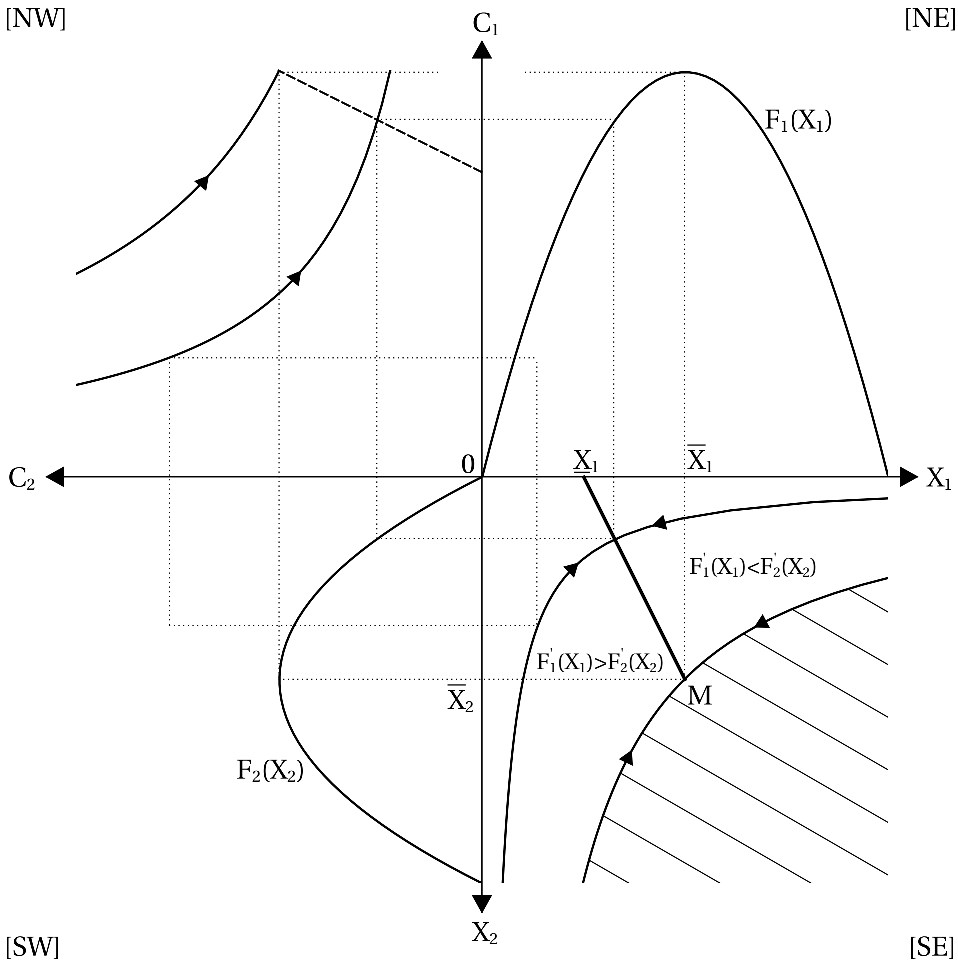


Figure 1: Graphical representation for the neoclassical benchmark

mutandis) north-east of the equilibrium line. For any state North-West of the hatched area, maximin prices are positive.

From any state in the hatched area, South-East of the state trajectory through M, we conjecture that the maximin value is equal to $m\left(\bar{X}_1, \bar{X}_2\right)$. Stocks are redundant and have nil shadow values. The hatched area represents a plateau in the maximin value function; all paths starting from a state in this area are non-regular in the sense that utility can exceed the maximin value for some time, until the state reaches the steady state $\left(\bar{X}_1, \bar{X}_2\right)$ and the utility equals the maximin value. Along such paths (including the limiting case at the boundary of the hatched area), maximin shadow values are nil.

The interesting feature with two resources substitutable in utility is that some states which would be redundant when considering a single resource are not when considering substitutability with another resource. If one stock is below its production peak and the other one above, the stock above the production peak is used in order to let the other one grow. Substitutability "limits" non-regularity. A stock may have a positive shadow value even if it is above its production peak.

3.3 Limiting case: perfect substitutability

We now discuss the case in which the two resources are perfect substitutes, as, for example, two groundwater stocks with different qualities or two fish stocks with different nutritive values. This case also covers the more general problem of sustaining a small economy with trade, as long as the terms of trade are not changing.[15] The utility function is assumed to be linear in the consumption of each good; $U(c_1, c_2) = a_1 c_1 + a_2 c_2$.

Mathematical details are quite similar to that of the neoclassical benchmark, except that, with perfect substitutability, we cannot exclude nil consumption of one or the other good. Proofs of the following results are provided in the Supplementary Materials.

Result 7 (Perfect substitutability: Steady states)**.** *In an economy relying on two perfectly substitutable reproducible assets, states* $(X_1^*, X_2^*) \leq \left(\bar{X}_1, \bar{X}_2\right)$ *such that* $F_1'(X_1^*) = F_2'(X_2^*)$ *are optimal steady states, with consumption levels* $c_1 = F_1(X_1^*)$ *and* $c_2 = F_2(X_2^*)$*.*

Just as in the neoclassical benchmark, states with equal marginal productivities are steady states. Apart from a steady state, maximin paths follow a dynamics in which the less productive resource is used up and the more productive resource is built up, as in the neoclassical benchmark. With perfect substitutability, however, one gets a bang-bang solution in which the less productive stock only is consumed, with a nil consumption of the more productive resource.

[15] An economy exporting two goods at given prices has an utility monotonically increasing with the value of production, whatever the actual preference for consumption goods (even complementarity). It thus aims at sustaining the export values. This relation follows from the separating hyperplane theorem.

Result 8 (Perfect substitutability: Dynamics)**.** *In an economy relying on two perfectly substitutable reproducible assets, apart from a steady state, for any state such that* $F_i'(X_i) > F_j'(X_j)$*, the constant consumption path with* $c_j(t) = \frac{m(X_1(t),X_2(t))}{a_j} > F_j(X_j(t))$ *and* $c_i(t) = 0$ *is an optimal maximin path. Stock* X_j *decreases while stock* X_i *increases.*

With such a dynamic path, exhaustion of the less productive resource may be optimal. Let $F_i'(0) > F_j'(0)$, and denote by $\underline{X}_i > 0$ the stock level such that $F_i'(\underline{X}_i) = F_j'(0)$.

Result 9 (Perfect substitutability: Optimal exhaustion)**.** *In an economy relying on two perfectly substitutable reproducible assets, a maximin path starting from any state* (X_i, X_j) *such that* $F_i'(X_i) > F_j'(X_j)$ *and* $m(X_i, X_j) \leq a_i F_i(\underline{X}_i)$ *exhausts asset* X_j*.*

Contrary to the benchmark neoclassical case, the relative shadow value is different from the Marginal Rate of Substitution of utility, except at a steady state.

Result 10 (Perfect substitutability: Shadow values)**.** *In an economy relying on two perfectly substitutable reproducible assets, apart from a steady state, along an optimal path with* $F_i'(X_i(t)) > F_j'(X_j(t))$*, the relative shadow value satisfies*

$$\frac{\mu_i}{\mu_j} \geq \frac{a_i}{a_j} \ . \tag{12}$$

At a steady state with $F_1'(X_1^*) = F_2'(X_2^*)$*, one has* $\frac{\mu_1}{\mu_2} = \frac{a_1}{a_2}$*.*

The more productive resource is relatively more valued in maximin terms than in the utility. This has consequences on the iso-maximin curves, which do not have the same shape as the iso-utility curves.

Result 11 (Perfect substitutability: Iso-maximin curves)**.** *The iso-maximin curves are convex to the origin is the plane* (X_1, X_2)*, with* $-\frac{\mathrm{d}X_2}{\mathrm{d}X_1} = \frac{\mu_1}{\mu_2} \geq \frac{a_1}{a_2}$*.*

Graphical representation The maximin solution for the perfect substitutability case is plotted in Figure 2. The interpretation is the same as that of Figure 1, except that one has corner solutions for the consumption. A path exhausting resource X_2 illustrates Result 9.

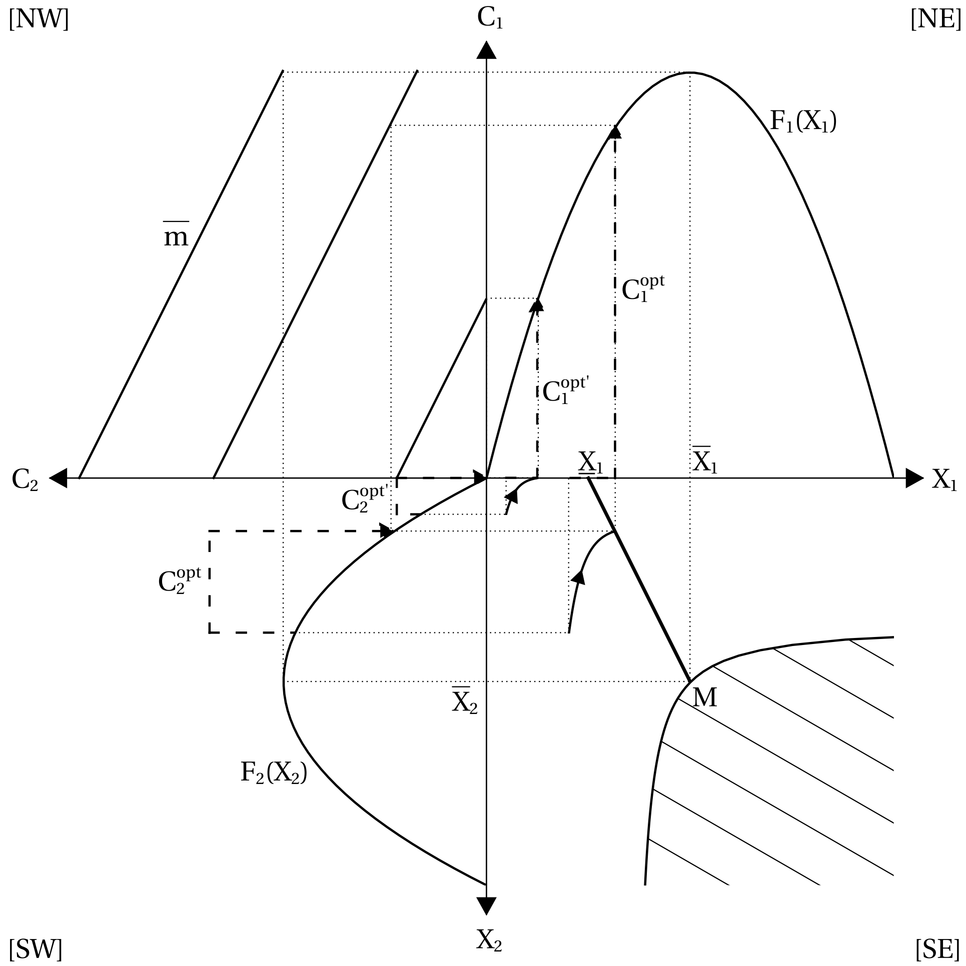


Figure 2: Perfect substitutes

An example We illustrate the previous results by providing an explicit solution for the maximin value and shadow values in a simple economic model with a renewable natural resource with logistic growth ($F_1(X_1) = rX_1(1 - X_1)$) and an A-K technology ($F_2(X_2) = AX_2$). The respective dynamics are $\dot{X}_1 = rX_1(1 - X_1) - c_1$ and $\dot{X}_2 = AX_2 - c_2$, with $A, r > 0$. Utility is $U(c_1, c_2) = a_1c_1 + a_2c_2$.

We assume that $r > A$, so that low resource stocks are more productive than the manufactured capital stock, and the natural resource is not exhausted. We give the maximin value for states (X_1, X_2) satisfying $F_1'(X_1) > F_2'(X_2)$, i.e., for $X_1(0) < \underline{X}_1 \equiv$

$\frac{1}{2}\left(1-\frac{A}{r}\right)$. All mathematical details as well as a graphical illustration are available in the Supplementary Materials.

Along the maximin path, the capital stock X_2 is consumed to let the natural resource X_1 grow (recovery). For paths that do not exhaust the manufactured capital stock, i.e, trajectories reaching a stationary state with $X_2^* > 0$ and $X_1^* = \underline{X}_1$, the maximin value function is given by $m(X_1, X_2) = a_1\Gamma\left(\frac{1}{X_1}-1\right)^{-\frac{A}{r}} + a_2AX_2$, with $\Gamma = \frac{r}{4}\left(1+\frac{A}{r}\right)^{1+\frac{A}{r}}\left(1-\frac{A}{r}\right)^{1-\frac{A}{r}}$. Associated shadow prices are $\mu_1 = a_1\frac{A}{r}\Gamma\left(1-X_1\right)^{-1-\frac{A}{r}}X_1^{-1+\frac{A}{r}}$ and $\mu_2 = a_2A$.

For states (X_1, X_2) such that $m(X_1, X_2) \leq a_1F_1(\underline{X}_1)$, maximin paths exhaust the manufactured capital stock and the maximin value function is implicitly given by $m = \frac{ra_1\left(1-\frac{a_2AX_2}{m}\right)^{\frac{r}{A}}\left(\frac{1}{X_1}-1\right)}{\left(1+\left(1-\frac{a_2AX_2}{m}\right)^{\frac{r}{A}}\left(\frac{1}{X_1}-1\right)\right)^2}$.

3.4 Limiting case: perfect complementarity

An issue raised by environmentalists and ecologists is the possibility that some natural assets do not have substitutes in well-being (Neumayer, 2010). If the two resources are perfect complements, the utility function is $U(c_1, c_2) = \min\{a_1c_1, a_2c_2\}$, where $a_1 > 0$ and $a_2 > 0$ are fixed proportions in which the resources are combined to provide utility. The maximin value in this case depends on the sustainable consumption of each stock, defined as follows.

Definition 2 (Sustainable consumption)**.** *The sustainable level of consumption for a current stock level X_i is denoted by $\bar{c}_i(X_i)$ and defined as*

$$\bar{c}_i\left(X_i\right) = \begin{cases} F_i(X_i) & \textit{if} \quad X_i < \bar{X}_i \\ F_i\left(\bar{X}_i\right) & \textit{if} \quad X_i \geq \bar{X}_i \end{cases} .$$

In the perfect complementarity case, the capital stock yielding the lowest value $a_i\bar{c}_i$ is limiting sustainable utility.

$$m(X_1, X_2) = \min\left\{a_1\bar{c}_1(X_1), a_2\bar{c}_2(X_2)\right\} .$$

The maximin value is determined by the "limiting resource". There are two ways for a resource to be limiting in our problem. (1) The resource can be relatively less productive in the long-run, notwithstanding short-term scarcity, and impose a limit on long-run utility. We use the terminology "globally limiting" then. (2) The resource can be scarce relative to the other resource given its current stock and impose a limit on current utility. We use the terminology "locally limiting" then.

To illustrate the influence of a *"globally limiting resource"*, consider large resource stocks (e.g., virginal natural resource stocks) characterized by $(X_1(0), X_2(0)) >> \left(\bar{X}_1, \bar{X}_2\right)$. Clearly, neither resource limits current utility, as under our assumptions the extraction at the initial time is not bounded for either stock. In this case, $\bar{c}_i(X_i(0)) = F_i(\bar{X}_i)$ for both resources. The maximin value for such an initial state is $m(X_1(0), X_2(0)) = \min\left\{a_1 F_1(\bar{X}_1), a_2 F_2(\bar{X}_2)\right\}$. This level is the upper bound for the maximin value over the whole state domain. It is a two-dimensional generalization of the Maximum Sustainable Yield in the single-resource case. We thus have a non-regularity of the redundant stock type (Asako, 1980; Doyen and Martinet, 2012). When $a_i F_i\left(\bar{X}_i\right) > a_j F_j\left(\bar{X}_j\right)$, stock j puts a tighter global limit on sustainable utility than stock i. The stock level $X_i^* < \bar{X}_i$ such that $a_i F_i(X_i^*) = a_j F_j\left(\bar{X}_j\right)$ is the smallest stock of resource i such that the MSY of X_j is limiting. (If stock X_i decreases below level X_i^*, resource i becomes locally limiting in place of the $F_j\left(\bar{X}_j\right)$). Increasing stock i above level X_i^* does not increase the maximin value, and stocks above this level have a nil shadow value, even if they are below the production peak.

A stock X_l below its production peak $\left(X_l(0) < \bar{X}_l\right)$ can be *"locally limiting"* if $a_l F_l(X_l(0)) < a_k \bar{c}_k(X_k(0))$. The current utility and the maximin value are given by the production of this locally limiting resource (which may not be the globally limiting one). The smallest stock for resource k such that resource l is limiting is denoted by $X_k^\sharp(X_l(0)) < X_k(0)$ such that $a_k F_k(X_k^\sharp) = a_l F_l(X_l(0))$. One has $m(X_k(0), X_l(0)) = m(X_k^\sharp(X_l(0)), X_l(0))$. The non-limiting resource has no marginal maximin value above stock $X_k^\sharp(X_l(0))$. Its stock is redundant, even though it is below its production peak.

Note that in the case of complementarity the condition for a globally or locally limiting resource involves total productivity levels F, and not marginal values F'. Marginal values do not govern the decision as in the previous cases. Since marginal values are nil, conditions on quantities and not prices are required.

As compared with the simple fishery (with one stock), having two complementary renewable resources increases the possibilities of redundancy for the stocks, either globally (one resource stock may be redundant if the other resource production peak is low) or locally (one stock is redundant if the other's current sustainable consumption is low).

The maximin value is a non-decreasing function in (X_i, X_j) with an upper bound $\bar{m} = a_j F_j\left(\bar{X}_j\right)$ for stocks above $\left(X_i^*, \bar{X}_j\right)$. Any state above this level is characterized by a double redundancy, with nil shadow values. When utility is limited locally by one of the resource stocks, for the non-limiting stock k, $\mu_k = 0$ but for the limiting stock l, $\mu_l > 0$. The limiting stock only has a positive maximin value.

Graphical representation These results are illustrated in Figure 3, with resource 2 being globally limiting $\left(a_1 F_1\left(\bar{X}_1\right) > a_2 F_2\left(\bar{X}_2\right)\right)$. All the representations are derived from the results above with $i = 1$ and $j = 2$. The hatched area South-East of point $M = \left(X_1^*, \bar{X}_2\right)$ corresponds to states in which both stocks are redundant. From such states, one can temporarily increase the utility above the maximin value, along a non-regular maximin path. Both stocks' shadow values are nil.

The curve $0M$, from $(0,0)$ to $\left(X_1^*, \bar{X}_2\right)$, represents the states for which $a_1F_1(X_1) = a_2F_2(X_2)$. For these states, both resources limit utility (no redundancy). The maximin solution is to stay at the steady state (X_1, X_2). The corresponding sustained level of utility (the maximin value) is given by $U(c_1, c_2) = a_1F_1(X_1) = a_2F_2(X_2)$. The maximin value increases along $0M$, from 0 at $(0,0)$ to $\bar{m} = m(X_1^*, \bar{X}_2)$. Both stocks have nil shadow values because increasing one stock alone does not increase the maximin value.

For all states East of the curve $0M$, resource 2 is locally limiting. A part of stock X_1 is redundant. Maximin value is given by the point on the curve directly west. Stock X_2 has a positive marginal maximin value. For states South of the curve, resource 1 is limiting, with a similar interpretation. Iso-maximin curves are given by perpendicular lines starting from any state on $0M$.

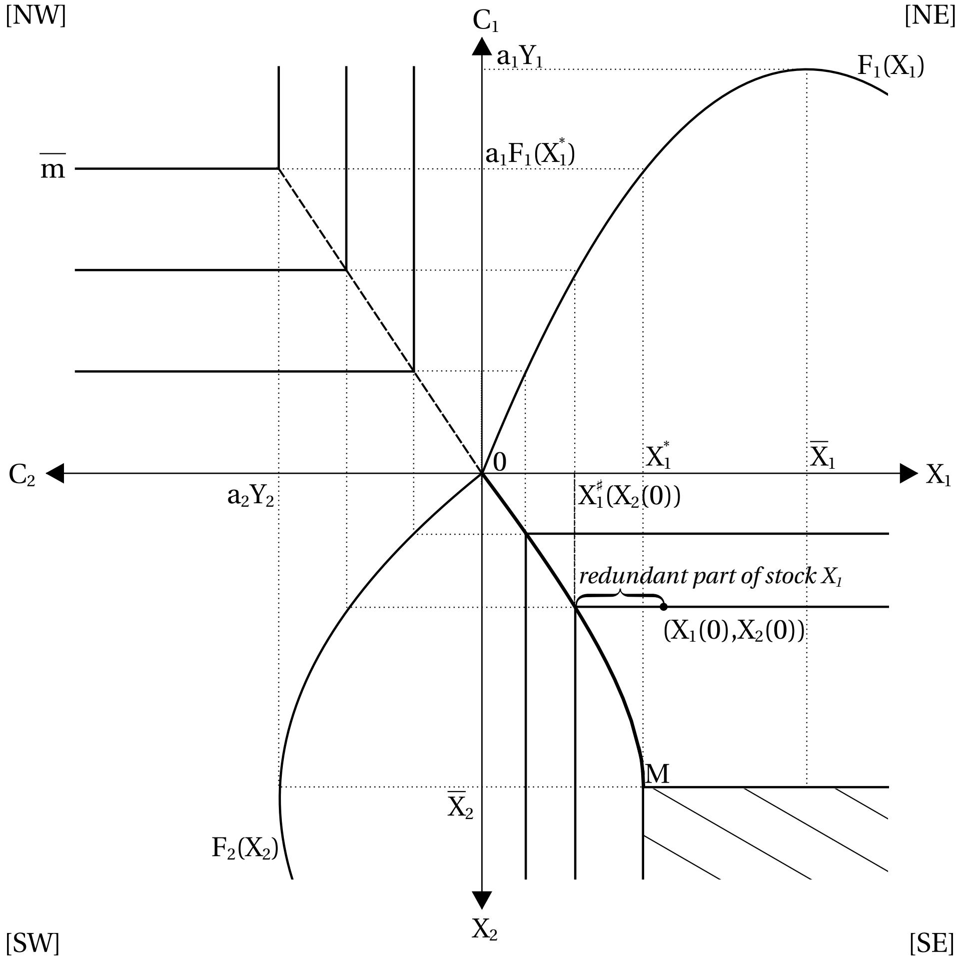


Figure 3: Perfect complements

4 Discussion: Accounting for sustainability improvement and decline

From the previous results on the maximin solution, we can discuss the consequences of consumption choices on the evolution of the sustainable level of utility. This provides insights for non-maximin paths and sustainability accounting.

Cairns and Martinet (2014) described the interplay between consumption, the maximin value and sustainability improvement, i.e., an increase over time of the sustainable level of utility, along any economic path. Net investment at maximin shadow values is a measure of sustainability improvement. For any economic state, it is possible to define the consumption levels resulting in a positive net investment. We consider regular paths for the neoclassical benchmark in the following discussion.

Assume that, for a given economic state (X_1, X_2), the maximin consumption levels (c_1^*, c_2^*) are known. These decisions, which satisfy $U(c_1^*, c_2^*) = m(X_1, X_2)$, can be used as a reference point. To do so, consider the iso-utility curve $U(c_1, c_2) = m(X_1, X_2)$. At point (c_1^*, c_2^*), one has $\frac{\mu_1}{\mu_2} = \frac{U'_{c_1}}{U'_{c_2}}$ (Result 1). From the definition of net investment (eq. 4), we derive the condition for non-negative net investment.

$$\dot{m} = \mu_1[F_1(X_1) - c_1] + \mu_2[F_2(X_2) - c_2] \geq 0 \quad \Leftrightarrow \quad c_2 \leq \frac{\mu_1}{\mu_2}[F_1(X_1) - c_1] + F_2(X_2) ,$$

where X_1, X_2, μ_1 and μ_2 are given. This gives us a linear relationship between c_1 and c_2 for the condition $\dot{m} = 0$.

Fig. 4 depicts the possible consumption decisions along with their consequences for sustainability improvement. In the consumption map (NW quadrant), the line $\dot{m} = 0$ is tangent to the iso-utility curve $U(c_1, c_2) = m(X_1, X_2)$ at (c_1^*, c_2^*). This delimits four areas, labeled 1, 2 and 3 on the Figure. Area 1 corresponds to consumption decisions associated with a sustainable utility ($U(c_1, c_2) < m(X_1, X_2)$) and a positive net investment at maximin shadow values, i.e., to sustainability improvement. Area 2 corresponds to decisions with $U(c_1, c_2) > m(X_1, X_2)$ and necessarily entails sustainability decline, i.e., $\dot{m} < 0$ (see Cairns and Martinet, 2014). The two areas labeled 3 correspond to consumption decisions that induce a sustainability decline in spite of the fact that the resulting utility is lower than the maximin value and thus would be sustainable or even allow an improvement of sustainability with different choices.

The same analysis can be done on the perfect substitutability and complementarity cases. The corresponding Figures are provided in the Appendix.

In our model where investment comes from forgone consumption, reducing current utility below the maximin level benefits future sustainability only if this 'sacrifice' results in an increase of the maximin value. Sustainability improvement is measured by net maximin investment. Maximin shadow values can be used as accounting prices along any path, and provide a valuable information on sustainability improvement.

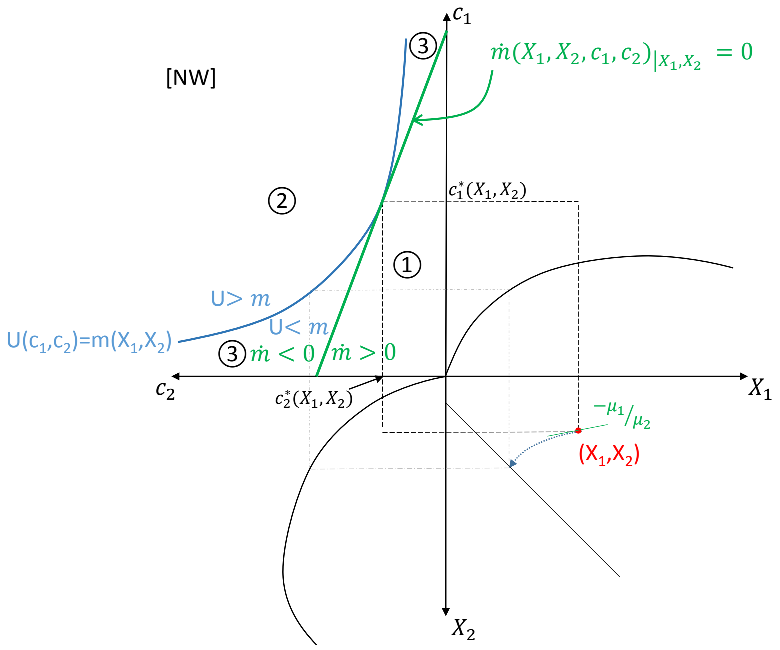


Figure 4: Utility, maximin and sustainability improvement

Of course, such an insight can be drawn only from an accounting system based on maximin shadow values, which requires that (i) these values are properly defined and (ii) these values are computable.

The analysis in this paper stresses that finding these values is a difficult task, as for any other optimization problem. But this challenge can be overtaken with proper numerical tools. Given the theoretical characterization of the maximin solution in this paper, and given the recursive structure of the maximin objective, a Bellman algorithm could be used to compute the (approximate) maximin values and shadow values. More generally, there are strong links between maximin and viability (Doyen and Martinet, 2012) and all the numerical tools for set-valued analysis could be used to solve maximin problems.

Moreover, the degree of substitutability of consumption goods in utility strongly influences maximin shadow values, in particular with respect to potential non-regularities.

Our results show that substitutability reduces the occurrence of non-regularities due to redundant stock for single-peak technologies. When one stock is below its production peak whereas another stock is over-abundant, the latter can be depleted to build up the former. This feature can be generalized to more than two sectors. Also, when one of the consumption goods is produced by a technology with no production peak (with $F'(X) > 0$ for all capital stocks, regardless of whether there is a finite asymptote), and this good has substitutes in utility, none of the capital stocks producing these substitutes can be redundant, whether or not they are produced by single-peaked technologies. There is no non-regularity is such an economy. This is encouraging for building a sustainability accounting system in the world described by the proponents of Weak Sustainability (Neumayer, 2010). On the contrary, whatever the type of technology (single-peaked or not), complementarity induces non-regularity. In the world depicted by the proponents of Strong Sustainability (Neumayer, 2010), a sustainability accounting system would be useless, and sustainability requires focusing only on the availability (i.e., quantities) of the limiting resource, possibly the environment.

5 Conclusion

The idea of sustainability has resonance with many people. "Progress" and not decline, has been sought by ambitious generations, but some have questioned whether continuing progress might some day give way to a maintenance objective, within natural constraints. These are economic notions and can be addressed using economic tools.

Sustaining utility is the very objective of a maximin problem. Even if the idea is not recent, maximin has been applied to only a handful of problems. In this paper, we characterize the maximin solution for an economy with two separate sectors (e.g., manufactured sectors or natural resources). The two sectors interact only indirectly, through the utility derived from the consumption of the two produced goods. We characterize the maximin trajectory and the maximin shadow values. This shadow values have an interest in themselves, independently of the maximin trajectory. They can be used *along any trajectory* to compute Genuine Savings at maximin prices, i.e., the evolution of the capacity of the economy to sustain economic well-being.

Non-regularity may be a persistent phenomenon in the study of sustainability. The investigation in the present paper appears to imply that non-regularity can arise where there is complementarity between different consumption assets. Environmentalists insist that produced capital is not substitutable for natural capital, and that this non-substitutability jeopardizes the sustainability of an economy relying on non-renewable assets. This view is confirmed in the maximin framework by Solow (1974), who showed that sustaining utility is possible in an economy with a non-renewable input only if the accumulated capital is a substitute for the resource. Our model extends these concerns to the case in which substitutability between natural and produced consumption goods is limited to provide utility.

Moreover, environmentalists may be chagrined to learn that, (a) where prices cannot be defined and decentralized, it is because there is a non-regularity in which some actions on quantities are non-unique and difficult to characterize, and that (b) seeking to attain maximum sustainable yields of renewable resources may not be even close to socially desirable and that (c) if indifference curves meet axes, extinction may be optimal.

It is clear that solving for the maximin level of utility for a modern economy, with all its various assets, consumption goods, production techniques, etc., would be a formidable task. For goods bought and sold in a market, it would not do to substitute market prices for the shadow values. Whatever may be maximized in a market is almost surely not the minimum level of all future levels of utility. In any case, market outcomes are obviously not optimal. Therefore, sustainability is practically and conceptually an extraordinarily difficult problem. Even if the shadow values could be found for the modern world, it is inconceivable that they could be implemented in a global political economy that cannot agree on or implement a price for even a single good, atmospheric carbon. Still, the study of maximin solutions is useful because it provides insight into how to evaluate the relative contributions of the different types of capital in attaining the objective. It systematically confronts the complicated interactions of assets and the difficulty of achieving sustainability. It indicates how shadow values may differ from market prices and hence how the latter may misdirect decisions in the economy.

A Appendix: Proofs, mathematical details and extensions

A.1 General results

Proof of Lemma 1. The dynamic path $\dot{X}_i = 0$ driven by decisions $c_i = F_i(X_i)$ is feasible and yields the constant utility $U(F_1(X_1), F_2(X_2))$. This provides a lower bound for the maximin value. ∎

Proof of Lemma 2. This is a direct result from Lemma 1 and the dynamics. ∎

A.2 The neoclassical benchmark: mathematical details and proofs

Taking the sustained utility level u as a control parameter and denoting the adjoint variables of the stocks by μ_i, one can write the Hamiltonian associated with Problem (2) as

$$\mathcal{H}(X, c, \mu) = \mu_1 \dot{X}_1 + \mu_2 \dot{X}_2 = \mu_1 \left(F_1(X_1) - c_1\right) + \mu_2 \left(F_2(X_2) - c_2\right) \ . \tag{13}$$

Let ω be the multiplier associated with the constraint (3), so as to write the Lagrangian as

$$\mathcal{L}(X, c, \mu, u, \omega) = \mathcal{H}(X, c, \mu) + \omega \left(U(c_1, c_2) - u\right) \ . \tag{14}$$

The necessary conditions are, for $i = 1, 2$ and for any time t, as follows.

$$\frac{\partial \mathcal{L}}{\partial c_i} = 0 \ ; \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial X_i} = -\dot{\mu}_i \ ; \tag{16}$$

$$\int_0^\infty \left(-\frac{\partial \mathcal{L}}{\partial u}\right) \mathrm{d}s = \int_0^\infty \omega(s)\, \mathrm{d}s < \infty \ ; \tag{17}$$

$$\lim_{t\to\infty} \mu_i X_i = 0 \ ; \tag{18}$$

$$\lim_{t\to\infty} \mathcal{H}(X, c, \mu) = 0 \ ; \tag{19}$$

along with the usual complementary slackness conditions ($\mu_i \geq 0,\ i = 1, 2$);

$$U(c_1, c_2) - u \geq 0 \ , \quad \omega \geq 0 \ , \quad \omega\left(U(c_1, c_2) - u\right) = 0 \ . \tag{20}$$

The optimality conditions above can be given economic meanings in a regular problem, when $(\mu_1, \mu_2, \omega) \neq (0, 0, 0)$.

Proof of Result 1. From the equation (15), we get for $i = 1, 2$;

$$\mu_i = \omega U'_{c_i} \ . \tag{21}$$

Dividing through both sides of the equation, it comes $\frac{\mu_1}{\mu_2} = \frac{U'_{c_1}}{U'_{c_2}}$. ∎

Proof of Result 2. From the equation (16), $\dot{\mu}_i = -\mu_i F'_i(X_i)$, therefore $\frac{\dot{\mu}_i}{\mu_i} = -F'_i(X_i),\ i = 1, 2$. ∎

Proof of Result 3. Taking the logarithmic derivative of eq. (21) gives $\frac{\dot{\mu}_i}{\mu_i} = \frac{\dot{\omega}}{\omega} + \frac{\dot{\left[U'_{c_i}\right]}}{U'_{c_i}}$. Substituting $\frac{\dot{\mu}_i}{\mu_i}$ by the expression obtain in Result 2, we obtain, for $i = 1, 2$

$$-\frac{\dot{\omega}}{\omega} = F'_i(X_i) + \frac{\dot{\left[U'_{c_i}\right]}}{U'_{c_i}} \ . \tag{22}$$

∎

Proof of Result 4. At the steady state (X_1^*, X_2^*), $\dot{c}_i = 0$ and hence $\dot{\left[U'_{c_i}\right]} = 0, i = 1, 2$. It follows from eq. (22) that $-\frac{\dot{\omega}}{\omega} = F'_1(X_1^*) = F'_2(X_2^*)$. ∎

Proof of Result 5. The converse of Hartwick's rule (eq. 5) implies that one stock increases while the other decreases, i.e., $c_i < F_i(X_i)$ and $c_j > F_j(X_j)$. Since the 'equity' shadow value ω is the same for all stocks, one has from eq. (22) the equality $F'_i(X_i) + \frac{\dot{\left[U'_{c_i}\right]}}{U'_{c_i}} = F'_j(X_j) + \frac{\dot{\left[U'_{c_j}\right]}}{U'_{c_j}}$, which gives us

$$F'_i - F'_j = \dot{c}_i \left(\frac{U''_{c_j, c_i}}{U'_{c_j}} - \frac{U''_{c_i, c_i}}{U'_{c_i}} \right) - \dot{c}_j \left(\frac{U''_{c_i, c_j}}{U'_{c_i}} - \frac{U''_{c_j, c_j}}{U'_{c_j}} \right) \ . \tag{23}$$

Under the properties of the utility function (increasing, concave, and $U''_{c_1,c_2} > 0$), the expressions in both parenthesis are positive. If $F'_i > F'_j$, then a simple rearrangement yields

$$\dot{c}_i > \dot{c}_j \left(\frac{U''_{c_i,c_j}}{U'_{c_i}} - \frac{U''_{c_j,c_j}}{U'_{c_j}} \right) \Big/ \left(\frac{U''_{c_j,c_i}}{U'_{c_j}} - \frac{U''_{c_i,c_i}}{U'_{c_i}} \right) .$$

This condition means that $\dot{c}_j > 0$ would imply $\dot{c}_i > 0$. However, both levels of consumption cannot increase along the maximin path, where utility is constant over time. On the other hand, having $\dot{c}_j < 0$ is compatible with having $\dot{c}_i > 0$. Eventually, either $F'_i = F'_j$, or else $c_i < F_i(X_i)$. (If $c_i > F_i(X_i)$ always, the stock is exhausted in finite time.) ∎

Proof of Result 6. Along the optimal path, for any state $(X_1(t), X_2(t))$, the partial derivatives of the maximin value equal the shadow price of the stocks, i.e., $m'_{X_i}|_t = \mu_i(t)$ (Cairns and Long, 2006). From condition (6), one gets $\left.\frac{m'_{X_1}}{m'_{X_2}}\right|_t = \frac{\mu_1}{\mu_2} = \frac{U'_{c_1}}{U'_{c_2}}$. At the optimum, the marginal rate of transformation equals the marginal rate of substitution as in all economic optimizations. Along the optimal path, $\mu_1 \dot{X}_1 + \mu_2 \dot{X}_2 = 0$ (Cairns and Long, 2006). By eq. (6), if $\dot{X}_i \neq 0$ for $i = 1, 2$,

$$\frac{\dot{X}_1}{\dot{X}_2} = -\frac{\mu_2}{\mu_1} = -\frac{U'_{c_2}}{U'_{c_1}} < 0 . \tag{24}$$

Along a regular maximin path, utility is constant. One has $\frac{\mathrm{d}U(c_1,c_2)}{\mathrm{d}t} = \dot{c}_1 U'_{c_1} + \dot{c}_2 U'_{c_2} = 0$. This gives us

$$\frac{\dot{c}_1}{\dot{c}_2} = -\frac{U'_{c_2}}{U'_{c_1}} < 0 . \tag{25}$$

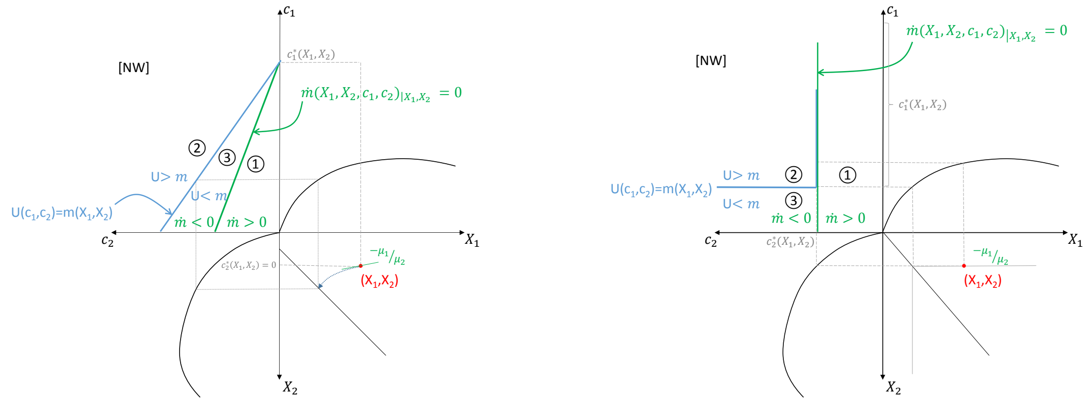
Combining conditions (24) and (25), one gets

$$\frac{\dot{X}_1}{\dot{X}_2} = \frac{\dot{c}_1}{\dot{c}_2} \quad \Leftrightarrow \quad \frac{\mathrm{d}X_1}{\mathrm{d}X_2} = \frac{\mathrm{d}c_1}{\mathrm{d}c_2} . \tag{26}$$

From Result 2 (eq. (9)), we know that, whenever resource X_i is more productive at the margin than resource X_j, i.e., $F'_i(X_i) > F'_j(X_j)$, then $\frac{\dot{\mu}_i}{\mu_i} - \frac{\dot{\mu}_j}{\mu_j} = -F'_i(X_i) + F'_j(X_j) < 0$. That gives us information on the shape of the iso-maximin values in the state space for regular paths. The previous equation tells us that the relative price $\frac{\mu_i}{\mu_j}$ has to decrease, so, from the condition (26), the tangents (in absolute value) to the paths in the state space (X_i, X_j) have to decrease as well.

∎

A.3 Sustainability improvement: graphical illustrations



(a) Perfect substitutes (b) Perfect complements

Figure 5: Utility, maximin and sustainability improvement on special cases

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B SUPPLEMENTARY MATERIALS

B.1 Perfect substitutability: mathematical details and proofs

Consider the case $U(c_1, c_2) = a_1c_1 + a_2c_2$. The Lagrangean associated with the maximin problem is linear in the decisions, which implies bang-bang solutions.

$$\mathcal{L}(X, c, \mu, u, \omega) = \mu_1 (F_1(X_1) - c_1) + \mu_2 (F_2(X_2) - c_2) + \omega (a_1c_1 + a_2c_2 - u) \ . \tag{27}$$

Optimality conditions The necessary conditions (15) become, for $i = 1, 2$:

$$\frac{\partial \mathcal{L}}{\partial c_i} = -\mu_i + a_i\omega \leq 0 \ , \ c_i \geq 0 \ , \ c_i \frac{\partial \mathcal{L}}{\partial c_i} = 0 \ . \tag{28}$$

The other conditions are unchanged. The complementary slackness reads

$$\omega \geq 0 \ , \ \omega (a_1c_1 + a_2c_2 - u) = 0 \ . \tag{29}$$

From conditions (16), we get

$$-\frac{\partial \mathcal{L}}{\partial X_i} = -\mu_i F_i'(X_i) = \dot{\mu}_i, \quad \Leftrightarrow \quad \frac{\dot{\mu}_i}{\mu_i} = -F_i'(X_i) \ . \tag{30}$$

By Lemma 1, $c_1 = c_2 = 0$ cannot be solution of the problem.

Proof of Result 7. For a regular path, $(\mu_1, \mu_2, \omega) \neq (0, 0, 0)$, $\omega > 0$ and thus $U(c_1, c_2) = m(X_1, X_2)$. Suppose that both consumption levels are strictly positive, i.e., $c_i > 0, i = 1, 2$. Eq (28) then implies that $\frac{\partial \mathcal{L}}{\partial c_i} = 0$ and thus that $\omega = \frac{\mu_1}{a_1} = \frac{\mu_2}{a_2}$. Taking the time derivative of ω, we get $\frac{\dot{\omega}}{\omega} = \frac{\dot{\mu}_1}{\mu_1} = \frac{\dot{\mu}_2}{\mu_2}$. Moreover, from eq. (30), we get the conditions $\frac{\dot{\mu}_i}{\mu_i} = -F_i'(X_i)$, $i = 1, 2$. Combining these conditions, we can state that a solution with $c_1 > 0$ and $c_2 > 0$ is possible only for states (X_1, X_2) such that $F_1'(X_1) = F_2'(X_2)$. This is dynamically possible only for a steady state, with $c_1 = F_1(X_1^*)$ and $c_2 = F_2(X_2^*)$. ∎

Proof of Result 8. Consider a state $(X_i, X_j) >> (0, 0)$ such that $F_i'(X_i) > F_j'(X_j)$ and $F_i'(X_i) > 0$. This last condition ensures that the more productive resource is below its production peak $\bar{X}_i$.[16] We shall demonstrate that, under these conditions, stock X_j is consumed alone while stock X_i builds up as long as the previous inequality holds by proving that the opposite is not possible.

Assume that $c_i > 0$ and $c_j = 0$. Let us denote the maximin value by m. Along such a maximin path, one would have $U(c_1, c_2) = a_i c_i = m$, which is constant. Therefore, c_i would be constant. By Lemmata 1 and 2, $c_i > F_i(X_i)$. Therefore, $\frac{\mathrm{d}X_i}{\mathrm{d}t} = F_i(X_i) - c_i < 0$. Also, $\frac{\mathrm{d}^2 X_i}{(\mathrm{d}X_i)^2} = F_i'(X_i)\frac{\mathrm{d}X_i}{\mathrm{d}t} - \frac{\mathrm{d}c_i}{\mathrm{d}t} = F_i'(X_i)\frac{\mathrm{d}X_i}{\mathrm{d}t} < 0$. Stock X_i would exhausted in finite time (τ). After that time, utility would be derived only from the sustained consumption of stock X_j. At exhaustion time, stock X_j would have increased to some level $X_j^* \equiv X_j(\tau)$ such that $F_j(X_j^*) = m/a_j$, allowing consumption c_j to sustain exactly the maximin utility. The equilibrium state would be $(0, X_j^*)$.

Let us make a step backward and examine the states through which the path goes just prior to exhaustion. The dynamics before exhaustion is

$$\dot{X}_i = F_i(X_i) - c_i \;\Leftrightarrow\; \mathrm{d}X_i = (F_i(X_i) - m/a_i)\,\mathrm{d}t\;,$$
$$\dot{X}_j = F_j(X_j) \;\Leftrightarrow\; \mathrm{d}X_j = F_j(X_j)\mathrm{d}t\;.$$

Consider an infinitesimal time lapse $\mathrm{d}t$. At time $\tau - \mathrm{d}t$, stock X_i is equal to $\tilde{X}_i = \mathrm{d}X_i = \frac{m}{a_i}\mathrm{d}t$. Stock X_j is equal to $\tilde{X}_j = X_j^* - \mathrm{d}X_j = X_j^* - \frac{m}{a_j}\mathrm{d}t$.

By Lemma 1, we know that the maximin value at time $\tau - \mathrm{d}t$ is greater than or equal to the equilibrium utility of state $(\tilde{X}_i, \tilde{X}_j)$. Let us denote this utility level by $\tilde{U} = U\left(\tilde{X}_i, \tilde{X}_j\right) = a_i F_i(\frac{m}{a_i}\mathrm{d}t) + a_j F_j(X_j^* - \frac{m}{a_j}\mathrm{d}t)$. We have $m(\tau - \mathrm{d}t) \geq \tilde{U}$. By substracting

[16] If both marginal productivities are negative, both stocks are above their MSY value $\bar{X}$ and we are in a non-regular case in which the two stocks decrease and converge to a steady state at $\left(\bar{X}_i, \bar{X}_j\right)$. Result 8 is relevant only if one stock is below its MSY level.

$m(\tau)$ from both sides of the equation, we obtain the following.

$$\begin{aligned}
m(\tau - \mathrm{d}t) - m(\tau) &\geq \tilde{U} - m(\tau) \ ; \\
&\geq a_i F_i\left(\frac{m}{a_i}\mathrm{d}t\right) + a_j F_j\left(X_j^* - \frac{m}{a_j}\mathrm{d}t\right) - a_j F_j(X_j^*) \ ; \\
&\geq a_i\left(F_i\left(0 + \frac{m}{a_i}\mathrm{d}t\right) - F_i(0)\right) + a_j\left(F_j\left(X_j^* - \frac{m}{a_j}\mathrm{d}t\right) - F_j(X_j^*)\right) \ ; \\
&\geq m\mathrm{d}t\frac{F_i\left(0 + \frac{m}{a_i}\mathrm{d}t\right) - F_i(0)}{\frac{m}{a_i}\mathrm{d}t} - m\mathrm{d}t\frac{F_j\left(X_j^* - \frac{m}{a_j}\mathrm{d}t\right) - F_j(X_j^*)}{-\frac{m}{a_j}\mathrm{d}t} \ .
\end{aligned}$$

Let us write $\epsilon_i = \frac{m}{a_i}\mathrm{d}t$ and $\epsilon_j = -\frac{m}{a_j}\mathrm{d}t$, as well as $\tilde{\tau} = \tau - \mathrm{d}t$ (and thus $\tau = \tilde{\tau} + \mathrm{d}t$). We get

$$\begin{aligned}
m(\tilde{\tau}) - m(\tilde{\tau} + \mathrm{d}t) &\geq m\mathrm{d}t\frac{F_i(0+\epsilon_i) - F_i(0)}{\epsilon_i} - m\mathrm{d}t\frac{F_j(X_j^* + \epsilon_j) - F_j(X_j^*)}{\epsilon_j} \ ; \\
\Leftrightarrow \quad \frac{1}{m}\left(\frac{m(\tilde{\tau}+\mathrm{d}t) - m(\tilde{\tau})}{\mathrm{d}t}\right) &\leq \frac{F_j(X_j^* + \epsilon_j) - F_j(X_j^*)}{\epsilon_j} - \frac{F_i(0+\epsilon_i) - F_i(0)}{\epsilon_i} \ .
\end{aligned}$$

By taking the limits $\epsilon_i, \epsilon_j, \mathrm{d}t \to 0$, we obtain

$$\frac{\dot{m}}{m} \leq F_j' - F_i' < 0 \ .$$

As the maximin value cannot decrease along a maximin path, we get a contradiction. We thus can say that if $F_i'(X_i) > F_j'(X_j)$, $c_i = 0$ and $c_j > 0$.

By regularity, $m = U(c_1, c_2) = a_1c_1 + a_2c_2$. Thus, $c_j(t) = \frac{m(X_1(t), X_2(t))}{a_j}$. ∎

Proof of Result 9. According to Result 8, if $F_i'(X_i) > F_j'(X_j)$, $c_i = 0$ and $c_j = \frac{m}{a_j}$. Assume that there is a steady state such that $U(\cdot) = m \leq a_iF_i(\underline{X}_i)$, then this steady state must satisfy $X_j^* = 0$ and $X_i^* \leq \underline{X}_i$. Otherwise, it would satisfy $F_i'(X_i) = F_j'(X_j)$ and we would have $U \geq a_iF_i(\underline{X}_i)$. Given that $X_i \leq \underline{X}_i$, we have $F_i'(X_i) > F_j'(X_j)$ and we exhaust X_j. ∎

Proof of Result 10. From the necessary conditions, we have $\omega = \frac{\mu_j}{a_j}$ and $\frac{\dot{\omega}}{\omega} = \frac{\dot{\mu}_j}{\mu_j} = -F_j'(X_j)$. One also has $-\mu_i + a_i\omega \leq 0 \Leftrightarrow \mu_i \geq \frac{a_i}{a_j}\mu_j \Leftrightarrow \frac{\mu_i}{\mu_j} \geq \frac{a_i}{a_j}$. Given that $\mu_i = m'_{X_i}$ and $a_i = U'_{c_i}$, $i = 1, 2$, one gets that the marginal rate of transformation of maximin value is greater than the marginal rate of substitution of consumption.

Let us define the relative price $\pi \equiv \frac{\mu_i}{\mu_j}$ and examine how this relative price evolves over time. Given conditions (30), we obtain

$$\frac{\dot{\pi}}{\pi} = \frac{\dot{\mu}_i}{\mu_i} - \frac{\dot{\mu}_j}{\mu_j} = -F_i'(X_i) + F_j'(X_j) < 0 \; . \tag{31}$$

The relative price of stock j rises. Since one harvests only the stock j, $F_j'(X_j)$ increases and $F_i'(X_i)$ decreases until the marginal levels of growth are equal.[17] In other words, if one begins with a resource relatively abundant and with a low relative (shadow) price, one has to harvest only that resource at a level that keeps utility constant. Meanwhile the other one grows until the steady state at which both resources have the same marginal productivity. ∎

Proof of Result 11. From condition (26), the optimal path is characterized by

$$-\frac{\mathrm{d}X_2}{\mathrm{d}X_1} = \frac{\mu_1}{\mu_2} \; , \tag{32}$$

but contrary to the general case, here the utility function and the maximin value do not have the same shape. ∎

B.2 An example: Perfect substitutability with AK-technology and a renewable resource

We have a linear utility function; $U(c_1, c_2) = a_1 c_1 + a_2 c_2$; a reneawable resource; $F_1(X_1) = rX_1(1 - X_1))$; and a manufactured capital; $F_2(X_2) = AX_2$. The steady state condition for a non-exhaustion ($F_i' = F_j'$) is $\underline{X}_1 = \frac{1}{2}\left(1 - \frac{A}{r}\right)$.

We have two types of paths;

1. the ones that end up with $X_2^* > 0, X_1^* = \underline{X}_1$;
2. the ones that end up with $X_2^* = 0, X_1^* \leq \underline{X}_1$ (manufactured capital is exhausted).

[17] One cannot pass from $\frac{\mu_i}{\mu_j} > \frac{a_i}{a_j}$ to $\frac{\mu_i}{\mu_j} < \frac{a_i}{a_j}$ since the equality is a steady state of the system.

Case 1

Let m be the maximin value. We know that in such a case; $c_1 = 0$ and $c_2 = \frac{m}{a_2}$, i.e. $\dot{X}_1 = rX_1(1 - X_1)$ and $\dot{X}_2 = AX_2 - \frac{m}{a_2}$. It may be checked that the solution of theses equations is of the form $X_1(t) = \frac{X_1(0)}{X_1(0)+\mathrm{e}^{-rt}(1-X_1(0))}$ and $X_2(t) = \left(X_2(0) - \frac{m}{a_2 A}\right)\mathrm{e}^{At} + \frac{m}{a_2 A}$.

From $X_1(t)$, we may obtain the inverse function $t^* = -\frac{1}{r}\ln\left(\frac{1-X_1}{X_1}\frac{X_1(0)}{1-X_1(0)}\right)$ and compute it at the steady state $t^* = -\frac{1}{r}\ln\left(\frac{1+\frac{A}{r}}{\left(1-\frac{A}{r}\right)\left(\frac{1}{X_1(0)}-1\right)}\right)$, which may serve to compute $X_2^* = X_2(t^*)$ as;

$$X_2^* = \left(X_2(0) - \frac{m}{a_2 A}\right)\left(\frac{1+\frac{A}{r}}{\left(1-\frac{A}{r}\right)\left(\frac{1}{X_1(0)}-1\right)}\right)^{-\frac{A}{r}} + \frac{m}{a_2 A}\ . \tag{33}$$

Besides, at the steady state, we have $c_1^* = F_1(X_1^*) = r\frac{1}{2}\left(1-\frac{A}{r}\right)\left(1-\frac{1}{2}\left(1-\frac{A}{r}\right)\right) = \frac{r}{4}\left(1-\frac{A^2}{r^2}\right)$. Then, we may compute the maximin value at the steady state

$$m^* = a_1 c_1^* + a_2 c_2^* = a_1\frac{r}{4}\left(1-\frac{A^2}{r^2}\right) + a_2 A X_2^*\ . \tag{34}$$

We substitue (33) into (34) to get the maximin value function;

$$m(X_1(0), X_2(0)) = a_1\Gamma\left(\frac{1}{X_1(0)}-1\right)^{-\frac{A}{r}} + a_2 A X_2(0)\ ,$$

with $\Gamma = \frac{r}{4}\left(1+\frac{A}{r}\right)^{1+\frac{A}{r}}\left(1-\frac{A}{r}\right)^{1-\frac{A}{r}}$.

Knowing that shadow prices are partial derivatives of the maximin value function, we have;

$$\mu_1 = \frac{\partial m(X_1(0), X_2(0))}{\partial X_1(0)} = a_1\frac{A}{r}\Gamma\left(1-X_1(0)\right)^{-1-\frac{A}{r}} X_1(0)^{-1+\frac{A}{r}}\ ;$$

$$\mu_2 = \frac{\partial m(X_1(0), X_2(0))}{\partial X_2(0)} = a_2 A\ .$$

Case 2

We want to find paths that end up with $X_2^* = 0$.

$$X_2^* = 0 \quad \Leftrightarrow \quad \left(X_2(0) - \frac{m}{a_2 A}\right) \mathrm{e}^{At^*} + \frac{m}{a_2 A} = 0 \quad \Rightarrow \quad t^* = \frac{1}{A} \ln\left(\frac{m}{m - a_2 A X_2(0)}\right) .$$

The corresponding resource stock is

$$X_1^* = \frac{X_1(0)}{X_1(0) + \mathrm{e}^{-\frac{r}{A}\ln\left(\frac{m}{m-a_2AX_2(0)}\right)}(1 - X_1(0))} \quad \Leftrightarrow \quad X_1^* = \frac{1}{1 + \left(1 - \frac{a_2AX_2(0)}{m}\right)^{\frac{r}{A}}\left(\frac{1}{X_1(0)} - 1\right)} .$$

Besides, we have; $m = a_1 F_1(X_1^*)$, which becomes

$$m = r\frac{a_1}{1 + \left(1 - \frac{a_2AX_2(0)}{m}\right)^{\frac{r}{A}}\left(\frac{1}{X_1(0)} - 1\right)}\left(1 - \frac{1}{1 + \left(1 - \frac{a_2AX_2(0)}{m}\right)^{\frac{r}{A}}\left(\frac{1}{X_1(0)} - 1\right)}\right) ;$$

$$\Leftrightarrow \quad m = \frac{ra_1\left(1 - \frac{a_2AX_2(0)}{m}\right)^{\frac{r}{A}}\left(\frac{1}{X_1(0)} - 1\right)}{\left(1 + \left(1 - \frac{a_2AX_2(0)}{m}\right)^{\frac{r}{A}}\left(\frac{1}{X_1(0)} - 1\right)\right)^2} .$$

Even in this simple framework, if the manufactured capital is exhausted, maximin value function and associated shadow prices are hard to find.

The figure 6 plots the non-exhaustion case (1) and the exhaustion one (2).[18]

[18] The dashed lines represent consumption paths, denoted respectively $C_{i,j}^{(1)}$ and $C_{i,j}^{(2)}$.

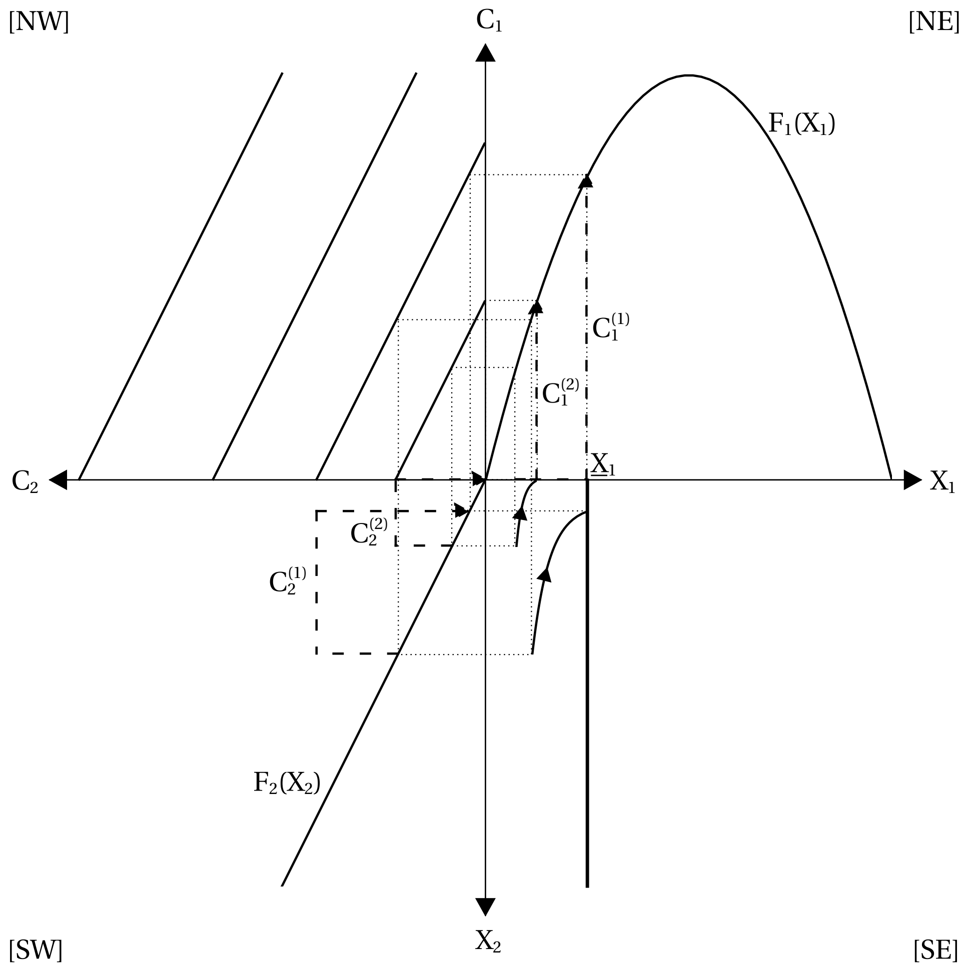

Figure 6: Example on A-K technology and a renewable resource