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Self-Allocation in Contests

Abstract

We consider contestants who must choose exactly one contest, out of several, to participate in. We show that when the contest technology is of a certain type, or when the number of contestants is large, a self-allocation equilibrium, i.e., one where no contestant would wish to change his choice of contest, results in the allocation of players to contests that maximizes aggregate equilibrium effort. For a class of oligopoly models that are equivalent to contests, this implies output maximization.

JEL-Codes: C720, D430, D440, D720, D740, L130.

Keywords: contests, self-allocation, effort maximization, quantity competition.

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1 Introduction

Many economically important activities take the form of contests, i.e., games where players invest effort or resources in trying to increase their probability of winning a prize or prizes. Examples include, but are not limited to, armed conflict (see, e.g., Skaperdas [8]), rent seeking (Tullock [13] is the seminal contribution), competition in R&D (see, e.g., Fullerton and McAfee [2]), sports competitions (see, e.g., Szymanski [11]), litigation (see, e.g., Tullock [12]), and certain types of imperfect competition in product markets. A large literature (see, e.g., Konrad [4] for a comprehensive survey) studies theoretical and empirical aspects of contests.

In this paper, we consider the hitherto relatively overlooked question of how players select which contest to enter, if they can only enter one contest out of several. Azmat and Möller [1] study a similar problem, but focus on how different contests might attract contestants of different abilities. In contrast, we study how the self-allocation of identical contestants affects global equilibrium effort, in particular when the number of contestants is large. Other papers concerned with large contests and effort maximization are Olszewski and Siegel [7, 6]. The problem studied in this paper is also related to the literature on how to allocate prizes in contests so as to maximize effort (see, e.g., Moldovanu and Sela [5]).

We can think of many settings in which the contest-selection problem might arise, such as the following.

- Athletes consider which out of several sports competitions to enter.
- Firms have to select a market to be active in.
- Lobbyists choose a bureaucrat or regulator to influence.
- Researchers in the cut-throat, publish-or-perish nightmare of academia select a topic to work on.

We find, in particular, that a self-allocation equilibrium may be such that it maximizes aggregate effort across contests—something that may be good or bad, from a welfare standpoint, depending on the intended application. If the contest technology is the "lottery" model introduced by Tullock [13], the effort-maximizing allocation of contestants to contests is always the equilibrium outcome. For a more general class of contests that has the property that it makes sense to talk about adding and subtracting players, it is always the equilibrium result in the limit as the number of contestants approaches infinity. That is, a large number of contestants who allocate themselves to contests spontaneously will do so in a fashion that leads to approximately the same aggregate equilibrium effort as in the allocation that maximizes global effort across contests.

2 Examples

2.1 Lottery contests

Consider a set of N identical risk neutral individuals, where N is "sufficiently large" in a sense to be made precise later on. Suppose there are K contests with strictly positive prizes $v_1, v_2, \dots, v_K$. Each individual can enter exactly one contest. If individual i and $N_k - 1$ others enter contest k, and i expends effort or resources x_{ki}, his probability of winning is

$$p_{ki}(x_{k1}, x_{k2}, \dots) = \begin{cases} x_{ki} / \sum_j x_{kj} & \text{if } \sum_j x_{kj} > 0 \\ 1/N_k & \text{otherwise.} \end{cases}$$

That is, the contest success function is the lottery contest made popular by Tullock [13].

Suppose x_{ki} comes at unit cost. Then contestant i's payoff function in contest k is

$$u_{ki} = p_{ki}(x_{k1}, x_{k2}, \dots) v_k - x_{ki}.$$

Assuming there are at least two contestants in contest k, there cannot be an equilibrium in which nobody expends anything. For suppose everyone else spends nothing. Then contestant i wins with probability $1/N_k$ if he also spends nothing, but with certainty if he spends some arbitrarily small amount. Since there must be some such small amount that would make spending profitable, it cannot be a best reply for contestant i to also spend nothing. Hence there is no equilibrium in which nobody spends anything.

We are therefore justified in characterizing contestant i's best-reply spending by the first-order condition

$$\frac{\partial u_{ki}}{\partial x_{ki}} = \frac{\sum_{j \neq i} x_{kj}}{\left(\sum_j x_{kj}\right)^2} v_k - 1 = 0.$$

Rearrangement reveals that each contestant's best reply depends only on aggregate spending. Hence there is a symmetric equilibrium where each individual spends

$$x_k = \frac{N_k - 1}{N_k^2} v_k,$$

and hence in aggregate

$$N_k x_k = \frac{N_k - 1}{N_k} v_k,$$

enjoying individual equilibrium utility

$$\frac{v_k}{N_k^2}.$$

In order for everyone to be satisfied with their choice of contest, it cannot be the case that a contestant would strictly prefer to participate in a different contest. Hence, ignoring the integer problem, in a self-allocation equilibrium it has to hold that

$$\frac{v_1}{N_1^2} = \frac{v_2}{N_2^2} = \ldots = \frac{v_K}{N_K^2}.$$

Consider now an allocation of individuals to contests that maximizes aggregate equilibrium effort

$$Z = \sum_k \frac{N_k - 1}{N_k} v_k$$

subject to the constraint

$$\sum_k N_k = N.$$

We can find the solution by forming the Lagrangian

$$L = \sum_k \frac{N_k - 1}{N_k} v_k - \lambda \sum_k N_k$$

with associated first-order conditions

$$\frac{\partial L}{\partial N_k} = \frac{v_k}{N_k^2} - \lambda = 0 \text{ for all } k.$$

Since this implies that equilibrium utility is the same in all contests, we have found that the self-allocation equilibrium maximizes aggregate effort.

Ignoring the integer problem is not innocuous when we are dealing with small numbers of contestants. Consider a setting with two contests with the same prize v and three contestants. If only integer allocations of players to contests are possible, in equilibrium two players will enter one contest and the single remaining player the other. Nobody would want to switch, since the player in the single-player contest would get an equilibrium utility of $v/9$ rather than v, and a player in the two-player contest could not improve his utility by switching. Aggregate effort across contests is $v/2$, but would have been $2v/3$ if all three players had been in the same contest. Hence under the integer restriction, self-allocation here not only does not maximize effort, but, in fact, minimizes it. (We owe this example to Jingfeng Lu.)

2.2 Oligopoly

The effort-maximization result has interesting implications for a class of oligopoly models. Consider Cournot, or quantity, competition in markets with isoelastic demand, i.e., where if q_{ki} is the output of firm i in market k, inverse demand in market k is given by

$$P_k(q_{k1}, q_{k2}, \ldots) = \frac{A_k}{\sum_j q_{kj}},$$

with $A_k > 0$.

Assume there are no fixed costs, and that all firms have access to the same production technology once they are in market k. If the constant average and marginal cost of producing in market k is $c_k > 0$, firm i's profit function in market k is then

$$\pi_{ki}(q_{k1}, q_{k2}, \ldots) = \frac{A_k}{\sum_j q_{kj}} q_{ki} - c_k q_{ki}.$$

Maximizing this with respect to q_{ki}, taking the output of other firms as given, is equivalent to maximizing the transformed profit function

$$\frac{\pi_{ki}(q_{k1}, q_{k2}, \ldots)}{c_k} = \frac{q_{ki}}{\sum_j q_{kj}} \frac{A_k}{c_k} - q_{ki}.$$

Letting $q_{ki} = x_{ki}$ for all i and $A_k/c_k = v_k$ for all k, we see that this is the same game as the lottery contest studied above. Hence under equilibrium self-allocation into markets, firms will (one is tempted to say "as if led by an invisible hand") allocate themselves so as to maximize aggregate output across markets, suggesting a kind of welfare theorem for this class of oligopoly models (again, of course, provided there are sufficiently many firms that ignoring the integer restriction is not problematic).

3 The general case

3.1 Contests

As before, suppose there are K contests with strictly positive prizes $v_1, \ldots, v_K$ and N risk neutral individuals who must each choose to enter one of the contests. If individual i and $N_k - 1$ others enter contest k, and i expends effort or resources x_{ki}, we now assume his probability of winning is given by the more general success function

$$p_{ki}(x_{k1}, x_{k2}, \ldots) := \begin{cases} f(x_{ki}) / \sum_j f(x_{kj}) & \text{if } \sum_j f(x_{kj}) > 0 \\ 1/N_k & \text{otherwise,} \end{cases}$$

where f is three times continuously differentiable and such that $f' > 0$, $f(0) = 0$, and $f'' \leq 0$.

This success function is close to the class axiomatized by Skaperdas [9], and seems the most general one that still allows us to conveniently scale a contest up or down by adding or subtracting identical players.

Individual i's payoff function is

$$u_i(k, x) := p_{ki} v_k - x_{ki}.$$

His first order condition for an interior maximum is

$$\frac{\partial u_i}{\partial x_{ki}} = \frac{f'(x_{ki}) \sum_{j \neq i} f(x_{kj})}{\left(\sum_j f(x_{kj})\right)^2} v_k - 1 = 0.$$

Szidarovszky and Okuguchi [10] show that there is a unique equilibrium in this setting. Furthermore, since contest k is symmetric it has a symmetric equilibrium where individual i sets x_{ki} such that it satisfies

$$\frac{f'(x_{ki})}{f(x_{ki})} \frac{N_k - 1}{N_k^2} v_k = 1.$$

We can define a solution function g implicitly by

$$\frac{f(g(N_k, v_k))}{f'(g(N_k, v_k))} = \frac{N_k - 1}{N_k^2} v_k.$$

Since we have $f' > 0$ and $f'' \leq 0$, the expression $f(x_{ki})/f'(x_{ki})$ is strictly increasing in x_{ki}, so $g(N_k, v_k)$ is well defined for all N_k, v_k. Furthermore, $g(N_k, v_k)$ is increasing in v_k and decreasing in N_k if $N_k \geq 2$, since

$$\frac{\partial}{\partial N_k} \left(\frac{N_k - 1}{N_k^2} \right) = \frac{2 - N_k}{N_k^3}$$

By definition of g the equilibrium utility in contest k is

$$\frac{v_k}{N_k} - g(N_k, v_k),$$

and the total effort in contest k is

$$N_k g(N_k, v_k).$$

3.2 Self-allocation equilibrium

We now consider the implications of allowing individuals to enter any one contest they like, costlessly. In equilibrium, no individual must then strictly prefer to join a different contest, and hence equilibrium utility has to be the same in all contests.

Ignoring the integer problem, assume N_k individuals enter contest k. If there is an equilibrium where at least one individual enters each contests then this *self-allocation equilibrium* $(N_1,\ldots,N_K)$ is characterized by

$$\frac{v_k}{N_k}-g(N_k,v_k)=\frac{v_{k+1}}{N_{k+1}}-g(N_{k+1},v_{k+1}),$$

$$\sum_{k=1}^{K}N_k=N,$$

and

$$N_1,N_2,\ldots,N_K\geq 1,$$

where N is the given total number of individuals.

Proposition 1 *Given K and $v_1,\ldots,v_K$ there is some N^* such that for all $N\geq N^*$ there is a unique self-allocation equilibrium where all contests are non-empty. For $N<N^*$ there is no such equilibrium. Furthermore, we have that*

$$N^*\leq\sum_{k=1}^{K}\frac{v_k}{\min\{v_1,\ldots,v_K\}}.$$

Proof. Define V_1 by

$$V_1(N_1,v_1):=\frac{v_1}{N_1}-g(N_1,v_1).$$

Then

$$\frac{\partial V_1(N_1,v_1)}{\partial N_1}=-\frac{v_1}{N_1^2}-\frac{\partial g(N_1,v_1)}{\partial N_1}.$$

If

$$\frac{\partial g(N_1,v_1)}{\partial N_1}\leq-\frac{v_1}{N_1^2},$$

then

$$\begin{aligned}\frac{\partial}{\partial N_1}\frac{f(g(N_1,v_1))}{f'(g(N_1,v_1))} &= \left(1-\frac{f(g(N_1,v_1))f''(g(N_1,v_1))}{f'(g(N_1,v_1))^2}\right)\frac{\partial g(N_1,v_1)}{\partial N_1}\\ &\leq -\frac{v_1}{N_1^2},\end{aligned}$$

where the inequality uses that $f \geq 0$ and $f'' \leq 0$ by assumption. Therefore we cannot have that $\partial g(N_1,v_1)/\partial N_1 \leq -v_1/N_1^2$ since the right hand side and the left hand side of the equation that defines g must have the same partial derivative with respect to N_1 and

$$\frac{\partial}{\partial N_1}\left(\frac{N_1-1}{N_1^2}v_1\right) = \frac{2-N_1}{N_1^3}v_1 > -\frac{v_1}{N_1^2}.$$

In a self-allocation equilibrium with just two contests and prices v_1 and v_2 we have to have that

$$V_1(N_1,v_1) = V_1(N_2,v_2), N_1+N_2 = N, \text{ and } N_1, N_2 \geq 1. \tag{1}$$

Assume, without loss of generality that $v_1 \geq v_2$. If

$$V_1(N-1,v_1) > v_2 = V_1(1,v_2),$$

then there is no solution to (1) since

$$V_1(N_1,v_1) \geq V_1(N-1,v_1) > V_1(1,v_2)$$

for all $N_1 \leq N-1$. If $V_1(N-1,v_1) \leq v_2$, then there is a unique solution to (1) since $V_1(N_k,v_k)$ is strictly and continuously decreasing in N_k and

$$\begin{aligned} V_1(N-1,v_1) &\leq V_1(1,v_2),\\ V_1(1,v_1) &> V_1(N-1,v_2).\end{aligned}$$

The equilibrium is found by setting $N_1 = N-1$, $N_2 = N-N_1$ and then decreasing N_1 until the values of the payoffs in the two contests equalize. If we define N_1^* to be such that $V_1(N_1^*,v_1) = v_2$, then we have proven the result for $K = 2$.

Notice also that the payoff that the individuals get in the two contest equilibrium is continuously decreasing in N since both N_1 and N_2 will be higher in equilibrium if N increases.

Assume that we have proven the result for k contests. Then we can define $V_k(N, v_1, \ldots, v_k)$ to denote the payoff that an individual gets in any contest in the unique self-allocation equilibrium with N individuals and k contests with prices $v_1, \ldots v_k$. We also know that there is some N_k^* such that $V_k(N, v_1, \ldots, v_k)$ is well defined and continuously decreasing in N for all $N \geq N_k^*$. In an equilibrium with $k+1$ contests we have to have that

$$V_k(N - N_{k+1}, v_1, \ldots, v_k) = V_1(N_{k+1}, v_{k+1}). \tag{2}$$

If

$$V_k(N-1, v_1, \ldots, v_k) > v_{k+1} = V_1(1, v_{k+1})$$

then there is no such equilibrium with $N_{k+1} \geq 1$ since $V_k(N - N_{k+1}, v_1, \ldots, v_k)$ increases when N_{k+1} increases. Similarly, we cannot find an equilibrium where all contest are non empty if $V_k(N_k^*, v_1, \ldots, v_k) < V_1(N - N_k^*, v_{k+1})$. But, for all N that are sufficiently large such that

$$\begin{aligned} V_k(N-1, v_1, \ldots, v_k) &\leq v_{k+1}, \text{ and} \\ V_k(N_k^*, v_1, \ldots, v_k) &\geq V_1(N - N_k^*, v_{k+1}) \end{aligned}$$

there is a unique choice of $N_{k+1} \geq 1$ that satisfies (2) and thus a unique self-allocation equilibrium with $k+1$ contests.

To show the last part of the result assume that $N \geq \sum_{k=1}^{K} (v_k / \min\{v_1, \ldots, v_K\})$, and without loss of generality assume that $v_k = \min\{v_1, \ldots, v_K\}$ for $k = 1, \ldots, \bar{k}$. We then have that

$$V_{\bar{k}}(\bar{k}, v_1, \ldots, v_{\bar{k}}) = v_1.$$

Set $N_k = 1$ for $k = 1, \ldots, \bar{k}$, and for $k = \bar{k}+1, \ldots, K$ pick N_k such that

$$V_1(N_k, v_k) = \frac{v_k}{N_k} - g(N_k, v_k) = v_1.$$

Then, by construction, $N_1, \ldots, N_K$ is a self-allocation equilibrium for the number $\sum_{k=1}^{K} N_k$ of individuals since the payoff in each contest is v_1. It follows that

$V_K(N', v_1, \ldots, v_K)$ is well defined for $N' \geq \sum_{k=1}^K N_k$. Since $v_k/N_k \geq v_1$ for all k we also have that

$$\sum_{k=1}^K N_k \leq \sum_{k=1}^K \frac{v_k}{v_1} \leq N,$$

so in particular $V_K(N, v_1, \ldots, v_K)$ is well defined. □

3.3 Effort maximization

In this subsection, we consider how to allocate individuals to contests so as to maximize total effort across contests.

Define $Z_k := N_k g(N_k, v_k)$ and total global effort

$$Z =: \sum_k Z_k,$$

and let $h(x) := f(x)/f'(x)$.[1]

Proposition 2 *If h is convex, then the problem of maximizing Z subject to the constraint that $\sum_k N_k = N$ has a unique solution that is characterized by the equations $\partial Z_k/\partial N_k = \partial Z_1/\partial N_1$ for all k together with the constraint.*

Proof. We have that

$$h'(x) = 1 - \frac{f(x)f''(x)}{f'(x)^2},$$

so if h is convex then

$$\frac{d}{dx}\left(1 - \frac{f(x)f''(x)}{(f'(x))^2}\right)^{-1} \leq 0$$

Since g is defined implicitly by

$$\frac{f(g(N,v))}{f'(g(N,v))} - \frac{N-1}{N^2} v = 0,$$

the implicit function theorem implies that

$$\frac{\partial g(N,v)}{\partial N} = \frac{\frac{2-N}{N^3}}{\frac{(f')^2 - ff''}{(f')^2}} v = \frac{2-N}{N^3}\left(1 - \frac{ff''}{(f')^2}\right)^{-1} v,$$

[1] For more on the role of the h function in contests, see Wärneryd [14] and Inderst *et al* [3].

where the functions f, f', f'' are evaluated at the point $g(N, v)$. So

$$\frac{\partial Z_1}{\partial N_1} = g(N_1, v_1) + N_1 \frac{\partial g(N_1, v_1)}{\partial N_1} = g(N_1, v_1) + \frac{2 - N_1}{N_1^2} \left(1 - \frac{f f''}{(f')^2}\right)^{-1} v_1,$$

and

$$\frac{\partial^2 Z_1}{\partial N_1^2} = \frac{\partial g(N_1, v_1)}{\partial N_1} + \frac{N_1 - 4}{N_1^3} \left(1 - \frac{f f''}{(f')^2}\right)^{-1} v_1 + \frac{2 - N_1}{N_1^2} \frac{\partial}{\partial N_1} \left(1 - \frac{f f''}{(f')^2}\right)^{-1} v_1$$

Suppose $N_1 \geq 2$. Then $g(N_1, v_1)$ is decreasing in N_1, and since h is convex this implies that

$$\frac{\partial}{\partial N_1} \left(1 - \frac{f(g(N_1, v_1)) f''(g(N_1, v_1))}{(f'(g(N_1, v_1)))^2}\right)^{-1} \geq 0.$$

Since $(2 - N_1)/N_1^2 \leq 0$ it follows that

$$\begin{aligned}
\frac{\partial^2 Z_1}{\partial N_1^2} &\leq \frac{\partial g(N_1, v_1)}{\partial N_1} + \frac{N_1 - 4}{N_1^3} \left(1 - \frac{f f''}{(f')^2}\right)^{-1} v_1 \\
&= \frac{2 - N_1}{N_1^3} \left(1 - \frac{f f''}{(f')^2}\right)^{-1} v_1 + \frac{N_1 - 4}{N_1^3} \left(1 - \frac{f f''}{(f')^2}\right)^{-1} v_1 \\
&= \frac{-2}{N_1^3} \left(1 - \frac{f f''}{(f')^2}\right)^{-1} v_1 < 0.
\end{aligned}$$

If instead $N_1 \in [1, 2)$, then $g(N_1, v_1)$ is increasing in N_1 and

$$\begin{aligned}
\frac{\partial}{\partial N_1} \left(1 - \frac{f(g(N_1, v_1)) f''(g(N_1, v_1))}{(f'(g(N_1, v_1)))^2}\right)^{-1} &\leq 0, \text{ and} \\
\frac{2 - N_1}{N_1^2} &> 0
\end{aligned}$$

which also implies that $\partial^2 Z_1 / \partial N_1^2 < 0$.

Since $\partial^2 Z / \partial N_k^2 < 0$ for all k, and since $\partial^2 Z / \partial N_k \partial N_i = 0$ if $k \neq i$, total effort Z is a strictly concave function of $(N_1, \ldots, N_k)$. It follows that the problem of maximizing effort subject to the constraint that $\sum_k N_k = N$ has a unique solution given by the first order conditions. □

The following result shows that if $f''=0$ then three things happen: (i) the self-allocation equilibrium maximizes effort; (ii) the self-allocation equilibrium has the lowest possible number of people in the contest with the highest value (i.e., there are no other functions f for which there are fewer people in the contest with the highest value in equilibrium); and (iii) the allocation that maximizes effort also has the lowest possible number of people in the contest with the highest value.

Proposition 3 *Suppose that h is convex, and assume without loss of generality that $v_1 \geq v_2 \geq \cdots \geq v_K$. Then, if N is sufficiently large, both the self-allocation equilibrium and the allocation that maximizes effort are such that*

$$N_i^2 \geq \frac{v_i}{v_{i+1}} N_{i+1}^2$$

for $i=1,\ldots,K-1$. If $f''=0$, then the weak inequalities are equalities and the self-allocation equilibrium maximizes effort.

Proof. Suppose there are only two contests, assume that N_1 and N_2 are such that $v_1/N_1^2 = v_2/N_2^2$, and set $\gamma = v_1/N_1^2$. Then

$$\begin{aligned}\frac{f(g(N_1,v_1))}{f'(g(N_1,v_1))} &= h(g(N_1,v_1)) = (N_1-1)\gamma, \text{ and}\\ \frac{f(g(N_2,v_2))}{f'(g(N_2,v_2))} &= h(g(N_2,v_2)) = (N_2-1)\gamma.\end{aligned}$$

So,

$$h(g(N_1,v_1)) - h(g(N_2,v_2)) = \gamma(N_1-N_2). \tag{3}$$

Since $h' \geq 1$ it follows that

$$g(N_1,v_1) - g(N_2,v_2) \leq \gamma(N_1-N_2)$$

which implies

$$\frac{v_1}{N_1} - \frac{v_2}{N_2} = \gamma(N_1-N_2) \geq g(N_1,v_1) - g(N_2,v_2).$$

If $f''=0$, then $h'=1$ and the weak inequality is an equality and we have found the equilibrium. Otherwise N_1 must be increased to reach the equilibrium.

Since h is convex, and since $g(N_1,v_1)\geq g(N_2,v_2)$, it follows from (3) that

$$h'(g(N_1,v_1))(g(N_1,v_1)-g(N_2,v_2))\geq \gamma(N_1-N_2). \tag{4}$$

Assume that $(N-2)^2/4\geq v_1/v_2$. Then $v_1/N_1^2=v_2/N_2^2$ and $N_1+N_2=N$ implies $N_2\geq 2$. Since $h'(g(N_1,v_1))/h'(g(N_2,v_2))\geq 1$ it follows that

$$\begin{aligned}\left(N_2\frac{\partial g(N_2,v_2)}{\partial N_2}-N_1\frac{\partial g(N_1,v_1)}{\partial N_1}\right)h'(g(N_1,v_1)) \\ =\gamma(2-N_2)\frac{h'(g(N_1,v_1))}{h'(g(N_2,v_2))}-\gamma(2-N_1) \\ \leq\gamma(2-N_2)-\gamma(2-N_1) \\ =\gamma(N_1-N_2).\end{aligned} \tag{5}$$

Together (4) and (5) imply

$$\frac{\partial Z_1}{\partial N_1}=g(N_1,v_1)+N_1\frac{\partial g(N_1,v_1)}{\partial N_1}\geq g(N_2,v_2)+N_2\frac{\partial g(N_2,v_2)}{\partial N_2}=\frac{\partial Z_2}{\partial N_2}.$$

If $h'=1$, then the weak inequality is an equality and we have found the allocation that maximizes effort. Otherwise N_1 must be increased to equalize $\partial Z_1/\partial N_1$ and $\partial Z_2/\partial N_2$ and maximize effort.

We have proven the result for $K=2$ but the same argument applies for N_i and N_{i+1} if $v_i\geq v_{i+1}$. □

Proposition 4 *The self-allocation equilibrium maximizes effort for all N and all values $v_1,\ldots,v_K$ if and only if $f''=0$.*

Proof. It remains only to prove the "only if" part of this proposition. If the self-allocation equilibrium maximizes effort, then $V_1(N_1,v_1)=V_1(N_2,v_2)$ and $\partial Z_1(N_1,v_1)/\partial N_1=\partial Z_2(N_2,v_2)/\partial N_2$. We have that

$$\begin{aligned}-\frac{\partial^2 Z_1/\partial N_1^2}{\partial^2 Z_1/(\partial N_1\partial v_1)} &= -\frac{2\frac{\partial g(N_1,v_1)}{\partial N_1}+N_1\frac{\partial^2 g(N_1,v_1)}{\partial N_1^2}}{\frac{\partial g(N_1,v_1)}{\partial v_1}+N_1\frac{\partial^2 g(N_1,v_1)}{\partial N_1\partial v_1}}, \text{ and}\\ -\frac{\partial V_1/\partial N_1}{\partial V_1/\partial v_1} &= -\frac{-\frac{v_1}{N_1^2}-\frac{\partial g(N_1,v_1)}{\partial N_1}}{\frac{1}{N_1}-\frac{\partial g(N_1,v_1)}{\partial v_1}}.\end{aligned}$$

If $f'' < 0$, then $h' > 1$ and $-(\partial^2 Z_1/\partial N_1^2)/(\partial^2 Z_1/\partial N_1 \partial v_1) \neq -(\partial V_1/\partial N_1)/(\partial V_1/\partial v_1)$. This is easy to see at $N_1 = 2$ where $\partial g(N_1, v_1)/\partial N_1 = 0$ and the expressions simplify to

$$\begin{aligned}
-\frac{\partial^2 Z_1/\partial N_1^2}{\partial^2 Z_1/(\partial N_1 \partial v_1)} &= -\frac{N_1 \frac{\partial}{\partial N_1}\left(\frac{2-N_1}{N_1^3}\right)\frac{v_1}{h'}}{\frac{N_1-1}{N_1^2}\frac{1}{h'}} = -\frac{-\frac{v_1}{4h'}}{\frac{1}{4h'}} = v_1, \text{ and} \\
-\frac{\partial V_1/\partial N_1}{\partial V_1/\partial v_1} &= -\frac{\frac{-v_1}{4}}{\frac{1}{N_1} - \frac{N_1-1}{N_1^2}\frac{1}{h'}} = \frac{\frac{v_1}{4}}{\frac{1}{2} - \frac{1}{4}\frac{1}{h'}} < v_1.
\end{aligned}$$

Since $-(\partial^2 Z_1/\partial N_1^2)/(\partial^2 Z_1/\partial N_1 \partial v_1) \neq -(\partial V_1/\partial N_1)/(\partial V_1/\partial v_1)$ the functions V_1 and $\partial Z_1/\partial N_1$ do not have the same level sets and thus the equations $V_1(N_1, v_1) = V_1(N_2, v_2)$ and $\partial Z_1(N_1, v_1)/\partial N_1 = \partial Z_2(N_2, v_2)/\partial N_2$ do not have the same solutions. □

We have already established a common lower bound for the number of people in the the contest with the highest value. We can also establish a common upper bound by arguing intuitively that effort should be higher in contests with higher prices both in the equilibrium and in the allocation that maximizes effort, and that this implies that the number of people in the contest with the highest value cannot be too high since a high number of people discourages effort.

Proposition 5 *Assume without loss of generality that $v_1 \geq v_2 \geq \cdots \geq v_K$. Then, both the self-allocation equilibrium and the allocation that maximizes effort are such that $g(N_i, v_i) \geq g(N_{i+1}, v_{i+1})$ for $i = 1, \ldots, K-1$ which is equivalent to*

$$\frac{N_i^2}{N_i - 1} \leq \frac{v_i}{v_{i+1}} \frac{N_{i+1}^2}{N_{i+1} - 1}.$$

In the self-allocation equilibrium we also have that $N_i \leq (v_i/v_{i+1})N_{i+1}$.

Proof. Suppose there are only two contests and assume that N_1 and N_2 are such that $v_1/N_1 = v_2/N_2$. Then

$$\frac{f(g(N_1, v_1))}{f'(g(N_1, v_1))} = \frac{N_1 - 1}{N_1^2} v_1 = \frac{N_1 - 1}{N_1} \frac{v_2}{N_2} \geq \frac{N_2 - 1}{N_2} \frac{v_2}{N_2} = \frac{f(g(N_2, v_2))}{f'(g(N_2, v_2))},$$

and since f/f' is increasing it follows that $g(N_1, v_1) \geq g(N_2, v_2)$ and

$$\frac{v_1}{N_1} - g(N_1, v_1) \leq \frac{v_2}{N_2} - g(N_2, v_2).$$

If the inequality is an equality we have found the equilibrium and otherwise N_1 must be reduced to reach it. Hence the self-allocation equilibrium is such that $N_1 \leq (v_1/v_2)N_2$ and $g(N_1, v_1) \geq g(N_2, v_2)$.

Let N_1 and N_2 be such that $v_1(N_1-1)/N_1^2 = v_2(N_2-1)/N_2^2$ and set $\alpha = v_1(N_1-1)/N_1^2$. Then $g(N_1, v_1) = g(N_2, v_2)$ and $h'(g(N_2, v_2))/h'(g(N_1, v_1)) = 1$ which implies

$$\begin{aligned}\left(N_2\frac{\partial g(N_2, v_2)}{\partial N_2} - N_1\frac{\partial g(N_1, v_1)}{\partial N_1}\right)h'(g(N_2, v_2)) &= \frac{v_2}{N_2^2} - \frac{v_1}{N_1^2}\\ &\geq 0.\end{aligned}$$

Since $g(N_1, v_1) = g(N_2, v_2)$ it follows that

$$\frac{\partial Z_1}{\partial N_1} = g(N_1, v_1) + N_1\frac{\partial g(N_1, v_1)}{\partial N_1} \leq g(N_2, v_2) + N_2\frac{\partial g(N_2, v_2)}{\partial N_2} = \frac{\partial Z_2}{\partial N_2}$$

and thus we have either found the allocation that maximizes effort or N_1 must be decreased to equalize $\partial Z_1/\partial N_1$ and $\partial Z_2/\partial N_2$.

We have proven the result for $K = 2$ but the same argument applies for N_i and N_{i+1} if $v_i \geq v_{i+1}$. □

Inderst *et al* [3] show that if h is convex, then the dissipation rate in a contest increases with the number of contestants. If the dissipation rate is increasing and bounded by 1, then it must converge. We show that as the number of contestants in contest k increases, the dissipation rate in contest k approaches $1/h'(0)$. For large N total effort in both the self-allocation equilibrium and the allocation that maximizes effort thus depends only on the slope of h at the origin.

Proposition 6 *Assume that* $v_K = \min\{v_1, \ldots, v_K\}$ *and define* $\kappa(N) := \frac{1}{N}\sum_{k=1}^{K}\frac{v_k}{v_K}$. *Let* $Z^{eq}(N)$ *and* $Z^*(N)$ *denote, resepectively, total effort in the self-allocation* *equilibrium and total effort in the allocation that maximizes effort as functions of N. Then, if h is convex, both Z^{eq} and Z^* are increasing in N and furthermore*

$$(1-\kappa(N))\sum_{k=1}^{K}\frac{v_k}{h'(v_k\kappa(N))} \leq Z^{eq}(N) \leq Z^*(N) \leq \sum_{k=1}^{K}\frac{v_k}{h'(0)}, \textit{ and}$$

$$\lim_{N\to\infty} Z^{eq}(N) = \lim_{N\to\infty} Z^*(N) = \sum_{k=1}^{K}\frac{v_k}{h'(0)}.$$

Proof. Since h is convex and since $h(0)=0$, we have that $h'(x)x \geq h(x)$. So

$$g(N_1, v_1) \geq \frac{h(g(N_1, v_1))}{h'(g(N_1, v_1))} = \frac{N_1-1}{N_1^2}\frac{v_1}{h'(g(N_1, v_1))}. \tag{6}$$

It follows from (6) that $\partial Z_1/\partial N_1 = g(N_1, v_1) - (N_1-2)v_1/(N_1^2 h'(g(N_1, v_1))) > 0$. More generally $\partial Z_k/\partial N_k > 0$ for all k which implies that both Z^{eq} and Z^* are increasing in N.

Since f is three times continuously differentiable, h^{-1} is two times continuously differentiable and we can use a Taylor approximation of h^{-1}. We have that $h(0)=0$ and thus

$$h^{-1}(x) = x/h'(0) + x^2 B(x)$$

where $B(x)$ is bounded for x close to 0. Hence

$$\begin{aligned} N_1 g(N_1, v_1) &= N_1 h^{-1}\left(\frac{N_1-1}{N_1^2}v_1\right) \\ &= \frac{N_1-1}{N_1}\frac{v_1}{h'(0)} + \frac{(N_1-1)^2}{N_1^3}v_1^2 B\left(\frac{N_1-1}{N_1^2}v_1\right). \end{aligned} \tag{7}$$

Since $(N_1-1)^2 v_1^2/N_1^3 \to 0$ as $N_1 \to \infty$, and since $(N_1-1)/N_1 \to 1$, it follows from (7) that $N_1 g(N_1, v_1) \to v_1/h'(0)$ as $N_1 \to \infty$. More generally, $N_k g(N_k, v_k) \to v_k/h'(0)$ as $N_k \to \infty$. Proposition 5 implies that $N_k \to \infty$ as $N \to \infty$ for all k both for the self-allocation equilibrium and for the allocation that maximizes effort and thus $Z^{eq}(N) \to \sum_{k=1}^{K}\frac{v_k}{h'(0)}$ and $Z^*(N, v_1, \ldots, v_K) \to \sum_{k=1}^{K}\frac{v_k}{h'(0)}$ as $N \to \infty$.

To see that $(1-\kappa(N))\sum_{k=1}^{K}\frac{v_k}{h'(v_k\kappa(N))} \leq Z^{eq}(N)$ we first assume without loss of generality that $v_i \geq v_{i+1}$ for $i=1,\ldots,K-1$. Then, by Proposition 5, we have that $N_{i+1} \geq N_i(v_{i+1}/v_i)$ in the self-allocation equilibrium which implies

$$N_K \geq \frac{N}{\sum_{k=1}^{K}\frac{v_k}{v_K}} = \frac{1}{\kappa(N)}. \tag{8}$$

Since $h'(g(N_k,v_k))g(N_k,v_k) \geq h(g(N_k,v_k)) = (N_k-1)v_k/N_k^2$ we have that

$$\begin{aligned} N_k g(N_k,v_k) &\geq (1-\frac{1}{N_k})\frac{v_k}{h'(g(N_k,v_k))} \\ &\geq (1-\kappa(N))\frac{v_k}{h'(g(N_k,v_k))} \end{aligned} \tag{9}$$

where the second inequality uses (8) and that $N_k \geq N_K$ for all k in the self-allocation equilibrium. We do not know exactly what $g(N_k,v_k)$ is but we do know that $g(N_k,v_k) \leq v_k/N_k$ since otherwise $N_k g(N_k,v_k) > v_k \geq v_k/h'(0)$ which contradicts that $N_k g(N_k,v_k)$ increases towards $v_k/h'(0)$ as N_k increases. Thus $h'(g(N_k,v_k)) \leq h'(v_k/N_k) \leq h'(v_k\kappa(N))$ so it follows from (9) that

$$N_k g(N_k,v_k) \geq (1-\kappa(N))\frac{v_k}{h'(v_k\kappa(N))}$$

as we wanted to show. □

An immediate corollary of Proposition 6 is that there is a bound on the difference between total effort in the self-allocation equilibrium and total effort in the effort maximizing allocation that depends on the curvature of h and how large the number of contestants N is.

Proposition 7 *Suppose that h is convex. The difference between total effort in the self-allocation equilibrium and total effort in the effort maximizing allocation is then less than*

$$\sum_{k=1}^{K}\left(\frac{v_k}{h'(0)} - \frac{v_k(1-\kappa(N))}{h'(v_k\kappa(N))}\right). \tag{10}$$

As $N \to \infty$, we have that $\kappa(N) = \frac{1}{N}\sum_{k=1}^{K}\frac{v_k}{v_K} \to 0$ and the difference in (10) tends to 0.

The difference in (10) depends on the curvature of h because the less curved h is, i.e., the closer h'' is to 0, the smaller is the difference between $h'(0)$ and $h'(v_k\kappa(N))$. But the difference tends to 0 for large N irrespective of how convex h is. In this sense the result that the self-allocation equilibrium maximizes total effort holds approximately for large N.

We conclude with an example. Suppose there are 100 contestants in two contests with $v_1 = 9$ and $v_2 = 1$. From Proposition 3 we know that $(N_1/N_2)^2 \geq 9$ which implies $N_1 \geq 75$. From Proposition 5 we know that $9(N_1-1)/N_1^2 \geq (N_2-1)/N_2^2$ which implies $N_1 \leq 91$. Both the self-allocation equilibrium and the allocation that maximizes effort are such that dissipation rates are lower than $1/h'(0)$ and thus total effort is less than $10/h'(0)$. However, $\kappa(N) = 1/10$ so total effort is always at least $(1-1/10)*10/h'(9/10)) = 9/h'(0.9)$ so the dissipation rates are at least $0.9/h'(0.9)$.

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