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Voting in Hiring Committees: Which "Almost" Rule is Optimal?

Abstract

We determine the scoring rule that is most likely to select a high-ability candidate. A major result is that neither the widely used plurality rule nor the inverse-plurality rule are ever optimal, and that the Borda rule is hardly ever optimal. Furthermore, we show that only the almost-plurality, the almost-inverse-plurality, and the almost-Borda rule can be optimal. Which of the "almost" rules is optimal depends on the likelihood that a candidate has high ability and how likely committee members are to correctly identify the abilities of the different candidates.

JEL-Codes: D710.

Keywords: committee decisions, scoring rules, "Almost" voting rules.

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1 Introduction

Voting rules are typically analyzed on an axiomatic basis. This study adopts a more applied approach by considering the problem where a firm establishes a three-member committee to hire a single individual from three possible candidates. Each candidate can have either high or low ability. The number of high-ability candidates is uncertain, and the candidates' abilities cannot be verified before a choice is made. However, each committee member receives noisy signals of every candidate's ability and ranks the candidates accordingly. As the committee uses a scoring rule to make its choice,[1] each committee member assigns a fixed score to the candidate he ranks first, a (possibly different) fixed score to the candidate he ranks second, and a (possibly different) fixed score to the one he ranks third. The candidate receiving the highest total score is selected. Among all scoring rules, we determine the rule that is most likely to select a high-ability candidate.[2]

Surprisingly, we find that the widely used plurality rule is never optimal. The explanation is that the plurality rule ignores relevant information when it leads to a tie and the final selection therefore is made by a draw among the tied candidates. In particular, the plurality rule is dominated by what we will refer to as the *almost-plurality rule*. The latter is similar

[1] Scoring rules are also referred to as "positional rules" (Núñez and Laslier, 2014) or as "point-voting schemes" (Nitzan, 1985). For an analysis of scoring rules, see Saari (2001) and Nurmi (2002). The advantages of scoring rules include, among other things, their ability to guarantee the existence of an outcome (Sen, 1970, and Mueller, 2003); their ability to satisfy desirable properties (Young, 1975, Chebotarev and Shamis, 1998, Baharad and Nitzan, 2002, García-Lapresta et al., 2010, and Llamazares and Peña, 2015); and the existence of a metric according to which the selected alternative is the closest to the preference profiles (Lerer and Nitzan, 1985).

[2] For an axiomatization of the Borda rule, see Young (1974), Nitzan and Rubinstein (1981), and Saari (1990); for the plurality rule, see Richelson (1978) and Ching (1996); and for the inverse plurality rule, see Baharad and Nitzan (2005). The latter rule is also referred to as "anti-plurality" (Saari, 1995) and as "negative voting" (Myerson, 2002). We focus our analysis on the desirability of different *rigid* scoring rules. Thus, we do not consider flexible scoring rules such as approval voting (see Brams and Fishburn, 1978) and single-approval multiple-rejection voting (Baharad and Nitzan, 2016). See Ahn and Oliveros (2016) for an analysis that includes both rigid scoring rules and approval voting. They show that under strategic voting, approval voting dominates rigid scoring rules. While Ahn and Oliveros (2016) study large electorates and choose the best scoring rule in their setting, we analyze voting in small committees and compare the desirability of different possible scoring rules.

to the plurality rule, except that a committee member also gives a small positive score to the candidate that he ranks second. As a consequence, the almost-plurality rule leads to the same outcome as the plurality rule except for when the outcome of the plurality rule is determined by a draw. The additional information used by the almost-plurality rule makes it more likely that a high-ability candidate will be selected.

Similarly, the inverse-plurality rule is never optimal as it also ignores relevant information when the final selection is made by a draw.[3] In particular, the inverse-plurality rule is dominated by what we will refer to as the *almost-inverse-plurality rule*, which is similar to the inverse-plurality rule except that a committee member gives a slightly smaller score to his second-ranked candidate than to his first-ranked candidate. The almost-inverse-plurality rule therefore leads to the same outcome as the inverse-plurality rule except when the outcome of the inverse-plurality rule is determined by a draw. The additional information used by the almost-inverse-plurality rule would then increase the likelihood that a high-ability candidate is selected.

We also establish that the well-known Borda rule is hardly ever optimal. The explanation is that the outcome of the Borda rule is an equal mixture of the outcomes of the almost-plurality rule and the *almost-Borda rule,* where the latter is similar to the Borda rule except that a committee member assigns his second-ranked candidate a slightly higher weight than under the Borda rule. The Borda rule can therefore be optimal only if these two other rules are simultaneously optimal, which we will show is practically impossible.[4]

In general, we establish that the only scoring rules that can be optimal are the three "almost" rules: the almost-plurality, the almost-Borda, and the almost-inverse-plurality

[3] Under the inverse-plurality rule, committee members vote for every candidate except their least preferred one and the candidate with the highest total score is selected. When the voting leads to a tie, we assume that the final selection is made by a draw among the tied candidates.

[4] We also show that what we refer to as the two-third rule (formally defined below) is never optimal as it is an equal mixture of the outcomes of the almost-inverse-plurality rule and the almost-Borda rule.

rule. Which of these "almost" rules is optimal depends on the likelihood that a candidate has a high-ability and on the likelihoods that committee members correctly identify the high- and low-ability candidates. For instance, if committee members are sufficiently more likely to correctly identify a high-ability than a low-ability candidate, then the almost-inverse-plurality rule is optimal, while if the opposite is true, then the almost-Borda rule is optimal.

The main objective of the paper is thus to rank the different "almost" rules based on their optimality under the various possible combinations of probabilities that a candidate has high ability and that high- and low-ability candidates are identified as such. We therefore shed light on the old social-choice questions regarding the identification of the best voting rule and the comparison of particular voting rules.

For the sake of simplicity we present our results in a three candidates – three committee members setting.[5] The combinatorial expressions with more candidates and/or committee members are more complicated and do not give additional insight. Indeed, as discussed later, the intuition behind the optimality of the "almost-" rules carries over to any number of candidates and committee members. Likewise, we later argue that our results are robust to allowing for candidate abilities to take more than two values.

2 The Model

Consider a firm that is in the process of hiring a new worker. We focus on the final decision, in which there are three qualifying candidates. Every candidate can have either high or low ability. The firm wishes to hire a high-ability candidate, and has established a three-member committee that has to carry out the hiring decision. The problem facing the committee is that the abilities of the different candidates may not be perfectly observable. Each committee member does, however, receive noisy signals of the candidates' abilities.

[5] Saari (1999), Saari and Tataru (1999), Saari and Valognes (1999), Myerson (2002) and Giles and Postl (2014) also utilize a three-alternative setting.

The firm guarantees strategy-proofness by nominating external committee members who are anonymous to each other. This is commonly the case with, for example, referee reports, recommendation letters, and expert witnesses. One member's voting will then not be influenced by that of the others. Furthermore, in due course, the quality of all the candidates will be commonly known, and each member will be evaluated according to the correctness of his ranking of the candidates. Thus, we can henceforth assume sincere voting.

Every candidate is associated with a probability $p \in (0,1)$ of having either high or low ability. This probability is independent of the other candidates' abilities. If a candidate is a high-ability worker, then a committee member receives a correct signal of the candidate's ability with probability $x \in [\frac{1}{2}, 1)$, and an incorrect signal that the candidate has low ability with probability $1 - x$. Similarly, if a candidate is a low-ability worker, then a committee member receives a correct signal of the candidate's ability with probability $z \in [\frac{1}{2}, 1)$, and an incorrect signal that the candidate has high ability with probability $1 - z$. We assume that $\frac{1}{4} < xz$ and that the signals are conditionally independent over candidates and committee members.

Each committee member independently ranks the candidates according to the signals he has received. Should a member receive the same signal for more than one candidate, then the ranking among these candidates is determined randomly. The committee makes the hiring decision using a scoring mechanism where each member assigns one point to the candidate he ranks first, $\alpha \in [0,1]$ point to the candidate he ranks second, and zero point to the last-ranked candidate. The candidate with the highest total score is chosen. If there are two or three candidates with the same highest total score, the selected candidate is determined by a lottery among those candidates.

Since the committee considers three candidates and consists of three members, there are seven equivalent classes of scoring rules in the sense that all scoring rules within the same class yield the same outcome (see Appendix A for a formal proof). We refer to the seven equivalent classes as follows:

- Plurality rule ($\alpha = 0$): The scoring vector is $(1, \alpha_p, 0)$, where $\alpha_p = 0$.
- Almost-plurality rule ($0 < \alpha < \frac{1}{2}$): The scoring vector is $(1, \alpha_{ap}, 0)$, where $\alpha_{ap} \in (0, \frac{1}{2})$.
- Borda rule ($\alpha = \frac{1}{2}$): The scoring vector is $(1, \alpha_B, 0)$, where $\alpha_B = \frac{1}{2}$.
- Almost-Borda rule ($\frac{1}{2} < \alpha < \frac{2}{3}$): The scoring vector is $(1, \alpha_{aB}, 0)$, where $\alpha_{aB} \in (\frac{1}{2}, \frac{2}{3})$.
- Two-third rule ($\alpha = \frac{2}{3}$): The scoring vector is $(1, \alpha_{tt}, 0)$, where $\alpha_{tt} = \frac{2}{3}$.
- Almost-inverse-plurality rule ($\frac{2}{3} < \alpha < 1$): The scoring vector is $(1, \alpha_{aip}, 0)$, where $\alpha_{aip} \in (\frac{2}{3}, 1)$.
- Inverse-plurality rule ($\alpha = 1$): The scoring vector is $(1, \alpha_{ip}, 0)$, where $\alpha_{ip} = 1$.

The plurality, Borda, and inverse-plurality rules are well known, while we have named the two-third rule and the three "almost" rules.

The optimal scoring rule maximizes the probability of selecting a high-ability candidate. Which scoring rule is optimal depends on how certain the committee members are about the correctness of their ranking and on the probability that a candidate has high ability. Suppose, for example, that it were known that there is exactly one high-ability candidate. Then, the more a committee member believes that his first-ranked candidate is the correct choice, and hence that his second-ranked one is an incorrect choice, the less interested he is in having this second-ranked candidate chosen. Accordingly, a class of scoring rules with smaller value(s) of α would be preferred in order to reduce the likelihood that his second-ranked candidate will be chosen. As another example, suppose that it were known that there are exactly two high-ability candidates. Then, the more a committee member believes that his first- and second-ranked candidates have high ability, the more he wants to avoid the choice of his third-ranked candidate. Thus, a class of scoring rules with higher value(s) of α would be preferred in order to reduce the likelihood that his last-ranked candidate is chosen. Hence, which scoring rule is optimal depends both on the probability p that a candidate has high ability and on the probabilities x and z that committee members receive correct signals of the high or low ability of a candidate.

Let $M(\alpha, i, x, z)$ denote the probability that a scoring rule which assigns the score α to the second-ranked candidate selects a high-ability candidate, given that there are $i \in \{0, 1, 2, 3\}$ high-ability ones among the three applicants. Since the probability that there is exactly zero, one, two, or three high-ability candidates equals $(1-p)^3$, $3p(1-p)^2$, $3p^2(1-p)$, and p^3, respectively, the probability of choosing a high-ability candidate for a given scoring rule is

$$(1-p)^3 M(\alpha, 0, x, z) + 3p(1-p)^2 M(\alpha, 1, x, z) + 3p^2(1-p) M(\alpha, 2, x, z) + p^3 M(\alpha, 3, x, z). \quad (1)$$

As it is not possible to choose a high-ability candidate if no such candidate applies, it follows that $M(\alpha, 0, x, z) = 0$ for every scoring rule, and as a high-ability candidate is necessarily chosen if only high-ability candidates apply, it also follows that $M(\alpha, 3, x, z) = 1$ for every scoring rule. Accordingly, the cases in which there are either zero or three high-ability candidates do not affect which scoring rule is chosen. Therefore, selecting a scoring rule to maximize (1) is equivalent to selecting a scoring rule to maximize the probability of choosing a high-ability candidate if either one or two high-ability candidates have applied, i.e., to maximize[6]

$$A(p, \alpha, x, z) \equiv (1-p) M(\alpha, 1, x, z) + p M(\alpha, 2, x, z). \quad (2)$$

Let the three candidates be denoted by a, b, and c. Further, let a ranking profile be defined as a *non-ordered* three-tuple of the committee members' rankings. For example, the

[6] Note that $(1-p)/p$ is the probability that there is one high-ability candidate relative to the probability that there are two high-ability candidates.

ranking profile $\{abc, acb, bca\}$ indicates that one committee member ranks a higher than b, and b higher than c; one committee member ranks a higher than c, and c higher than b; and one committee member ranks b higher than c, and c higher than a. However, since the ranking profile is non-ordered, it does not associate a particular committee member with a specific ranking. Hence, corresponding to this ranking profile, there are six *ordered* three-tuples of the committee members' rankings. If, instead, the ranking profile is $\{abc, abc, bca\}$, there would be only three ordered three-tuples of the committee members' rankings.

3 One High-Ability Candidate

Suppose that there is one high-ability and two low-ability candidates. Let a denote the high-ability candidate, and b and c the low-ability ones. Further, let q_j denote the probability that a committee member ranks a at the jth position, $j \in \{1, 2, 3\}$. Since b and c are low-ability candidates whose signals are identically and independently distributed, it follows, for example, that the probability that a is ranked first, b is ranked second, and c is ranked third, is the same as the probability that a is ranked first, c is ranked second, and b is ranked third.[7]

In Appendix B we show that

$$
\begin{aligned}
q_1 &= \tfrac{1}{3}(xz + x + z^2), \\
q_2 &= \tfrac{1}{3}(-2xz + x - 2z^2 + 3z), \\
q_3 &= \tfrac{1}{3}(-2x + xz + z^2 - 3z + 3).
\end{aligned}
$$

Table 1 lists all the possible ranking profiles, the number of ordered 3-tuples, the probability of each ordered 3-tuple, and the probability that a high-ability candidate is selected for a given scoring rule.

[7] Similarly, the probability that a is ranked second and b is ranked first is the same as the probability that a is ranked second and c is ranked first. The probability that a is ranked third and b is ranked first is the same as the probability that a is ranked third and c is ranked first.

3.1 The Probability of Choosing the High-Ability Candidate

Based on the values given in Table 1, we can compute the probability $M(\alpha, 1, x, z)$ that a scoring rule selects the sole high-ability candidate. Table 2 shows the probabilities for the different scoring rules both as functions of q_1, q_2, and q_3, and of x and z.

In Figure 1, the curve labelled $m^1_{ap,aB}$ plots the combinations of x and z for which the almost-plurality rule and the almost-Borda rule (as well as the Borda rule) are equally satisfactory and strictly dominate the other rules given that there is one high-ability candidate. Likewise, the curve labelled $m^1_{aB,aip}$ plots the combinations of x and z for which the almost-Borda rule and the almost-inverse-plurality rule (as well as the two-third rule) are equally satisfactory and strictly dominate the other rules given that there is one high-ability candidate. The almost-plurality rule is strictly preferred to all other rules for combinations of x and z in the area to the left of the $m^1_{ap,aB}$-curve; the almost-Borda rule is strictly preferred to all other rules for combinations of x and z in the area between the $m^1_{ap,aB}$- and $m^1_{aB,aip}$-curves; and the almost-inverse-plurality rule is strictly preferred to all other rules for combinations of x and z in the area to the right of the $m^1_{aB,aip}$-curve.

The intuition for Figure 1 can be understood by examining the relationship between the value of α and the accuracy of the information available to the committee members, and thus the correctness of their ranking of the candidates. More specifically, for a given x, the higher z is, the better will a committee member be able to identify the low-ability candidates and, therefore, the less likely it is that one of the low-ability candidates is ranked first and the high-ability candidate is ranked second. Thus, the higher is z, the smaller is the value α that should be assigned to the second-ranked candidate. For example, suppose the committee member has a very high probability x of correctly identifying a high-ability candidate. When z is relatively small, this committee member is more likely to incorrectly perceive a low-ability candidate as having high ability. In order to correctly reflect the committee members' preferences, the optimal scoring rule should assign the candidate that is ranked second a higher weight, i.e., a higher α (making the almost-inverse-plurality the optimal rule).

4 Two High-Ability Candidates

Suppose now that there are two high-ability candidates and one low-ability candidate. The probabilities in Table 2 of the different scoring rules choosing the only high-ability candidate can be used to derive the probabilities of the different scoring rules choosing one of the two high-ability candidates as follows: Since $1 - z$ is the probability that the single low-ability candidate is erroneously thought to have high ability, and $1 - x$ is the probability that any one of the two high-ability candidates is erroneously thought to have low ability, $M(\alpha, 1, 1-z, 1-x)$ is the probability that the low-ability candidate is chosen. Accordingly, the probability that one of the high-ability candidates is selected is given by $M(\alpha, 2, x, z) = 1 - M(\alpha, 1, 1-z, 1-x)$. The value of $M(\alpha, 2, x, z)$ is calculated in Table 3 both as functions of r_1, r_2, and r_3, where r_i denotes the probability that the low-ability candidate is ranked at the ith place, and as functions of x and z.

In Figure 2, the curve labelled $m^2_{ap,aB}$ represents the combinations of x and z for which the almost-plurality and almost-Borda rules (as well as the Borda rule) are equally satisfactory and strictly dominate the other rules, given that there are two high-ability candidates. Similarly, the curve labelled $m^2_{aB,aip}$ represents the combinations of x and z for which the almost-Borda and almost-inverse-plurality rules (as well as the two-third rule) are equally satisfactory and strictly dominate the other rules, given that there are two high-ability candidates. Thus, if there are two high-ability candidates, the almost-plurality rule is strictly preferred to all other rules for combinations of x and z in the area to the left of the $m^2_{ap,aB}$-curve; the almost-Borda rule is strictly preferred to all other rules for combinations of x and z in the area between the $m^2_{ap,aB}$- and $m^2_{aB,aip}$-curves; and the almost-inverse-plurality rule is strictly preferred to all other rules for combinations of x and z in the area to the right of the $m^2_{aB,aip}$-curve.

5 Ranking of Scoring Rules

In this section we compare the different scoring rules and rank their performance in selecting a high-ability candidate if there are 0,1,2 or 3 of such candidates. Our analysis takes into account that each of the three candidates has a high ability with an independent probability p. Since we have shown that maximizing the probability of selecting a high-ability candidate is equivalent to maximizing the probability of choosing a high-ability candidate given there are one or two high-ability candidates, the mechanism for ranking the different rules is based on a comparison of $A(p, \alpha, x, z)$ (as defined in equation (2)) for different α values.

For the same reason as with exactly one or exactly two high-ability candidates (the cases discussed in Sections 3 and 4), the plurality rule is dominated by the almost-plurality rule, and the inverse-plurality rule is dominated by the almost-inverse-plurality rule. Furthermore, the Borda rule is an equal-weighted mixture of the almost-plurality rule and the almost-Borda rule. The Borda rule is therefore hardly ever optimal and is strictly inferior to one of these rules except in the borderline case where the almost-plurality and almost-Borda rules are equally likely to choose a high-ability candidate. Similarly, the two-third rule is an equal-weighted mixture of the almost-Borda rule and the almost-inverse-plurality rule and is hardly ever optimal.

5.1 The Strict Inferiority of the Plurality Rule

A comparison between the plurality rule and the almost-plurality rule yields

Proposition 1: $A(p, \alpha_p, x, z) < A(p, \alpha_{ap}, x, z)$.

Proof: It follows from Table 2 that

$$M(\alpha_p, 1, x, z) - M(\alpha_{ap}, 1, x, z)$$

$$= \frac{1}{6}(1-z)(x+z-1)\left(xz-z+z^2-1\right)\left(x+xz+z^2\right)$$

which has the same sign as $xz - z + z^2 - 1$ and is therefore negative. Hence, $M(\alpha_p, 1, x, z) < M(\alpha_{ap}, 1, x, z)$.

It follows from Table 3 that

$$\begin{aligned} & M(\alpha_p, 2, x, z) - M(\alpha_{ap}, 2, x, z) \\ = & \frac{1}{6}x\left(-2x - z + xz + x^2\right)(x+z-1)\left(-3x - 2z + xz + x^2 + 3\right) \end{aligned}$$

which has the same sign as $-2x-z+xz+x^2$ and is therefore negative. Hence, $M(\alpha_p, 2, x, z) < M(\alpha_{ap}, 2, x, z)$.

Since $A(p, \alpha_p, x, z)$ is the weighted average of $M(\alpha_p, 1, x, z)$ and $M(\alpha_p, 2, x, z)$, and $A(p, \alpha_{ap}, x, z)$ is the weighted average of $M(\alpha_{ap}, 1, x, z)$ and $M(\alpha_{ap}, 2, x, z)$ with the same weights, it follows that $A(p, \alpha_p, x, z) < A(p, \alpha_{ap}, x, z)$. □

This proposition shows that the almost-plurality rule strictly dominates the plurality rule. The intuition is that for some profiles relevant information is lost when using the plurality rule. Thus, when two candidates are tied for the first place under the plurality rule, then a lottery between them is applied to determine the winner. That is, no importance is given to whether a candidate is ranked second or third. In contrast, the almost-plurality rule uses the information about a candidate's ranking. Since it is more likely that a high-ability candidate is ranked second than third, it follows that the almost-plurality rule performs better than the plurality rule.

5.2 The Strict Inferiority of the Inverse Plurality Rule

A comparison between the inverse-plurality rule and the almost-inverse-plurality rule yields

Proposition 2: $A(p, \alpha_{ip}, x, z) < A(p, \alpha_{aip}, x, z)$.

Proof: It follows from Table 2 that

$$\begin{aligned} & M(\alpha_{ip},1,x,z)-M(\alpha_{aip},1,x,z) \\ = & \frac{1}{16}(q_2-q_1)\left(4q_1q_2+8q_1q_3+4q_2q_3+q_1^2+q_2^2\right), \end{aligned}$$

which has the same sign as q_2-q_1. Since

$$\begin{aligned} & x+z>1 \\ \Rightarrow\ & (x+z-1)(1-z)>0 \\ \Rightarrow\ & \tfrac{1}{3}(-2xz+x-2z^2+3z)<\tfrac{1}{3}(xz+x+z^2) \\ \Rightarrow\ & q_2-q_1<0, \end{aligned}$$

it follows that $M(\alpha_{ip},1,x,z)<M(\alpha_{aip},1,x,z)$.

It follows from Table 3 that

$$\begin{aligned} & M(\alpha_{ip},2,x,z)-M(\alpha_{aip},2,x,z) \\ = & -\frac{1}{216}\left[W_p(x,z)+W_n(x,z)\right], \end{aligned} \tag{3}$$

where

$$\begin{aligned} W_p(x,z) \equiv\ & 66x+159z+45xz^3+252x^3z+3x^5z+192x^2z^2+x^3z^3+3x^4z^2+171xz \\ & +324x^2+237x^4+23x^6+36, \\ W_n(x,z) \equiv\ & -210xz^2-348x^2z-69x^4z-21x^2z^3-66x^3z^2-473x^3-69x^5-39z^2-z^3. \end{aligned}$$

We want to show that (3) is negative and therefore need to verify that $W_p(x,z)+W_n(x,z)$ is positive. Since $W_p(x,z)$ is positive and increases in x and z, and $W_n(x,z)$ is negative and decreases in x and z, it is sufficient to show that $W_p(x,x)+W_n(x+\epsilon,x+\epsilon)>0$, and by continuity, that $W_p(x,x)+W_n(x,x)>0$. Now,

$$W_p(x,x)+W_n(x,x)$$

$$
\begin{aligned}
&= 3\left(75x + 152x^2 - 344x^3 + 242x^4 - 75x^5 + 10x^6 + 12\right) \\
&= 3(75x - 75x^5 + 10x^6 + 12) + 6x^2(76 - 172x + 121x^2) \\
&> 6x^2(76 - 172x + 121x^2) \\
&> 0,
\end{aligned}
$$

where the latter inequality follows from the fact that $76 - 172x + 121x^2$ reaches its minimum value of 14.876 at $x = 86/121$. Hence, (3) is negative.

Since $A(p, \alpha_{ip}, x, z)$ is the weighted average of $M(\alpha_{ip}, 1, x, z)$ and $M(\alpha_{ip}, 2, x, z)$, and $A(p, \alpha_{aip}, x, z)$ is the weighted average of $M(\alpha_{aip}, 1, x, z)$ and $M(\alpha_{aip}, 2, x, z)$ with the same weights, it follows that $A(p, \alpha_{ip}, x, z) < A(p, \alpha_{aip}, x, z)$. □

Hence, the almost-inverse-plurality rule dominates the inverse-plurality rule, and the reason is similar to why the almost-plurality rule dominates the plurality rule. That is, if two candidates are tied for the first place under the inverse-plurality rule, then a lottery between these two candidates determines the winner, with no importance given to whether a candidate is ranked first or second. In contrast, the almost-inverse-plurality rule uses the information about a candidate's ranking, and since it is more likely that a high-ability candidate is ranked first than second, it follows that the almost-inverse-plurality rule outperforms the inverse-plurality rule.

5.3 The Inferiority of the Borda Rule

A comparison between the Borda rule and the almost-plurality and almost-Borda rules yields

Proposition 3: $A(p, \alpha_B, x, z) \leq \max\{A(p, \alpha_{ap}, x, z), A(p, \alpha_{aB}, x, z)\}$.

Proof: When the almost-plurality and the almost-Borda rules yield the same outcome, then this outcome is also obtained by the Borda rule. If the outcomes of the almost-plurality and almost-Borda rules are different, then the Borda rule yields an outcome that is equal to either the one of the almost-plurality rule or to the one of the almost-Borda rule with equal probabilities. Hence, the Borda rule is an equal mixture of the almost-plurality and the almost-Borda rules, and it is therefore weakly dominated by at least one of these rules. In order to prove that the Borda rule is strictly inferior to one of these rules, we need to show that $A(p, \alpha_{ap}, x, z) \neq A(p, \alpha_{aB}, x, z)$, which we proceed to do in the following.

From Table 2 we get

$$M(\alpha_B, 1, x, z) = \tfrac{1}{2}[M(\alpha_{ap}, 1, x, z) + M(\alpha_{aB}, 1, x, z)],$$

from which it follows that $M(\alpha_B, 1, x, z) < \max\{M(\alpha_{ap}, 1, x, z), M(\alpha_{aB}, 1, x, z)\}$ if $M(\alpha_{ap}, 1, x, z) \neq M(\alpha_{aB}, 1, x, z)$, and that $M(\alpha_B, 1, x, z) = M(\alpha_{ap}, 1, x, z)$ if $M(\alpha_{ap}, 1, x, z) = M(\alpha_{aB}, 1, x, z)$.

Similarly, from Table 3 we obtain

$$M(\alpha_B, 2, x, z) = \tfrac{1}{2}[M(\alpha_{ap}, 2, x, z) + M(\alpha_{aB}, 2, x, z)],$$

from which it follows that $M(\alpha_B, 2, x, z) < \max\{M(\alpha_{ap}, 2, x, z), M(\alpha_{aB}, 2, x, z)\}$ if $M(\alpha_{ap}, 2, x, z) \neq M(\alpha_{aB}, 2, x, z)$, and that $M(\alpha_B, 2, x, z) = M(\alpha_{ap}, 2, x, z)$ if $M(\alpha_{ap}, 2, x, z) = M(\alpha_{aB}, 2, x, z)$.

Note that (i) $M(\alpha_{ap}, 1, \frac{1}{2}, \frac{1}{2}) = M(\alpha_{aB}, 1, \frac{1}{2}, \frac{1}{2}) = M(\alpha_B, 1, \frac{1}{2}, \frac{1}{2})$ and $M(\alpha_{ap}, 2, \frac{1}{2}, \frac{1}{2}) = M(\alpha_{aB}, 2, \frac{1}{2}, \frac{1}{2}) = M(\alpha_B, 2, \frac{1}{2}, \frac{1}{2})$, and (ii) $M(\alpha_{ap}, 1, 1, 1) = M(\alpha_{aB}, 1, 1, 1) = M(\alpha_B, 1, 1, 1)$ and $M(\alpha_{ap}, 2, 1, 1) = M(\alpha_{aB}, 2, 1, 1) = M(\alpha_B, 2, 1, 1)$ entail that if x and z were both equal to $\frac{1}{2}$, or both equal to 1, the Borda rule would yield the same outcome as the almost-plurality and almost-Borda rules. Since Lemma 1 (Appendix C) shows that $M(\alpha_{ap}, 1, x, z) = M(\alpha_{aB}, 1, x, z)$ traces z as an increasing and strictly convex function of x, while Lemma 2 (Appendix C) shows that $M(\alpha_{ap}, 2, x, z) = M(\alpha_{aB}, 2, x, z)$ traces z as an increasing and strictly concave function of x, it follows that it is practically always the case that $A(p, \alpha_{ap}, x, z) \neq A(p, \alpha_{aB}, x, z)$. Consequently, the Borda rule is practically always strictly inferior to either the almost-plurality or the almost-Borda rule and always weakly inferior to one of these two rules. □

The outcome of the Borda rule is an equal mixture of the outcome of the almost-Borda and the almost-plurality rule and hence can never outperform both of these “almost” rules.

In fact, Proposition 3 shows that the Borda rule is inferior to either the almost-plurality or the almost-Borda rule.

5.4 The Inferiority of the Two-Third Rule

A comparison between the two-third rule and the almost-inverse-plurality and the almost-Borda rules yields

Proposition 4: $A(p, \alpha_{tt}, x, z) \leq \max\{A(p, \alpha_{aB}, x, z), A(p, \alpha_{aip}, x, z)\}$.

Proof: The proof is similar to that of Proposition 3. When the almost-Borda and the almost-inverse-plurality rules yield the same outcome, then this outcome is also obtained by the two-third rule. If the outcomes of the almost-Borda and almost-inverse-plurality rules are different, then the two-third rule yields either the same outcome as with the almost-Borda rule or as with the almost-inverse-plurality rule, with equal probabilities. Hence, the two-third rule is an equal mixture of the almost-Borda and the almost-inverse-plurality rules, and it is therefore weakly dominated by (at least one of) these rules. In order to prove that the two-third rule is strictly inferior to one of these rules, we need to show that $A(p, \alpha_{aB}, x, z) \neq A(p, \alpha_{aip}, x, z)$.

From Table 2 we get

$$M(\alpha_{tt}, 1, x, z) = \tfrac{1}{2}[M(\alpha_{aB}, 1, x, z) + M(\alpha_{aip}, 1, x, z)],$$

from which it follows that $M(\alpha_{tt}, 1, x, z) < \max\{M(\alpha_{aB}, 1, x, z), M(\alpha_{aip}, 1, x, z)\}$ if $M(\alpha_{aB}, 1, x, z) \neq M(\alpha_{aip}, 1, x, z)$, and that $M(\alpha_{tt}, 1, x, z) = M(\alpha_{aB}, 1, x, z)$ if $M(\alpha_{aB}, 1, x, z) = M(\alpha_{aip}, 1, x, z)$.

Similarly, from Table 3 we obtain

$$M(\alpha_{tt}, 2, x, z) = \tfrac{1}{2}[M(\alpha_{aB}, 2, x, z) + M(\alpha_{aip}, 2, x, z)],$$

from which it follows that $M(\alpha_{tt}, 2, x, z) < \max\{M(\alpha_{aB}, 2, x, z), M(\alpha_{aip}, 2, x, z)\}$ if $M(\alpha_{aB}, 2, x, z) \neq M(\alpha_{aip}, 2, x, z)$, and that $M(\alpha_{tt}, 2, x, z) = M(\alpha_{aB}, 2, x, z)$ if $M(\alpha_{aB}, 2, x, z) = M(\alpha_{aip}, 2, x, z)$.

Now, observe that

$$
\begin{aligned}
& M(\alpha_{aB}, 1, x, z) = M(\alpha_{aip}, 1, x, z) \\
\Leftrightarrow\ & \frac{1}{12}(x+z-1)\left(7xz^2 + x^2z - 14xz^3 + 6xz^4 - 4x^2z^2 + 3x^2z^3 - x^2 + 9z^3 - 10z^4 + 3z^5\right) = 0 \\
\Leftrightarrow\ & 7xz^2 + x^2z - 14xz^3 + 6xz^4 - 4x^2z^2 + 3x^2z^3 - x^2 + 9z^3 - 10z^4 + 3z^5 = 0, \\
\Leftrightarrow\ & x = -\frac{\frac{3}{2}z^{3/2}(4-3z)^{1/2} + \frac{7}{2}z^2 - 7z^3 + 3z^4}{z - 4z^2 + 3z^3 - 1},
\end{aligned}
$$

which is only satisfied for specific combinations of x and z. It then follows that

$$
\begin{aligned}
& A(p, \alpha_{aB}, x, z) = A(p, \alpha_{aip}, x, z) \\
\Leftrightarrow\ & (1-p)M(\alpha_{aB}, 1, x, z) + pM(\alpha_{aB}, 2, x, z) = (1-p)M(\alpha_{aip}, 1, x, z) + pM(\alpha_{aip}, 2, x, z),
\end{aligned}
$$

(the equality entails that the two-third rule, the almost-Borda rule, and the almost-inverse-plurality rule will choose a high-ability candidate with the same probability) will hardly ever be the case as it occurs only for a zero measure of combinations of x, z, and p. □

6 The Optimal Rule: The General Case

Figure 3 is obtained by combining Figures 1 and 2. For all values of p, the almost-plurality rule is preferred in the area which is to the left of the $m^2_{ap,aB}$-curve (since the $m^2_{ap,aB}$-curve is to the left of the $m^1_{ap,aB}$-curve), the almost-Borda rule is preferred in the area which is to the right of the $m^1_{ap,aB}$-curve and to the left of the $m^2_{aB,aip}$-curve, and the almost-inverse-plurality rule is preferred in the area which is to the right of the $m^1_{aB,aip}$-curve (since the $m^1_{aB,aip}$-curve is to the right of the $m^2_{aB,aip}$-curve). In the area enclosed by the $m^2_{ap,aB}$-, $m^2_{aB,aip}$- and $m^1_{ap,aB}$-curves, the almost-plurality rule is preferred if there is one high-ability candidate while the almost-Borda rule is preferred if there are two high-ability candidates. Since the probability of choosing a high-ability candidate if only one high-ability candidate has applied relative to the probability of choosing a high-ability candidate if two high-ability candidates have applied decreases with p, it follows that for a given (x, z) in this area there exists a critical value of p such that the almost-plurality rule is preferred if p exceeds this critical value while the almost-Borda rule is preferred if p is less than this critical value.

Similarly, in the area to the left of the $m^2_{ap,aB}$-curve and to the right of the $m^2_{aB,aip}$-curve, the almost-plurality rule is preferred if there is one high-ability candidate while the almost-Borda rule is preferred if there are two high-ability candidates. It follows that for a given (x, z) in this area there exists a critical value of p such that the almost-Borda rule is preferred if p exceeds this critical value while the almost-inverse-plurality rule is preferred if p is less than this critical value.

Finally, in the area enclosed by the $m^1_{ap,aB}$-, $m^1_{aB,aip}$- and $m^2_{aB,aip}$-curves, the almost-Borda rule is preferred if there is one high-ability candidate while the almost-inverse-plurality rule is preferred if there are two high-ability candidates. Therefore, for a given (x, z) in this area there exists a critical value of p such that the almost-inverse-plurality rule is preferred if p exceeds this critical value while the almost-Borda rule is preferred if p is less than this critical value.

7 Conclusion

This study has considered a three-member committee established to choose the best out of three candidates whose abilities are not known with certainty. Each committee member votes according to the signals he has obtained regarding the ability of each candidate. The hiring committee uses a scoring rule to aggregate its members' preferences when selecting a candidate.

We have determined the scoring rule that is most likely to select a high-ability candidate. A major result is that neither the widely-used plurality rule nor the inverse-plurality rule are ever optimal, and that the Borda rule is hardly ever optimal. Furthermore, we show that the set of optimal rules is comprised of the almost-plurality, the almost-inverse-plurality, and the almost-Borda rules. In particular, the optimality of a specific "almost" rule depends on how likely committee members are to correctly identify the abilities of the different candidates. If a low-ability candidate is correctly identified with a high likelihood, then the almost-plurality rule is optimal. If a high-ability candidate is correctly identified with a high likelihood, then the almost-inverse-plurality rule is optimal. In some intermediate cases the almost-Borda rule is optimal.

The intuition behind the superiority of the almost-plurality rule over the plurality rule and the almost-inverse-plurality rule over the inverse-plurality rule is related to the occurrence of ties, which are the only cases where the "almost" rules may yield an outcome that is different from the corresponding plurality and inverse-plurality rules. In such cases, the lottery that breaks the tie ignores relevant information about the committee members' preferences regarding the candidates that are ranked second. This information is conveyed in the "almost" rules that eliminate the possibility of ties (except in the case of cyclical preferences) and thus avoid the undesirable use of a lottery. For example, under the plurality rule where each candidate receives exactly one vote, each of these candidates will be chosen with the same probability. Thus, this rule ignores the information embodied in the committee members' ranking of the candidates, and, in particular, the distinction between the second- and third- ranked ones. However, this information is being utilized by the almost-plurality rule in a way that increases the probability that a higher-ability candidate is selected.

Concerning the almost-Borda rule, the intuition for its superiority over the Borda rule is that the latter is an equal mixture of the almost-plurality and the almost-inverse-plurality rules. Hence, the Borda rule is optimal only if both of these other two rules are simultaneously optimal, which is hardly ever the case.

We have considered two possible types of candidates: high- and low-ability ones. One may wonder whether expanding the range of possible candidate types (e.g., considering intermediate-ability type of candidates) would change the results. The answer to this question is negative. Allowing for more than two ability types does not change the superiority of the “almost” rules relative to the “classical” rules, since every committee member has predetermined scores (for a given rule) that he assigns to the candidates, and these scores are not affected by the possible range of abilities. The reason is that this superiority is caused by the possibility of ties which is more likely under the classical rules both if the abilities of the candidates can take only two values and if they can take many different values.[8] We therefore use only two types of abilities to convey the message.

The superiority of the “almost” rules holds also with more than three candidates and/or more than three committee members, since the likelihood of ties is then higher under the classical rules. This is true even though the difference between the two types of rules decreases with the number of candidates and the number of committee members. The intuition is that: (1) relative to the plurality rule, under the almost-plurality rule committee members reveal more information regarding their preferences concerning the full ranking of the candidates. This decreases the probability of a tie; (2) relative to the Borda rule, the almost-Borda rule is asymmetric in that being ranked twice at the second place is not equivalent to being ranked once at the first place. This also decreases the probability of a tie; (3) relative to the inverse-plurality rule, it is not only the case that under the almost-inverse-plurality rule committee members reveal more information regarding their preferences concerning the full ranking of the candidates, but also that the almost-inverse-plurality rule is asymmetric in that being ranked once at the second place is not equivalent to being ranked once at the first place. However, our main objective was not to focus on the advantages of the “almost” rules that use more information, but to study when each of the “almost” rules is optimal.

In this paper we have focused on the special case of applying voting theory to the case of a firm that is in the process of hiring workers. However, our results are applicable and readily implementable in other areas requiring aggregation of decision makers’ signals, such

[8] However, a wider range of candidate abilities would improve the performance of both the classical and the “almost” rules since it reduces the likelihood of ties under both rules.

as policy making, project selection, and investment decisions.

Appendix A

The Seven Equivalent Classes of Scoring Rules

We examine the case where two candidates, say a and b, are tied with the same highest score. Let S_a^1 and S_a^2 denote the number of committee members who rank candidate a first and second, respectively, and similarly S_b^1 and S_b^2 the number of committee members who rank candidate b first and second, respectively. Since a and b are tied for the first place

$$S_a^1 \cdot 1 + S_a^2 \cdot \alpha = S_b^1 \cdot 1 + S_b^2 \cdot \alpha$$

$$\Rightarrow \quad S_a^1 - S_b^1 = (S_b^2 - S_a^2) \cdot \alpha$$

If $S_b^2 = S_a^2$, then $S_b^1 = S_a^1$, in which case either $S_b^2 = S_a^2 = 0$, which rules out the possibility of a tie, or $S_b^2 = S_a^2 = 1$, which, given that a and b are tied, implies cyclical preferences. In such a case, all rules yield a lottery between the three candidates.

If $S_b^2 \neq S_a^2$, then $\alpha = (S_a^1 - S_b^1)/(S_b^2 - S_a^2)$. Since $S_a^1, S_a^2, S_b^1, S_b^2 \in \{0, 1, 2, 3\}$, the possible values for α are $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$. The case of $\alpha = \frac{1}{3}$ is ruled out since a and b are then not selected. The case of $\alpha = 0$ corresponds to the plurality rule, the case of $\alpha = \frac{1}{2}$ to the Borda rule, the case of $\alpha = \frac{2}{3}$ to the two-third rule, and the case of $\alpha = 1$ to the inverse-plurality rule. The intermediate cases where $0 < \alpha < \frac{1}{2}$ correspond to the almost-plurality rule, where $\frac{1}{2} < \alpha < \frac{2}{3}$ to the almost-Borda rule, and where $\frac{2}{3} < \alpha < 1$ to the almost-inverse-plurality rule.

Appendix B

Derivation of q_1

To determine q_1, observe that a is ranked first in only the following cases:

- The committee member receives correct signals from all three candidates, which happens with probability xz^2.

- The committee member receives a correct signal from candidate a, and from only one of the candidates b and c, which happens with probability $2xz(1-z)$. The ranking between candidate a and the candidate from whom an incorrect signal was received is determined randomly. Hence, a is ranked first with probability $\frac{1}{2}\cdot 2xz(1-z) = xz(1-z)$.

- The committee member receives a correct signal from candidate a and incorrect signals from candidates b and c, which happens with probability $x(1-z)^2$. The ranking between candidate a and the other two candidates is determined randomly. Hence, a is ranked first with probability $\frac{1}{3}x(1-z)^2$.

- The committee member receives an incorrect signal from candidate a and correct signals from candidates b and c, which happens with probability $(1-x)z^2$. The ranking between candidate a and the other two candidates is determined randomly. Hence, a is ranked first with probability $\frac{1}{3}(1-x)z^2$.

Accordingly, q_1 is given by

$$\begin{aligned} q_1 &= xz^2 + xz(1-z) + \tfrac{1}{3}x(1-z)^2 + \tfrac{1}{3}(1-x)z^2 \\ &= \tfrac{1}{3}(xz + x + z^2). \end{aligned}$$

Derivation of q_2

To determine q_2, observe that a is ranked second in only the following cases:

- The committee member receives a correct signal from candidate a, and from only one of the candidates b and c, which happens with probability $2xz(1-z)$. The ranking between candidate a and the candidate from whom an incorrect signal was received is determined randomly. Hence, a is ranked second with probability $\frac{1}{2} \cdot 2xz(1-z) = xz(1-z)$.

- The committee member receives a correct signal from candidate a and incorrect signals from candidates b and c, which happens with probability $x(1-z)^2$. The ranking between candidate a and the other two candidates is determined randomly. Hence, a is ranked second with probability $\frac{1}{3}x(1-z)^2$.

- The committee member receives an incorrect signal from candidate a and correct signals from candidates b and c, which happens with probability $(1-x)z^2$. The ranking between candidate a and the other two candidates is determined randomly. Hence, a is ranked second with probability $\frac{1}{3}(1-x)z^2$.

- The committee member receives an incorrect signal from candidate a, and a correct signal from only one of candidates b and c, which happens with probability $2(1-x)z(1-z)$. The ranking between candidate a and the candidate from whom a correct signal was received is determined randomly. Hence, a is ranked second with probability $\frac{1}{2} \cdot 2(1-x)z(1-z) = (1-x)z(1-z)$.

It follows that q_2 is given by

$$\begin{aligned} q_2 &= xz(1-z) + \tfrac{1}{3}x(1-z)^2 + \tfrac{1}{3}(1-x)z^2 + (1-x)z(1-z) \\ &= \tfrac{1}{3}(-2xz + x - 2z^2 + 3z). \end{aligned}$$

Derivation of q_3

Finally, to determine q_3, observe that a is ranked third in only the following cases:

- The committee member receives a correct signal from candidate a and incorrect signals from candidates b and c, which happens with probability $x(1-z)^2$. The ranking between candidate a and the other two candidates is determined randomly. Hence, a is ranked third with probability $\frac{1}{3}x(1-z)^2$.

- The committee member receives an incorrect signal from candidate a and correct signals from candidates b and c, which happens with probability $(1-x)z^2$. The ranking between candidate a and the other two candidates is determined randomly. Hence, a is ranked third with probability $\frac{1}{3}(1-x)z^2$.

- The committee member receives an incorrect signal from candidate a, and a correct signal from only one of candidates b and c, which happens with probability $2(1-x)z(1-z)$. The ranking between candidate a and the candidate from whom a correct signal was received is determined randomly. Hence, a is ranked third with probability $\frac{1}{2} \cdot 2(1-x)z(1-z) = (1-x)z(1-z)$.

- The committee member receives incorrect signals from all candidates, Hence, a is ranked third with probability $(1-x)(1-z)^2$.

Thus, q_3 is given by

$$\begin{aligned} q_3 &= \tfrac{1}{3}x(1-z)^2 + \tfrac{1}{3}(1-x)z^2 + (1-x)z(1-z) + (1-x)(1-z)^2 \\ &= \tfrac{1}{3}(-2x + xz + z^2 - 3z + 3). \end{aligned}$$

Appendix C

The Relationship Between $Z(\cdot)$ and x

Let $Z(\alpha_i, \alpha_{i'}, j, x)$, where $i \neq i'$, denote the value of z for which the firm is indifferent between the rules for which the scores are $(1, \alpha_i, 0)$ and $(1, \alpha_{i'}, 0)$, given that there are $j \in \{1, 2\}$ high-ability candidates. Formally, $Z(\alpha_i, \alpha_{i'}, j, x)$ is defined by

$$M[\alpha_i, j, x, Z(\alpha_i, \alpha_{i'}, j, x)] = M[\alpha_{i'}, j, x, Z(\alpha_i, \alpha_{i'}, j, x)].$$

Lemma 1: *$Z(\alpha_{ap}, \alpha_{aB}, 1, x)$ is an increasing and strictly convex function of x.*

Proof: For the sake of simplicity, in the following we omit the arguments of $Z(\alpha_i, \alpha_{i'}, j, x)$. We have that

$$\begin{aligned} & M(\alpha_{ap}, 1, x, Z) = M(\alpha_{aB}, 1, x, Z) \\ \Leftrightarrow \quad & \tfrac{1}{12}(x + Z - 1)(x + xZ + Z^2)(-x + xZ + 2Z^2 - Z^3 - xZ^2) = 0. \end{aligned}$$

By setting $-x + xZ + 2Z^2 - Z - xZ^2 = 0$, we obtain

$$x = \frac{2Z^2 - Z^3}{1 - Z + Z^2}.$$

Hence,

$$\frac{dx}{dZ} = \frac{(Z^2 - Z + 4)(1 - Z)Z}{(Z^2 - Z + 1)^2},$$

which is positive, and

$$\frac{d^2x}{dZ^2} = \frac{2(Z + 1)(2Z - 1)(Z - 2)}{(Z^2 - Z + 1)^3},$$

which is negative. Therefore, $\lim_{Z \to 1/2} x = \frac{1}{2}$ and $\lim_{Z \to 1} x = 1$, so that x is an increasing strictly concave function of Z. Hence, Z (and the values of z for which $z = Z$) is an increasing and strictly convex function of x. $\square$

Lemma 2: *$Z(\alpha_{ap}, \alpha_{aB}, 2, x)$ is an increasing and strictly concave function of x.*

Proof: We have that

$$
\begin{aligned}
& M(\alpha_{ap}, 2, x, Z) = M(\alpha_{aB}, 2, x, Z) \\
\Leftrightarrow \quad & \tfrac{1}{12}\left(-3x - 2Z + xZ + x^2 + 3\right)\left(Z + x^2 Z - xZ - 2x^2 + x^3\right)(x + Z - 1) = 0 \\
\Leftrightarrow \quad & \left(-3x - 2Z + xZ + x^2 + 3\right)\left(Z + x^2 Z - xZ - 2x^2 + x^3\right) = 0 \\
\Leftrightarrow \quad & Z + x^2 Z - xZ - 2x^2 + x^3 = 0 \\
\Leftrightarrow \quad & Z = \frac{2x^2 - x^3}{1 - x + x^2},
\end{aligned}
$$

where we have used that $-3x - 2Z + xZ + x^2 + 3 > 0$ since its left-hand side decreases in x and converges to zero as x converges to 1.

Now, $d^2\left[(2x^2 - x^3)/(1 - x + x^2)\right]/dx^2$ has the same sign as $2x^3 - 3x^2 - 3x + 2$, which is negative since $2x^3 - 3x^2 - 3x + 2 = 0$ for $x = \frac{1}{2}$, and decreases in x. Hence, $M(\alpha_{ap}, 2, x, Z) = M(\alpha_{aB}, 2, x, Z)$ traces Z (and the values of z for which $z = Z$) as an increasing and strictly concave function of x. $\square$

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Table 1: The Probability That a High-Ability Candidate Is Selected

Ranking Profile	No. of Profiles	Probability	Scoring Rules						
			p	ap	B	aB	tt	aip	ip
{*abc*, *abc*, *abc*}	1	$\frac{1}{8}q_1^3$	1	1	1	1	1	1	$\frac{1}{2}$
{*abc*, *abc*, *acb*}	3	$\frac{1}{8}q_1^3$	1	1	1	1	1	1	1
{*abc*, *abc*, *bac*}	3	$\frac{1}{8}q_1^2q_2$	1	1	1	1	1	1	$\frac{1}{2}$
{*abc*, *acb*, *bac*}	6	$\frac{1}{8}q_1^2q_2$	1	1	1	1	1	1	1
{*acb*, *acb*, *bac*}	3	$\frac{1}{8}q_1^2q_2$	1	1	1	1	1	1	1
{*abc*, *abc*, *bca*}	3	$\frac{1}{8}q_1^2q_3$	1	1	$\frac{1}{2}$	0	0	0	0
{*abc*, *abc*, *cba*}	3	$\frac{1}{8}q_1^2q_3$	1	1	1	1	$\frac{1}{2}$	0	0
{*abc*, *acb*, *bca*}	6	$\frac{1}{8}q_1^2q_3$	1	1	1	1	1	1	$\frac{1}{3}$
{*abc*, *bac*, *cab*}	6	$\frac{1}{8}q_1q_2^2$	$\frac{1}{3}$	1	1	1	1	1	1
{*acb*, *bac*, *cba*}	6	$\frac{1}{8}q_1q_2q_3$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
{*acb*, *bac*, *bac*}	3	$\frac{1}{8}q_1q_2^2$	0	0	$\frac{1}{2}$	1	1	1	1
{*bca*, *abc*, *cba*}	6	$\frac{1}{8}q_1q_3^2$	$\frac{1}{3}$	0	0	0	0	0	0
{*cba*, *abc*, *bac*}	6	$\frac{1}{8}q_1q_2q_3$	$\frac{1}{3}$	0	0	0	0	0	0
{*bac*, *cab*, *cab*}	3	$\frac{1}{8}q_2^3$	0	0	0	0	$\frac{1}{2}$	1	1
{*abc*, *bac*, *bac*}	3	$\frac{1}{8}q_1q_2^2$	0	0	0	0	0	0	$\frac{1}{2}$
{*bac*, *bac*, *bac*}	1	$\frac{1}{8}q_2^3$	0	0	0	0	0	0	$\frac{1}{2}$
{*abc*, *cab*, *cba*}	6	$\frac{1}{8}q_1q_2q_3$	0	0	0	0	0	0	$\frac{1}{3}$
(*bac*, *bca*, *cab*}	6	$\frac{1}{8}q_2^2q_3$	0	0	0	0	0	0	$\frac{1}{3}$

Note: Due to symmetry between candidates *b* and *c*, permutations of *b* and *c* are equi-probable and, for simplicity, not listed. These permutations are of course taken into account when calculating the probabilities in Table 2.

Legend:
p - plurality rule
ap - almost-plurality rule
B - Borda rule
aB - almost-Borda rule
tt - two-third rule
aip - almost-inverse-plurality rule
ip - inverse-plurality rule

Table 2: The Probability That a Sole High-Ability Candidate Is Selected

Scoring Rule	$M(\alpha,1,x,z)$ as a Function of q_1,q_2, and q_3	$M(\alpha,1,x,z)$ as a Function of x and z
Plurality	$\frac{1}{2}q_1\left(6q_1q_2+6q_1q_3+2q_2q_3+2q_1^2+q_2^2+q_3^2\right)$	$\frac{1}{18}\left(4x-x^2+4xz+4z^2-z^4-2xz^2-2x^2z-2xz^3-x^2z^2+3\right)\left(x+xz+z^2\right)$
Almost-Plurality	$\frac{1}{2}q_1\left(6q_1q_2+6q_1q_3+q_2q_3+2q_1^2+3q_2^2\right)$	$\frac{1}{18}\left(7x+3z+7xz-x^2+10z^2-9z^3+2z^4-14xz^2-5x^2z+4xz^3+2x^2z^2\right)$ $\left(x+xz+z^2\right)$
Borda	$\frac{1}{8}q_1\left(24q_1q_2+21q_1q_3+4q_2q_3+8q_1^2+15q_2^2\right)$	$\frac{1}{72}\left(25x+12z+34xz-x^2+46z^2-45z^3+11z^4-68xz^2-23x^2z+22xz^3+11x^2z^2\right)$ $\left(x+xz+z^2\right)$
Almost-Borda	$\frac{1}{4}q_1\left(12q_1q_2+9q_1q_3+2q_2q_3+4q_1^2+9q_2^2\right)$	$\frac{1}{36}\left(11x+6z+20xz+x^2+26z^2-27z^3+7z^4-40xz^2-13x^2z+14xz^3+7x^2z^2\right)$ $\left(x+xz+z^2\right)$
Two-Third	$\frac{1}{8}\left(4q_1q_2q_3+8q_1^3+3q_2^3+18q_1q_2^2+24q_1^2q_2+15q_1^2q_3\right)$	$\frac{1}{72}(12xz+19x^2+5x^3+39z^3-5z^4-15z^5+5z^6+107xz^2+68x^2z-52xz^3-27x^3z-$ $30xz^4+15xz^5-74x^2z^2-15x^2z^3+15x^2z^4+5x^3z^3)$
Almost-Inverse-Plurality	$\frac{1}{4}\left(2q_1q_2q_3+4q_1^3+3q_2^3+9q_1q_2^2+12q_1^2q_2+6q_1^2q_3\right)$	$\frac{1}{36}(8x^2-31z^4+4x^3+33z^3+6xz+12z^5-2z^6+64xz^2+37x^2z-71xz^3-$ $15x^3z+30xz^4-6xz^5-55x^2z^2+24x^2z^3+6x^3z^2-6x^2z^4-2x^3z^3)$
Inverse-Plurality	$\frac{1}{8}\left(q_1+q_2\right)\left(14q_1q_2+4q_1q_3+7q_1^2+7q_2^2\right)$	$\frac{z}{216}(216x^2-84x^3+390xz+11z^5-525xz^2-372x^2z+228xz^3+54x^3z-33xz^4+$ $195x^2z^2-33x^2z^3-11x^3z^2)$

Table 3: The Probability That One Out of Two High-Ability Candidates Is Selected

Scoring Rule	$M(\alpha,2,x,z)$ as a Function of r_1, r_2, r_3	$M(\alpha,2,x,z)$ as a Function of x and z
Plurality	$1-\frac{1}{2}r_1\left(6r_1r_2+6r_1r_3+2r_2r_3+2r_1^2+r_2^2+r_3^2\right)$	$\frac{1}{18}\left(x^2-3x-2z+xz+5\right)\left(3x+2z-xz-x^2\right)^2$
Almost-Plurality	$1-\frac{1}{2}r_1\left(6r_1r_2+6r_1r_3+r_2r_3+2r_1^2+3r_2^2\right)$	$\frac{1}{18}(6xz^3-41xz^2-47x^2z-8x^3z+21x^4z-6x^5z+8x^2z^2+3x^2z^3+15x^3z^2-2x^3z^3-6x^4z^2+51xz+27x^2-12x^3-10x^4+9x^5-2x^6+20z^2-8z^3)$
Borda	$1-\frac{1}{8}r_1\left(24r_1r_2+21r_1r_3+4r_2r_3+8r_1^2+15r_2^2\right)$	$\frac{1}{72}(9z-131xz^2-119x^2z+15xz^3-116x^3z+129x^4z-33x^5z-19x^2z^2+21x^2z^3+96x^3z^2-11x^3z^3-33x^4z^2+177xz+90x^2-3x^3-82x^4+54x^5-11x^6+65z^2-26z^3)$
Almost-Borda	$1-\frac{1}{4}r_1\left(12r_1r_2+9r_1r_3+2r_2r_3+4r_1^2+9r_2^2\right)$	$\frac{1}{36}(9z-49xz^2-25x^2z+3xz^3-100x^3z+87x^4z-21x^5z-35x^2z^2+15x^2z^3+66x^3z^2-7x^3z^3-21x^4z^2+75xz+36x^2+21x^3-62x^4+36x^5-7x^6+25z^2-10z^3)$
Two-Third	$1-\frac{1}{8}\left(4r_1r_2r_3+8r_1^3+3r_2^3+18r_1r_2^2+24r_1^2r_2+15r_1^2r_3\right)$	$\frac{1}{72}(27z-101xz^2-92x^2z+12xz^3-83x^3z+75x^4z-15x^5z-16x^2z^2+15x^2z^3+60x^3z^2-5x^3z^3-15x^4z^2+141xz+81x^2-3x^3-55x^4+30x^5-5x^6+38z^2-17z^3)$
Almost-Inverse-Plurality	$1-\frac{1}{4}\left(2r_1r_2r_3+4r_1^3+3r_2^3+9r_1r_2^2+12r_1^2r_2+6r_1^2r_3\right)$	$\frac{1}{36}(18z-52xz^2-67x^2z+9xz^3+17x^3z-12x^4z+6x^5z+19x^2z^2-6x^3z^2+2x^3z^3+6x^4z^2+66xz+45x^2-24x^3+7x^4-6x^5+2x^6+13z^2-7z^3+36)$
Inverse-Plurality	$1-\frac{1}{8}\left(r_1+r_2\right)\left(14r_1r_2+4r_1r_3+7r_1^2+7r_2^2\right)$	$\frac{1}{216}(9xz^3-66x-51z-102xz^2-54x^2z-150x^3z-3x^4z+33x^5z-78x^2z^2+21x^2z^3+30x^3z^2+11x^3z^3+33x^4z^2+225xz-54x^2+329x^3-195x^4+33x^5-11x^6+117z^2-41z^3+180)$

Note: Table 3 is obtained from Table 2 by replacing q_i with r_i, M with 1-M, x with 1-z, and z with 1-x.

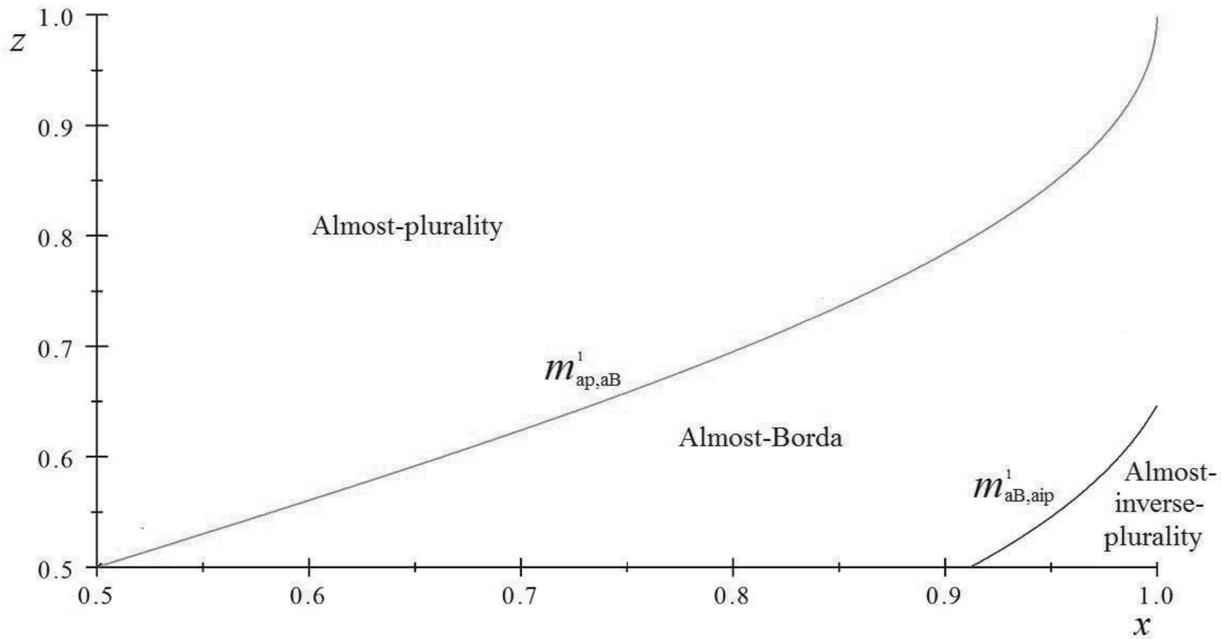

Figure 1: The preferred rule when there is one high-ability candidate

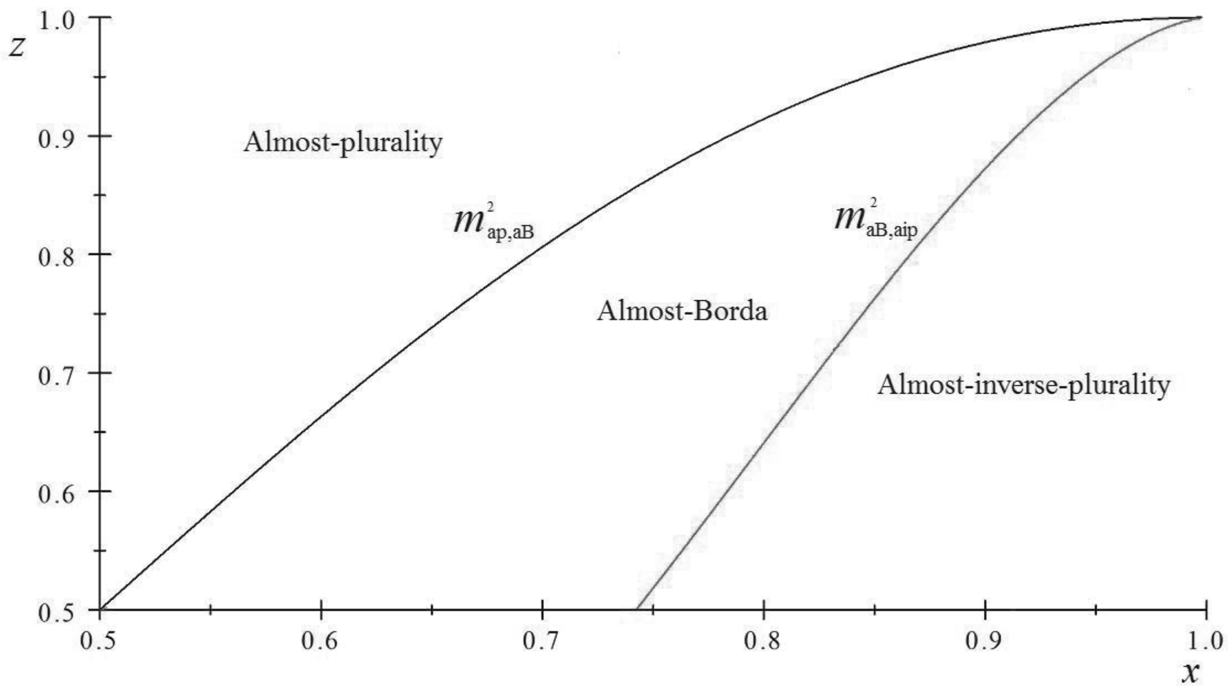

Figure 2: The preferred rule when there are two high-ability candidates

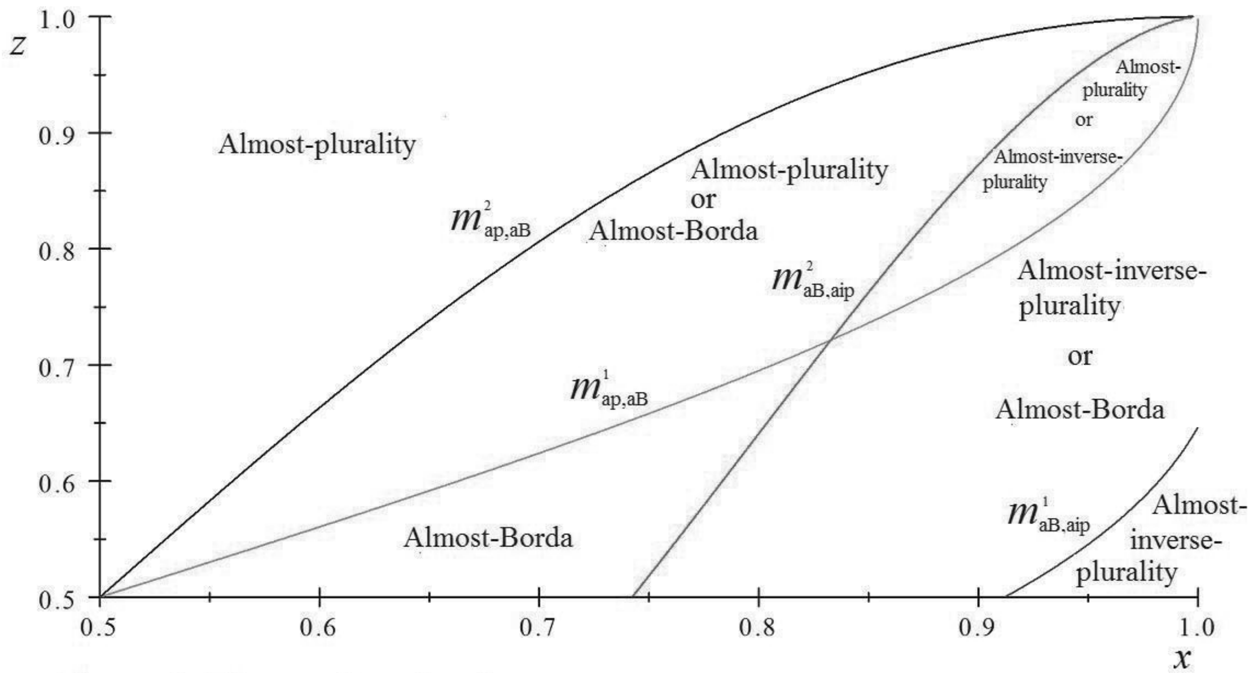

Figure 3: The preferred rule